On a stochastic wave equation in two space dimensions: regularity of the solution and its density

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Abstract

We pursue the investigation started in a recent paper by A. Millet & M. Sanz-Solé concerning a non-linear wave equation driven by a Gaussian noise white in time and correlated in the two-dimensional space variable. Under more restrictive conditions on the covariance function of the noise, we prove Hölder-regularity properties for both the solution and its density. For the latter, we adapt the method used in a paper by P.L. Morien, based on the Malliavin calculus.

Keywords and phrases: stochastic partial differential equation, wave equation, Gaussian noise, Malliavin calculus.

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1 Introduction

In this paper we study the stochastic wave equation with two-dimensional space variable:

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = \sigma(u(t,x))\, F(dt,dx) + b(u(t,x)), \quad t \in]0,+\infty[,\ x \in \mathbb{R}^2\,, \\ u(0,x) = u_0(x)\,, \\ \frac{\partial u}{\partial t}(0,x) = v_0(x)\,. \end{cases} \tag{1.1}$$

Here the noise $F(t,x)$ is assumed to be a generalized Gaussian field with covariance

$$\mathbb{E}(F(t,x)F(s,y)) = \delta_0(t-s)\ f(|x-y|), \tag{1.2}$$

where δ_0 denotes the Dirac delta function and $f :]0,+\infty[\to \mathbb{R}_+$ is continuous and satisfies some integrability condition on a neighbourhood of 0.

In [3], R. Dalang and N. Frangos proved that, if f is continuous, satisfies $\int_{0^+} r f(r) dr < \infty$ and, in addition, if the bilinear form $J : \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^2) \times \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^2) \to \mathbb{R}$ given by

$$J(\varphi,\psi) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy\, \varphi(t,x)\, \psi(t,y)\, f(|x-y|) \tag{1.3}$$

is non-negative definite, then such a Gaussian process $F(t,x)$ exists. Moreover, these authors obtained existence and uniqueness of the (local) solution of (1.1) up to a time t_0 (depending on f) assuming that σ and b are Lipschitz, that $u_0 = v_0 = 0$, and that f satisfies

$$\int_{0^+} r\, f(r)\, \ln\left(\frac{1}{r}\right)\, dr < \infty. \tag{1.4}$$

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Finally, they showed that if the following stronger integrability condition holds

$$(\mathbf{H}_\beta) \qquad \int_{0+} r^{1-\beta} f(r)\, dr < \infty,$$

for some $\beta \in]0,1[$, then $u(t,x)$ possesses a b-Hölder-continuous version for any $b < \frac{\beta}{4}$.

In [5], A. Millet and M. Sanz-Solé obtained a global result of existence and uniqueness for the solution of (1.1) under the same condition (1.4) for Lipschitz coefficients σ and b and assuming $u_0 \in C_b^1(\mathbb{R}^2)$ and $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$, for some $q_0 \in]2,+\infty[$.

In addition, under a condition slightly weaker than $(\mathbf{H}_\beta)$, Millet and Sanz-Solé [5] proved that, if u_0 has $\dfrac{\beta}{2(1+\beta)}$-Hölder-continuous partial derivatives and if $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$, for some $q_0 \in]4,+\infty[$, then the solution $u(t,x)$ admits a γ- Hölder-continuous modification for every $\gamma \in]0, \dfrac{\beta}{2(1+\beta)}[$. Finally, using a Malliavin calculus with respect to the Gaussian noise F, when σ, b are of class C^∞ with bounded derivatives, A. Millet and M. Sanz-Solé proved that, under a hypoellipticity condition on σ and certain (technical) hypotheses on u_0 and v_0, the law of $u(t,x)$ admits a density $p_{t,x}(y)$ which is smooth w.r.t y.

Note that a global existence and uniqueness result has since been proved - by means of the Fourier transform - for the (distribution-valued) solution of a stochastic semilinear wave equation in any dimension, using an infinite-dimensional calculus by S. Peszat and J. Zabczyk [9], and using martingale measures by R. Dalang [2].

We will deal with the wave equation in the plane. Let S denote the Green function of the deterministic equation $\left(\dfrac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = 0$. The arguments in [3] depend on upper estimates of the covariance of the Gaussian process

$$W(t,x) := \int_0^t\int_{\mathbb{R}^2} S(t-r, x-y)\, F(dr,dy)$$

and of the covariance of time and space increments of W. These estimates have been improved in [5] where lower estimates of the covariance of W is also proved.

The main purpose of the present paper is to investigate the Hölder regularity of the map $(t,x) \in [0,T] \times \mathbb{R}^2 \to p_{t,x}(y)$. The method is inspired by that of P.L. Morien [6], and is based on the Malliavin calculus with respect to the Gaussian noise F. It requires precise estimates of "disymmetric" integrals of the form $J(S(t,x), [S(t+h,y) - S(t,y)])$. Note that, unlike the case of parabolic SPDEs, the Hölder regularity of the density is better than that of the process, but not necessarily twice as good (depending on the property of the correlation function of the noise F).

As a by-product of the study of integrals like $J(S(t,x), [S(t+h,y) - S(t,y)])$, we improve the upper estimates of the covariance of increments of W, and thus improve the Hölder regularity of u with respect to that proved in [5].

The paper is organized as follows. In Section 2 we give the setting of our work and state the main results. In Section 3 we recall how the integrals defined by (A.10)-(A.13) enable to evaluate the Hölder regularity of $u(t,x)$. Section 4 is devoted to prove the Hölder regularity of the density, which is the main topic of the paper. Finally, an appendix gathers the results and proofs of technical lemmas which are used throughout the work.

2 General framework and statement of the results

Let $F(t,x)$ be a Gaussian centered noise on $\mathbb{R}_+ \times \mathbb{R}^2$ with covariance given by (1.2). We assume that the function $f :]0,+\infty[\longrightarrow \mathbb{R}_+$ is continuous and satisfies

$$\int_{0^+} r\, f(r)\, dr < \infty.$$

In addition, we suppose that the functional defined by (1.3) is nonnegative definite.

We shall say that condition $(\mathbf{H}_\beta)$ holds if

$$(\mathbf{H}_\beta) \qquad \int_{0^+} r^{1-\beta}\, f(r)\, dr < \infty \text{ for some } \beta > 0.$$

Consider the stochastic wave equation defined in (1.1). We assume that $\frac{\partial u}{\partial t}(0,x)$ is a measure with density $v_0(x)$. Following the method of Walsh [10], a natural way to give a rigorous meaning to (1.1) is in terms of the following evolution equation

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^2} S(t,x-y)\, v_0(y)\, dy + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^2} S(t,x-y)\, u_0(y)\, dy\right) \\ &\quad + \int_0^t\!\!\int_{\mathbb{R}^2} S(t-s,x-y)\,[\sigma(u(s,y))\, F(ds,dy) + b(u(s,y))\, dsdy]\,, \end{aligned} \tag{2.1}$$

where S is the fundamental solution of the deterministic wave equation associated with (1.1) and is given by

$$S(t,x) = \frac{1}{2\pi}\,(t^2 - |x|^2)^{-\frac{1}{2}}\, 1_{\{|x|<t\}}\,. \tag{2.2}$$

The map J can be extended to act on functions $\varphi : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ such that the right handside of (1.3) is defined. Notice that, if we set $\mathcal{F}_t = \sigma(F([0,s] \times A\,;\ 0 \le s \le t\,,\ A \in \mathcal{B}(\mathbb{R}^2))$ for any $t \ge 0$, then the stochastic integral in (2.1) is defined with respect to the $\mathcal{F}_t$-martingale measure

$$M_t(A) = F([0,t] \times A)\,,$$

for $t \ge 0$ and $A \in \mathcal{B}(\mathbb{R}^2)$ (see [10] for details). The following result has been obtained by A. Millet & M. Sanz-Solé in [5] (Theorem 1.2):

Theorem 2.1 (Existence and uniqueness) *Let $u_0 : \mathbb{R}^2 \to \mathbb{R}$ be of class C^1 and bounded, $v_0 : \mathbb{R}^2 \to \mathbb{R}$ be such that $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in]2,+\infty[$, and let σ, b be globally Lipschitz functions. We assume that the function f associated with the noise F satisfies*

$$\int_{0^+} r\, f(r)\, \ln\left(\frac{1}{r}\right)\, dr < +\infty.$$

Then equation (2.1) (and hence (1.1)) has a unique solution. Furthermore, for any fixed time $T > 0$ and $p \in [1,+\infty[$,

$$\sup_{x\in\mathbb{R}^2}\ \sup_{0\le t\le T}\ \mathbb{E}(|u(t,x)|^p) < \infty. \tag{2.3}$$

In the present work, we are first interested in the Hölder-regularity of $u(t,x)$ with respect to its parameters t and x. The problem has been addressed in [5] (Proposition 1.4); our result simply improves that proved in this previous work:

Theorem 2.2 *Assume that condition* $(\mathbf{H}_\beta)$ *holds for some* $\beta > 0$. *Let* $u_0 : \mathbb{R}^2 \to \mathbb{R}$ *be of class* C^1*, bounded, with* $\frac{1}{2}(\beta \wedge 1)$*-Hölder-continuous partial derivatives, and such that* $|\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$ *for some* $q_0 \in]2, +\infty]$*, let* $v_0 : \mathbb{R}^2 \to \mathbb{R}$ *be such that* $|v_0| \in L^{q_1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ *for some* $q_1 \in \left[\frac{2}{1-(\beta\wedge 1)}, +\infty\right[$ *and* $p \in]4, +\infty[$*, and let* σ*,* b *be globally Lipschitz functions. Then, for every compact set* $K \subset \mathbb{R}^2$*, the trajectories of* u *are* γ*-Hölder-continuous in* $(t, x) \in [0, T] \times K$ *for* $\gamma < \frac{1}{2}(\beta \wedge 1)$*.*

On the other hand, the following result has been obtained in [5] (Theorem 3.1):

Theorem 2.3 (Existence of a smooth density) *Fix* $t > 0$ *and pairwise different points* $x_1, \ldots, x_d$ *of* $\mathbb{R}^2$*. Let* $u(t, \bar{x}) := (u(t, x_1), \ldots, u(t, x_d))$*. Assume that*
(i) There exist $a_1 \geq a_2 > 0$ *such that* $2(1 + a_2)(a_1 - a_2) < a_2 \leq a_1 < 2$*, positive constants* C_1 *and* C_2 *such that for* $t \in [0, T]$*,*

$$C_1\, t^{a_1} \leq \int_0^t r\, f(r)\, \ln\left(1 + \frac{t}{r}\right) dr \leq C_2\, t^{a_2}. \tag{2.4}$$

(ii) $u_0 : \mathbb{R}^2 \to \mathbb{R}$ *is of class* C^1 *and bounded, with* $\frac{a_2}{2(1+a_2)}$*-Hölder-continuous partial derivatives,* $v_0 : \mathbb{R}^2 \to \mathbb{R}$ *be such that* $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$ *for some* $q_0 \in]4, +\infty[$ *.*
(iii) σ *and* b *are* C^∞ *with bounded derivatives of any order* $i \geq 1$*.*
(iv) There exists $\rho > 0$ *such that* $|\sigma(u(t, x_j))| \geq \rho$ *for any* $i = 1, \ldots, d$*.*
Then the law of the random vector $u(t, \bar{x})$ *admits a* C^∞ *density w.r.t. to the Lebesgue measure on* $\mathbb{R}^d$*.*

Remark 2.3: 1) Note that the proof of Theorem 3.1 in [5] depends on the $\frac{\beta}{1+\beta}$-Hölder regularity of the trajectories of $u(t, x)$. Since Theorem 2.2 refines this regularity, we can replace conditions (i) and (ii) in Theorem 2.3 by weaker ones as follows:

(i') There exist $0 < a_2 \leq a_1 < 2$ such that $2(a_1 - a_2) < (a_2 \wedge 1)$, $(\mathbf{H}_{\mathbf{a_2}})$ holds and

$$C_1\, t^{a_1} \leq \int_0^t r\, f(r)\, \ln\left(1 + \frac{t}{r}\right) dr\,, \quad t \in [0, T], \tag{2.5}$$

for some positive constant C_1.
(ii') $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1, bounded, with $\frac{1}{2}(a_2 \wedge 1)$-Hölder-continuous partial derivatives, such that $|\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in]2, +\infty]$, $v_0 : \mathbb{R}^2 \to \mathbb{R}$ is such that $|v_0| \in L^{q_1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for some $q_1 \in \left[\frac{2}{1-(a_2\wedge 1)}, +\infty\right]$ and $p \in]4, +\infty[$.

2) As will be shown in the sequel (cf. Lemma 4.3), if assumption (iv) of Theorem 2.3 is reinforced in

(iv") There exists a constant $\rho > 0$ such that $\sigma(r) \geq \rho$ for any $r \in \mathbb{R}$,

then conditions (i) and (ii) can be weakened as follows:

(i") There exist $0 < a_2 < a_1$ such that $a_1 < \inf(1 + 2a_2\,,\, 2 + a_2)$, $(\mathbf{H}_{\mathbf{a_2}})$ and (2.5) hold.
(ii") $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 and bounded, $v_0 : \mathbb{R}^2 \to \mathbb{R}$ is such that $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in]2, +\infty]$.

We are interested in the Hölder-regularity of the density $p_{t,x}(y)$ of $u(t,x)$ w.r.t. its parameters t and x. Such a problem has been addressed by P.L. Morien in [6] for a white-noise-driven parabolic equation in $\mathbb{R}$. The main result of the present paper is the following:

Theorem 2.4 *Assume that:*
(i) There exist $a_1 \geq a_2 > 0$ such that $a_1 < \inf(1+2a_2\,,\,2+a_2)$, $(\mathbf{H_{a_2}})$ and (2.5) hold.
(ii) $u_0 : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1, bounded, with $\frac{1}{2}(a_2 \wedge 1)$-Hölder-continuous partial derivatives and $|\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in]2,+\infty]$, $v_0 : \mathbb{R}^2 \to \mathbb{R}$ is such that $|v_0| \in L^{q_1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for some $q_1 \in \left[\dfrac{2}{(1-2a_2)^+}, +\infty\right[$ and $p \in]4,+\infty[$, (with the convention $1/0 = +\infty$).
(iii) σ and b are C^∞ with bounded derivatives of any order.
(iv) There exists $\rho > 0$ such that $|\sigma(r)| \geq \rho$ for any $r \in \mathbb{R}$.
Then, for every compact subset K of $\mathbb{R}^2$ and every $\tau > 0$ the density $p_{t,x}(y)$ of $u(t,x)$ is γ-Hölder-continuous in $(t,x) \in [\tau, T] \times K$, for $\gamma < a_2 \wedge \frac{1}{2}$.

We point out that, as in the case of parabolic SPDEs, the Hölder regularity of the density is twice that of the process if $a_2 \leq 1/2$. However, if $a_2 > 1/2$, the Hölder regularity of the density is still better than that of u, but no longer twice as good.

The next sections are devoted to the proofs of Theorems 2.2 and 2.4.

3 Proof of Theorem 2.2

We refer to the proof of Proposition 1.4 in [5] for the notations and several inequalities. Our contribution is the improvement of the upper inequalities of the integrals (A.10)-(A.13) defined in the appendix. In order to be self-contained, we recall here the main features of the proof and concentrate on time-regularity (space-regularity is left to the reader). Let $0 \leq t \leq T$, $h > 0$ be such that $t+h \leq T$, $x \in K$. We write

$$\mathbb{E}|u(t+h,x) - u(t,x)|^p \leq C_p \sum_{i=1}^{4} |R_i|^p,$$

where

$$R_1 = \int_{\mathbb{R}^2} (S(t+h, x-y) - S(t, x-y))\, v_0(y)\, dy\,, \tag{3.1}$$

$$R_2 = \frac{\partial}{\partial s}\left(\int_{\mathbb{R}^2} S(s, x-y)\, u_0(y)\, dy\right)_{s=t+h} - \frac{\partial}{\partial s}\left(\int_{\mathbb{R}^2} S(s, x-y)\, u_0(y)\, dy\right)_{s=t}, \tag{3.2}$$

$$R_3 = \int_0^{t+h}\int_{\mathbb{R}^2} S(t+h-s, x-y)\,\sigma(u(s,y))\, F(ds,dy) - \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y)\,\sigma(u(s,y))\, F(ds,dy), \tag{3.3}$$

$$R_4 = \int_0^{t+h}\int_{\mathbb{R}^2} S(t+h-s, x-y)\, b(u(s,y))\, dsdy - \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y)\, b(u(s,y))\, dsdy. \tag{3.4}$$

Let q the conjugate exponent of q_0. The calculations in [5] yield

$$|R_1| \;\leq\; C\|v_0\|_{q_0} \cdot h^{(\frac{1}{q}-\frac{1}{2})}, \tag{3.5}$$

$$
\begin{aligned}
|R_2| &\leq Ch^{\frac{1}{2}(\beta\wedge 1)}, &(3.6)\\
\mathbb{E}|R_3|^p &\leq C_p\{h^{\frac{p}{2}}+\mu_{t,h}^{\frac{p}{2}}+\tilde{\mu}_{t,h}^{\frac{p}{2}}\}, &(3.7)\\
\mathbb{E}|R_4|^p &\leq C_p(\nu_{t,h}^p+\tilde{\nu}_{t,h}^p)\leq C_p h^{\frac{p}{2}}, &(3.8)
\end{aligned}
$$

where $\mu_{t,h}$, $\tilde{\mu}_{t,h}$, $\nu_{t,h}$ and $\tilde{\nu}_{t,h}$ are the integrals defined by (A.10), (A.11), (A.5) and (A.6). Then it only remains to apply Lemmas A.2 and A.6 *via* Remark A.6 to get the desired result, provided $\frac{1}{q}-\frac{1}{2}\geq\frac{1}{2}(\beta\wedge 1)$, i.e., $q_0\geq\frac{2}{1-(\beta\wedge 1)}$. □

4 Proof of Theorem 2.4

Let $\mathcal{E}$ denote the inner product space of measurable functions $\varphi:\mathbb{R}^2\longrightarrow\mathbb{R}$ such that

$$
\int_{\mathbb{R}^2}dx\int_{\mathbb{R}^2}dy\,\varphi(x)\,f(|x-y|)\,\varphi(y)<\infty
$$

endowed with the inner product

$$
\langle\varphi,\psi\rangle_{\mathcal{E}}=\int_{\mathbb{R}^2}dx\int_{\mathbb{R}^2}dy\,\varphi(x)\,f(|x-y|)\,\psi(y)\,,
$$

and let $\mathcal{H}$ denote the completion of $\mathcal{E}$. Set $\mathcal{H}_T=L^2([0,T];\mathcal{H})$; then $\mathcal{H}_T$ is a separable Hilbert space isomorphic to the reproducing kernel Hilbert space of the Gaussian noise F, and F can be identified with a Gaussian process $(W(h)\,;\,h\in\mathcal{H}_T)$ as follows. Let $(e_j,j\in\mathbb{N})\subset\mathcal{E}$ be a complete orthonormal basis of the Hilbert space $\mathcal{H}$. Then

$$
W_j(t)=\int_0^t\int_{\mathbb{R}^2}e_j(x)\,F(ds,dx)\,,\quad j\in\mathbb{N}\,,\ t\in[0,T]\,, \tag{4.1}
$$

is a sequence of independent Brownian motions such that

$$
F(\varphi)=\sum_{j\geq 0}\int_0^T\langle\varphi(s,.)\,,\,e_j\rangle_{\mathcal{H}}\,dW_j(s)\,,\ \varphi\in\mathcal{D}([0,T]\times\mathbb{R}^2)\,.
$$

For $h\in\mathcal{H}_T$, we set

$$
W(h)=\sum_{j\geq 0}\int_0^T\langle h(s)\,,\,e_j\rangle_{\mathcal{H}}\,dW_j(s)\,. \tag{4.2}
$$

Therefore, we can use the framework of the Malliavin calculus described in [8] and [7], and we use these references for all questions concerning this stochastic calculus of variations.

For $h\in\mathcal{H}_T$, set $D_hX=\langle DX\,,\,h\rangle_{\mathcal{H}_T}$ and for $r\in[0,T]$, $\varphi\in\mathcal{H}$, set $D_{r,\varphi}=\langle D_{r,\bullet}\,,\,\varphi\rangle_{\mathcal{H}}$, where $D_{r,\bullet}:=DX(r)\in\mathcal{H}$.

Our result will be deduced from the following proposition:

Proposition 4.1 *Suppose that the assumptions of Theorem 2.4 are satisfied. Let $\tau\in]0,T[$ and K be a compact subset of $\mathbb{R}^2$. There exists some real number $C_{\tau,K}$ such that for all $t\in[\tau,T]$, $x\in K$ and $h>0$, $\xi\in\mathbb{R}^2$ satisfying $t+h\leq T$ and $x-\xi\in K$, for all $g\in C_b^3(\mathbb{R})\cap L^1(\mathbb{R})$ and $\Phi_g(s)=\int_{-\infty}^s g(r)\,dr$, we have for $\gamma<a_2\wedge\frac{1}{2}$,*

$$
\begin{aligned}
|\mathbb{E}[\,g(u(t+h,x))-g(u,t,x))\,]| &\leq C_{\tau,K}\|\Phi_g\|_\infty h^\gamma\,, &(4.3)\\
|\mathbb{E}[\,g(u(t,x-\xi))-g(u,t,x))\,]| &\leq C_{\tau,K}\|\Phi_g\|_\infty|\xi|^\gamma\,. &(4.4)
\end{aligned}
$$

Suppose that Proposition 4.1 is proved. Let then $\{g_{n,y}\}_n$ be a sequence of Gaussian kernels of mean y and whose variance decreases towards zero as n increases and set $\Phi_{n,y} = \Phi_{g_{n,y}}$; then $\sup_{n,y} \|\Phi_{n,y}\|_\infty = 1$, and on the other hand: $\forall (t,x)$, $\lim_{n\to\infty} \mathbb{E}[g_{n,y}(u(t,x))] = p_{t,x}(y)$. Hence, taking limits in (4.3) and (4.4), we get the result of Theorem 2.4.

Since Itô's formula cannot be used for the evolution equation (2.1), a natural tool to obtain estimates (4.3) and (4.4) is a Taylor expansion with integral remainder. Terms involving g, g', g'' will then appear, that will have to be replaced by other ones involving Φ_g. A natural way to do so is the following integration-by-parts formula (in the sense of Malliavin calculus): for any random variable G, set

$$\|G\|_{N,p} = \left[\mathbb{E}(\,|G|^p\,) + \sum_{i=1}^{N} \mathbb{E}\left(\|D^i G\|^p_{\mathcal{H}_T^{\otimes i}}\right)\right]^{\frac{1}{p}} .$$

Then

$$\mathbb{E}\,(U\,g(X)) = \mathbb{E}\left(\Phi_g(X)\,\delta\left(\frac{U\cdot DX}{\|DX\|^2}\right)\right), \tag{4.5}$$

where δ denotes the Skorohod integral. Of course, such a method is relevant if and only if there is no loss of regularity with respect to the time increment h along the way. This is ensured by the following result, which can be found in P.L. Morien [6] (Corollary 4.2).

Proposition 4.2 *Let g be in $C_b^{r_0}(\mathbb{R})$ and Φ_g be an antiderivative of g. For $r_0 \le 3$, if Z and ξ are random variables such that $Z \in \mathbb{D}^\infty$ and ξ satisfies*

$$\xi \in \mathbb{D}^\infty,\ \ \|D\xi\|^{-1}_{\mathcal{H}_T} \in \bigcap_{p\ge 2} L^p(\Omega)\,. \tag{4.6}$$

Then, for $r \le r_0$

$$\left|\mathbb{E}[g^{(r)}(\xi)\,Z]\right| \le C_r\,\|\Phi_g\|_\infty\,\|Z\|_{r+1,2^{2r+2}}\,,$$

where $C_r = K_r\left\{\|\xi\|_{2(r+1),2^{2r+2}} + \mathbb{E}\left(\|D\xi\|^{-\kappa_r}_{\mathcal{H}_T}\right)\right\}$, with a constant K_r and an integer κ_r depending only on r.

Theorem 2.2 in [5] shows that for any $p \in [1,+\infty[$, $N \ge 1$,

$$\sup_{x\in\mathbb{R}^2}\sup_{t\in[0,T]} \|D^N u(t,x)\|_{L^p(\Omega,\mathcal{H}_T^{\otimes N})} < \infty\,, \tag{4.7}$$

and the proof of Theorem 3.1 in [5] shows that for every $t \ge 0$, $x \in \mathbb{R}^2$, $u(t,x)$ satisfies the classical criterion

$$\|Du(t,x)\|^{-1}_{\mathcal{H}_T} \in \bigcap_{2\le p<\infty} L^p(\Omega). \tag{4.8}$$

However, we need Proposition 4.2 to provide estimates which are uniform in $(t,x) \in [\tau,T]\times K$. Therefore we prove a version of Theorem 3.1 in [5] which is uniform in (t,x): the strict ellipticity of σ allows us to weaken the conditions on u_0, v_0, since the Hölder regularity of u is no longer used in the proof.

Lemma 4.3 *Let u_0, v_0 satisfy the assumptions of Theorem 2.4. For any $\tau \in]0,T]$, $p \in [2,+\infty[$, there exists a constant $C_{p,\tau}$ such that*

$$\sup_{\tau\le t\le T}\sup_{x\in\mathbb{R}^2} \mathbb{E}(\|Du(t,x)\|^{-p}_{\mathcal{H}_T}) \le C_{p,\tau} < \infty. \tag{4.9}$$

The proof of Lemma 4.3 uses the following result proved in [5], Theorem 2.2.

Lemma 4.4 *Let u_0, v_0 satisfy the assumptions of Theorem 2.4. For every $p \in [1,+\infty[$, there exists a constant C_p such that for $0 \le s \le t \le T$,*

$$\sup_{s\le\rho\le t}\sup_{x\in\mathbb{R}^2} \mathbb{E}\Big(\Big|\int_s^t \|D_{r,\bullet}u(\rho,x)\|^2_{\mathcal{H}}\,dr\Big|^p\Big) \le C_p\,\mu^p_{t-s}\,, \tag{4.10}$$

where μ_t is defined by (A.2).

Proof of Lemma 4.3: As usual (see e.g. [7], Lemma 2.33), this reduces to checking that for all $\tau \in]0,T]$, there exists $\lambda > 0$ such that for all $q \in [1,+\infty[$, there exists $\varepsilon_0(\tau,q)$ for which if $0 < \varepsilon \le \varepsilon_0(\tau,q)$,

$$P(\varepsilon) = \sup_{\tau\le t\le T}\sup_{x\in\mathbb{R}^2} \mathbb{P}(\|Du(t,x)\|^2_{\mathcal{H}_T} \le \varepsilon) \le C_q\,\varepsilon^{\lambda q}. \tag{4.11}$$

We proceed as in the proof of Theorem 3.1 in [5] (see also [1]), and include the argument for the sake of completeness. To simplify the notations, for φ, $\psi : [0,T]\times\mathbb{N}\to\mathbb{R}$, set

$$\langle\varphi\,,\,\psi\rangle_{\mathcal{H}(t,\varepsilon^\delta)} := \int_{t-\varepsilon^\delta}^t \langle\varphi(r)\,,\,\psi(r)\rangle_{\mathcal{H}}\,dr\,, \qquad \|\varphi\|^2_{\mathcal{H}(t,\varepsilon^\delta)} = \langle\varphi\,,\,\varphi\rangle_{\mathcal{H}(t,\varepsilon^\delta)}\,.$$

For $t \ge \tau$ and ε small enough ($\varepsilon^\delta < \tau$), $t-\varepsilon^\delta \ge 0$ and for $(t,x)\in[\tau,T]\times K$,

$$\|Du(t,x)\|^2_{\mathcal{H}_T} \ge \|Du(t,x)\|^2_{\mathcal{H}(t,\varepsilon^\delta)}\,.$$

Furthermore, the Malliavin derivative of u satisfies the following evolution equation (see [5], (2.9)): for $\varphi\in\mathcal{H}$ and $r\in[0,t]$,

$$\begin{aligned} D_{r,\varphi}u(t,x) &= \langle S(t-r,x-\bullet)\,\sigma(u(r,\bullet))\,,\,\varphi\rangle_{\mathcal{H}} + \int_r^t\int_{\mathbb{R}^2} S(t-s,x-y) \\ &\qquad \times D_{r,\varphi}u(s,y)\,[\,\sigma'(u(s,y)\,F(ds,dy) + b'(u(s,y))\,dsdy\,]\,, \end{aligned} \tag{4.12}$$

and $D_{r,\varphi}u(t,x) = 0$ for $r > t$. Since $\langle\cdot\,,\,\cdot\rangle_{\mathcal{H}(t,\varepsilon^\delta)}$ is a (possibly degenerate) positive bilinear form, for $\varepsilon \le \varepsilon_0(\tau)$, $t \ge \tau$,

$$\begin{aligned} \|Du(t,x)\|^2_{\mathcal{H}(t,\varepsilon^\delta)} &\ge \frac{1}{2}\|(S(t-\cdot,x-\bullet)\,\sigma(u(\cdot,\bullet))\|^2_{\mathcal{H}(t,\varepsilon^\delta)} \\ &\qquad -\|Du(t,x) - S(t-\cdot,x-\bullet)\,\sigma(u(\cdot,\bullet))\|^2_{\mathcal{H}(t,\varepsilon^\delta)}\,, \end{aligned}$$

so that $P(\varepsilon) \le P_1(\varepsilon) + P_2(\varepsilon)$, where

$$\begin{aligned} P_1(\varepsilon) &= \sup_{\tau\le t\le T}\sup_{x\in\mathbb{R}^2} \mathbb{P}(\|\,S(t-\cdot,x-\bullet)\,\sigma(u(\cdot,\bullet))\,\|^2_{\mathcal{H}(t,\varepsilon^\delta)} \le 4\varepsilon)\,, \\ P_2(\varepsilon) &= \sup_{\tau\le t\le T}\sup_{x\in\mathbb{R}^2} \mathbb{P}(\|Du(t,x) - S(t-\cdot,x-\bullet)\,\sigma(u(\cdot,\bullet))\,\|^2_{\mathcal{H}(t,\varepsilon^\delta)} \ge \varepsilon). \end{aligned}$$

Assumptions (i), (iv), along with (A.4) yield for $t-\varepsilon^\delta \ge \tau/2$, $x\in\mathbb{R}^2$:

$$\begin{aligned} &\|S(t-\cdot,x-\bullet)\,\sigma(u(\cdot,\bullet))\,\|^2_{\mathcal{H}(t,\varepsilon^\delta)} \\ &= \int_{t-\varepsilon^\delta}^t dr\int_{\mathbb{R}^2}dy\int_{\mathbb{R}^2}dz\,S(t-s,x-y)\,\sigma(u(s,y))\,f(|y-z|)\,S(t-s,x-z)\,\sigma(u(s,z)) \\ &\ge \rho^2\,\mu_{\varepsilon^\delta} \ge C_1\,\rho^2\,\varepsilon^{\delta(1+a_1)}. \end{aligned}$$

Thus, there exists $\varepsilon_0(\tau) > 0$ such that, for $0 < \varepsilon \le \varepsilon_0(\tau)$,

$$P_1(\varepsilon) = 0 \ \text{ if } \ \delta(1 + a_1) < 1 \,. \tag{4.13}$$

On the other hand, Chebychev's inequality implies that for $q \in [2, +\infty[$,

$$P_2(\varepsilon) \le C_q\, \varepsilon^{-q} \Big(T_{2,1}(\varepsilon) + T_{2,2}(\varepsilon) \Big) \,,$$

where

$$\begin{aligned} T_{2,1}(\varepsilon) &= \sup_{\tau \le t \le T} \sup_{x \in \mathbb{R}^2} \mathbb{E}\Big(\Big\| \int_r^t \int_{\mathbb{R}^2} S(t-s, x-y)\, \sigma'(u(s,y))\, Du(s,y)\, F(ds,dy) \Big\|_{\mathcal{H}(t,\varepsilon^\delta)}^{2q} \Big) \,, \\ T_{2,2}(\varepsilon) &= \sup_{\tau \le t \le T} \sup_{x \in \mathbb{R}^2} \mathbb{E}\Big(\Big\| \int_r^t \int_{\mathbb{R}^2} S(t-s, x-y)\, b'(u(s,y))\, Du(s,y)\, dsdy \Big\|_{\mathcal{H}(t,\varepsilon^\delta)}^{2q} \Big). \end{aligned}$$

Let $(\varphi_k\,,\, k \ge 0)$ be a CONS of $\mathcal{H}(t, \varepsilon^\delta)$; then since $D_{r,\bullet}u(s,y) = 0$ if $s < r$, the Burkholder-Davis-Gundy inequality for Hilbert-valued martingales (see Métivier, [4], E.2., p. 212), Hölder's inequality and Parseval's identity imply that for any $q \in [1, +\infty[$,

$$\begin{aligned} & T_{2,1}(\varepsilon) \\ & \le C_q \sup_{\tau \le t \le T} \sup_{x \in \mathbb{R}^2} \mathbb{E}\Big(\Big| \sum_{k \ge 0} \int_{t-\varepsilon^\delta}^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, S(t-s, x-y)\, \sigma'(u(s,y)) \\ & \qquad \times D_{\varphi_k} u(s,y)\, f(|y-z])\, S(t-s, x-z)\, \sigma'(u(s,z))\, D_{\varphi_k} u(s,z) \Big|^q \Big) \\ & \le C_q\, \|\sigma'\|_\infty^{2q} \sup_{\tau \le t \le T} \sup_{x \in \mathbb{R}^2} \mathbb{E}\Big(\Big| \int_{t-\varepsilon^\delta}^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, S(t-s, x-y)\, f(|y-z|) \\ & \qquad \times S(t-s, x-z)\, \langle Du(s,y)\,,\, Du(s,z) \rangle_{\mathcal{H}(t,\varepsilon^\delta)} \Big|^q \Big) \\ & \le C_q\, \mu_{\varepsilon^\delta}^{q-1}\, \|\sigma'\|_\infty^{2q} \sup_{\tau \le t \le T} \sup_{x \in \mathbb{R}^2} \int_{t-\varepsilon^\delta}^t ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} dz\, S(t-s, x-y)\, f(|y-z|) \\ & \qquad \times S(t-s, x-z) \Big\{ \mathbb{E}\|Du(s,y)\|_{\mathcal{H}(t,\varepsilon^\delta)}^{2q}\ \mathbb{E}\|Du(s,z)\|_{\mathcal{H}(t,\varepsilon^\delta)}^{2q} \Big\}^{\frac{1}{2}} . \end{aligned}$$

Thus, (4.10), assumption ($\mathbf{H_{a_2}}$) and (A.4) along with Remark (A.6) imply that for $0 < \varepsilon \le \varepsilon_0(\tau)$,

$$T_{2,1}(\varepsilon) \le C_q\, \mu_{\varepsilon^\delta}^{2q} \le C_q\, \varepsilon^{2q\delta(1+a_2)}. \tag{4.14}$$

A similar (easier) computation yields for $0 < \varepsilon \le \varepsilon_0(\tau)$

$$\begin{aligned} T_{2,2}(\varepsilon) &\le C_q \left(\int_{t-\varepsilon^\delta}^t ds \int_{\mathbb{R}^2} dy\, S(t-s, x-y) \right)^q \mu_{\varepsilon^\delta}^q \\ &\le C_q\, \varepsilon^{2q\delta}\, \varepsilon^{\delta q(1+a_2)} \,. \end{aligned} \tag{4.15}$$

Hence, for $0 < \varepsilon \le \varepsilon_0(\tau)$, (4.14) and (4.15) imply

$$P_2(\varepsilon) \le C_q \varepsilon^{-q} \Big\{ \varepsilon^{2q\delta(1+a_2)} + \varepsilon^{\delta q(3+a_2)} \Big\} \,. \tag{4.16}$$

Using (4.13) and (4.16), we conclude that (4.11) holds if one can find $\delta > 0$ such that

$$\begin{cases} \delta(1 + a_1) < 1, \\ 2\delta(1 + a_2) > 1, \delta(3 + a_2) > 1. \end{cases}$$

Assumption (i) ensures that these constraints can be fulfilled. □

In the sequel, we prove (4.3) (the proof of (4.4), which is similar, is left to the reader).

Assume $t \in [\tau, T]$, $x \in K$ and $h > 0$ with $t + h \le T$. Taylor's expansion of g around $u(t,x)$ and Fubini's theorem imply: $\mathbb{E}[g(u(t+h,x)) - g(u(t,x))] := T_1 + T_2$, where

$$\begin{aligned} T_1 &= \mathbb{E}[g'(u(t,x))\,(u(t+h,x) - u(t,x))], \\ T_2 &= \int_0^1 (1-v)\mathbb{E}\left[(u(t+h,x) - u(t,x))^2 g''(Y_{t,x}(h,v))\right] dv, \end{aligned}$$

and $Y_{t,x}(h,v) = u(t,x) + v(u(t+h,x) - u(t,x))$.

4.1 Bound for T_1

Let R_1 and R_2 be defined as in (3.1) and (3.2); then $T_1 = \sum_{i=1}^5 T_{1i}$, where:

$$\begin{aligned} T_{11} &= \mathbb{E}\left[g'(u(t,x))\,R_1\right], \\ T_{12} &= \mathbb{E}\left[g'(u(t,x))\,R_2\right], \\ T_{13} &= \mathbb{E}\Big[g'(u(t,x))\Big(\int_0^t\int_{\mathbb{R}^2}[S(t+h-s,x-y) - S(t-s,x-y)]\,b(u(s,y))\,dyds \\ &\qquad + \int_t^{t+h}\int_{\mathbb{R}^2} S(t+h-s,x-y)\,b(u(s,y))\,dyds\Big)\Big], \\ T_{14} &= \mathbb{E}\Big[g'(u(t,x))\int_0^t\int_{\mathbb{R}^2}[S(t+h-s,x-y) - S(t-s,x-y)]\,\sigma(u(s,y))\,F(dy,ds)\Big], \\ T_{15} &= \mathbb{E}\Big[g'(u(t,x))\int_t^{t+h}\int_{\mathbb{R}^2} S(t+h-s,x-y)\,\sigma(u(s,y))\,F(dy,ds)\Big]. \end{aligned}$$

It is clear that $T_{15} = 0$ since $g'(u(t,x))$ is $\mathcal{F}_t$-measurable and

$$M_u = \int_0^u\int_{\mathbb{R}^2} S(t+h-s,x-y)\sigma(u(s,y))F(dy,ds), \quad u \le t+h,$$

is an L^2-bounded $\mathcal{F}_u$-martingale.

If $a_2 < 1/2$, the condition $a_2 \wedge \frac{1}{2} \le \frac{1}{q} - \frac{1}{2}$ is equivalent to $q_0 \ge \frac{2}{1-2a_2}$, while if $a_2 \ge 1/2$ it is equivalent to $q_0 = +\infty$. In either case, (3.5) implies $|R_1|^p \le C\|v_0\|_{q_0}^p\, h^{p(a_2\wedge\frac{1}{2})}$. Therefore Proposition 4.2 applied with $Z = \int_{\mathbb{R}^2}(S(t+h,x-y) - S(t,x-y))v_0(y)dy$ and $\xi = u(t,x)$ yields $T_{11} \le C_\tau\|\Phi_g\|_\infty h^{a_2\wedge\frac{1}{2}}$. (Indeed, Z being deterministic, its Malliavin derivative is zero and (4.7) and (4.8) ensure that $u(t,x)$ satisfies (4.6).)

The term T_{12} is similar: using (3.6) and (ii) we get $|R_2|^p \le Ch^{p(a_2\wedge\frac{1}{2})}$. Since R_2 is deterministic, Proposition 4.2 then yields $|T_{12}| \le C_\tau\|\Phi_g\|_\infty\, h^{a_2\wedge\frac{1}{2}}$.

On the other hand, if R_4 is defined as in (3.4)

$$T_{13} = \mathbb{E}[g'(u(t,x))\,R_4],$$

and (3.8) yields $\mathbb{E}(|R_4|)^p \le C\,h^{p/2}$. Furthermore, (4.7), (4.9) and the upper estimate (A.7) of $\nu_{t,h} + \tilde{\nu}_{t,h}$ clearly imply that $\|R_4\|_{2,p} \le Ch^{1/2}$ for any $p \ge 2$. Therefore, Proposition 4.2 yields $|T_{13}| \le C_\tau\|\Phi_g\|_\infty h^{1/2}$.

We finally turn to T_{14} which is the most difficult term because of the stochastic integral, main source of loss of regularity. Since the integrand is adapted, this term is a Skorohod integral and the duality formula between δ and D yields:

$$\begin{aligned} T_{14} &= \mathbb{E}\Big[\langle D(g'(u(t,x))\,,\, [S(t+h-\cdot, x-\bullet) - S(t-\cdot, x-\bullet))]\,\sigma(u(\cdot,\bullet)\rangle_{\mathcal{H}_T}\Big] \\ &= \mathbb{E}\left[g''(u(t,x))\,\langle Du(t,x)\,,\, B(t,x)\rangle_{\mathcal{H}_T}\right]\,, \end{aligned}$$

where for $\tau \le \bar{t}$, $\bar{x} \in \mathbb{R}^2$,

$$B(\bar{t},\bar{x}) = [S(\bar{t}+h-\cdot, \bar{x}-\bullet) - S(\bar{t}-\cdot, \bar{x}-\bullet)]\,\sigma(u(\cdot,\bullet)) \in \mathcal{H}_T\,.$$

Recall that $Du(t,x)$ satisfies the evolution equation (4.12). Then Fubini's theorem (for deterministic and stochastic integrals) implies

$$\begin{aligned} \langle Du(t,x)\,,\, B(\bar{t},\bar{x})\rangle_{\mathcal{H}_T} &= \langle S(t-\cdot, x-\bullet)\,\sigma(u(\cdot,\bullet))\,,\, B(\bar{t},\bar{x})\rangle_{\mathcal{H}_T} \\ &\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s,x-y)\,\sigma'(u(s,y))\,\langle Du(s,y)\,,\, B(\bar{t},\bar{x})\,\rangle_{\mathcal{H}_T}\, F(ds,dy) \\ &\quad + \int_0^t ds \int_{\mathbb{R}^2} dy\, S(t-s,x-y)\, b'(u(s,y))\langle\, Du(s,y)\,,\, B(\bar{t},\bar{x})\,\rangle_{\mathcal{H}_T}\,. \end{aligned}$$

The Burkholder-Davis-Gundy inequality for Hilbert-valued martingales, Hölder's inequality applied to the measure $S(t-s,x-y)\, f(|y-z|)\, S(t-s,x-z)\, dydz$ (which is finite by Lemma A.1) and to the finite measure $S(t-s,x-y)\,dy$ yield for $p \in [2,+\infty[$,

$$\begin{aligned} \sup_{x\in\mathbb{R}^2} \mathbb{E}\left(|\langle Du(t,x)\,,\, B(\bar{t},\bar{x})\rangle_{\mathcal{H}_T}|^p\right) &\le C_p\,\mathbb{E}\Big(\,|\,\langle S(t-\cdot, x-\bullet)\,\sigma(u(\cdot,\bullet))\,,\, B(\bar{t},\bar{x})\rangle_{\mathcal{H}_T}\,|^p\Big) \\ &\quad + C_p \int_0^t \sup_{y\in\mathbb{R}^2} \mathbb{E}\left(|\langle Du(s,y), B(\bar{t}\,,\,\bar{x})\rangle_{\mathcal{H}_T}|^p\right)\, ds. \end{aligned}$$

Thus, Parseval's identity and Gronwall's lemma imply that there exists a constant C_p (independent of $\bar{t}$ and $\bar{x}$) such that for $p \in [2,+\infty[$

$$\begin{aligned} \sup_{x\in\mathbb{R}^2} \mathbb{E}\left(|\langle Du(t,x), B(\bar{t},\bar{x})\rangle_{\mathcal{H}_T}|^p\right) &\le C_p\mathbb{E}\Big(\Big|\int_0^t ds \int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2} dydz\, S(t-s,x-y)\,\sigma(u(s,y)) \\ &\quad \times f(|y-z|)\,[S(\bar{t}+h-r,\bar{x}-z) - S(\bar{t}-r,\bar{x}-z)]\,\sigma(u(s,z))\Big|^p\Big)\,. \end{aligned}$$

The linear growth property of σ, (2.3) and Hölder's inequality imply that if $A_{t,x}(h) := \langle Du(t,x)\,,\, B(t,x)\rangle_{\mathcal{H}_T}$, then

$$\mathbb{E}(|A_{t,x}(h)|^p) \le C_p\,|\Delta_{t,h}|^p\,, \tag{4.17}$$

where $\Delta_{t,h}$ is defined by (A.8). Therefore, (A.33) in Lemma A.6 implies that $\mathbb{E}(|A_{t,x}(h)|^p) \le C_p h^{p\gamma}$ with $\gamma < a_2 \wedge \frac{1}{2}$. The computations for the derivatives of $A_{t,x}(h)$, which are similar, are omitted. Proposition 4.2 then yields that for $\gamma < a_2 \wedge \frac{1}{2}$, one has $|T_{14}| \le C_{p,\tau}\|\Phi_g\|_\infty\, h^\gamma$.

We then conclude that, for $\gamma < a_2 \wedge \frac{1}{2}$,

$$|T_1| \le C_{p,\tau}\,\|\Phi_g\|_\infty\, h^\gamma\,. \tag{4.18}$$

4.2 Bound for T_2

Recall that

$$T_2 = \int_0^1 (1-v)\mathbb{E}\left[(u(t+h,x)-u(t,x))^2 g''(Y_{t,x}(h,v))\right] dv\,.$$

The method used in the proof of Theorem 2.2 along with (4.7) easily prove that for fixed $q \in \mathbb{N}$, $p \in [2,+\infty[$, one has

$$\|u(t+h,x)-u(t,x)\|_{q,p} \le Ch^\gamma, \qquad \gamma < \frac{1}{2}(a_2 \wedge 1).$$

Thus Schwarz's inequality implies that

$$\|(u(t+h,x)-u(t,x))^2\|_{q,p} \le Ch^\gamma, \qquad \gamma < a_2 \wedge 1. \tag{4.19}$$

On the other hand, the convexity of $\|\cdot\|^p_{\mathcal{H}_T}$ and $\|\cdot\|^{-p}_{\mathcal{H}_T}$, (4.7) and (4.9) imply that

$$\sup_{\tau \le t \le t+h \le T} \sup_{x\in\mathbb{R}^2} \sup_{v\in[0,1]} \mathbb{E}\left(\|DY_{t,x}(h,v)\|^p_{\mathcal{H}_T} + \|DY_{t,x}(h,v)\|^{-p}_{\mathcal{H}_T}\right) = C_{p,\tau} < \infty. \tag{4.20}$$

Therefore, we can apply Proposition 4.2 with $r=2$, $\xi = Y_{t,x}(h,v)$ and $Z = (u(t+h,x) - u(t,x))^2$, which yields

$$|T_2| \le C_{p,\tau}\, \|\Phi_g\|_\infty\, h^\gamma, \qquad \gamma < a_2 \wedge 1\,. \tag{4.21}$$

Finally, the upper estimates (4.18) and (4.21) show that for $t \in [\tau, T]$, $x \in K$, $\gamma < a_2 \wedge \frac{1}{2}$,

$$|E[g(u(t+h,x)) - g(u(t,x))]| \le C_{p,\tau}\, \|\Phi_g\|_\infty\, h^\gamma\,. \tag{4.22}$$

A Appendix

This section contains the proofs of various estimates on the fundamental solution S of the deterministic wave equation in the plane. The first result is proved in [5], Lemma A.1:

Lemma A.1 *For $s,t>0$, set*

$$J(s) := \int_{|y|<|x|<s} S(s,x)\, f(|x-y|)\, S(s,y)\, dxdy\,, \tag{A.1}$$

$$\mu_t := \int_0^t \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy S(s,x)\, f(|x-y|)\, S(s,y) = 2\int_0^t J(s) ds\,. \tag{A.2}$$

Then, for any fixed $T>0$, there exist positive constants $C_1(T)$, $C_2(T)$ such that:
(a) for every $0 \le s \le T$,

$$C_1(T) \int_0^s r\, f(r)\, \ln\left(1+\frac{s}{r}\right) dr \le J(s) \le C_2(T) \int_0^{2s} r\, f(r)\, \ln\left(1+\frac{s}{r}\right) dr\,. \tag{A.3}$$

(b) for all $0 \le t \le T$,

$$C_1(T)\, t \int_0^{\frac{t}{3}} r\, f(r)\, \ln\left(1+\frac{t}{r}\right) dr \le \mu_t \le C_2(T)\, t \int_0^{2t} r\, f(r)\, \ln\left(1+\frac{t}{r}\right) dr\,. \tag{A.4}$$

For every $t \in [0,T]$, set

$$\nu_{t,h} := \int_0^t ds \int_{|y|<s} dy\, [S(s,y) - S(s+h,y)]\,, \tag{A.5}$$

$$\tilde{\nu}_{t,h} := \int_0^t ds \int_{s<|y|<s+h} dy\, S(s+h,y)\,. \tag{A.6}$$

Then (cf. [5], (1.26)):

$$\nu_{t,h} + \tilde{\nu}_{t,h} \le h^{1/2}. \tag{A.7}$$

We now prove upper estimates for integrals of increments of the Green function S with respect to the function f defining the covariance structure of the Gaussian noise F. For $t \in [0,1]$, $h > 0$, $\xi \in \mathbb{R}^2$, set

$$\Delta_{t,h} := \int_0^t ds \iint_{\mathbb{R}^2\times\mathbb{R}^2} dydz\, S(s,y)\, f(|y-z|)\, |S(s,z) - S(s+h,z)|\,, \tag{A.8}$$

$$\Theta_{t,\xi} := \int_0^t ds \iint_{\mathbb{R}^2\times\mathbb{R}^2} dydz\, S(s,y)\, f(|y-z|)\, |S(s,z) - S(s,z-\xi)|\,, \tag{A.9}$$

$$\mu_{t,h} = \int_0^t ds \iint_{|y|\vee|z|<s} dydz\, [S(s,y) - S(s+h,y)]\, f(|y-z|)\, [S(s,z) - S(s+h,z)]\,, \tag{A.10}$$

$$\tilde{\mu}_{t,h} := \int_0^t ds \int_{s<|y|<s+h} dy \int_{s<|z|<s+h} dz\, S(s+h,y)\, f(|y-z|)\, S(s+h,z)\,, \tag{A.11}$$

$$M_{t,\xi} = \int_0^t ds \int_{\substack{|y|<s\\ |y-\xi|>s}} dy \int_{\substack{|z|<s\\ |z-\xi|>s}} dz\, S(s,y)\, f(|y-z|)\, S(s,z), \tag{A.12}$$

$$N_{t,\xi} = \int_0^t ds \iint_{\substack{|y|\vee|z|<s\\ |y-\xi|\vee|z-\xi|<s}} dydz\, |S(s,y) - S(s,y-\xi)|\, f(|y-z|)\, |S(s,z) - S(s,z-\xi)|\,. \tag{A.13}$$

In all what follows, we assume that the condition $(\mathbf{H}_\beta)$ holds for some $\beta > 0$. First, we improve the estimate of $\tilde{\mu}_{t,h}$ (resp. of $M_{t,\xi}$) proved in [5], Lemma A.5:

Lemma A.2 *For h small enough, $|\xi| = h$,*

$$\tilde{\mu}_{t,h} + M_{t,\xi} \le C h^{\tilde{\beta}}, \tag{A.14}$$

where

$$\begin{cases} \tilde{\beta} = \beta \text{ if } \beta < 1, \\ \tilde{\beta} < 1 \text{ if } \beta \ge 1\,. \end{cases} \tag{A.15}$$

Proof: First, we concentrate on $\tilde{\mu}_{t,h}$; Fubini's theorem and the change of variables $u = s+h$ imply

$$\tilde{\mu}_{t,h} = C \iint_{|y|-h<|z|<|y|<t+h} dydz f(|y-z|)\, I(h,y,z,t)\,,$$

where

$$I(h,y,z,t) = \int_{|y|\vee h}^{(|z|+h)\wedge(t+h)} \frac{du}{\sqrt{u^2-|y|^2}\,\sqrt{u^2-|z|^2}} \le \int_{|y|\vee h}^{|z|+h} \frac{du}{\sqrt{u^2-|y|^2}\,\sqrt{u^2-|z|^2}}.$$

The change of variables $u^2 = v$ and the identity (28) in [3] yield

$$\begin{aligned} I(h,y,z,t) &\leq \frac{C}{|y|}\ln\left(1+\frac{2[(|z|+h)^2-|y|^2]+2\sqrt{((|z|+h)^2-|y|^2)(((|z|+h)^2-|z|^2)}}{|y|^2-|z|^2}\right) \\ &\leq \frac{C}{|y|}\ln\left(1+\frac{Ch}{|y|^2-|z|^2}\right), \end{aligned}$$

since $|z|+h < |y|+h$ and $((|z|+h)^2-|z|^2) = h(2|z|+h) \leq Ch$. Thus

$$\tilde{\mu}_{t,h} \leq C \iint_{|y|-h<|z|<|y|<t+h} dydz \frac{f(|y-z|)}{|y|}\ln\left(1+\frac{Ch}{|y|^2-|z|^2}\right).$$

Set $y = (\rho\cos\theta_0, \rho\sin\theta_0)$, $y-z = (r\cos(\theta+\theta_0), r\sin(\theta+\theta_0))$, $v = \cos\theta - \frac{r}{2\rho}$. This change of variables yields

$$\tilde{\mu}_{t,h} = \int_0^{h^\gamma} d\rho \int_0^{2\rho} rf(r)dr \int_0^{1-\frac{r}{2\rho}} \ln\left(1+\frac{Ch}{\rho r v}\right) \frac{dv}{\sqrt{1-\left(\frac{r}{2\rho}+v\right)^2}}.$$

For $0<b<1$ and $x \geq 0$ one has $\ln(1+x) \leq Cx^b$ so that, if $b \leq \beta$, by Fubini's theorem

$$\begin{aligned} \tilde{\mu}_{t,h} &\leq Ch^b \int_0^{2(t+h)} r^{1-b}f(r)dr \int_{\frac{r}{2}}^{t+h} \rho^{-b}d\rho \int_0^{1-\frac{r}{2\rho}} \frac{dv}{v^b\left(1-\left(\frac{r}{2\rho}+v\right)\right)^{1/2}} \\ &\leq Ch^b(\tilde{\mu}^1_{t,h}+\tilde{\mu}^2_{t,h}), \end{aligned}$$

where

$$\begin{aligned} \tilde{\mu}^1_{t,h} &= \int_0^{2(t+h)} r^{1-b}f(r)dr \int_{r/2}^{t+h} \rho^{-b}\left(1-\frac{r}{2\rho}\right)^{-\frac{1}{2}} d\rho \int_0^{\frac{1}{2}\left(1-\frac{r}{2\rho}\right)} v^{-b}dv, \\ \tilde{\mu}^2_{t,h} &= \int_0^{2(t+h)} r^{1-b}f(r)dr \int_{r/2}^{t+h} \rho^{-b}\left(1-\frac{r}{2\rho}\right)^{-b} d\rho \int_{\frac{1}{2}\left(1-\frac{r}{2\rho}\right)}^{1-\frac{r}{2\rho}} \left(1-\left(\frac{r}{2\rho}+v\right)\right)^{-1/2} dv. \end{aligned}$$

Integrating both terms with respect to v, we obtain that

$$\begin{aligned} \tilde{\mu}_{t,h} &\leq Ch^b \int_0^{2(t+h)} r^{1-b}f(r)dr \int_{r/2}^{t+h} \rho^{-b}\left(1-\frac{r}{2\rho}\right)^{\frac{1}{2}-b} d\rho \\ &\leq Ch^b \int_0^{2(t+h)} r^{1-b}f(r)dr \int_{r/2}^{t+h} (2\rho-r)^{\frac{1}{2}-b}\, d\rho \\ &\leq Ch^b. \end{aligned}$$

Optimizing on $b \leq \beta$ we obtain for $\tilde{\beta} = \beta$ if $\beta < 1$ and $\tilde{\beta} = 1-\varepsilon$, with $\varepsilon > 0$, if $\beta \geq 1$

$$\tilde{\mu}_{t,h} \leq Ch^{\tilde{\beta}}. \tag{A.16}$$

We then deal with $M_{t,\xi}$. For $h = |\xi|$, it is clear that

$$M_{t,\xi} \leq C \iint_{|y|-h<|z|<|y|<t+h} f(|y-z|)\left(\int_{|y|}^{(|z|+h)\wedge(t+h)} \frac{ds}{\sqrt{(s^2-|y|^2)(s^2-|z|^2)}}\right) dydz.$$

Then the calculations made for $\tilde{\mu}_{t,x}$ can be applied, which concludes the proof. $\square$

We now turn to the evaluation of $\Delta_{t,h}$ (resp. $\Theta_{t,\xi}$). The following result, similar to Lemma 4 in [3], will be crucial ; its proof is omitted:

Lemma A.3 *Suppose* $0 \le a \vee c \le b \le t$. *Then*

$$\mathcal{I} := \int_{\sqrt{b}}^{t} (s^2-c)^{-1/2} \left[(s^2-b)^{-1/2} - (s^2-a)^{-1/2} \right] ds \le \frac{1}{2\sqrt{b}} \ln\left(1 + \frac{b-a}{b-c} + 2\sqrt{\frac{b-a}{b-c}} \right) . \tag{A.17}$$

The following lemma gives an upper estimate for $\Delta_{t,h}$ (resp. $\Theta_{t,\xi}$) in terms of $\mu_{t,h}$ and $\tilde{\mu}_{t,h}$ (resp. $N_{t,\xi}$ and $M_{t,\xi}$).

Lemma A.4 *Suppose that* $(\mathbf{H}_\beta)$ *holds; then for any* $\varepsilon > 0$,

$$\Delta_{t,h} \le C(\varepsilon) \left(h^{\beta \wedge \left(\frac{1-\varepsilon}{2}\right)} + \sqrt{\mu_{t,h}\, \tilde{\mu}_{t,h}} \right) , \tag{A.18}$$

and

$$\Theta_{t,h} \le C(\varepsilon) \left(h^{\beta \wedge \left(\frac{1-\varepsilon}{2}\right)} + \sqrt{M_{t,\xi}\, N_{t,\xi}} \right) . \tag{A.19}$$

Proof: We at first prove (A.18). Set $F(s,y,z) := S(s,y)\, [S(s,z) - S(s+h,z)]$. For $s \ge |y| \ge |z|$, a simple computation shows that $F(s,y,z) \le F(s,z,y)$. Therefore $\Delta_{t,h} \le 2I_1(h) + I_2(h)$, where

$$\begin{aligned} I_1(h) &= \int_0^t \iint_{|y|<|z|<s} S(s,y)\, [S(s,z) - S(s+h,z)]\, f(|y-z|)\, dydz, \\ I_2(h) &= \int_0^t \iint_{|y|<s<|z|<s+h} S(s,y)\, S(s+h,z)\, f(|y-z|)\, dydz\, . \end{aligned}$$

Fubini's theorem and Lemma A.3 applied with $a = |z|^2 - 2|z|h - h^2$, $b = |z|^2$ and $c = |y|^2$ imply

$$\begin{aligned} & I_1(h) \\ \le\ & C \iint_{|y|<|z|<t} f(|y-z|) dydz \int_{|z|}^{t} \frac{1}{\sqrt{s^2-|y|^2}} \left[\frac{1}{\sqrt{s^2-|z|^2}} - \frac{1}{\sqrt{s^2-(|z|^2-2th-h^2)}} \right] ds \\ \le\ & C \iint_{|y|<|z|<t} \frac{f(|y-z|)}{|z|} \ln\left(1 + \frac{C\sqrt{h}}{\sqrt{|z|^2-|y|^2}} + \frac{Ch}{|z|^2-|y|^2} \right) dydz := C I_1'(h). \end{aligned} \tag{A.20}$$

The change of variables $z = (\rho\cos\theta_0, \rho\sin\theta_0)$, $z - y = (r\cos(\theta+\theta_0), r\sin(\theta+\theta_0))$, $v = \cos\theta - \frac{r}{2\rho}$ gives $I_1'(h) \le C(I_{1,1}(h) + I_{1,2}(h))$, where

$$\begin{aligned} I_{1,1}(h) &:= \int_0^{2t} r\, f(r)\, dr \int_{\frac{r}{2}}^{t} d\rho \int_0^{1-\frac{r}{2\rho}} \ln\left(1 + \frac{C\sqrt{h}}{\sqrt{\rho r v}} \right) \frac{dv}{\sqrt{1 - \left(\frac{r}{2\rho} + v\right)^2}}, \\ I_{1,2}(h) &:= \int_0^{2t} r\, f(r)\, dr \int_{\frac{r}{2}}^{t} d\rho \int_0^{1-\frac{r}{2\rho}} \ln\left(1 + \frac{Ch}{\rho r v} \right) \frac{dv}{\sqrt{1 - \left(\frac{r}{2\rho} + v\right)^2}}. \end{aligned}$$

For $0 < b < 1$, using the method of Lemma A.2 (proof of (A.16)), one easily gets

$$I_{1,1}(h) \le C h^{\frac{b}{2}} \int_0^{2t} r^{1-\frac{b}{2}} f(r)\, dr \le C h^{\frac{b}{2}},$$

if $\frac{b}{2} \le \beta$. Optimizing over b under the constraints $b \le 2\beta$ and $b < 1$ yields $I_{1,1}(h) \le C(\varepsilon)\, h^{\beta \wedge \left(\frac{1-\varepsilon}{2}\right)}$ for any $\varepsilon > 0$.

A similar computation for $I_{1,2}(h)$, replacing b by $\frac{b}{2}$, yields the same estimate. Therefore,

$$I_1(h) \le C(\varepsilon) h^{\beta \wedge \left(\frac{1-\varepsilon}{2}\right)}. \tag{A.21}$$

On the other hand, Schwarz's inequality implies that

$$I_2(h) \le \sqrt{\mu_{t,h}\, \tilde{\mu}_{t,h}} + I_3(h), \tag{A.22}$$

where

$$I_3(h) = \int_0^t ds \iint_{|y|<s<|z|<s+h} S(s+h, y)\, S(s+h, z)\, f(|y-z|)\, dydz\,.$$

Fubini's theorem and the change of variables $u = s^2$ yield

$$\begin{aligned}
I_3(h) &\le C \iint_{|y|<|z|<t+h} f(|y-z|)\, dydz \int_{|z|}^{|z|+h} \frac{ds}{\sqrt{s^2-|y|^2}\sqrt{s^2-|z|^2}} \\
&\le C \iint_{|y|<|z|<t+h} \frac{f(|y-z|)}{|z|}\, dydz \int_{|z|^2}^{(|z|+h)^2} \frac{du}{\sqrt{u-|y|^2}\sqrt{u-|z|^2}} \\
&\le C \iint_{|y|<|z|<t+h} \frac{f(|y-z|)}{|z|} \ \ln\left(1 + \frac{C\sqrt{h}}{\sqrt{|z|^2-|y|^2}} + \frac{Ch}{|z|^2-|y|^2}\right) dydz\,.
\end{aligned}$$

The inequalities (A.20) and (A.21) imply that for any $\varepsilon > 0$

$$I_3(h) \le C(\varepsilon) h^{\beta \wedge \left(\frac{1-\varepsilon}{2}\right)}\,, \tag{A.23}$$

and the inequalities (A.21) and (A.23) complete the proof of (A.18).
We now turn to the study of $\Theta_{t,\xi}$. We have $\Theta_{t,\xi} := \Theta_1(\xi) + \Theta_2(\xi) + \Theta_3(\xi)$, where

$$\begin{aligned}
\Theta_1(\xi) &= \int_0^t ds \int_{|y|<s} dy \int_{|z|\vee|z-\xi|<s} dz\, S(s,y)\, |S(s,z) - S(s,z-\xi)|\, f(|y-z|)\,, \\
\Theta_2(\xi) &= \int_0^t ds \int_{|y|<s} dy \int_{\substack{|z|<s \\ |z-\xi|\ge s}} dz\, S(s,y)\, S(s,z)\, f(|y-z|)\,, \\
\Theta_3(\xi) &= \int_0^t ds \int_{|y|<s} dy \int_{\substack{|z|\ge s \\ |z-\xi|<s}} dz\, S(s,y)\, S(s,z-\xi)\, f(|y-z|)\,.
\end{aligned}$$

Decompose $\Theta_1(\xi) = \Theta_{11}(\xi) + \Theta_{12}(\xi)$, where

$$\begin{aligned}
\Theta_{11}(\xi) &= \int_0^t ds \int_{|y|<s} dy \int_{|z-\xi|<|z|<s} dz\, S(s,y)\, |S(s,z) - S(s,z-\xi)|\, f(|y-z|), \\
\Theta_{12}(\xi) &= \int_0^t ds \int_{|y|<s} dy \int_{|z|<|z-\xi|<s} dz\, S(s,y)\, |S(s,z) - S(s,z-\xi)|\, f(|y-z|)\,.
\end{aligned}$$

Fubini's theorem yields

$$\Theta_{11}(\xi) = \iint_{\mathcal{D}_{1,1}} dydz\, f(|y-z|) \left(\int_{|y|\vee|z|}^{t} S(s,y)\, |S(s,z) - S(s,z-\xi)|\, ds \right) ,$$

where $\mathcal{D}_{1,1} = \{|y| < t, |z-\xi| < |z| < t\}$. Splitting the inner integral into two terms (depending whether $|y| > |z|$ or not), we get $\Theta_{11}(\xi) = \Theta_{111}(\xi) + \Theta_{112}(\xi)$. Lemma 4 in [3] easily yields

$$\Theta_{111}(\xi) \le C \iint_{\mathcal{D}_{1,1} \bigcap \{|y|>|z|\}} dydz\, \frac{f(|y-z|)}{|y|}\, \ln\left(1 + \frac{C|\xi|}{|y|^2 - |z|^2} \right) .$$

Computations similar to those made to obtain an upper estimate of $I_{1,2}(h)$ prove that for any $\varepsilon > 0$,

$$\Theta_{111}(\xi) \le C(\varepsilon)|\xi|^{\beta\wedge(1-\varepsilon)}. \tag{A.24}$$

Using Lemma A.3, we see that

$$\Theta_{112}(\xi) \le C \iint_{\mathcal{D}_{11} \bigcap \{|y|>|z|\}} dydz\, \frac{f(|y-z|)}{|y|}\, \ln\left(1 + \frac{C_1\sqrt{|\xi|}}{\sqrt{|y|^2 - |z|^2}} + \frac{C_1|\xi|}{|y|^2 - |z|^2} \right) ;$$

the upper estimate of $I_1'(h)$ yields

$$\Theta_{112}(\xi) \le C(\varepsilon)|\xi|^{\beta\wedge\left(\frac{1-\varepsilon}{2}\right)}. \tag{A.25}$$

On the other hand,

$$\Theta_{12}(\xi) = \iint_{\mathcal{D}_{1,2}} dydz\, f(|y-z|) \left(\int_{|y|\vee|z|}^{t} S(s,y) \cdot |S(s,z) - S(s,z-\xi)| ds \right) ,$$

where $\mathcal{D}_{1,2} = \{|y| < t, |z| < |z-\xi| < t\}$. Then, proceeding as for $\Theta_{12}(\xi)$, we obtain for every $\varepsilon > 0$,

$$\Theta_{12}(\xi) \le C(\varepsilon)|\xi|^{\beta\wedge\left(\frac{1-\varepsilon}{2}\right)}. \tag{A.26}$$

We now deal with $\Theta_2(\xi)$; Fubini's theorem implies

$$\Theta_2(\xi) \le \iint_{|y|\vee|z|<t} dydz\, f(|y-z|) \left(\int_{|z|}^{|z|+|\xi|} \frac{ds}{\sqrt{(s^2-|y|^2)(s^2-|z|^2)}} \right) ,$$

and the calculations made for $I_3(h)$ yield for $\varepsilon > 0$,

$$\Theta_2(\xi) \le C(\varepsilon)|\xi|^{\beta\wedge\left(\frac{1-\varepsilon}{2}\right)}. \tag{A.27}$$

Finally, $\Theta_3(\xi) := \Theta_{31}(\xi) + \Theta_{32}(\xi)$, where

$$\begin{aligned}
\Theta_{31}(\xi) &= \int_0^t ds \int_{|y|<s} dy \int_{\substack{|z|\ge s \\ |z-\xi|<s}} dz\, |S(s,y) - S(s,y-\xi)|\, f(|y-z|)\, S(s,z-\xi) , \\
\Theta_{32}(\xi) &= \int_0^t ds \int_{|y|<s} dy \int_{\substack{|z|\ge s \\ |z-\xi|<s}} dz\, S(s,y-\xi)\, f(|y-z|)\, S(s,z-\xi) .
\end{aligned}$$

Since $S(s, y-\xi) = 0$ if $|y-\xi| \geq s$, we have

$$\Theta_{31}(\xi) = \int_0^t ds \int_{\substack{|y|<s \\ |y-\xi|<s}} dy \int_{\substack{|z|\geq s \\ |z-\xi|<s}} dz \, |S(s,y) - S(s,y-\xi)| \, f(|y-z|) \, S(s,z-\xi)\,,$$

and Schwarz's inequality implies

$$\Theta_{31}(\xi) \leq \sqrt{M_{t,\xi}\, N_{t,\xi}}. \tag{A.28}$$

A similar computation yields

$$\Theta_{32}(\xi) \leq \int_0^t ds \int_{|y-\xi|<s} dy \int_{\substack{|z|\geq s \\ |z-\xi|<s}} S(s,y-\xi) \, f(|y-z|) \, S(s,z-\xi)\,,$$

which means that $\Theta_{32}(\xi)$ can be dealt with as $\Theta_2(\xi)$, so that for $\varepsilon > 0$:

$$\Theta_{3,2}(\xi) \leq C(\varepsilon)|\xi|^{\beta \wedge \left(\frac{1-\varepsilon}{2}\right)}. \tag{A.29}$$

The inequalities (A.24)-(A.29) complete the proof. □

The next lemma establishes an inequality relating $\mu_{t,h}$, $\Delta_{t,h}$ and $\tilde{\mu}_{t,h^\delta}$ (resp. $N_{t,\xi}$, $\Theta_{t,\xi}$ and $M_{t,\xi}$) for $0 < \delta < 1$:

Lemma A.5 *Suppose that* ($\mathbf{H}_\beta$) *holds; then for* $0 < \delta < 1$ *and* $t \in]0,T]$, *one has*

$$\mu_{t,h} \leq C \left\{ h^{1-\delta} \, \Delta_{t,h} + \sqrt{\mu_{t,h}} \, \sqrt{\tilde{\mu}_{t-h^\delta, h^\delta}} + h^{\delta(1+\beta)+\tilde{\beta}(1-\delta)} \right\} \tag{A.30}$$

and

$$N_{t,\xi} \leq C \left\{ h^{1-\delta} \, \Theta_{t,\xi} + \sqrt{N_{t,\xi}} \, \sqrt{M_{t-h^\delta,\eta}} + h^{\delta(1+\beta)+\tilde{\beta}(1-\delta)} \right\} \tag{A.31}$$

where $|\eta| \leq |\xi|^\delta$, $\tilde{\beta} = \beta$ *if* $\beta < 1$ *and* $\tilde{\beta} < 1$ *if* $\beta \geq 1$.

Proof: First, we prove (A.30). Let $0 < \delta < 1$; then for $s > h^\delta$ and $|y| < s - h^\delta$, one has $0 \leq S(s,y) - S(s+h,y) \leq C h^{1-\delta} S(s,y)$. Indeed,

$$0 \leq S(s,y) - S(s+h,y) = \frac{h^2 + 2sh}{\sqrt{(s^2-|y|^2)((s+h)^2-|y|^2)}\,(\sqrt{s^2-|y|^2} + \sqrt{(s+h)^2-|y|^2})},$$

and, if $|y| < s - h^\delta$, one has

$$s^2 - y^2 \geq 2sh^\delta - h^{2\delta} \geq sh^\delta;$$

since $h^2 \leq Csh$, we obtain the required estimate for the increment of the Green function.

Hence, if $t \geq h^\delta$, then $\mu_{t,h} \leq C\,\{J_1(t,h) + J_2(t,h) + J_3(t,h)\}$, where

$$\begin{aligned}
J_1(t,h) &:= h^{1-\delta} \int_{h^\delta}^t ds \iint_{|y|<(s-h^\delta)\wedge|z|} S(s,y)\, f(|y-z|) \, [S(s,z) - S(s+h,z)] \, dydz\,, \\
J_2(t,h) &:= \int_{h^\delta}^t ds \iint_{\substack{s-h^\delta<|y|\wedge|z| \\ |y|\vee|z|\leq s}} [S(s,y) - S(s+h,y)] \, f(|y-z|) \\
&\qquad\qquad \times [S(s,z) - S(s+h,z)] \, dydz\,, \\
J_3(t,h) &= \int_0^{h^\delta} ds \iint_{|y|<|z|<s} [S(s,y) - S(s+h,y)] \, f(|y-z|) \, S(s,z) \, dydz\,.
\end{aligned}$$

The definition of $\Delta_{t,h}$ and Schwarz's inequality imply

$$\begin{aligned} J_1(t,h) &\leq Ch^{1-\delta}\Delta_{t,h}\,, \\ J_2(t,h) &\leq \sqrt{\mu_{t,h}}\sqrt{\tilde{\mu}_{t-h^\delta,h^\delta}}\,. \end{aligned}$$

Furthermore, Lemma 4 in [3] implies that

$$J_3(t,h) \leq C \iint_{|y|<|z|<h^\delta} \frac{f(|y-z|)}{|z|} \ln\left(1+\frac{Ch^{1+\delta}}{|z|^2-|y|^2}\right) dydz.$$

The change of variables used in the proof of Lemma A.4 yields

$$J_3(t,h) \leq C\int_0^{2h^\delta} rf(r)dr \int_{\frac{r}{2}}^{h^\delta} d\rho \int_0^{1-\frac{r}{2\rho}} \ln\left(1+\frac{Ch^{1+\delta}}{\rho r v}\right) \frac{dv}{\sqrt{1-\left(\frac{r}{2\rho}+v\right)^2}}\,.$$

The method used to prove (A.16) yields

$$\begin{aligned} J_3(t,h) &\leq Ch^{b(1+\delta)} \int_0^{2h^\delta} r^{1-b} f(r)\,dr \int_{\frac{r}{2}}^{h^\delta} (2\rho-r)^{\frac12 - b}\,d\rho \\ &\leq Ch^{b(1+\delta)+\delta(1-b)+\delta(\beta-b)}\,. \end{aligned}$$

Optimizing over $b<1$, $b\leq\beta$, we obtain

$$J_3(t,h)\leq Ch^{\delta(1+\beta)+\tilde\beta(1-\delta)}\,,$$

where $\tilde\beta=\beta$ if $\beta<1$ and $\tilde\beta<1$ if $\beta\geq 1$. This concludes the proof of (A.30) if $t\geq h^\delta$. Finally, if $t\leq h^\delta$, then $\mu_{t,h}\leq C\,J_3(t,h)$ and (A.30) still holds.

To prove (A.31), we remark that if we assume $|y|<|y-\xi|$, then, for η such that $|\eta|<s-|\xi|^\delta$,

$$|S(s,y)-S(s,y-\xi)|\leq C\,|\xi|^{1-\delta}\,S(s,\eta).$$

The proof can then be carried on as that of (A.30). □

¿From Lemmas A.2, A.4 and A.5, we conclude the following

Lemma A.6 *Assume* $(\mathbf{H}_\beta)$ *holds. Then, for* $|\xi|\leq h$ *small enough, one has*

$$\textbf{(i)} \qquad \mu_{t,h}+N_{t,\xi}\leq Ch^\gamma,\ \textit{with}\ \gamma<\beta\wedge 1\,; \qquad \text{(A.32)}$$

$$\textbf{(ii)} \qquad \Delta_{t,h}+\Theta_{t,\xi}\leq Ch^\gamma,\ \textit{with}\ \gamma<\beta\wedge\frac12\,. \qquad \text{(A.33)}$$

Proof: We only prove the estimates for $\mu_{t,h}$ and $\Delta_{t,h}$. From (A.14), (A.18) and (A.30), we conclude that if $\tilde\beta$ is defined as in Lemma A.2, for $0<\delta<1$, $\varepsilon>0$ we have

$$\mu_{t,h}\leq C\left\{h^{\delta(1+\beta)+\tilde\beta(1-\delta)} + h^{1-\delta}\left(h^{\beta\wedge\left(\frac{1-\varepsilon}{2}\right)}+\sqrt{\mu_{t,h}}\,h^{\frac{\tilde\beta}{2}}\right)+\sqrt{\mu_{t,h}}\,h^{\frac{\delta\tilde\beta}{2}}\right\}\,. \qquad \text{(A.34)}$$

Since $\delta<1$ one has $1-\delta+\frac{\beta}{2}\geq\frac{\delta\tilde\beta}{2}$. Set $X=\sqrt{\mu_{t,h}}$; then $X\geq 0$ and there exist positive constants c_1 and c_2 such that, for $\alpha_1=\frac{\delta\tilde\beta}{2}$ and $\alpha_2=\left[\delta(1+\beta)+\tilde\beta(1-\delta)\right]\wedge\left[1-\delta+\left(\beta\wedge\left(\frac{1-\varepsilon}{2}\right)\right)\right]$,

$$X^2-c_1\,h^{\alpha_1}\,X-c_2\,h^{\alpha_2}\leq 0\,.$$

Thus $X \leq X_1$, where

$$X_1 = \frac{1}{2}\left[c_1\, h^{\alpha_1} + \sqrt{c_1^2\, h^{2\alpha_1} + 4\, c_2\, h^{\alpha_2}}\right] \leq C\, h^{\alpha_1 \wedge \frac{\alpha_2}{2}}\,,$$

which in turn implies

$$\mu_{t,h} \leq C h^{(2\alpha_1)\wedge\alpha_2}\,. \tag{A.35}$$

Obviously, $\delta\tilde{\beta} \leq \delta(1+\beta) \leq \delta(1+\beta)+\tilde{\beta}(1-\delta)$. If $\beta < 1/2$, since $\delta < 1$ we have $\delta\beta \leq 1-\delta+\beta$ so that if $\delta \sim 1$,

$$\mu_{t,h} \leq C\, h^{\gamma},\ \text{with}\ \gamma < \beta = \tilde{\beta}\,. \tag{A.36}$$

If $\beta \geq 1/2$, again for $0 < \delta < 1$, $\delta\tilde{\beta} < 1 - \delta + \frac{1}{2}$ and for $\delta \sim 1$, $\varepsilon > 0$, we conclude that

$$\mu_{t,h} \leq Ch^{\gamma},\ \text{with}\ \gamma < \tilde{\beta}\,. \tag{A.37}$$

The inequalities (A.36) and (A.37) imply (A.32) for $\mu_{t,h}$. Substituting (A.32) and (A.14) in (A.18), we conclude that (A.33) holds. □

Remark A.6: Under the condition $(\mathbf{H}_\beta)$, one has, for t small enough

$$\int_0^t r f(r) \ln\left(1+\frac{t}{r}\right) dr \leq Ct^{\beta}\,.$$

Therefore, Lemmas A.2 and A.6 improve (A.26) in [5]. Indeed,

$$\mu_{t,h} + \tilde{\mu}_{t,h} \leq Ch^{\gamma},\ \text{with}\ \gamma < \beta \wedge 1\,. \tag{A.38}$$

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