A trajectorial proof of the Vortex method for the 2d Navier-Stokes equations

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ABSTRACT: We consider the Navier-Stokes equation in dimension 2 and more precisely the vortex equation satisfied by the curl of the velocity field. We give a probabilistic interpretation of this equation in terms of a weak solution of a nonlinear stochastic differential equation. Next we construct an interacting particle system for which we have a propagation of chaos result: the laws of the empirical measures tend, as the number of particles tends to infinity, to a deterministic law for which marginals are solutions of the vortex equation. This pathwise result justifies completely the vortex method introduced by Chorin to simulate the solutions of the vortex equation and considerably improves the results already obtained by Marchioro and Pulvirenti [7]. The main difficulty is to control the possible explosions of the drift term appearing in the nonlinear stochastic differential equation.
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1 Introduction

In this paper we consider the equation describing the velocity of an incompressible fluid in dimension two. This equation on the velocity field of the fluid is the well known Navier-Stokes equation and in dimension two, it can be expressed as an equation for the curl of the velocity. On this simplified

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form, the Navier-Stokes appears as a McKean-Vlasov equation, in which the coefficient of the drift term K can explode.

About twenty years ago, Chorin proposed a vortex method to simulate the solutions of this equation. Its method consists in considering some "cutoffed" drift kernels and in assuming, first the convergence of the cutoffed model to the original model and second, in using (for the simulation) some particle approximations for the cutoffed model. Its approach was not mathematically proved and many authors tried to give a proof of the convergence.

In 1982, Marchioro and Pulvirenti proved a partial result of convergence, by using an approximation of K by some cutoffed drift coefficients, and for bounded integrable initial conditions.

In 1984, Osada proved a propagation of chaos result for an interacting particle system with an analytical method based on generators of generalized divergence form, but only on the case of large viscosity and for a bounded density initial data. The tightness of the laws of the particle systems is always true and the constraint on the viscosity appears in the identification of the limit laws.

In this paper, we consider the framework introduced by Marchioro and Pulvirenti, and give a complete proof of the vortex method, by showing a propagation of chaos result for a stochastic interacting particle system. For each size n, The n-particle system is defined with a cutoffed drift kernel, where the cutoff depends on n in a precise asymptotics, given by the computations. One proves that the laws of the particle systems converge, and also the empirical measures, as probability measures on the path space, to a probability measure whose time marginals have density functions with respect to the Lebesgue measure, which are solution of the Navier-Stokes equation. One obtains a trajectorial rate of convergence, which is unfortunately certainly not optimal. Marchioro and Pulvirenti obtained just the convergence of the expectation of the empirical measures. Our results, performed directly on the path space, yield the asymptotics of path functionals such as hitting times or suprema over time intervals.

We consider here as initial condition for the Navier-Stokes equation a bounded integrable function, and not only a bounded density function. We are inspired by a paper of Jourdain [4], which shows how to take into account bounded signed measures instead of probability measures as initial conditions in a McKean-Vlasov context. The idea is the following: one considers on the path space probability measures under which a signed weight depending on the initial condition is given to each sample-path. This trick allows us to have a simpler approach than in Marchioro-Pulvirenti.

Notation - We will denote by $|.|$ the euclidian norm in $\mathbb{R}^2$, by $\|.\|_\infty$ the L^∞-norm and by $\|.\|_1$ the L^1-norm in $\mathbb{R}^2$.
- The letter C will denote a real constant which can change from line to line.

1.1 The Vortex Equation

Let us consider the velocity flow $v(t,x), t \in \mathbb{R}_+, x \in \mathbb{R}^2$ of a viscous and incompressible fluid in the whole plane. The governing equation of this motion is the Navier-Stoke equation given by

$$\begin{aligned} &\frac{\partial v}{\partial t}(t,x) + (v.\nabla)v(t,x) = \nu\Delta v(t,x) - \nabla p \ ; \\ &\nabla.v(t,x) = 0 \ ; \ v(x,t) \to 0 \text{ as } |x| \to +\infty, \qquad \text{for } 0 \le t < +\infty, \end{aligned} \tag{1.1}$$

where p is the pressure and $\nu > 0$ the viscosity (assumed to be constant).

It is well known that the Navier-Stokes equation in dimension two can be reformulated in terms of the vorticity flow $w(t,x) = \text{curl } v(t,x)$. More precisely, the flow w is solution of the nonlinear partial differential equation

$$\frac{\partial w}{\partial t}(t,x) + (v.\nabla)w(t,x) = \nu\Delta w(t,x) \ ; \tag{1.2}$$

$$v(t,x) = \int_{\mathbb{R}^2} K(x-y)w(t,y)dy \tag{1.3}$$

for an initial data w_0, where $K(x)$ is the kernel of Biot-Savart defined by

$$\forall x = (x_1,x_2) \in \mathbb{R}^2, K(x) = \frac{1}{2\pi}\frac{1}{(x_1^2+x_2^2)}(-x_2,x_1). \tag{1.4}$$

Note that $\nabla.K = 0$, implying $\nabla.v = 0$.

The main difficulty of the kernel K is its explosion at 0. However, as it can be seen in the following lemma, its integrability near 0 allows us to consider the set $L^\infty \cap L^1$ in $\mathbb{R}^2$ as a good class of functions solutions of the equation (1.2).

Lemma 1.1 *The function K is bounded at infinity and integrable near zero.*

Let us introduce

$$K_1 = \int_{B(0,1)} |K(y)|dy \ ; \ K_\infty = \sup_{y \in B(0,1)^c} |K(y)|.$$

Then for every function $g \in L^1 \cap L^\infty$ *and* $x \in I\!R^2$,

$$|K * g(x)| \leq \|g\|_\infty K_1 + \|g\|_1 K_\infty. \tag{1.5}$$

The proof of this lemma is very simple. This property of the kernel K allows us to obtain an existence and uniqueness result for the solution of (1.2) if the initial function is assumed to be bounded and integrable.

Theorem 1.2 *Let us consider* $w_0 \in L^1 \cap L^\infty$, $\nu > 0$ *and* $T > 0$, *then*

1) there exists a unique solution $t \to w_t \in L^1 \cap L^\infty(I\!R^2)$, *solution of (1.2) in the weak sense, that means that* $\forall \phi \in C_b^2(I\!R^2)$,

$$\begin{aligned} \int_{I\!R^2} \phi(x) w_t(x) dx &= \int_{I\!R^2} \phi(x) w_0(x) dx + \nu \int_0^t \int_{I\!R^2} \Delta\phi(x) w_s(x) dx ds \\ &+ \int_0^t \int_{I\!R^2} (K * w_s(x).\nabla\phi(x)) w_s(x) dx ds. \end{aligned} \tag{1.6}$$

Moreover,

$$\sup_{t \in [0,T]} \|w_t\|_\infty \leq \|w_0\|_\infty \quad ; \quad \|w_t\|_1 \leq \|w_0\|_1 \ \forall t \in [0,T], \tag{1.7}$$

2) If the initial function w_0 *is nonnegative, then it is the same for each* $w_t, t \geq 0$.

3) The function w *is a.s. solution of the evolution equation*

$$w(t,x) = G_t^\nu * w_0(x) + \int_0^t \int_{I\!R^2} \nabla_x G_{t-s}^\nu (x-y).K * w_s(y) w_s(y) dy ds. \tag{1.8}$$

where G_t^ν *is the heat kernel in* $I\!R^2$ *defined by* $G_t^\nu(x) = \frac{1}{4\pi t\nu} e^{-\frac{|x|^2}{4t\nu}}$.

Proof. 1) The existence and uniqueness of the weak solution of (1.2) with initial data w_0 in $L^1 \cap L^\infty$ is proved for example in Marchioro-Pulvirenti [7], as (1.7) and the positivity.

2) By using Fubini's Theorem (allowed by Lemma 1.1 and applied to w_s), we can easily prove that for every function $\psi(t,x) \in C_b^{1,2}(I\!R_+ \times I\!R^2)$,

$$\int_{I\!R^2} \psi(t,x) w_t(x) dx = \int_{I\!R^2} \psi(0,x) w_0(x) dx + \nu \int_0^t \int_{I\!R^2} \Delta\psi(s,x) w_s(x) dx ds$$
$$+ \int_0^t \int_{I\!R^2} \psi'_s(s,x) w_s(x) dx ds + \int_0^t \int_{I\!R^2} (K * w_s(x).\nabla_x \psi(s,x)) w_s(x) dx ds.$$

Then by choosing for a fixed time t, $\psi(s,x) = \int_{\mathbb{R}^2} G^\nu_{t-s}(x-y)\phi(y)dy$ for $\phi \in C^2_b(\mathbb{R}^2)$, we obtain the evolution equation (1.8). In fact, a direct proof of the existence and uniqueness of the solution of (1.2) in $L^1 \cap L^\infty$ can be given by studying (1.8) (see Giga-Miyakawa-Osada [2]). $\square$

Let us remark that a simple computation gives an estimate of the norm in L^1 of the gradient of G^ν_t:

Lemma 1.3

$$\|\nabla_x G^\nu_t\|_{L^1(\mathbb{R}^2)} \leq \frac{K}{\sqrt{\nu t}} \tag{1.9}$$

where K is a real constant and then $\int_0^t \|\nabla_x G^\nu_{t-s}\|_{L^1(\mathbb{R}^2)} ds < +\infty$.

2 The nonlinear martingale problem and the stochastic differential equation associated with the 2d Navier-Stokes equation

We are in a McKean-Vlasov context. If the initial function w_0 is a density function on $\mathbb{R}^2$, is is natural to associate with the weak form (1.6) of the Navier-Stokes equation a nonlinear martingale problem and a nonlinear stochastic differential equation, by interpreting a solution as the time-marginal flow of a probability measure on $C([0,T],\mathbb{R}^2)$. Here, we will use a trick due to Jourdain [4] to pass from a density function to every bounded integrable function.

In all the following, we will consider an initial bounded and integrable function w_0 on $\mathbb{R}^2$. We assume moreover that w_0 is not the function identically equal to 0.

We define the bounded function h from the initial condition w_0 by

$$\forall x \in \mathbb{R}^2 \ , \quad h(x) = \frac{w_0(x)\|w_0\|_1}{|w_0(x)|}, \text{ with the convention } \frac{0}{0} = 0. \tag{2.1}$$

Let us remark that for each $x \in \mathbb{R}^2$,

$$-\|w_0\|_1 \leq h(x) \leq \|w_0\|_1. \tag{2.2}$$

Let us also remark that

$$w_0(x)dx = h(x)\frac{|w_0(x)|}{\|w_0\|_1}dx.$$

where $\frac{|w_0|}{\|w_0\|_1}$ is a density of probability measure.

Now, for Q a probability measure on $C([0,+\infty),I\!R^2)$, we define the flow $(\tilde{Q}_t)_{t\geq 0}$ of signed measures on $I\!R^2$ by

$$\forall B \text{ Borel subset of } I\!R^2, \tilde{Q}_t(B) = E^Q(1_B(X_t)h(X_0)), \tag{2.3}$$

where X denotes the canonical process on $C([0,+\infty),I\!R^2)$. (One associate with each sample-path a signed weight depending on the initial position).

The following lemma will be usefull in the following.

Lemma 2.1 *1) For each $t \geq 0$, the signed measure $\tilde{Q}_t$ is bounded, and the total mass is less than $\|w_0\|_1$.*

2) If Q_t is absolutely continuous with respect to the Lebesgue measure, then it is the same for $\tilde{Q}_t$, and its density is integrable.

Proof. Since the function h is bounded by $\|w_0\|_1$ and using (2.3), the lemma is obvious. $\square$

The equation (1.2) understood in its weak form leads naturally to the following definition

Definition 2.2 *The probability measure $P \in \mathcal{P}(C(I\!R_+,I\!R^2))$ is solution of the nonlinear martingale problem $(\mathcal{M})$ if for each $\phi \in C_b^2(I\!R^2)$,*

$$\phi(X_t) - \phi(X_0) - \int_0^t K * \tilde{P}_s(X_s).\nabla\phi(X_s)ds - \nu\int_0^t \Delta\phi(x_s)ds$$

is a P-martingale, where X is the canonical process on $C(I\!R_+,I\!R^2)$, P_0 has the law $\frac{|w_0(x)|}{\|w_0\|_1}dx$, and $P_s = P \circ X_s^{-1}$.

This nonlinear martingale problem is related to the following nonlinear stochastic differential equation.

Definition 2.3 *Let us consider a random $I\!R^2$-valued variable X_0 with distribution $\frac{|w_0(x)|}{\|w_0\|_1}dx$ and B be a 2-dimensional Brownian motion independent of X_0. A solution $X \in C(I\!R_+,I\!R^2)$ of the nonlinear stochastic differential equation satisfies $\forall t \in I\!R_+$*

$$\begin{aligned} &X_t = X_0 + \sqrt{2\nu}B_t + \int_0^t K * \tilde{P}_s(X_s)ds, \\ &P_s \text{ is the marginal at time } s \text{ of the law of } X_s. \end{aligned} \tag{2.4}$$

Notations: We denote by $\hat{\mathcal{P}}_\infty(C(I\!R_+, I\!R^2))$ the space of probability measures on $C(I\!R_+, I\!R^2)$ whose marginals P_s at time s are absolutely continuous with respect to the Lebesgue measure on $I\!R^2$, for every $s \in I\!R_+$, and have a density in $L^\infty(I\!R^2)$. For such probability measures, there exists a measurable function $(s,x) \to \tilde{p}(s,x)$ in L^∞ such that $\forall s \in [0,T]$, $\tilde{P}_s(dx) = \tilde{p}(s,x)dx$. (cf. Meyer [9] p.194). We call this function $\tilde{p}$ a measurable version of the densities of $(\tilde{P}_s)$. We will prove the following theorem.

Theorem 2.4 *Let us consider a bounded integrable function w_0. Then there exists a unique solution $P \in \hat{\mathcal{P}}_\infty(C(I\!R_+, I\!R^2))$ to the martingale problem $(\mathcal{M})$ such that $P_0(dx) = \frac{|w_0(x)|}{\|w_0\|_1}dx$.*

Moreover, for each $t \in I\!R_+$, the signed bounded measure $\tilde{P}_t$ is absolutely continuous with respect to the Lebesgue measure, and each measurable version of its density is in L^∞ and is almost surely equal to the function w_t defined in Theorem 1.2.

Proof. 1) If $P \in \hat{\mathcal{P}}_\infty(C([0,T], I\!R^2))$ is a solution of $(\mathcal{M})$ and $\tilde{p}_t$ a measurable version of the densities of $\tilde{P}_t$, then by multiplying by $h(X_0)$ and by taking the expectation in $(\mathcal{M})$, we obtain that $\tilde{p}$ is a weak solution of (1.2) with initial condition w_0. Then by the uniqueness given in Theorem 1.2, for each $t \in I\!R_+$, $\tilde{p}_t(x) = w_t(x)$ *a.s.*.

2) let us consider this unique weak solution w of (1.2) issued from w_0. Then w_t is for each t a bounded integrable function. We say that $P^w \in \mathcal{P}(C(I\!R_+, I\!R^2))$ is solution of the classical martingale problem $(\mathcal{M}^w)$ if for each $\phi \in C_b^2(I\!R^2)$,

$$\phi(X_t) - \phi(X_0) - \nu \int_0^t \Delta\phi(X_s)ds - \int_0^t K * w_s(X_s).\nabla\phi(X_s)ds$$

is a P^w-martingale and $P_0^w(dx) = \frac{|w_0(x)|}{\|w_0\|_1}dx$. This martingale problem is well-posed. Indeed, by Lemma 1.1 and (1.7), for each $s \geq 0$,

$$\|K * w_s\|_\infty \leq K_1\|w_s\|_\infty + K_\infty\|w_s\|_1 \leq K_1\|w_0\|_\infty + K_\infty\|w_0\|_1 = \hat{K}(w_0) \quad (2.5)$$

so the drift coefficient is bounded, and by Girsanov's Theorem, we get the existence and uniqueness of the solution of $(\mathcal{M}^w)$. Moreover, every time marginal of P^w admits a density p_s^w, and if we multiply by $h(X_0)$ all the terms of the martingale problem and take expectations, we obtain immediately that the flow $(\tilde{p}_s^w)_{s\in I\!R_+}$ is solution of the weak equation: for each

$\phi \in C_b^2(I\!R^2)$,

$$\begin{aligned}\int_{I\!R^2} \phi(x)\tilde{p}_t^w(x)dx &= \int_{I\!R^2} \phi(x)w_0(x)dx + \nu \int_0^t \int_{I\!R^2} \Delta\phi(x)\tilde{p}_s^w(x)dxds \\ &+ \int_0^t \int_{I\!R^2} (K * w_s(x).\nabla\phi(x)\tilde{p}_s^w(x)dxds. \qquad (2.6)\end{aligned}$$

Then the flow $\tilde{p}_t$ is solution of

$$\frac{\partial \tilde{p}^w}{\partial t} + (K * w.\nabla)\tilde{p}^w = \nu\Delta\tilde{p}^w \ ; \quad \tilde{p}_0^w = w_0.$$

Since the divergence of the drift term is equal to zero, we can adapt the proof of Theorem 6.2.2 in Lewandowski [6] and obtain that for each $t \in I\!R_+$, for each $q \in I\!R_+^*$,

$$\|\tilde{p}_t^w\|_q \leq \|w_0\|_q,$$

and since w_0 belongs to $L^\infty \cap L^1$, we conclude by taking $q \to +\infty$ that for each $t \in [0,T]$,

$$\|\tilde{p}_t^w\|_\infty \leq \|w_0\|_\infty. \qquad (2.7)$$

3) Let us now prove the existence and uniqueness of a solution of (2.6) in $L^\infty([0,T], L^1 \cap L^\infty)$. We obtain as before by Fubini's Theorem and thanks to Lemma 1.1 that $\tilde{p}^w$ is solution of the following evolution equation

$$\tilde{p}^w(t,x) = G_t^\nu * w_0(x) + \int_0^t \int \nabla_x G_{t-s}^\nu(x-y)K * w_s(y)\tilde{p}^w(s,y)dy. \qquad (2.8)$$

Let us prove the uniqueness of the solution of (2.8) in $L^\infty([0,T], L^\infty)$. Let q and q' be two solutions. Then

$$q(t,x) - q'(t,x) = \int_0^t \int \nabla_x G_{t-s}^\nu(x-y)K * w_s(y)(q(s,y) - q'(s,y))dyds.$$

Thus

$$\begin{aligned}\|q(t,.) - q'(t,.)\|_\infty &\leq \int_0^t \| \int \nabla_x G_{t-s}^\nu(x-y)K * w_s(y)(q(s,y) - q'(s,y))dy\|_\infty ds \\ &\leq \hat{K}(w_0) \int_0^t \|q(s,.) - q'(s,.)\|_\infty \|\nabla_x G_{t-s}^\nu\|_{\mathbb{L}^1} ds \\ &\leq \frac{\hat{K}(w_0)}{2\sqrt{\pi\nu}} \int_0^t \frac{\|q(s,.) - q'(s,.)\|_\infty}{\sqrt{t-s}} ds.\end{aligned}$$

($\hat{K}(w_0)$ is defined in (2.5)). We are almost in a standard Gronwall's situation. By an iteration, we obtain

$$\begin{aligned}
\|q(t,.)-q'(t,.)\|_\infty &\leq C\int_0^t \frac{1}{\sqrt{t-s}}\int_0^s \frac{\|q(u,.)-q'(u,.)\|_\infty}{\sqrt{s-u}}duds \\
&\leq C\int_0^t \|q(u,.)-q'(u,.)\|_\infty \int_u^t \frac{1}{\sqrt{t-s}\sqrt{s-u}}dsdu \\
&\leq C\int_0^t \|q(u,.)-q'(u,.)\|_\infty du.
\end{aligned}$$

Therefore,

$$\sup_{t\in[0,T]}\|q(t,.)-q'(t,.)\|_\infty = 0$$

and we have uniqueness in (2.8). Hence $\tilde{p}^w$ is the unique solution of (2.8) and since w satisfies this equation, then

$$\sup_{t\in[0,T]}\|\tilde{p}^w(t,.)-w(t,.)\|_\infty = 0.$$

So the probability measure P^w is solution of the nonlinear martingale problem $(\mathcal{M})$.

4) Let us now prove the uniqueness of a solution of this martingale problem. Let P and Q be two solutions. By the same reasoning as above, it is easy to prove that $(\tilde{P}_t)_{t\geq 0}$ and $(\tilde{Q}_t)_{t\geq 0}$ are equal to $(w(t,x)dx)_{t\geq 0}$. Hence P and Q are solutions of the classical well-posed martingale problem $(\mathcal{M}^w)$ and are then equal, and Theorem 2.4 is proved. □

Remark 2.5 *The function $K*w$ is bounded but not Lipschitz. So we have existence and uniqueness in law for the nonlinear SDE (2.4), but we don't have a strong uniqueness result.*

3 An approximating system

3.1 An approximating system with a non explosing kernel

Let us consider exactly as in Marchioro-Pulvirenti [7] a cutoffed kernel K_ε defined in the following way. We denote by $g(r) = -\frac{1}{2\pi}\ln r$ the fundamental solution of the Poisson equation. One knows that for $x\in\mathbb{R}^2$, $K(x) = \nabla^\perp g(|x|)$. For each $\varepsilon>0$, we consider g_ε defined as $g_\varepsilon(r) = g(r)$ if $|r|\geq\varepsilon$ and arbitrarily extended to an even $C^2(\mathbb{R})$ function such that $|g'_\varepsilon(r)|\leq|g'(r)|$

and $|g''_\varepsilon(r)| \leq |g''(r)|$. Then we define

$$K_\varepsilon(x) = \nabla^\perp g_\varepsilon(|x|).$$

The function K_ε is then Lipschitz continuous and bounded and we denote by M_ε the maximum value of K_ε on $\mathbb{R}^2$ and by L_ε a Lipschitz constant.

We now define the interacting particle system we are interested in.

Definition 3.1 *Consider a sequence $(B^i)_{i\in\mathbb{N}}$ of independent Brownian motions on $\mathbb{R}^2$ and a sequence of independent variables $(Z_0^i)_{i\in\mathbb{N}}$ with values in $\mathbb{R}^2$ distributed according $\frac{|w_0(x)|}{\|w_0\|_1}dx$, and independent of the Brownian motions. For a fixed ε, for each $n \in \mathbb{N}^*$, and $1 \leq i \leq n$, let us consider the interacting processes defined by*

$$Z_t^{in,\varepsilon} = Z_0^i + \sqrt{2\nu}B_t^i + \int_0^t K_\varepsilon * \tilde{\mu}_s^{n,\varepsilon}(Z_s^{in,\varepsilon})ds \tag{3.1}$$

where $\mu^{n,\varepsilon} = \frac{1}{n}\sum_{j=1}^n \delta_{Z^{jn,\varepsilon}} \in \mathcal{P}(C(\mathbb{R}_+,\mathbb{R}^2))$ is the empirical measure of the system, and then

$$\tilde{\mu}_s^{n,\varepsilon} = \frac{1}{n}\sum_{j=1}^n h(Z_0^j)\delta_{Z_s^{jn,\varepsilon}}$$

We also define the limiting independent processes by

$$\bar{Z}_t^{i,\varepsilon} = Z_0^i + \sqrt{2\nu}B_t^i + \int_0^t K_\varepsilon * \tilde{P}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})ds, \tag{3.2}$$

where P_s^ε is the law of $\bar{Z}_s^{i,\varepsilon}$.

Proposition 3.2 *1) For each $T > 0$ and for each n, there exits a unique (pathwise) solution to the interacting particle system (3.1) in $C([0,T],\mathbb{R}^{2n})$ and a unique (pathwise) solution to the nonlinear equation (3.2) in $C([0,T],\mathbb{R}^2)$.*

2) For each $T > 0$,

$$E(\sup_{t\leq T}|Z_t^{in,\varepsilon} - \bar{Z}_t^{i,\varepsilon}|) \leq \frac{M_\varepsilon}{\sqrt{n}L_\varepsilon}\exp(\|w_0\|_1 T L_\varepsilon). \tag{3.3}$$

Proof. By the boundedness of h, the proof of the first assertion is standard and can be adapted from Sznitman [12] Theorem 1.1 and the second assertion comes from an easy adaptation of the computations in Jourdain-Méléard [3] Proposition 2.3. See also Jourdain [4]. $\square$

3.2 The approximating interacting particle system

We now consider $T > 0$ and a sequence (ε_n) tending to 0 such that

$$\lim_n \frac{M_{\varepsilon_n}}{nL_{\varepsilon_n}} \exp(\|w_0\|_1 T L_{\varepsilon_n}) = 0. \tag{3.4}$$

For each n and given independent Brownian motions $(B^i)_{1\leq i\leq n}$, we consider a coupling between the n particle system $(Z^{in} = Z^{in,\varepsilon_n})$ defined with the drift K_{ε_n} as in (3.1), and the corresponding n independent limiting processes $\bar{Y}^{in} = \bar{Z}^{i,\varepsilon_n}$ defined for each $t \leq T$ and n by

$$\bar{Y}_t^{in} = Z_0^i + \sqrt{2\nu} B_t^i + \int_0^t K_{\varepsilon_n} * \tilde{P}_s^n(\bar{Y}_s^{in}) ds, \tag{3.5}$$

where P_s^n is the common law of $\bar{Y}_s^{in}$.

By the same arguments as before, P_s^n admits a density function p_s^n and then $\tilde{P}_s^n$ admits a density function $\tilde{p}_s^n$, which is a weak solution of the equation

$$\frac{\partial \tilde{p}^n}{\partial t} = \nu \Delta \tilde{p}^n - (K_{\varepsilon_n} * \tilde{p}^n \cdot \nabla)\tilde{p}^n \ ; \ p_0^n = w_0. \tag{3.6}$$

Indeed, to prove it, we compute $\phi(\bar{Y}_t^{in})$ for a smooth function ϕ by Itô's formula, we multiply it by $h(Z_0^i)$ and take expectations. By the same reasoning as in the proof of Theorem 2.4 and since $w_0 \in L^1 \cap L^\infty$ and $K_{\varepsilon_n} * \tilde{p}^n$ is bounded by M_{ε_n} and has a divergence equal to 0, we can prove that

Proposition 3.3

$$\sup_n \sup_{t\in[0,T], x\in \mathbb{R}^2} |\tilde{p}_t^n(x)| \leq \|w_0\|_\infty. \tag{3.7}$$

It is moreover easy to prove by standard arguments that $\tilde{p}^n$ is solution of the evolution equation

$$\tilde{p}_t^n(x) = G_t^\nu * w_0(x) + \int_0^t \nabla_x G_{t-s}^\nu * (\tilde{p}_s^n . K_{\varepsilon_n} * \tilde{p}_s^n)(x) ds. \tag{3.8}$$

Let us now introduce for each n the coupling of processes $(Z^{in}, \bar{Y}^{in}, \bar{X}^i)_{1\leq i\leq n}$, where $(\bar{X}^i)$ are independent copies of X defined as in (2.4) on a certain probability space and $Z^{in}, \bar{Y}^{in}$ are driven, for each i respectively, following the same Brownian motion as $\bar{X}^i$. We will now compare the two processes $\bar{Y}^{in}$ and $\bar{X}^i$. So we need to estimate $w - \tilde{p}^n$. Using (1.8) and (3.8), we obtain

$$\tilde{p}_t^n(x) - w_t(x) = \int_0^t \int_{\mathbb{R}^2} \nabla_x G_{t-s}^\nu (x-y). \Big(K_{\varepsilon^n} * \tilde{p}_s^n(y) \tilde{p}_s^n(y) - K * w_s(y) w_s(y) \Big) dy ds$$

$$ \tag{3.9} $$

We will prove the

Theorem 3.4 *Let us define the norm*

$$|||f||| = \|f\|_1 + \|f\|_\infty.$$

The space of real functions f on $\mathbb{R}^2$ such that $|||f||| < +\infty$ is obviously a complete space.

We have

$$\sup_{t\le T} |||\tilde{p}_t^n - w_t||| \le \frac{C_T}{\nu}\varepsilon_n(\|w_0\|_\infty + \|w_0\|_1)\exp C_1(w_0)T, \tag{3.10}$$

where $C_1(w_0)$ and C_T are real constants.

The proof begins with a lemma.

Lemma 3.5 *For each $t \le T$, for each $x \in \mathbb{R}^2$,*

$$\left|\int_{\mathbb{R}^2}(K_{\varepsilon_n}(x-y) - K(x-y))\tilde{p}_t^n(y)dy\right| \le 2\varepsilon_n\|w_0\|_\infty. \tag{3.11}$$

and then

$$\|K_{\varepsilon_n} * \tilde{p}_t^n - K * w_t\|_\infty \le 2\varepsilon_n\|w_0\|_\infty + (K_1 + K_\infty)|||\tilde{p}_t^n - w_t|||. \tag{3.12}$$

Proof. 1) Since K_{ε_n} and K coincide for $|x| \ge \varepsilon_n$, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^2}(K_{\varepsilon_n}(x-y) - K(x-y))\tilde{p}_t^n(y)dy\right| &\le \int_{|x-y|\le\varepsilon_n} |K_{\varepsilon_n}(x-y) - K(x-y)||\tilde{p}_t^n(y)|dy \\
&\le \int_{|x-y|\le\varepsilon_n} (|K_{\varepsilon_n}(x-y)| + |K(x-y)|)|\tilde{p}_t^n(y)|dy \\
&\le 2\int_{|x-y|\le\varepsilon_n} |K(x-y)||\tilde{p}_t^n(y)|dy \\
&\le 2\|w_0\|_\infty \int_{|z|\le\varepsilon_n} |K(z)|dz \quad \text{by (3.7)} \\
&\le 2\varepsilon_n\|w_0\|_\infty \quad \text{by an easy computation.}
\end{aligned}$$

We have used that by definition, $|K_{\varepsilon_n}(x-y)| \le |K(x-y)|$.

2) For $x \in \mathbb{R}^2$,

$$\begin{aligned}
|K_{\varepsilon_n} * \tilde{p}_t^n(x) - K * w_t(x)| &\le 2\varepsilon_n\|w_0\|_\infty + \int_{\mathbb{R}^2}|K(x-y)||\tilde{p}_t^n(y) - w_t(y)|dy \\
&\le 2\varepsilon_n\|w_0\|_\infty + K_\infty\|\tilde{p}_t^n - w_t\|_1 + K_1\|\tilde{p}_t^n - w_t\|_\infty.
\end{aligned}$$

□

Let us now prove Theorem 3.4.

Proof. We consider (3.9). Then,

$$
\begin{aligned}
& |\tilde{p}_t^n(x) - w_t(x)| \\
\leq \ & \Big|\int_0^t \int_{I\!R^2} \nabla_x G_{t-s}^\nu(x-y).\Big(\tilde{p}_s^n(y)(K_{\varepsilon_n} * \tilde{p}_s^n(y) - K * w_s(y)) + K * w_s(y)(\tilde{p}_s^n(y) - w_s(y))\Big) dyds\Big| \\
\leq \ & \int_0^t \int_{I\!R^2} |\nabla_x G_{t-s}^\nu(x-y)| \Big(|\tilde{p}_s^n(y)|(2\varepsilon_n \|w_0\|_{infty} + (K_\infty + K_1)|||\tilde{p}_s^n - w_s|||) + \hat{K}(w_0)|||\tilde{p}_s^n - w_s|||\Big) dyds \\
& \quad \text{by (3.12), (2.5) and (1.5)} \\
\leq \ & \frac{K}{\sqrt{\nu}} \int_0^t \frac{1}{\sqrt{t-s}} \Big(\|w_0\|_\infty (2\varepsilon_n \|w_0\|_{infty} + (K_\infty + K_1)|||\tilde{p}_s^n - w_s|||) + \hat{K}(w_0)|||\tilde{p}_s^n - w_s|||\Big) ds \\
\leq \ & \frac{A_1}{\sqrt{\nu}} \Big(\varepsilon_n \|w_0\|_\infty^2 \sqrt{T} + \int_0^t \frac{1}{\sqrt{t-s}} |||\tilde{p}_s^n - w_s||| ds\Big),
\end{aligned}
$$

where A_1 is a real constant.

We consider now the L^1-norm of $\tilde{p}_t^n - w_t$ and by the same way, we obtain

$$
\begin{aligned}
& \int_{I\!R^2} |\tilde{p}_t^n(x) - w_t(x)| dx \\
\leq \ & \int_0^t \frac{1}{\sqrt{t-s}} \|\tilde{p}_s^n\|_1 (2\varepsilon_n + |||\tilde{p}_s^n - w_s|||) ds + \int_0^t \frac{1}{\sqrt{t-s}} \|K * w_s\|_\infty \|\tilde{p}_s^n - w_s\|_1 ds \\
\leq \ & \frac{A_2}{\nu} \Big(\varepsilon_n \|w_0\|_1 \sqrt{T} + \int_0^t \frac{1}{\sqrt{t-s}} |||\tilde{p}_s^n - w_s||| ds\Big),
\end{aligned}
$$

A_2 being a real contant. By associating the two previous results, we obtain that

$$|||\tilde{p}_t^n - w_t||| \leq \frac{A}{\nu} \Big((\|w_0\|_\infty + \|w_0\|_1)\varepsilon_n \sqrt{T} + A \int_0^t \frac{1}{\sqrt{t-s}} |||\tilde{p}_s^n - w_s||| ds\Big).$$

We iterate twice this inequality and obtain finally by Gronwall's Lemma that

$$\sup_{t \leq T} |||\tilde{p}_t^n - w_t||| \leq \frac{C_T}{\nu} \varepsilon_n (\|w_0\|_\infty + \|w_0\|_1) \exp(C_1(w_0)T).$$

□

Corollary 3.6 *We deduce that for each $t \leq T$*

$$\|K_{\varepsilon_n} * \tilde{p}_t^n - K * w_t\|_\infty \leq A_T \varepsilon_n, \tag{3.13}$$

where A_T is a real constant depending on ν, $\|w_0\|_\infty$ and $\|w_0\|_1$.

Proof. By using the two previous results, we obtain

$$\begin{aligned}
\|K_{\varepsilon_n} * \tilde{p}_t^n - K * w_t\|_\infty &\leq 2\varepsilon_n \|w_0\|_\infty + (K_\infty + K_1)|||\tilde{p}_t^n - w_t||| \\
&\leq 2\varepsilon_n \|w_0\|_\infty + \frac{C_T}{\nu}\varepsilon_n(\|w_0\|_\infty + \|w_0\|_1)\exp(C_1(w_0)T) = A_T \varepsilon_n.
\end{aligned}$$

□

We are now able to obtain our main theorem.

Theorem 3.7 *Let us consider a sequence of limiting independent processes solutions on a certain probability space of the stochastic differential equation*

$$\bar{X}_t^i = Z_0^i + \sqrt{2\nu} B_t^i + \int_0^t K * \tilde{p}_s(\bar{X}_s^i) ds, \tag{3.14}$$

where $(B^i)_{i \in \mathbb{N}}$ *are independent Brownian motions on* $\mathbb{R}^2$ *and* $(Z_0^i)_{i \in \mathbb{N}}$ *are independent of the Brownian motions and are identically distributed independent random variables ,with values in* $\mathbb{R}^2$ *and with law* $P_0(dx) = \frac{|w_0(x)|}{\|w_0\|_1} dx$*, where* w_0 *is a bounded integrable function on* $\mathbb{R}^2$*. The probability measure* P_s *is the common law of the* $\bar{X}_s^i$*, and* $\tilde{p}_s$ *the density of the signed measure* $\tilde{P}_s$ *associated with* P_s *by (2.3)*

Let us in the other hand consider $T > 0$ *and a sequence* $(\varepsilon_n)_{n \in \mathbb{N}}$ *of real numbers such that*

$$\lim_n \frac{M_{\varepsilon_n}}{\sqrt{n} L_{\varepsilon_n}} \exp(\|w_0\|_1 T L_{\varepsilon_n}) = 0.$$

We consider on the same probability space the coupled n*-particle system* $(Z^{in})_{1 \leq i \leq n}$ *defined as solutions of*

$$Z_t^{in} = Z_0^i + \sqrt{2\nu} B_t^i + \int_0^t \frac{1}{n} \sum_{j=1}^n h(Z_0^{jn}) K_{\varepsilon_n}(Z_s^{in} - Z_s^{jn}) ds. \tag{3.15}$$

Then, for each $1 \leq i \leq n$*,*

$$\lim_{n \to +\infty} E\left(\sup_{t \leq T} |Z_t^{in} - \bar{X}_t^i| \right) = 0, \tag{3.16}$$

(in the precise asymptotics given by (3.20).

That implies the propagation of chaos and the convergence in law (uniformly in time), of the "signed empirical measures" $\tilde{\mu}_s^n = \frac{1}{n} \sum_{i=1}^n h(Z_0^{in}) \delta_{Z_s^{in}}$ *to* $\tilde{P}_s$ *associated with the law of* $\bar{X}^i$*, and* $\tilde{P}_s$ *has a density equal to the solution* w *of the Navier-Stokes equation with initial condition* w_0*, as seen in Theorem 2.4.*

Proof.

$$\begin{aligned}
E\Big(\sup_{t\le T}|Z_t^{in}-\bar{X}_t^i|\Big) &\le E\Big(\sup_{t\le T}|Z_t^{in}-\bar{Y}_t^{i,n}|\Big)+E\Big(\sup_{t\le T}|\bar{Y}_t^{i,n}-\bar{X}_t^i|\Big)\\
&\le \frac{M_{\varepsilon_n}}{nL_{\varepsilon_n}}\exp(\|w_0\|_1TL_{\varepsilon_n})+E\Big(\sup_{t\le T}|\bar{Y}_t^{i,n}-\bar{X}_t^i|\Big).
\end{aligned} \tag{3.17}$$

But

$$|\bar{Y}_t^{i,n}-\bar{X}_t^i|\le\int_0^t|K_{\varepsilon_n}*\tilde{p}_s^n(\bar{Y}_s^{i,n})-K_{\varepsilon_n}*\tilde{p}_s^n(\bar{X}_s^i)|ds+\int_0^t|K_{\varepsilon_n}*\tilde{p}_s^n(\bar{X}_s^i)-K*w_s(\bar{X}_s^i)|ds.$$

The second right hand-side term is controled thanks to Corollary 3.6. It remains to study the first right hand-side term. By the same proof as Lemma 3.1 in Marchioro-Pulvirenti [7], we obtain that for x and z in $I\!R^2$,

$$|K_{\varepsilon_n}*\tilde{p}_s^n(x)-K_{\varepsilon_n}*\tilde{p}_s^n(z)|\le C_0(\|w_0\|_1+\|w_0\|_\infty)\phi(x,z),$$

where

$$\phi(x,z)=\tilde{\phi}(|x-z|),$$

and $\tilde{\phi}(r)=r(1-\ln r)$ if $0<r<1$ and $\tilde{\phi}(r)=1$ if $r\ge1$. Let us remark that the the function $\tilde{\phi}$ is non-decreasing and concave.

Then, by noting $C=C_0(\|w_0\|_1+\|w_0\|_\infty)$, we deduce that

$$|\bar{Y}_t^{i,n}-\bar{X}_t^i|\le A_T\varepsilon_n+C\int_0^t\phi(\bar{Y}_s^{i,n},\bar{X}_s^i)ds.$$

Then,

$$\begin{aligned}
E\Big(\sup_{u\le t}|\bar{Y}_u^{i,n}-\bar{X}_u^i|\Big) &\le A_T\varepsilon_n+C\int_0^tE\Big(\sup_{u\le s}\phi(\bar{Y}_u^{i,n},\bar{X}_u^i)\Big)ds\\
&\le A_T\varepsilon_n+C\int_0^tE\Big(\sup_{u\le s}\phi(\bar{Y}_u^{i,n},\bar{X}_u^i)\Big)ds\\
&\le A_T\varepsilon_n+C\int_0^tE\Big(\tilde{\phi}(\sup_{u\le s}|\bar{Y}_u^{i,n}-\bar{X}_u^i|)\Big)ds\\
&\qquad\text{since } \tilde{\phi} \text{ is non decreasing}\\
&\le A_T\varepsilon_n+C\int_0^t\tilde{\phi}(E\Big(\sup_{u\le s}|\bar{Y}_u^{i,n}-\bar{X}_u^i|\Big))ds\quad\text{by concavity of } \tilde{\phi}
\end{aligned}$$

Let us denote by $H(t)=E\Big(\sup_{u\le t}|\bar{Y}_u^{i,n}-\bar{X}_u^i|\Big)$. Then by the previous computations,

$$H(t)\le A_T\varepsilon_n+C\int_0^t\tilde{\phi}(H(s))ds. \tag{3.18}$$

As in [7], we introduce the solution $h(x_0, t)$ of the equation

$$z'(t) = C\tilde{\phi}(z(t)) \quad ; \quad z(0) = x_0 > 0.$$

Then, if $x_0 < 1$, and if $t_0 = \inf\{t, h(x_0, t) > 1\}$,

$$\begin{aligned} h(x_0, t) &= x_0^{\exp(-Ct)} \exp(1 - e^{-Ct}) \text{ if } h(x_0, t) < 1 \ , \ t < t_0 \\ &= 1 + C(t - t_0) \text{ if } h(x_0, t) \geq 1, \ t \geq t_0 \end{aligned}$$

and if $x_0 \geq 1$, $h(x_0, t) = x_0 + Ct$. Hence, by (3.18), we have

$$H(t) \leq h(A_T \varepsilon_n, t).$$

But since ε_n tends to 0 as n tends to infinity, for n sufficiently large, we deduce that

$$H(t) \leq (A_T \varepsilon_n)^{\exp(-Ct)} \exp(1 - e^{-Ct})$$

and finally

$$E\left(\sup_{u \leq T} |\bar{Y}_u^{i,n} - \bar{X}_u^i| \right) \leq (A_T \varepsilon_n)^{\exp(-CT)} \exp(1 - e^{-CT}) \tag{3.19}$$

for n sufficiently large.

Now, by (3.17), and (3.19), we finally deduce that

$$E\left(\sup_{t \leq T} |Z_t^{in} - \bar{X}_t^i| \right) \leq \frac{M_{\varepsilon_n}}{\sqrt{n} L_{\varepsilon_n}} \exp(\|w_0\|_1 T L_{\varepsilon_n}) + (A_T \varepsilon_n)^{\exp(-CT)} \exp(1 - e^{-CT}), \tag{3.20}$$

where $C = C_0(\|w_0\|_1 + \|w_0\|_\infty)$ and then tends to 0 when n tends to infinity.
$\square$

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