*Semi-classical limit for the Binegar-Zierau quantization of the minimal nilpotent orbits of $SO_o(2p,2)$

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Abstract

Let n be an odd integer greater than 3 and X be the $(n+1)$-dimensional anti-de Sitter spacetime. Binegar and Zierau have constructed in [1] a unitary representation $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ of the conformal group $SO_o(n+1,2)$ of X. They have shown that this particular representation is, in a certain algebraic sense, a quantization of the union, $\mathcal{O}^{min}_{n+1}=\mathcal{O}^{min+}_{n+1}\cup\mathcal{O}^{min-}_{n+1}$, of the two minimal nilpotent orbits in $so(n+1,2)^*$ [1]. In spite of this, it is known how to obtain $\mathcal{H}$ from $\mathcal{O}^{min}_{n+1}$ using a known quantization procedure. One reason for this interest in $\mathcal{H}$ is that, in the case $n=3$, $\mathcal{H}_+$ is the representation carried by the one-particle sector of the massless scalar field on the anti-de Sitter space-time X. In this paper, we strengthen this link between the co-adjoint orbits $\mathcal{O}^{min\pm}_{n+1}$ and $\mathcal{H}_\pm$, by studying the semi-classical limit of the latter. In this way, we clarify their appearance in the massless theory and corroborate the existing evidence that $\mathcal{H}_\pm$ is the "correct" quantization of $\mathcal{O}^{min\pm}_{n+1}$. As a preliminary, we show that the projection onto $so(n,2)^*$ of $\mathcal{O}^{min\pm}_{n+1}$ is the union of the two co-adjoint orbits $\mathcal{O}^{o\pm}_n$ and $\mathcal{O}^{min\pm}_n$ (Proposition IV.1); one of those is the phase space of the classical massless particle on X. We then show (Theorem VI.1 and Corollary VI.3) that the semi-classical limit of the spectral counting function of the generator of the $SO(2)$ subgroup of $SO_o(n,2)$ in the representation $\mathcal{H}_\pm$ is dominated by a Weyl term, expressed naturally in terms of the symplectic volume of a compact portion of the classical phase space $\mathcal{O}^{o\pm}_n$. Furthermore, we show (Theorem VI.4) that the highest weight vectors of the representation coincide in the semi-classical limit with the BKW functions constructed starting from $\mathcal{O}^{o\pm}_n$. We show in addition that, eventhough the orbit method applied to $\mathcal{O}^{min\pm}_{n+1}$ does not yield $\mathcal{H}_\pm$, it nevertheless establishes a natural relation between them. Namely, the simple $SO(n+1)\times S0(2)$-modules appearing in $\mathcal{H}_+$ are those we obtain if we apply the orbit method to integral $SO(n+1)\times S0(2)$-orbits contained in the projection on $(so(n+1)\oplus so(2))^*$ of $\mathcal{O}^{min\pm}_{n+1}$ (section VII). As a byproduct of our analysis, we study the restriction to $SO_o(n,2)$ of $\mathcal{H}$ (Proposition VIII.1) and we show that the unitary structure on $\mathcal{H}$ is exacly a Klein-Gordon scalar product on X (Proposition IX.2).

I Introduction.

Let n be an integer greater than 3 and $\mathcal{O}^{min}_{n+1}=\mathcal{O}^{min+}_{n+1}\cup\mathcal{O}^{min-}_{n+1}$ be the union of the two minimal nilpotent orbits of $so(n+1,2)^*$, the vector dual of the Lie algebra $so(n+1,2)$ of the Lie group $G_c\equiv SO_o(n+1,2)$. In [1] Binegar and Zierau have studied a direct sum $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ of two unitary irreducible representations $\mathcal{H}_\pm$ of G_c. In particular, they showed that the annihilator of this representation in the enveloping algebra of $so(n+1,2)^{\mathbf{C}}$ is the so-called Joseph ideal, which is the maximal primitive ideal associated to $\mathcal{O}^{min\,\mathbf{C}}_{n+1}$, the complexification of $\mathcal{O}^{min}_{n+1}$ [1]-[2]. This suggests that the representation $\mathcal{H}$ is naturally associated to the (complexified) minimal orbit, or in the language of the physics literature, that $\mathcal{H}$ is a quantization of $\mathcal{O}^{min}_{n+1}$.

Furthermore, the $(n+1)$-dimensional real hyperboloid

$$X=\{(x_1,\cdots,x_{n+2})\in\mathbf{R}^{n+2}\mid\sum_{j=1}^{n}x_j^2-x_{n+1}^2-x_{n+2}^2=-1\}$$

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is a Lorentzian manifold known as the generalization of the classical anti-de Sitter space-time. The restriction of the representation $\mathcal{H}_+$ of $SO_o(n+1,2)$ to its isometry group $G \equiv SO_o(n,2)$ defines the one-particle sector of the scalar massless free field on X [3]-[4]-[5]-[6]-[7]. This also suggests it is the correct quantization of the phase space of the classical massless free particle on X, which is a nilpotent co-adjoint orbit $\mathcal{O}_n^{o+}$ of the isometry group $SO_o(n,2)$ of X. Our interest in the notion of masslessness on X, which is closely related to the notion of conformal invariance on X, is motivated by the fact that massless particles play a crucial role in many physical theories on X (see, for example, [3]-[5]-[7] for more detail).

In spite of all this, no known method of quantization actually leads to a construction of $\mathcal{H}$ (respectively $\mathcal{H}_\pm$), starting from the coadjoint orbit $\mathcal{O}_{n+1}^{min}$ (respectively $\mathcal{O}_n^{o\pm}$).

On the other hand, if the links established in the above works are really natural, we expect to observe their manifestation also in the semi-classical limit. It is the primary goal of this paper (section VI) to study the semi-classical limit of $\mathcal{H}_+$ and to show it naturally leads to $\mathcal{O}_{n+1}^{min+}$ and to $\mathcal{O}_n^{o+}$ (the case of $\mathcal{H}_-$ being similar). We note that the case $n=1$, which is somewhat singular, was treated in [8].

To prepare our analysis, we first establish in sections II to IV the link between $\mathcal{O}_{n+1}^{min+}$ and the classical phase space $\mathcal{O}_n^{o+}$ of the free massless particle on X. Here $\mathcal{O}_n^{o+}$ is obtained as the symplectic reduction of the zero-mass hyperboloid $\,_0^+$ in TX. We show in particular (Proposition IV.1) that $\mathcal{O}_n^{o+}$ is open and dense in $\mathcal{R}_{so(n,2)}(\mathcal{O}_{n+1}^{min+})$, where $\mathcal{R}_{so(n,2)}$ stands for the natural projection of $so(n+1,2)^*$ on $so(n,2)^*$.

Then, in section V, we recall the definition of the Binegar-Zierau representation $\mathcal{H}$ of the group G_c. In fact, $\mathcal{H}$ is realized as the kernel of the Laplace-Beltrami operator acting on some space of homogeneous functions on the cone :

$$C_{n+1} = \{(x_0, \cdots, x_{n+2}) \in \mathbf{R}^{n+3} \setminus \{0\} \mid \sum_{j=0}^{n} x_j^2 - x_{n+1}^2 - x_{n+2}^2 = 0\}.$$

A crucial point for our analysis will be the realization of $\mathcal{H}$ as a space of functions on X. More precisely, we consider these homogeneous functions on C_{n+1} as functions on X by taking their restriction to the hypersurface $x_0 = 1$ of C_{n+1} (V.3).

Then the analysis of the semi-classical behaviour of the representation $\mathcal{H}_\pm$ is given in section VI. In particular, we show (Theorem VI.1) that, for all functions $f : \mathbf{R} \to \mathbf{R}$, Riemann integrable on any interval and of sufficiently fast decrease, we have :

$$Tr(f(-i\hbar X_{n+1n+2})) = \frac{1}{(2\pi\hbar)^n} \int_{\mathcal{O}_n^{o+}} (f \circ J_{n+1n+2}) \omega_o^n + o(\hbar^{-n}). \qquad (*)$$

Here X_{n+1n+2} denotes the generator in $\mathcal{H}$ of $SO(2)$ and J_{n+1n+2} is the Hamiltonian of the $SO(2)$-co-adjoint action on $\mathcal{O}_n^{o+}$. The operator $-i\hbar X_{n+1n+2}$ has a nice physical interpretation (see section VI). It is the analog of an "energy operator" and its spectrum has a lower bound. In particular, as a corollary (Corollary VI.3), we obtain that if f is the characteristic function of a fixed interval $[a,b]$ then $(*)$ reads :

$$\sharp([a,b] \cap Spec(-i\hbar X_{n+1n+2})) = \frac{1}{(2\pi\hbar)^n} volume(J_{n+1n+2}^{-1}([a,b])) + o(\hbar^{-n})$$

which is exactly the Weyl's law for the operator $-i\hbar X_{n+1n+2}$. Morevover, using the realization, given in section V, of the representation $\mathcal{H}$ of G_c as a space of functions on the hyperboloid X, we study the semi-classical behaviour of some highest weight vectors of the simple K_c-modules in $\mathcal{H}$. Actually, we establish a link, in the semi-classical limit $\hbar \to 0$, between a level surface $\mathcal{L}_e$ of J_{n+1n+2} and some eigenfunctions of the operator $-i\hbar X_{n+1n+2}$ acting on $\mathcal{H}$. More precisely, we show (Theorem VI.4) that, in the semi-classical limit $\hbar \to 0$, these eigenfunctions are localized on $\pi(\mathcal{L}_e)$, where π is the natural projection of TX on X.

After Theorems $VI.1$ and $VI.4$, a third link between the unitary irreducible representation $\mathcal{H}_\pm$ of the group G_c and the orbit $\mathcal{O}_{n+1}^{min\pm}$ of G_c is given in section VII. We show that, eventhough the orbit method applied to $\mathcal{O}_{n+1}^{min\pm}$ does not yield $\mathcal{H}_\pm$, it nevertheless establishes a natural relation between them. Namely, the simple $SO(n+1)\times S0(2)$-modules appearing in $\mathcal{H}_+$ are those we obtain if we apply the orbit method to integral $SO(n+1)\times S0(2)$-orbits contained in the projection on $(so(n+1)\oplus so(2))^*$ of $\mathcal{O}_{n+1}^{min\pm}$ $((VII.5)-(VII.6))$.

Finally, as a byproduct of our analysis, we compare the unitary structure on $\mathcal{H}$ with the Klein-Gordon scalar product on X. First, we compute explicitely the restriction to the group G of $\mathcal{H}$. And we obtain (Proposition $VIII.1$) that, as a representation of G, $\mathcal{H}_\pm$ is the direct sum $W_{\pm,0}\oplus W_{\pm,1}$ of two unitary irreducible representations of G. It turns out (Proposition $IX.1$) that not all of the restriction to G of $\mathcal{H}_\pm$ belongs to $L^2(X)$, the usual space of square integrable functions on X. In particular, in the direct sum $W_{\pm,0}\oplus W_{\pm,1}$, only $W_{\pm,1}$ belongs to $L^2(X)$. Then it is completely natural to look for a nice geometric expression for the inner product, different from the usual L^2-integral. We show (Proposition $IX.2$) that the unitary structure (V.11) defined by Binegar and Zierau on $\mathcal{H}$ is exactly the Klein-Gordon scalar product on the hyperboloid X.

It is of interest to compare our results with the well known harmonic analysis of X [9]-[10]-[11]-[12]-[13]. One has the following decomposition of the quasi-regular representation of G on X

$$L^2(X)=\int_{t>0}^{\oplus}\mathcal{H}_t\,dm(t)+\sum_{\rho\in\mathbf{Z},\ \rho<-\frac{n}{2}}\mathcal{H}_\rho \tag{I.1}$$

in a continuous series and a discrete series of unitary irreducible representations, where dm is the Plancherel measure on $\mathbf{R}$. The $\mathcal{H}_\mu$'s are realized as eigenspaces of the Laplace-Beltrami operator Δ_X of X,

$$\Delta_X f=-\mu(\mu+n)f\quad\text{if } f\in\mathcal{H}_\mu. \tag{I.2}$$

There is a nice and suggestive relation, inspired by the philosophy behind the orbit method, and explained in some more detail in the next section, between the direct integral decomposition $(I.1)$ and the G-orbits in the tangent bundle TX of X. More precisely, if one identifies the cotangent bundle of X with its tangent bundle TX in the usual way, then there is a natural symplectic structure on TX. Moreover, the symplectic reduction of a G-orbit in TX of X is identified, via the moment map, with a co-adjoint orbit of G. Consider now the co-adjoint orbits obtained through the symplectic reduction of the G-orbits in TX. Apart from the trivial orbit, they consist of three families described in the next section. Two of them are semisimple and applying the orbit method to them yields precisely the relative continous and discrete series appearing in $(I.1)$. The discrete series is associated to the orbits obtained by symplectic reduction of the mass hyperboloids in TX and those representations show up in the one-particle sectors of the massive free scalar fields on X. In short, quantizing, via the orbit method, all semi-simple orbits obtained by reduction from the orbits in TX, one obtains all representations showing up in the Plancherel formula for $L^2(X)$. This is a rather pleasing picture, where it is not for the third family, which consists of the two nilpotent orbits $\mathcal{O}_n^{o\pm}$ described above. One remarks that, in the correspondence between the decomposition of $L^2(X)$ given in $(I.1)$ and the G-orbits in TX, they do not seem to play a role. We have argued in this paper that the representation associated to them is $\mathcal{H}_\pm$. A first argument in favour of this lies in the observation that $\mathcal{O}_n^{o\pm}$ is the (locally conformally invariant) phase space of the massless particle on X, combined with the work on the conformally coupled massless scalar quantum field theory on X [3]-[4]-[5]-[6]-[7] which shows (for $n=3$) that the one particle sector of this field is given by the completely reducible $\mathcal{H}\pm=W_{\pm,0}\oplus W_{\pm,1}$, realized as a subspace of the space $E_{\frac{n^2-1}{4}}(X)$ of distributional solutions of the conformally invariant wave equation on X :

$$\Delta_X\phi=\frac{n^2-1}{4}\phi.$$

The work of [1] and our semi-classical analysis give yet another argument in favour of this thesis.

But then, since only $W_{\pm,1}$ belongs to $L^2(X)$, with $W_{\pm,1} = \mathcal{H}_\rho$ and $\rho = -\frac{n}{2} - \frac{1}{2} \in \mathbf{Z}$, it is clear why the nilpotent orbits do not "contribute" to the Plancherel formula.

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II The symplectic reduction of the G-orbits in TX.

Let $n \geq 3$ be an integer. We consider the bilinear form $< \cdot, \cdot >$ on $\mathbf{R}^{n+2}$ defined by :

$$< x, y >= \sum_{i,j=1}^{n+2} \eta_{ij} x_i y_j \equiv \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1} - x_{n+2} y_{n+2} \;\; \forall (x,y) \in \mathbf{R}^{n+2} \times \mathbf{R}^{n+2}, \qquad \text{(II.1)}$$

with the pseudo-orthogonal group $O(n,2)$ as group of isometries. Let X be the one-sheeted hyperboloid

$$X = \{x \in \mathbf{R}^{n+2} \,|< x, x >= -1\}. \qquad \text{(II.2)}$$

It is clear that the natural action of $O(n,2)$ on $\mathbf{R}^{n+2}$ induces on X a transitive action of the identity component $G \equiv SO_o(n,2)$ of $O(n,2)$. Moreover, we shall identify the tangent bundle of X with the submanifold of $\mathbf{R}^{n+2} \times \mathbf{R}^{n+2}$:

$$TX = \{(x,y) \in \mathbf{R}^{n+2} \times \mathbf{R}^{n+2} \,|< x, x >= -1 \text{ and } < x, y >= 0\}. \qquad \text{(II.3)}$$

The natural action of G on TX is no longer transitive, its orbits are just $X \times \{0\}$ and the connected components of the $\;_\lambda$'s, $\lambda \in \mathbf{R}$, where :

$$_\lambda = \{(x,y) \in TX \mid y \neq 0 \text{ and } < y, y >= \lambda\}. \qquad \text{(II.4)}$$

So, the decomposition of TX into G-orbits follows :

$$TX = [\cup_{\lambda > 0} \;_\lambda] \cup [\cup_{\lambda \leq 0} \;^\pm_\lambda] \cup [X \times \{0\}]. \qquad \text{(II.5)}$$

Note that $\;_\lambda$ is a $2n+1$-dimensional submanifold of TX which is connected if $\lambda > 0$, and has two connected components $\;^\pm_\lambda$ if $\lambda \leq 0$. For convenience, we shall use $< \cdot, \cdot >$ to identify TX with the cotangent bundle T^*X of X, so that there is a symplectic structure on TX given by the symplectic form $\omega = \sum_{i=1}^{n+2} \eta_{ii} dx_i \wedge dy_i$. The restriction ω_λ of ω to $\;_\lambda$ equips $\;_\lambda$ with the structure of a presymplectic submanifold of TX. The symplectic reduction of $\;_\lambda$ is easily computed using the moment map. More precisely, let $\{X_{ij}\}_{1 \leq i,j \leq n+2}$ be the standard basis of the Lie algebra $\;\equiv so(n,2)$ of G, with commutation relations :

$$[X_{ij}, X_{kl}] = \eta_{ik} X_{jl} + \eta_{jl} X_{ik} - \eta_{il} X_{jk} - \eta_{jk} X_{il}. \qquad \text{(II.6)}$$

The generators of the natural action of G on TX are the hamiltonian vector fields $\{\tilde{X}_{ij}\}_{1 \leq i,j \leq n+2}$ given by :

$$\tilde{X}_{ij}(x,y) = \eta_{jj} x_i \frac{\partial}{\partial x_j} - \eta_{ii} x_j \frac{\partial}{\partial x_i} + \eta_{jj} y_i \frac{\partial}{\partial y_j} - \eta_{ii} y_j \frac{\partial}{\partial y_i} \quad \forall (x,y) \in TX \qquad \text{(II.7)}$$

with hamiltonians :

$$J_{ij}(x,y) = x_i y_j - x_j y_i \;\; \forall (x,y) \in TX \;\; \forall 1 \leq i,j \leq n + II. \qquad \text{(II.8)}$$

If * denotes the vector dual of and $\{X^*_{ij}\}_{1\leq i,j\leq n+2}$ the dual basis of $\{X_{ij}\}_{1\leq i,j\leq n+2}$, then the Ad^*-equivariant moment map $J : TX \to {}^*$ associated to the (strongly hamiltonian) action of G on TX is given by :

$$J(x,y) = \frac{1}{2}\sum_{i,j=1}^{n+2} J_{ij}(x,y)X^*_{ij} \quad \forall (x,y)\in TX. \tag{II.9}$$

In particular, this implies that the image under J of the G-orbits $^{\pm}_{\lambda}$ (resp. $_{\lambda}$) are $2n$-dimensional co-adjoint orbits $\mathcal{O}^{\lambda\pm}_n$ (resp. $\mathcal{O}^{\lambda}_n$) of G if $\lambda \leq 0$ (resp $\lambda > 0$). Actually, we shall identify, via the moment map J, the orbits $\mathcal{O}^{\lambda\pm}_n$ (resp. $\mathcal{O}^{\lambda}_n$) with the symplectic reduction of $^{\pm}_{\lambda}$ (resp. $_{\lambda}$) if $\lambda \leq 0$ (resp $\lambda > 0$). For the sequel, it is important to note that among the orbits $\mathcal{O}^{\lambda\pm}_n$ (resp. $\mathcal{O}^{\lambda}_n$), $\lambda \leq 0$ (resp $\lambda > 0$), only $\mathcal{O}^{o\pm}_n$ are nilpotent.

The orbit method [14] establishes, via the moment map J, a link between the decompositions $(I.1)$ and $(II.5)$. More precisely, it associates to the orbits $\mathcal{O}^{\lambda\pm}_n$, $\lambda < 0$ integer, the relative discrete series of X, and to the orbits $\mathcal{O}^{\lambda}_n$, $\lambda > 0$ real, the relative continuous series of X. However, this method associates to the orbits $\mathcal{O}^{o\pm}_n$ and $\{0\}$, unitary representations of G which do not belong to $L^2(X)$. More precisely, for the trivial orbit one obtains the trivial representation and for $\mathcal{O}^{o\pm}_n$ one obtains an irreducible unitary representation realized as a subspace of the distributional solutions of the equation $\Delta_X f = -\frac{n}{2}f$ on X [15]. In both cases the representations do not appear in the decomposition $(I.1)$ of $L^2(X)$ and they do not admit a local $so(n+1,2)$-action. In this sense, they are not the "right" ones.

III The conformal compactification of X.

To understand the action of the group $SO_o(n+1,2)$ on the G-orbits $\mathcal{O}^{o\pm}_n$, we recall the conformal compactification of the hyperboloid X [16]. The bilinear form $< \cdot,\cdot >$ on $\mathbf{R}^{n+2}$ induces, by restriction, a Lorentz metric η on X. We shall say that a vector field ξ on X is conformal if the Lie derivative $\mathcal{L}(\xi)\eta$ of η along ξ is equal to $f\eta$ where f is a real smooth function on X. It is well known that the set of conformal vector fields on X generate a Lie algebra isomorphic to the Lie algebra $so(n+1,2)$ of the $SO_o(n+1,2)$ [17]. For this reason we call $SO_o(n+1,2)$ the conformal group of X and denote it (resp. its Lie algebra) by G_c (resp. $_c$). In analogy with $(II.1)$, note that G_c is just the identity component of the isometry group of the bilinear form $< \cdot,\cdot >_c$ on $\mathbf{R}^{n+3}$ defined by :

$$< x,y >_c = \sum_{i,j=0}^{n+2} \beta_{ij}x_i y_j \equiv \sum_{i=0}^{n} x_i y_i - x_{n+1}y_{n+1} - x_{n+2}y_{n+2} \;\; \forall (x,y)\in \mathbf{R}^{n+3}\times\mathbf{R}^{n+3}. \tag{III.1}$$

It will be useful for us to realize G as the closed subgroup of G_c fixing the first coordinate x_o, that is :

$$i_G : G \hookrightarrow G_c,\; g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}. \tag{III.2}$$

Let us define the asymptotic cones $C_n \subset \mathbf{R}^{n+2}$ and $C_{n+1} \subset \mathbf{R}^{n+3}$ associated respectively to $< \cdot,\cdot >$ and $< \cdot,\cdot >_c$ by :

$$C_n = \{x \in \mathbf{R}^{n+2} \mid < x,x > = 0 \text{ and } x \neq 0\} \;\text{ and }\; C_{n+1} = \{x \in \mathbf{R}^{n+3} \mid < x,x >_c = 0 \text{ and } x\neq 0\}. \tag{III.3}$$

On the other hand, let C^o_{n+1} and $C^{'}_{n+1}$ be the subsets of C_{n+1} defined by :

$$C^o_{n+1} = \{x \in C_{n+1} \mid x_o = 0\} \simeq C_n \;\text{ and }\; C^{'}_{n+1} = \{x\in C_{n+1} \mid x_o \neq 0\}. \tag{III.4}$$

Let $\mathbf{PR}^{n+3}$ be the projective space of $\mathbf{R}^{n+3}\setminus\{0\}$ and p the canonical surjection :

$$p : \mathbf{R}^{n+3}\setminus\{0\} \to \mathbf{PR}^{n+3},\; x \mapsto [x]. \tag{III.5}$$

We denote by $\mathbf{P}C_{n+1}$ (resp. $\mathbf{P}C^o_{n+1}$, $\mathbf{P}C^{'}_{n+1}$) the image under p of C_{n+1} (resp. C^o_{n+1}, $C^{'}_{n+1}$). Then it is easy to see that the map :

$$X \rightarrow \mathbf{P}C^{'}_{n+1}, \; x \mapsto [(1, x)] \tag{III.6}$$

is a diffeomorphism. $\mathbf{P}C^{'}_{n+1}$ is called the projectivisation of X. Moreover, $\mathbf{P}C^{'}_{n+1}$ is a dense open subset of $\mathbf{P}C_{n+1}$, which is compact. We say that $\mathbf{P}C_{n+1}$ is the conformal compactification of X and we will denote it by $\overline{X}$. This terminology is justified by the fact that the group G_c acts transitively on $\mathbf{P}C_{n+1}$ but not on $X \simeq \mathbf{P}C^{'}_{n+1}$. However G_c acts locally on X and we can make more explicit this action. The generators of the G_c-action on C_{n+1} are the right-invariant vector fields $\{V_{ij}\}_{0\leq i,j\leq n+2}$ defined by :

$$V_{ij}(x) = \eta_{jj} x_i \frac{\partial}{\partial x_j} - \eta_{ii} x_j \frac{\partial}{\partial x_i} \quad \forall x \in C_{n+1}. \tag{III.7}$$

Then the generators $\{V^c_{ij}\}_{0\leq i,j\leq n+2}$ of the local action of G_c on X are given by the projection of $\{V_{ij}\}_{0\leq i,j\leq n+2}$:

$$V^c_{ij}(x) = X_{ij}(x) \;\; \forall x \in X \;\; \forall 1 \leq j \leq n+2 \tag{III.8}$$

and

$$V^c_{j0}(x) = \sum_{i=1}^{n+2} x_i X_{ij}(x) \;\; \forall x \in X \;\; \forall 1 \leq i,j \leq n+2 \tag{III.9}$$

where the X_{ij}'s are the generators of the natural G-ation on $\mathbf{R}^{n+2}$, that is the projection on X of the $\tilde{X}_{ij}$'s defined by $(II.7)$.

IV The local conformal structure on $\mathcal{O}^{o+}_n$.

In this section we analyze in detail the conformal structure of the nilpotent orbits $\mathcal{O}^{o\pm}_n$. First note that there is also a symplectic structure on $T\overline{X}$, since there is a unique G_c-invariant symplectic form $\overline{\omega}$ on $T\overline{X}$ which coincides with ω on TX. Next the injection i_G of G in G_c defined by $(III.2)$ induces a projection :

$$\mathcal{R} \;\; : \;\; {}^{*}_{c} \longrightarrow \;\; {}^{*}. \tag{IV.1}$$

In particular we have :

$$\mathcal{R} \;\; (\mathcal{O}^{min+}_{n+1}) \supset \mathcal{O}^{o+}_n. \tag{IV.2}$$

Indeed, from the injection $(III.6)$, we define the injection $\tilde{i}$ of X in $\overline{X}$ by :

$$X \ni x \mapsto [\tilde{x}] = p(\tilde{x}) \in \overline{X} \subset \mathbf{P}C_{n+1}. \tag{IV.3}$$

We denote by $d\tilde{i} : TX \rightarrow T\overline{X}$ its differential. Let (x^o, y^{o+}) be a point of $\;{}^{+}_{o} \subset TX$ such that $J((x^o, y^{o+})) = X^o \equiv (X^{1n+1} + X^{n+1n+2}) \in \;\; ^{*}$. It is easy to see that [15] :

$$J_c(d\tilde{i}((x^o, y^{o+}))) = (-X^{01} + X^{0n+2} + X^{1n+1} + X^{n+1n+2}) \in \mathcal{O}^{min+}_{n+1} \tag{IV.4}$$

such that $\mathcal{R} \;\; (J_c(d\tilde{i}((x^o, y^{o+})))) = X^o$. Note that $dim \; \mathcal{O}^{min\pm}_{n+1} = dim \; \mathcal{O}^{o\pm}_n$, but the restriction of $\mathcal{R}$ to $\mathcal{O}^{min\pm}_{n+1}$ is not a diffeomorphism onto $\mathcal{O}^{o\pm}_n$ [15], as is true when $n = 1$ [8]. Actually $(IV.2)$ shows that $\mathcal{O}^{o+}_n$ is an open G-orbit in the projection on $\;\; ^{*}$ of $\mathcal{O}^{min+}_{n+1}$. The same arguments hold for $\mathcal{O}^{o-}_n$ and $\mathcal{O}^{min-}_{n+1}$.

On the other hand, the local action of G_c on X defined by $(III.8) - (III.9)$ induces a local action of G_c on $\;{}^{\pm}_{o} \subset TX$ and also on $\mathcal{O}^{o\pm}_n$. Actually we have :

Proposition IV.1 *The local action of G_c on $\mathcal{O}_n^{o\pm}$, induced by $(III.8)-(III.9)$, is not global. Moreover, as a G-orbit, the minimal nilpotent G_c-orbit $\mathcal{O}_{n+1}^{min\pm}$ splits in two G-orbits :*

$$\mathcal{R}\ (\mathcal{O}_{n+1}^{min\pm}) = \mathcal{O}_n^{o\pm} \cup \mathcal{O}_n^{min\pm}, \tag{IV.5}$$

where $\mathcal{O}_n^{min\pm}$ is the minimal nilpotent G-orbit defined as $\mathcal{O}_{n+1}^{min\pm}$ while replacing G_c by G.

In order to prove this proposition (section $IV.2$), it will be useful to see how to realize $\mathcal{O}_{n+1}^{min\pm}$ as the symplectic reduction of a certain presymplectic submanifold in the tangent bundle TC_{n+1} of the cone C_{n+1}.

IV.1 $\mathcal{O}_{n+1}^{min\pm}$ and symplectic reduction of G_c-orbits in TC_{n+1}.

Let $TC_{n+1} = \{(\sigma,\sigma^{'}) \in C_{n+1} \times \mathbf{R}^{n+3} \,|< \sigma, \sigma^{'} >_c = 0\}$. Then, obviously, TC_{n+1} is a smooth connected submanifold of $\mathbf{R}^{n+3} \times \mathbf{R}^{n+3}$ of dimension $2n+4$ on which the group G_c does not act transitively. Moreover, the restriction ω^c to TC_{n+1} of the canonical symplectic form ω^c on $\mathbf{R}^{n+3} \times \mathbf{R}^{n+3}$ equips TC_{n+1} with the structure of a presymplectic submanifold of $\mathbf{R}^{n+3} \times \mathbf{R}^{n+3}$. More precisely, one easily checks that the kernel of ω^c on TC_{n+1} is generated by the vector fields $X_1 = \sigma \frac{\partial}{\partial \sigma^{'}}$ and $X_2 = \sigma \frac{\partial}{\partial \sigma} - \sigma^{'} \frac{\partial}{\partial \sigma^{'}}$.

On the other hand, let E_o be the subset of TC_{n+1} defined by :

$$E_o = \{(\sigma,\sigma^{'}) \in TC_{n+1} \,|< \sigma, \sigma^{'} >_c = 0 \ \text{ and } \ \sigma^{'} \neq \beta\sigma \ \forall \beta \in \mathbf{R}\}. \tag{IV.6}$$

The following lemma describes the G_c-orbits in E_o and their symplectic reduction.

Lemma IV.2 *(i) E_o is a $2n+3$-dimensional smooth submanifold of TC_{n+1} which has two connected components $E_o^{\pm}$.*

(ii) G_c acts transitively on each connected component of E_o.

(iii) The kernel of the restriction ω_o^c of ω^c to E_o is generated by X_1, X_2 and $X_3 = \sigma^{'} \frac{\partial}{\partial \sigma}$.

(iv) Let $J_c : \mathbf{R}^{n+3} \times \mathbf{R}^{n+3} \to {}^{*}_{c}$ be the unique Ad^*-equivariant moment map associated to the (strongly hamiltonian) action of G_c on $\mathbf{R}^{n+3} \times \mathbf{R}^{n+3}$. Then the symplectic reduction of $E_o^{\pm}$ is given by :

$$\mathcal{O}_{n+1}^{min\pm} = J_c(E_0^{\pm}). \tag{IV.7}$$

Proof : For (i), the condition $\sigma^{'} \neq \beta\sigma$ for all $\beta \in \mathbf{R}$ implies that E_o is a $2n+3$-dimensional smooth submanifold of TC_{n+1}. On the other hand, since C_{n+1} is connected, E_o has as many connected components as the fibre $E_o(\tilde{x}^o)$ of the point $\tilde{x}^o = (1, x^o) = (1,0,0,...,1,0) \in C_{n+1}$ relatively to the bundle $TC_{n+1} \to C_{n+1}$. Then (i) follows immediately from :

$$E_o(\tilde{x}^o) = \{v \in \mathbf{R}^{n+3} \setminus \{0\} \mid v_o = v_{n+1} \text{ and } \sum_{i=1}^{n} v_i^2 - v_{n+2}^2 = \lambda\}. \tag{IV.8}$$

A straightforward calculation shows that $G_c(\tilde{x}^o)$, the isotropy group of $\tilde{x}^o$, acts transitively on each connected component of $E_o(\tilde{x}^o)$. (ii) follows from the fact that $G_c(\tilde{x}_o)$ acts transitively on C_{n+1}. For (iii), it suffices to remark that :

$$T_{(\sigma,\sigma^{'})}E_\lambda = \{(H,K) \in \mathbf{R}^{n+1,2} \times \mathbf{R}^{n+1,2} \mid \quad \begin{array}{l} < \sigma, H >_c = < \sigma^{'}, K >_c = 0 \\ < \sigma, K >_c + < H, \sigma^{'}, >_c = 0\} \end{array} \tag{IV.9}$$

and one checks that the kernel of ω_o^c is generated by X_1, X_2 and X_3. To prove (iv), first note that since $\forall\ \mu \in \mathbf{R}^*$, $J_c(\mu\sigma, \frac{1}{\mu}\sigma) = J_c(\sigma, \sigma^{'})$, one obtains $(IV.7)$. On the other hand the co-adjoint

orbits $\mathcal{O}_{n+1}^{min\pm}$ of G_c are nilpotent. Indeed, if $f \in \mathcal{O}_{n+1}^{min\pm}$, then there exists $(\sigma, \sigma^{'}) \in E_o^{\pm}$ such that $J_c(\sigma, \sigma^{'}) = f$. But $(\lambda\sigma, \lambda\sigma^{'})$ belongs to $E_o^{\pm}$ and $J_c(\lambda\sigma, \lambda\sigma^{'}) = \lambda^2 f$. Since $J_c(E_o^{\pm}) = \mathcal{O}_{n+1}^{min\pm}$, then $\lambda^2 f \in \mathcal{O}_{n+1}^{min\pm}$ $\forall\lambda \in \mathbf{R}^*$, which exactely says that $\mathcal{O}_{n+1}^{min\pm}$ is nilpotent. Actually $\mathcal{O}_{n+1}^{min\pm}$ are exactly the minimal nilpotent G_c-orbits in ${}^*_c$ [18]. Finally the symplectic reduction of $E_o^{\pm}$ is computed using the moment map J_c, in the same way as explained in the section II, by considering the transport, via J_c, of the canonical symplectic structure on the G_c-orbits $\mathcal{O}_{n+1}^{min\pm}$. It is clear that $\mathcal{O}_{n+1}^{min\pm}$ is $2n$-dimensional.
We have :

Proposition IV.3 *Let p be the canonical surjection defined by $(III.5)$ and dp its differential, then :*

- *(i) $dp(E_o^{\pm})$ are G_c-orbits in $T\overline{X}$.*
- *(ii) Equipped with the restriction of $\overline{\omega}$, $dp(E_o^{\pm})$ is a $2n+1$-dimensional presymplectic submanifold of $T\overline{X}$ with a $2n$-dimensional symplectic reduction.*
- *(iii) We have also :*

$$dp(E_o^{\pm}) \cap TX = {}^{\pm}_{o}. \tag{IV.10}$$

So ${}^{\pm}_{o}$ are G-orbits in TX admitting a local action of the conformal group G_c of X.

Proof : First note that p commutes with the action of G_c. (i) follows immediately from Lemma $IV.2$. For (ii) it suffices to remark that the vector field X_1, instead of X_2 et X_3, is in the kernel of dp. Then the point (ii) comes directely from (iii) of Lemma $IV.2$. Since the kernel of dp is generated by X_1 and $\sigma\frac{\partial}{\partial\sigma} + \sigma^{'}\frac{\partial}{\partial\sigma^{'}}$, which are tangent to $E_o^{\pm}$, (iii) of Lemma $IV.2$ implies the point (iii). Considering all the G-orbits in TX, then, except $X \times \{0\}$, ${}^{\pm}_{o}$ are the only G-orbits admitting a local action of G_c.
We turn know to the proof of Proposition $IV.1$.

IV.2 Proof of Proposition IV.1.

We shall only consider $\mathcal{O}_{n+1}^{min+}$ since our arguments will still hold for $\mathcal{O}_{n+1}^{min-}$. We show that $\mathcal{O}_{n+1}^{min+}$ is the disjoint union of two G-orbits given by $J_c(E_{01}^{+})$ and $J_c(E_{02}^{+})$ where :

$$E_{01}^{+} = \{(\sigma, \sigma^{'}) \in E_o^{+} \mid \sigma_o = \sigma_o^{'} = 0\} \tag{IV.11}$$

and

$$E_{02}^{+} = \{(\sigma, \sigma^{'}) \in E_o^{+} \mid \sigma_o\sigma_o^{'} \neq 0\}. \tag{IV.12}$$

Moreover $J_c(E_{01}^{+})$ is a $2n-1$-dimensional presymplectic submanifold of $\mathcal{O}_{n+1}^{min+}$ and $J_c(E_{02}^{+})$ is a $2n$-dimensional symplectic submanifold.

Indeed for $J_c(E_{01}^{+})$ it suffices to note that one can identify E_{01}^{+} with $\{(\sigma, \sigma^{'}) \in TC_{n+1} \mid < \sigma^{'}, \sigma^{'} >= 0,\ \sigma_{n+1}\sigma_{n+2}^{'} - \sigma_{n+2}\sigma_{n+1}^{'} > 0$ and $\sigma^{'} \neq \beta\sigma\ \forall\beta \in \mathbf{R}\}$. Hence E_{01}^{+} is itself a G-orbit. To show that $J_c(E_{02}^{+})$ is a G-orbit, it is useful to study the action of G on E_{02}^{+}. First observe that if $\sigma_* = (1, 0, ..., 0, 1)$ and $\sigma_*^{'} = (0, 1, 0..., -1, 0)$ then $(\sigma_*, \sigma_*^{'}) \in E_{02}^{+}$. Let $(\sigma, \sigma^{'}) \in E_{02}^{+}$. It suffices to show that

$$\exists g \in G \ \mid \ J_c(g \cdot (\sigma, \sigma^{'})) = J_c(\sigma_*, \sigma_*^{'}). \tag{IV.13}$$

We shall use the following relations :

$$\begin{aligned} J_c(\sigma, \sigma^{'}) &= J_c(-\sigma, -\sigma^{'}) \\ &= J_c(\frac{1}{\lambda}\sigma, \lambda\sigma^{'}) \ \ \forall\lambda \in \mathbf{R}^* \\ &= J_c(\sigma + t\sigma^{'}, \sigma^{'}) \ \ \forall t \in \mathbf{R} \\ &= J_c(\sigma, \sigma^{'} + s\sigma) \ \ \forall s \in \mathbf{R} \end{aligned} \tag{IV.14}$$

and also the Ad^*-equivariance of J_c. So let $(\sigma,\sigma^{'}) \in E_{02}^{+}$ and suppose first that $\sigma_o \neq 0$. Up to $(\sigma,\sigma^{'}) \mapsto (-\sigma,-\sigma^{'})$, one may suppose that $\sigma_o > 0$. With a rotation of $SO(n)$, a rotation in the plane $(n+1, n+2)$ and a pseudo-rotation in the plane $(1, n+2)$, change the point $(\sigma,\sigma^{'})$ to a point :

$$\sigma = \sigma_o(1,0,...,0,1) \;\; \sigma_o > 0$$

where we used $< \sigma,\sigma >_c \neq 0$. Finally one can suppose $\sigma_o = 1$, since we have $(IV.14)$. Using, now, $J_c(\sigma,\sigma^{'}+s\sigma)$ $\forall s \in \mathbf{R}$ and a rotation of $SO(n)$, ths, without changing σ, move the point $\sigma^{'}$ to a point :

$$\sigma^{'} = (0,\sigma_1^{'},0,...,\sigma_{n+1}^{'},\sigma_{n+2}^{'}) \;\; \sigma_1^{'} > 0$$

since $< \sigma^{'},\sigma^{'} >= 0$ and $\sigma^{'} \neq 0$. Moreover, because $< \sigma,\sigma^{'} >= 0$, we see that $\sigma_{n+2}^{'} = 0$. On the other hand, since $(\sigma,\sigma^{'}) \in E_0^{+}$, we know that $J_{n+1n+2}(\sigma,\sigma^{'}) = -\sigma_{n+1}^{'} > 0$ and $< \sigma^{'},\sigma^{'} >_c = 0 = \sigma_1^{'2} - \sigma_{n+1}^{'2}$, hence $\sigma^{'} = \sigma_1^{'}(0,1,0,...,-1,0)$ with $\sigma_1^{'} > 0$. Finally, a pseudo-rotation in the plane $(1, n+1)$ does not affect σ and take $\sigma^{'}$ to $\sigma_*^{'}$, which shows $(IV.13)$. If $\sigma_o = 0$ then necessarely $\sigma_o^{'} \neq 0$ and we come back to the previous case using the fact that $J_c(\sigma,\sigma^{'}) = J_c(-\sigma^{'},\sigma)$.

V The Binegar-Zierau representation of G_c.

In this section, we recall the construction by Binegar and Zierau of the representation $\mathcal{H}$ of G_c and its unitary structure given in [1].

We assume now that $G = SO_o(n,2)$ with $n \geq 3$ an odd integer. Let $C_o^{\infty}(C_{n+1},l)$ (resp. $C_1^{\infty}(C_{n+1},l)$) be the vector space of even (resp. odd) smooth complex functions on the cone C_{n+1} homogeneous of degree $l \in \mathbf{C}$. Let $\Omega_{n,2}$ be the G-invariant operator on $\mathbf{R}^{n+2}$ defined by :

$$\Omega_{n,2} = \sum_{i,j=1}^{n+2} \eta_{ii}\eta_{jj}X_{ij}^2. \tag{V.1}$$

It is easy to check that, for $\epsilon = 0$ or 1, the subspace

$$\mathcal{H}_\epsilon = \{\phi \in C_\epsilon^{\infty}(C_{n+1},l) \mid l = 1 - \frac{n+1}{2} \text{ and } \Omega_{n,2}\phi = \frac{n^2-1}{4}\phi\}, \tag{V.2}$$

is a G_c-submodule of $C_\epsilon^{\infty}(C_{n+1},l)$. On the other hand, consider the injective map :

$$\Psi : \phi \in C_0^{\infty}(C_{n+1},l) \mapsto \phi \mid_{x_o=1} \in C^{\infty}(X). \tag{V.3}$$

One easily checks that if $\phi \in C_0^{\infty}(C_{n+1},l)$ then $\Psi(\phi)$ satisfies the wave equation on X :

$$\Delta_X\phi \mid_X = \frac{n^2-1}{4}\phi \mid_X \quad \forall\phi \in \mathcal{H}_\epsilon. \tag{V.4}$$

Hence the representation $\mathcal{H}_\epsilon$ is realized as a space of solutions of the wave equation on X.

Binegar and Zierau showed in [1] that, when $\epsilon \equiv -\frac{n}{2} + \frac{1}{2}[2]$, $\mathcal{H}_\epsilon$ is completely reducible. More precisely, in this case there exists two unitary simple G_c-submodules $\mathcal{H}_+$ and $\mathcal{H}_-$ of $C_\epsilon^{\infty}(C_{n+1}, -\frac{n}{2}+\frac{1}{2})$ such that $\mathcal{H} \equiv \mathcal{H}_\epsilon = \mathcal{H}_+ \oplus \mathcal{H}_-$. To define $\mathcal{H}_+$ and $\mathcal{H}_-$, it is useful to study first the *K_c-types* of $\mathcal{H}$ where $K_c = SO(n+1) \times SO(2)$ is a maximal compact subgroup of G_c. From $(V.2)$, it suffices to look at the K_c-types of $C_\epsilon^{\infty}(C_{n+1}, -\frac{n}{2}+\frac{1}{2})$, which are well-known [19]. Let $\epsilon = 0$ or 1 and $l \in \mathbf{C}$. Let $H_{qr}^{(n+1)} = H_q^{(n+1)} \otimes H_r^{(2)}$, where $H_q^{(n+1)}$ denotes the vector space of harmonic polynomials on $\mathbf{R}^{n+1}$, homogeneous of degree q, and $H_r^{(2)}$ is generated by the harmonic polynomial $(x_{n+1}-ix_{n+2})^r$ if $r > 0$ and $(x_{n+1}+ix_{n+2})^r$ if $r < 0$. Let $\mid x \mid_{n+1} = \sqrt{\sum_{j=0}^{n} x_j^2}$ for all x in $\mathbf{R}^{n+3}$, and define a map : $j_l^{(n+1)} : H_{qr}^{(n+1)} \to C^{\infty}(C_{n+1},l) = \bigcup_{\epsilon=0,1} C_\epsilon^{\infty}(C_{n+1},l)$, $h_1 \otimes h_2 \mapsto j_l^{(n+1)}(h_1 \otimes h_2)$ by :

$$j_l^{(n+1)}(h_1 \otimes h_2)(x,y) = h_1(x)h_2(y) \mid x \mid_{n+1}^{l-q-r} . \tag{V.5}$$

for all $(x,y) \in \{(x,y) \in \mathbf{R}^{n+1} \times \mathbf{R}^2 \mid \mid x \mid_{n+1} = \mid y \mid_2\}$. If we denote by $C^\infty_\epsilon(C_{n+1}, l)^{K_c}$ the vector subspace of K_c-finite elements in $C^\infty_\epsilon(C_{n+1}, l)$, then, as K_c-modules, we have the following isomorphism :

$$C^\infty_\epsilon(C_{n+1}, l)^{K_c} \simeq \bigoplus_{\substack{(q,r)\in \mathbf{N}\times\mathbf{Z} \\ q+r\equiv\epsilon[2]}} j_l^{(n+1)}(H_{qr}^{(n+1)}). \tag{V.6}$$

From the definition $(V.2)$ of $\mathcal{H}_\epsilon$, it is easy to check that

$$j^{(n+1)}_{-\frac{n}{2}+\frac{1}{2}}(H_{qr}^{(n+1)}) \subset \mathcal{H} \Leftrightarrow (q + \frac{n-1}{2})^2 - r^2 = 0. \tag{V.7}$$

If we asssume now that $n \geq 3$ is odd, $l = -\frac{n}{2} + \frac{1}{2}$ and $\epsilon \equiv -\frac{n}{2} + \frac{1}{2}[2]$, then $\mathcal{H} \simeq \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_+ = \bigoplus_{\substack{(q,r)\in \mathbf{N}\times\mathbf{N} \\ q-r=-\frac{n}{2}+\frac{1}{2}}} j^{(n+1)}_{-\frac{n}{2}+\frac{1}{2}}(H_{qr}^{(n+1)}) \tag{V.8}$$

and

$$\mathcal{H}_- = \bigoplus_{\substack{(q,r)\in \mathbf{N}\times\mathbf{Z}^- \\ q+r=-\frac{n}{2}+\frac{1}{2}}} j^{(n+1)}_{-\frac{n}{2}+\frac{1}{2}}(H_{qr}^{(n+1)}). \tag{V.9}$$

In the sequel, we shall denote by $U_\pm$ the G_c-action on $\mathcal{H}_\pm$:

$$U_\pm(g)(\phi)(x) = \phi(g^{-1}x) \;\; \forall g \in G_c \; \forall \phi \in \mathcal{H}_\pm \; \forall x \in C_{n+1}. \tag{V.10}$$

We will also denote by $U_\pm$ the induced action of ${}_c$ on $\mathcal{H}_\pm$.

Finally, concerning the unitary structure of $\mathcal{H}$, an explicit formula for the invariant scalar product $< \cdot, \cdot >_{BZ}$ is given in [1]. More precisely, for all $m \in \mathbf{N}^*$ define $\mathcal{D}_m$ the operator $(\Omega_m + \frac{m-2}{2})^{\frac{1}{2}}$ acting on the $SO(m)$-finite elements of $L^2(S^{m-1})$. Then $< \cdot, \cdot >_{BZ}$ is defined on $\mathcal{H}$ by :

$$< \Phi_1, \Phi_2 >_{BZ} = ((\mathcal{D}_{n+1} + \mathcal{D}_2)\Phi_1, \Phi_2) - (\Phi_1, (\mathcal{D}_{n+1} - \mathcal{D}_2)\Phi_2) \;\; \forall \Phi_1 \text{ and } \Phi_2 \in \mathcal{H} \tag{V.11}$$

where $(\Phi_1, \Phi_2) = \int_{\Sigma_{n+1}} \Phi_1 \overline{\Phi_2} d\sigma$ is the usual scalar product on $L^2(\Sigma_{n+1})$ relatively to the normalized K_c-invariant measure $d\sigma$ on Σ_{n+1}, where

$$\Sigma_{n+1} = \{(x,y) \in \mathbf{R}^{n+1} \times \mathbf{R}^2 \mid \mid x \mid_{n+1} = \mid y \mid_2 = 1\} \simeq S^n \times S^1. \tag{V.12}$$

VI The semi-classical limit for $\mathcal{O}^{min\pm}_{n+1}$.

VI.1 Introduction.

Let us briefly recall what is meant by a quantization in physics. Let (M, ω) be a symplectic manifold and $\mathcal{A}$ a Lie algebra (for the Poisson structure relatively to ω) of real smooth functions on M. We will assume that $\mathcal{A}$ contains the constants. By a quantization of $(M, \omega, \mathcal{A})$, we mean the construction of the following objects [20] :

(i) A familly $\{\mathcal{H}_\hbar\}_\hbar$ of Hilbert spaces indexed by $\hbar \in I \subset \mathbf{R}^+ \setminus \{0\}$ with $0 \in \overline{I}$,

(ii) A linear map $f \in \mathcal{A} \mapsto Op^\hbar f$ where $Op^\hbar f$ is a symmetric operator on $\mathcal{H}_\hbar$.

(iii) If u and v are in $\mathcal{A}$ then there exists $w \in \mathcal{A}$ such that $[Op^\hbar u, Op^\hbar v] = Op^\hbar w$.

We will ignore the technical problems related to the domains of $Op^{\hbar}f$. However we ask (at least) the following properties :

$$Op^{\hbar}1 = Id_{\mathcal{H}_{\hbar}} \tag{VI.1}$$

and

$$\frac{1}{i\hbar}[Op^{\hbar}f, Op^{\hbar}g] - Op^{\hbar}\{f,g\} \overset{\hbar\to 0}{\longrightarrow} 0. \tag{VI.2}$$

One of the main problems in semi-classical analysis is to find relations between spectral properties of $Op^{\hbar}f$ when $\hbar \to 0$ and properties of the hamiltonian flow of f on (M,ω) [21].

On the other hand, when (M,ω) is the co-adjoint orbit of a Lie group equipped with its canonical symplectic structure, the orbit method provides under some assumptions on the group, a quantization of $(M,\omega,\mathcal{A})$ [14]-[8]. More precisely, let $\mathcal{O}$ be a co-adjoint orbit of a Lie group G. For $\mu > 0$, we put $\mu\mathcal{O} \equiv \{\mu f \mid f \in \mathcal{O}\}$. Assume that the orbit method associates to $\mu\mathcal{O}$ a unitary representation $(\mathcal{H}_\mu, U_\mu)$ of G, for $\mu \in \ J \subset \mathbf{R}^+$, with J unbounded. In this case, a quantization of $(M,\omega,\mathcal{A})$ with $(M,\omega) = \mathcal{O}$ and $\mathcal{A} = \ \ \oplus \mathbf{R}$ is done as follows (here the Lie algebra   of G is realized by evaluation functions on $\mathcal{O}$ and $\mathbf{R}$ is realized by the constant functions). Put $\hbar = \frac{1}{\mu}$, $\mathcal{H}_\hbar = \mathcal{H}_\mu$ and

$$Op^{\hbar}Y = i\hbar Y^{\mu} \ \ \forall Y \in$$

where $Y^{\mu} \equiv \frac{d}{dt}U_\mu(\exp(-tY)) \mid_{t=0}$. It is easy to check that :

$$\frac{1}{i\hbar}[Op^{\hbar}Y_1, Op^{\hbar}Y_2] = Op^{\hbar}\{Y_1, Y_2\}$$

so $(VI.2)$ is satisfied, even without taking the limit. Note that in the applications one wants to quantize more than $\ \oplus \mathbf{R}$, for example the symmetric algebra of . The orbit method does not give a satisfactory solution to this problem. The case of $\mathcal{O} = \mathcal{O}^{min\pm}_{n+1}$ is particular since $\mathcal{O}^{min\pm}_{n+1}$ are nilpotent, so $\mu\mathcal{O} = \mathcal{O}$ and $(\mathcal{H}_\mu, U_\mu) = (\mathcal{H}_\pm, U_\pm)$ for all $\mu \in \mathbf{R}^*_+$.

VI.2 The semi-classical behaviour of the spectrum of $U_\pm(X_{n+1n+2})$.

We shall consider the quantization of $\mathcal{O}^{min\pm}_{n+1}$ given by $\mathcal{H}_\hbar = \mathcal{H}_\pm$ and $Op^{\hbar}X_{ij} \equiv \widehat{X}^{\hbar}_{ij} = -i\hbar X_{ij}$. Recall that $U_\pm$ is the G_c-action on $\mathcal{H}_\pm$:

$$U_\pm(g)(\phi)(x) = \phi(g^{-1}x) \ \ \forall g \in G_c \ \forall \phi \in \mathcal{H}_\pm \ \forall x \in C_{n+1}. \tag{VI.3}$$

For any element Y of $\ _c$ consider the evaluation function $\mathcal{O}^{min\pm}_{n+1} \to \mathbf{R}$, $\psi \mapsto \psi(Y)$, which we still denote by Y. To this function we associate, for all real number $\hbar > 0$, the self-adjoint linear operator $\widehat{Y}^{\hbar}$ on $\mathcal{H}_\pm$ defined by :

$$\widehat{Y}^{\hbar} = i\hbar\frac{d}{dt}U_\pm(\exp(-tY)) \mid_{t=0} = -i\hbar Y \tag{VI.4}$$

such that

$$[\widehat{Y}^{\hbar}, \widehat{Y}'^{\hbar}] = i\hbar\widehat{\{Y, Y'\}}^{\hbar} \ \ \forall (Y, Y') \in \ _c \times \ _c. \tag{VI.5}$$

In particular, we have :

$$\widehat{X}^{\hbar}_{n+1n+2} = -i\hbar X_{n+1n+2}. \tag{VI.6}$$

The reason why we pay a special attention to $\widehat{X}^{\hbar}_{n+1n+2}$ is motivated by its physical interpretation. Indeed, the operator $\widehat{X}^{\hbar}_{n+1n+2}$ is the analog of an "energy" operator" (see [8] for the case $n = 1$). Moreover, using the results of section V, one gets that the spectrum of $\widehat{X}^{\hbar}_{n+1n+2}$ has a lower bound.

In the sequel, we will restrict our attention on the representation U_+ of G_c in $\mathcal{H}_+$ since our arguments will be analogous for U_-. Finally, if we consider ω_o the restriction to $\ _o$ of the canonical symplectic two-form ω on TX and still denote by ω_o the image of ω_o under the moment map J, we have :

Theorem VI.1 *For all real function f Riemann-integrable on any interval $[0,a]$ with $a \in \mathbf{R}_+^*$ and such that*

$$\exists \alpha > 1 \; \exists c > 0 \quad | f(x) | \leq c x^{-\alpha-n+1} \tag{VI.7}$$

we have :

$$Tr(f(\widehat{X}^{\hbar}_{n+1n+2})) = \frac{1}{(2\pi\hbar)^n} \int_{\mathcal{O}_n^{o+}} (f \circ J_{n+1n+2}) \omega_o^n + o(\hbar^{-n}). \tag{VI.8}$$

To proof this theorem the following simple lemma will be useful [15].

Lemma VI.2 *If g is a real function on $\mathbf{R}$ which is Riemann-integrable on any interval $[0,a]$ with $a \in \mathbf{R}_+^*$ and, if there exists $\alpha > 1$ and $c > 0$ such that $| g(x) | \leq c x^{-\alpha}$ then*

$$\int_0^{+\infty} g(x) dx = \lim_{N \to +\infty} \frac{1}{N} \sum_{k=0}^{+\infty} g(\frac{k}{N}).$$

Proof of Theorem VI.1 : Let us first study the left side of $(VI.8)$. We know that [19] :

$$dim \; H_{qr}^{(n+1)} = (n-1+2q)[\frac{(n-2+q)!}{(n-1)!q!}] \tag{VI.9}$$

then it follows that :

$$Tr(f(\widehat{X}^{\hbar}_{n+1n+2})) = \frac{2}{(n-1)!} \frac{1}{\hbar^{n-1}} \sum_{r>0} \hbar r(\hbar r - \hbar(\frac{n}{2} - \frac{3}{2})) \cdots (\hbar r + \hbar(\frac{n}{2} - \frac{3}{2})) f(\hbar r) \tag{VI.10}$$

since we have :

$$\mathcal{H}_+ |_K = \bigoplus_{\substack{(q,r) \in \mathbf{N} \times \mathbf{N} \\ q-r = -\frac{n}{2}+\frac{1}{2}}} (\bigoplus_{k=0}^{q} j^{(n+1)}_{-\frac{n}{2}+\frac{1}{2}}(H_{kr}^{(n)})). \tag{VI.11}$$

Actually we have to study the behaviour of

$$S_f(\hbar) = \frac{1}{\hbar^{n-1}} \sum_{r>0} \hbar r(\hbar r - \hbar(\frac{n}{2} - \frac{3}{2})) \cdots (\hbar r + \hbar(\frac{n}{2} - \frac{3}{2})) f(\hbar r) \tag{VI.12}$$

when $\hbar$ goes to 0. An easy calculation shows that [15] :

$$S_f(\hbar) = \frac{1}{\hbar^{n-1}} \sum_{r>0} \sum_{l=1}^{n-1} a_l (\hbar r)^l \hbar^{n-1-l} f(\hbar r) \tag{VI.13}$$

where $a_i \in \mathbf{Z}$ with $a_{n-1} = 1$, so that

$$\hbar^n S_f(\hbar) = \sum_{r>0} (\hbar r)^{n-1} f(\hbar r) \hbar + \sum_{l=1}^{n-2} a_l [\sum_{r>0} (\hbar r)^l f(\hbar r) \hbar] \hbar^{n-1-l}. \tag{VI.14}$$

From Lemma $VI.2$, each series $\sum_{r>0} (\hbar r)^l f(\hbar r) \hbar$, $l = 1, \cdots, n-1$ is convergent, with

$$\lim_{\hbar \to 0} \sum_{r>0} (\hbar r)^l f(\hbar r) \hbar = \int_0^{+\infty} f(x) x^l dx \quad l = 1, \cdots, n-1. \tag{VI.15}$$

Then we obtain

$$Tr(f(\widehat{X}^{\hbar}_{n+1n+2})) = \frac{2}{(n-1)!} \frac{1}{\hbar^n} \int_0^{+\infty} f(x) x^{n-1} dx + o(\hbar^{-n}). \tag{VI.16}$$

We turn now to the study of the right side of $(VI.8)$. To determine the integral $\int_{\mathcal{O}_n^{o+}}(f \circ J_{n+1n+2})\omega_o^n$, we introduce a system of coordinates adapted to $\mathcal{O}_n^{o+}$. Let $\ ^+_o$ be the connected component of $\ _o$ whose symplectic reduction is $\mathcal{O}_n^{o+}$ and let

$$S_o = \{(x,y) \in \ ^+_o \mid x_{n+2} = 0\}. \tag{VI.17}$$

Then S_o is a $2n$-dimensional submanifold of $\ ^+_o$ which has two connected components $S_o^{\pm}$. The restriction of moment map J to S_o is injective but $J(S_o) \neq \mathcal{O}_n^{o+}$. Actually we have $\overline{J(S_o)} = \mathcal{O}_n^{o+}$ (the complement of S_o in $\ ^+_o$ is of measure zero) [15].

Consider the coordinates $(\vec{x}, \phi)$ on X with $\vec{x} = (x_1, ..., x_n) \in \mathbf{R}^n$ and $\phi \in [0, 2\pi[$ such that :

$$x_{n+1} = Y\cos\phi \quad \text{and} \quad x_{n+2} = Y\sin\phi \tag{VI.18}$$

with $Y = \sqrt{1+\vec{x}^2} = \sqrt{x_{n+1}^2 + x_{n+2}^2}$. Hence this gives us coordinates $(\vec{x}, \phi, \vec{p}, p_\phi)$ on TX defined by :

$$y \cdot dx = \vec{p}d\vec{x} - p_\phi d\phi. \tag{VI.19}$$

In particular, we have on S_o :

$$< y, y >= 0 \Leftrightarrow p_\phi^2 = Y^2((\vec{p})^2 + (\vec{x}\cdot\vec{p})^2). \tag{VI.20}$$

Since $\overline{J(S_o)} = \mathcal{O}_n^{o+}$, $\omega_o^n \mid_{S_o} = d\vec{x} \wedge d\vec{p}$ and $p_\phi = J_{n+1n+2}(x,y)$, we obtain that :

$$\int_{\mathcal{O}_n^{o+}}(f \circ J_{n+1n+2})\omega_o^n = 2\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(Y\sqrt{(\vec{p})^2 + (\vec{x}\cdot\vec{p})^2})d\vec{x} \wedge d\vec{p}. \tag{VI.21}$$

Now if we let $I(\vec{x}) = \int_{\mathbf{R}^n} f(Y\sqrt{(\vec{p})^2 + (\vec{x}\cdot\vec{p})^2})d\vec{p}$ and if R is a rotation of $\mathbf{R}^n$, then it is easy to check that :

$$I(R\cdot\vec{x}) = I(\vec{x}) \ \ \forall \vec{x} \in \mathbf{R}^n. \tag{VI.22}$$

Thus, using the spherical coordinates on $\vec{x}$, with $\vec{x} = (a, 0, ..., 0)$, we get that :

$$\int_{\mathcal{O}_n^{o+}}(f \circ J_{n+1n+2})\omega_o^n = 2\int_0^{+\infty}\int_{\mathbf{R}^n} f(\sqrt{(1+a^2)((1+a^2)p_1^2 + ... + p_n^2)})a^{n-1}dad\vec{p}(\sigma(S^{n-1})) \tag{VI.23}$$

where $\sigma(S^{n-1})$ denotes the surface of the unit sphere S^{n-1} in $\mathbf{R}^n$. Finally if we put $z_1 = \sqrt{1+a^2}p_1$ and $z_i = p_i$ for $2 \leq i \leq n$, we have :

$$\int_{\mathcal{O}_n^{o+}}(f \circ J_{n+1n+2})\omega_o^n = 2\int_0^{+\infty}\int_{\mathbf{R}^n} \frac{a^{n-1}}{\sqrt{1+a^2}} f(\sqrt{(1+a^2)\vec{z}^2})dad\vec{z}(\sigma(S^{n-1})), \tag{VI.24}$$

or using the usual spherical coordinates on z_i where $(\vec{z})^2 = \rho^2$:

$$\int_{\mathcal{O}_n^{o+}}(f \circ J_{n+1n+2})\omega_o^n = 2\int_0^{+\infty}\int_0^{+\infty} \frac{a^{n-1}}{\sqrt{1+a^2}} f(\rho\sqrt{1+a^2})\rho^{n-1}dad\rho(\sigma(S^{n-1}))^2. \tag{VI.25}$$

Moreover, if $\beta = \rho\sqrt{1+a^2}$ we get :

$$\int_{\mathcal{O}_n^{o+}}(f \circ J_{n+1n+2})\omega_o^n = 2(\int_0^{+\infty}(\frac{1}{1+a^2})^{\frac{n+1}{2}}da)(\int_0^{+\infty} f(\beta)\beta^{n-1}d\beta)(\sigma(S^{n-1}))^2. \tag{VI.26}$$

But we know [22] that if $n \geq 3$ is odd then,

$$\int_0^{+\infty}(\frac{1}{1+a^2})^{\frac{n+1}{2}}da = \frac{(n-2)!!}{(n-1)!!}\frac{\pi}{2} \quad \text{and} \quad \sigma(S^{n-1}) = \frac{2^{\frac{n+1}{2}}\pi^{\frac{n-1}{2}}}{(n-2)!!} \tag{VI.27}$$

where $(n-1)!! = 2 \times 4 \times \cdots \times n-1$ and $(n-2)!! = 1 \times 3 \times \cdots \times n-2$. Hence $(VI.26)$ becomes :

$$\int_{\mathcal{O}_n^{o+}} (f \circ J_{n+1n+2}) \omega_o^n = \frac{2}{(n-1)!} (2\pi)^n \int_0^{+\infty} f(\beta) \beta^{n-1} d\beta, \tag{VI.28}$$

so

$$\frac{1}{(2\pi\hbar)^n} \int_{\mathcal{O}_n^{o+}} (f \circ J_{n+1n+2}) \omega_o^n = \frac{2}{(n-1)!} \frac{1}{\hbar^n} \int_0^{+\infty} f(\beta) \beta^{n-1} d\beta. \tag{VI.29}$$

$(VI.8)$ follows then immediatey from $(VI.16)$ and $(VI.29)$.

The nilpotency of the orbit $\mathcal{O}_n^{o+}$ plays an important role in the semi-classical behaviour $(VI.8)$ of $Tr(f(\widehat{X}^{\hbar}_{n+1n+2}))$. Indeed, consider the G-orbits $\mathcal{O}_n^{\lambda+}$ with $\lambda = -m^2$ and m is a positive real number. These G-orbits are not nilpotent and their symplectic form ω_{-m^2} is realized as the image under J of the restriction to $_{-m^2}$ of the two-form ω on TX. Then using same arguments as those of the proof of Theorem $VIII.1$ (see [15] for details), we obtain that for all positive real function f satisfying the hypothesis of the Theorem $VI.1$:

$$\begin{aligned} \int_{\mathcal{O}_n^{-m^2+}} (f \circ J_{n+1n+2}) \omega_{-m^2}^n \quad &\leq \quad \int_{\mathcal{O}_n^{o+}} (f \circ J_{n+1n+2}) \omega_o^n \\ &- \quad 2 \int_0^{+\infty} \int_{m\sqrt{1+a^2}}^{+\infty} \frac{a^{n-1}}{(1+a^2)^{\frac{n+1}{2}}} f(\beta) \beta^{n-1} da d\beta (\sigma(S^{n-1}))^2. \end{aligned} \tag{VI.30}$$

VI.3 The Weyl's law for the spectrum of $\widehat{X}^{\hbar}_{n+1n+2}$.

An immediate and interesting corollary of the Theorem $VI.1$ is the so-called Weyl's law for the operator $\widehat{X}^{\hbar}_{n+1n+2}$. This formula gives an asymptotic behaviour, when $\hbar$ goes to zero, of the number of eigenvalues of the operator $\widehat{X}^{\hbar}_{n+1n+2}$ contained in a fixed real interval $[a;b]$. More precisely,

Corollary VI.3 *Let $[a;b]$ be a real interval independant of $\hbar$ and let $Spec(\widehat{X}^{\hbar}_{n+1n+2})$ be the spectrum of the operator $\widehat{X}^{\hbar}_{n+1n+2}$. We have :*

$$\sharp([a,b] \cap Spec(\widehat{X}^{\hbar}_{n+1n+2})) = \frac{1}{(2\pi\hbar)^n} volume(J^{-1}_{n+1n+2}([a,b])) + o(\hbar^{-n}). \tag{VI.31}$$

Proof : Just apply Theorem $VI.1$ with $f = \chi_{[a;b]}$ the characteristic function of the interval $[a;b]$.

VI.4 The semi-classical behaviour of highest weight vectors.

The result of the previous subsections is about the spectrum of the operator $\widehat{X}^{\hbar}_{n+1n+2}$. Now, we want to prove a result concerning the eigenfunctions themselves. We shall only consider $\mathcal{H}_+$, the arguments will still hold for $\mathcal{H}_-$.

Let $e > 0$. We study a family of vectors $\psi_\hbar \in \mathcal{H}_+ \subset C^\infty(X)$ which are eigenvectors of $\hat{X}^{\hbar}_{n+1n+2}$

$$\hat{X}^{\hbar}_{n+1n+2} \psi_\hbar = e \psi_\hbar \tag{VI.32}$$

and which are also highest weight vectors in $H_q^{(n+1)}$, that is

$$\hat{X}^{\hbar}_{01} \psi_\hbar = (e - \hbar(\frac{n}{2} - \frac{1}{2})) \psi_\hbar. \tag{VI.33}$$

Obviously $(VI.32)$ could be satisfied only for values of $\hbar$ for which there exists $r \in \mathbf{N}$ such that

$$\hbar r = e. \tag{VI.34}$$

In the sequel, we will always consider this situation. Actually, it is easy to calculate $\psi_\hbar$ [15] :

$$\psi_\hbar(x) = (1+ix_1)^q (x_{n+1} - ix_{n+2})^r \mid x \mid_{n+1}^{l-q-r} \quad \forall x = (1, x_1, \cdots, x_{n+2}) \in C_{n+1} \tag{VI.35}$$

where $l = -\frac{n}{2} + \frac{1}{2}$ and $q = r + l$. This can be rewritten as :

$$\psi_\hbar(x) = \frac{(1+ix_1)^r}{\mid x \mid_{n+1}^r} \frac{(x_{n+1} - ix_{n+2})^r}{\mid x \mid_{n+1}^r} (1+ix_1)^l. \tag{VI.36}$$

On the other hand, let X^r be the subset of X defined by :

$$X^r = \{x \in X \mid x_i = 0 \ \ \forall\, 2 \leq i \leq n\}. \tag{VI.37}$$

Then X^r is a 2-dimensional hyperboloid imbeded in the $n+1$-dimensional hyperboloid X. Moreover let ${}^r_0$ be the subset of TX^r defined by :

$${}^r_0 = \{(x,y) \in TX^r \mid y \neq 0 \text{ and } <y,y>=0\}. \tag{VI.38}$$

Here ${}^r_0$ is naturally embeded in the G-orbit ${}_0$ in TX defined by $(II.4)$. Note that X^r and ${}^r_0$ play the same role as X and ${}_0$ but for the subgroup $SO_o(1,2)$ of G fixing the coordinates x_i, $2 \leq i \leq n$. In particular the symplectic reduction of ${}^r_0$ is a nilpotent coadjoint orbit of $SO_0(1,2)$. If e is the real number given in $(VI.32)$, we consider the level surface $\mathcal{L}_e$ in TX^r of the function J_{n+1n+2} on TX defined by $(II.8)$, that is :

$$\mathcal{L}_e = \{(x,y) \in TX^r \mid J_{n+1n+2}(x,y) = e\}. \tag{VI.39}$$

If we now consider the parametrization $(t,\phi) \in \mathbf{R} \times [0, 2\pi[$ of $\mathcal{L}_e$ given by :

$$\begin{aligned} x_1 &= et,\ x_2 = et\cos\phi - \sin\phi \ \text{ and } \ x_3 = -et\sin\phi - \cos\phi \\ y_1 &= e,\ y_2 = e\cos\phi \ \text{ and } \ y_3 = -e\sin\phi. \end{aligned} \tag{VI.40}$$

then it is easy to show that $\mathcal{L}_e$ is a Lagrangian submanifold of TX^r, with $y \cdot dx = d\phi$ on $\mathcal{L}_e$.

In fact the function J_{n+1n+2} on TX defined by $(II.8)$ is just the hamiltonian function associated to the hamiltonian vector field $\tilde{X}_{n+1n+2}$ on TX given by $(II.7)$. Then, it is natural, from the point of view of the semi-classical analysis [23], to look for a link, in the semi-classical limit $\hbar \to 0$, between the level surface $\mathcal{L}_e$ of J_{n+1n+2} and the eigenfunctions $\psi_\hbar$ of the operator $\widehat{X}^\hbar_{n+1n+2}$ with eigenvalue e. If we denote by π the natural projection of TX on X, a link between the family $\psi_\hbar$ and $\mathcal{L}_e$ is then given by :

Theorem VI.4 *Let $\psi_\hbar$ the function on X defined by $(VI.36)$.*
(i) If $x \notin \pi(\mathcal{L}_e)$, then

$$\mid \psi_\hbar(x) \mid \leq (1+x_1^2)^{-\frac{n}{2}+\frac{1}{2}} \exp(-\frac{e}{\hbar} ln(1 + \frac{x_\perp^2}{1+x_\perp^2})) \tag{VI.41}$$

where $x = (x_1, x_\perp, x_{n+1}, x_{n+2}) \in X$.
(ii) If $x \in \pi(\mathcal{L}_e)$, we have

$$\psi_\hbar(x) = i^r \exp(i\frac{e}{\hbar}\phi) \tag{VI.42}$$

where ϕ is defined in $(VI.40)$.

Proof : For (i), it suffices to remark that $x=(x_1,x_\perp,x_{n+1},x_{n+2})$ is not in $\pi(\mathcal{L}_e)$ if, and only if, $x_\perp\neq 0$, $(VI.41)$ follows then directely from $(VI.34)$ et $(VI.36)$. Finally, for (ii), note that from $(VI.40)$, we have :

$$\psi_\hbar(x)=\frac{(1+iet)^r}{(1+e^2t^2)^{\frac{r}{2}}}\frac{(et\exp(i\phi)-\sin\phi+i\cos\phi)^r}{(1+e^2t^2)^{\frac{r}{2}}}=i^r\exp(ir\phi). \qquad \text{(VI.43)}$$

So, we see that, in the semi-classical limit $\hbar\to 0$, the function $\psi_\hbar$ has its support on $\pi(\mathcal{L}_e)$. However, outside of $\pi(\mathcal{L}_e)$, $\psi_\hbar$ is exponentially decreasing in $\frac{1}{\hbar}$. Moreover, the phase appearing in $(VI.42)$ is, according to $(VI.40)$, precisely the generating function of the Lagrangian manifold $\mathcal{L}_e$. Hence, $\psi_\hbar$ is identically equal to the WKB solution [23] of $(VI.32)-(VI.33)$ on $\pi(\mathcal{L}_e)$.

VII K_c-modules in $\mathcal{H}_\pm$ and K_c-orbits in $\mathcal{O}^{min\pm}_{n+1}$.

In this section we want to give, via the orbit method, a another link between the orbit $\mathcal{O}^{min\pm}_{n+1}$ and the unitary irreducible representation $\mathcal{H}_\pm$ of G_c. We need first to determine the K_c-orbits in $\mathcal{O}^{min\pm}_{n+1}$. Actually we shall restrict our attention on $\mathcal{H}_+$ and $\mathcal{O}^{min+}_{n+1}$, since our arguments will still hold for $\mathcal{H}_-$ and $\mathcal{O}^{min-}_{n+1}$.

VII.1 The K_c-orbits in $\mathcal{O}^{min+}_{n+1}$.

Let $f\in\mathcal{O}^{min+}_{n+1}$. We write

$$f=\frac{1}{2}\sum_{i,j=0}^{n+2}f_{ij}X^{ij} \qquad \text{(VII.1)}$$

and introduce the following K_c-invariant polynomial function on $so(n+1,2)^*$:

$$\Omega_{n+1}:f\in so(n+1,2)^*\mapsto\frac{1}{2}\sum_{i,j=0}^{n}f_{ij}^2. \qquad \text{(VII.2)}$$

The following simple lemma will be useful :

Lemma VII.1

$$\Omega_{n+1}(f)=f^2_{n+1n+2}\ \ \forall f\in\mathcal{O}^{min+}_{n+1}. \qquad \text{(VII.3)}$$

Proof : Since $\mathcal{O}^{min+}_{n+1}=J_c(E^+_o)$, it suffices to observe that if $<\cdot,\cdot>_{n+1}$ denotes the usual scalar product on $\mathbf{R}^{n+1}$, then $<\sigma,\sigma>_{n+1}<\sigma',\sigma'>_{n+1}-<\sigma,\sigma'>^2_{n+1}=J^2_{n+1n+2}(\sigma,\sigma')$, for all $(\sigma,\sigma')\in TC_{n+1}$, with $<\sigma',\sigma'>_c=0$.

Recall now that if   is a Lie subalgebra of a Lie algebra , there is a natural projection $\mathcal{R}$: $^*\to$ *. Introduce, for all real numbers $e\geq 0$, the subset $\mathcal{S}(e)$ of $\mathcal{O}^{min+}_{n+1}$ defined by :

$$\mathcal{S}(e)=\{f\in\mathcal{O}^{min+}_{n+1}\mid f_{n+1n+2}=e\}, \qquad \text{(VII.4)}$$

such that $\mathcal{O}^{min+}_{n+1}=\bigcup_{e>0}\mathcal{S}(e)$. We have :

Proposition VII.2 *For each real $e\geq 0$, $\mathcal{S}(e)$ is a $(2n-1)$-dimensional submanifold of $\mathcal{O}^{min+}_{n+1}$ and a K_c-orbit. The co-adjoint orbits $\mathcal{R}_{so(n+1)}(\mathcal{S}(e))$ are all disjoint and of dimension $2n-2$.*

Proof : Let us recall that $\mathcal{O}^{min+}_{n+1}=J_c(E^+_o)$ $(IV.7)$. On the other hand, $(\sigma_*,\sigma'_*)\in J_c^{-1}(\mathcal{S}(e))$ where $\sigma_*=(1,0,...,1,0)$ and $\sigma'_*=(0,e,0,....,0,e)$. Hence, let $(\sigma,\sigma')\in J_c^{-1}(\mathcal{S}(e))$. It will be sufficient to show that there exists $g \in K_c$ such that $g \cdot (\sigma, \sigma') = (\sigma_*, \sigma'_*)$. First, an apropriate rotation of $SO(n+1)$, we can suppose that $\sigma = (\sigma_o, 0, ..., \sigma_{n+1}, \sigma_{n+2})$ with $\sigma_o > 0$. Using the transformation $(\sigma, \sigma') \mapsto (\sigma, \sigma' + s\sigma)$ which leaves invariant J_c $(IV.14)$, we can transform σ' in $\sigma' = (0, \sigma'_1, ..., \sigma'_n, \sigma'_{n+1}, \sigma'_{n+2})$. Then, a rotation of $SO(n)$ take us to the point $\sigma = (\sigma_o, 0, ..., 0, \sigma_{n+1}, \sigma_{n+2})$ and $\sigma' = (0, \sigma'_1, 0, ..., 0, \sigma'_{n+1}, \sigma'_{n+2})$ with $\sigma_o > 0$ and $\sigma'_1 > 0$. Finally, with a rotation of $SO(2)$ takes to $\sigma_{n+1} > 0$ and $\sigma_{n+2} = 0$, and using $< \sigma, \sigma >_c = 0$ together with $(IV.14)$ we take σ to σ_*. On the other hand, since $< \sigma, \sigma' >_c = < \sigma', \sigma' >_c = 0$ and $\sigma'_{n+1} \sigma'_{n+2} = e$, we conclude that $\sigma' = \sigma'_*$. To show the last assertion of the proposition, remark that the canonical symplectic form on $\mathcal{O}^{min+}_{n+1}$, restricted to $\mathcal{S}(e)$, has a one-dimenional kernel, and that the leaves of the associated foliation are $SO(2)$-orbits in $\mathcal{S}(e)$. Since K_c acts transitively on $\mathcal{S}(e)$, we conclude that $SO(n+1)$ acts transitively on the symplectic reduction of $\mathcal{S}(e)$. But this symplectic reduction is directely obtained using the moment map of the K_c-action on $\mathcal{S}(e)$. So, since $\mathcal{R}_{so(2)}(\mathcal{S}(e))$ is reduced to a point, we get that $\mathcal{R}_{so(n+1)}(\mathcal{S}(e))$ is only one co-adjoint orbit of $SO(n+1)$. On the other hand, from Lemma $VII.1$, we know that $\Omega_{n+1}(\mathcal{S}(e)) = e^2$. Hence, $\mathcal{R}_{so(n+1)}(\mathcal{S}(e)) \neq \mathcal{R}_{so(n+1)}(\mathcal{S}(e'))$ as soon as $e \neq e'$.

We turn now to a relation between the orbit $\mathcal{O}^{min+}_{n+1}$ and the unitary irreducible representation $\mathcal{H}_+$.

VII.2 K_c-modules in $\mathcal{H}_+$ and K_c-orbits in $\mathcal{O}^{min+}_{n+1}$.

From $(V.8)$ we know that

$$\mathcal{H}_+ = \oplus_{\substack{(q,r)\in \mathbf{N}\times\mathbf{N} \\ q-r=-\frac{n}{2}+\frac{1}{2}}} j_{-\frac{n}{2}+\frac{1}{2}}(H^{(n+1)}_{qr}). \tag{VII.5}$$

Moreover, from the previous subsection we know that :

$$\mathcal{O}^{min+}_{n+1} = \bigcup_{e>0} \mathcal{S}(e) \tag{VII.6}$$

where each $\mathcal{S}(e)$ is a K_c-orbit. The analogy between $(VII.5)$ and $(VII.6)$ is completely natural. Indeed, consider the irreducible representation $H^{(n+1)}_{qr}$ of K_c with $q - r = -\frac{n}{2} + \frac{1}{2}$. Choosing $\hbar r \to e$, we see that $\hbar q \to e$ when $\hbar \to 0$. On the other hand $H^{(n+1)}_{qr}$ is, in the usual way (via its highest weight) [24], associated to the co-adjoint orbit passing through $\hbar q X^{01} + \hbar r X^{n+1 n+2} \to e(X^{01} + X^{n+1n+2})$. But the co-adjoint orbit of K_c passing through $e(X^{01} + X^{n+1n+2})$ is precisely $\mathcal{R}_{k_c}(\mathcal{S}(e))$, where $_c$ is the Lie algebra of K_c.

VIII Determination of the restriction to G of $\mathcal{H}$.

In this section, we compute explicitely the restriction to the group G of the irrducible unitary representations $\mathcal{H}_\pm$ of the group G_c. We assume in the sequel that $n \geq 3$ is odd and we define for $\epsilon = 0$ or 1 :

$$W_{\pm,\epsilon} = \bigoplus_{\substack{(s,r)\in \mathbf{N}\times\mathbf{N} \\ k=\pm r-\frac{n}{2}+\frac{1}{2}-2s-\epsilon}} j^{(n+1)}_{-\frac{n}{2}+\frac{1}{2}}(H^{(n)}_{kr}). \tag{VIII.1}$$

Then we have :

Proposition VIII.1 *The G-modules $W_{\pm,\epsilon}$, $\epsilon = 0$ or 1, are simple and unitarizable. Moreover, the restriction to G of the unitary simple G_c-module $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) is the G-module $W_{+,0} \oplus W_{+,1}$ (resp. $W_{-,0} \oplus W_{-,1}$).*

Proof : The idea is to determine the restriction to $K = SO(n) \times SO(2)$ of the K_c-module $\mathcal{H}_+$ (resp. $\mathcal{H}_-$). Since the injection $(III.2)$ of G in G_c determines also an injection of K in K_c, it suffices to know the restriction of the $SO(n+1)$-module $H_q^{(n+1)}$ to the subgroup $SO(n)$ of $SO(n+1)$ fixed by this injection. From, [19] IX, §2.8, we know that :

$$H_q^{(n+1)} |_{SO(n)} = \sum_{k=0}^{q} H_k^{(n)}, \tag{VIII.2}$$

so the restriction to K of the K_c-module $H_q^{(n+1)} \otimes H_r^{(2)}$ is :

$$H_{qr}^{(n+1)} |_K = \sum_{k=0}^{q} H_{kr}^{(n)}. \tag{VIII.3}$$

Then, it follows that :

$$\mathcal{H}_+ |_K = \bigoplus_{\substack{(q,r)\in \mathbf{N}\times\mathbf{N} \\ q-r=-\frac{n}{2}+\frac{1}{2}}} \left(\bigoplus_{k=0}^{q} j^{(n+1)}_{-\frac{n}{2}+\frac{1}{2}}(H_{kr}^{(n)})\right) \tag{VIII.4}$$

and

$$\mathcal{H}_- |_K = \bigoplus_{\substack{(q,r)\in \mathbf{N}\times\mathbf{Z} \\ q+r=-\frac{n}{2}+\frac{1}{2}}} \left(\bigoplus_{k=0}^{q} j^{(n+1)}_{-\frac{n}{2}+\frac{1}{2}}(H_{kr}^{(n)})\right). \tag{VIII.5}$$

Rewriting, slightly, $(VIII.4)$ and $(VIII.5)$ and using the definition $(VIII.1)$, one easily gets that :

$$\mathcal{H}_+ |_K = \bigoplus_{\substack{(k,r)\in \mathbf{N}\times\mathbf{N} \\ k+r\equiv 0[2] \\ k-r\leq -\frac{n}{2}+\frac{1}{2}}} (j^{(n)}_{-\frac{n}{2}+\frac{1}{2}}(H_{kr}^{(n)})) \bigoplus \bigoplus_{\substack{(k,r)\in \mathbf{N}\times\mathbf{N} \\ k+r\equiv 1[2] \\ k-r\leq -\frac{n}{2}+\frac{1}{2}}} (j^{(n)}_{-\frac{n}{2}+\frac{1}{2}}(H_{kr}^{(n)})) \tag{VIII.6}$$

and

$$\mathcal{H}_- |_K = \bigoplus_{\substack{(k,r)\in \mathbf{N}\times\mathbf{Z}^- \\ k+r\equiv 0[2] \\ k+r\leq -\frac{n}{2}+\frac{1}{2}}} (j^{(n)}_{-\frac{n}{2}+\frac{1}{2}}(H_{kr}^{(n)})) \bigoplus \bigoplus_{\substack{(k,r)\in \mathbf{N}\times\mathbf{Z}^- \\ k+r\equiv 1[2] \\ k+r\leq -\frac{n}{2}+\frac{1}{2}}} (j^{(n)}_{-\frac{n}{2}+\frac{1}{2}}(H_{kr}^{(n)})), \tag{VIII.7}$$

hence :

$$\mathcal{H}_\pm |_G = W_{\pm,0} \oplus W_{\pm,1}. \tag{VIII.8}$$

Finally, from [10], $W_{\pm,\epsilon}$, $\epsilon = 0$ or 1, are unitarizable simple G-modules.

Remark VIII.2 *Note that for* $n = 3$, $W_{+,0} \oplus W_{+,1}$ *is exactly the unitarizable* G*-module* $D(1,0) \oplus D(2,0)$ *describing positive energy solutions of the wave equation* $\Delta_X f = 2f$ *on* X *in* [5]. *The* G*-module* $W_{-,0} \oplus W_{-,1}$ *is its analog for the negative energy solutions.*

From Proposition $VIII.1$, we know that :

$$\mathcal{H}_\pm |_G = W_{\pm,0} \oplus W_{\pm,1}. \tag{VIII.9}$$

On the other hand, from Proposition $IV.1$, we know that $\mathcal{R}\ (\mathcal{O}_{n+1}^{min\pm})$ is the union of two G-orbits :

$$\mathcal{R}\ (\mathcal{O}_{n+1}^{min\pm}) = \mathcal{O}_n^{o\pm} \cup \mathcal{O}_n^{min\pm}. \tag{VIII.10}$$

Comparing $(VIII.9)$ and $(VIII.10)$, one would like to associate $W_{\pm,0}$ to $\mathcal{O}_n^{o\pm}$ and $W_{\pm,1}$ to $\mathcal{O}_n^{min\pm}$ or vice versa. But it does not seem to be a natural way to do this.

IX The Klein-Gordon scalar product on $\mathcal{H}$.

Let us first give some motivation for the consideration of the Klein-Gordon scalar product on X.

IX.1 The intersection of $\mathcal{H}_\pm$ with $L^2(X, d\mu)$.

We use the notation of the previous section. Recall that the application Ψ is the restriction map defined by $(V.3)$, we have :

Proposition IX.1

$$\Psi(\mathcal{H}_\pm) \bigcap L^2(X, d\mu) = W_{\pm,1} \tag{IX.1}$$

and

$$W_{\pm,0} \bigcap L^2(X, d\mu) = \{0\}. \tag{IX.2}$$

Proof : Let Φ_{qr} be the element of $\mathcal{H}_+$ defined by :

$$\Phi_{qr}(x) \equiv h_q^{(n+1)}(\sigma) h_r^{(2)}(\tau) \mid \sigma \mid_{n+1}^{l-q-r} \quad x = (\sigma, \tau) \in \mathbf{R}^{n+1} \times \mathbf{R}^2 \tag{IX.3}$$

with $l = -\frac{n}{2} + \frac{1}{2}$, $q = r - \frac{n}{2} + \frac{1}{2}$ and $r \in \mathbf{N}$. Here $h_k^{(m)}$ denotes a harmonic polynomial on $\mathbf{R}^m$ homogeneous of degree k. We first look at the nature of the integral

$$\int_X \mid \Phi_{qr} \mid^2 d\mu. \tag{IX.4}$$

Since $d\mu = d\sigma_1 \wedge \cdots \wedge d\sigma_n d\phi$, where $\phi \in [0, 2\pi[$ is defined by :

$$\begin{aligned} x_{n+1} &= \sqrt{x_{n+1}^2 + x_{n+2}^2} \cos\phi \\ x_{n+2} &= \sqrt{x_{n+1}^2 + x_{n+2}^2} \sin\phi, \end{aligned} \tag{IX.5}$$

we can rewrite $(IX.4)$ as

$$\int_{\mathbf{R}^n} \mid h_q^{(n+1)}(1, \vec{\sigma}) \mid^2 (1 + \vec{\sigma}^2)^{-r} d^n\sigma. \tag{IX.6}$$

We shall show that this integral converges if $\Phi_{qr} \in W_{+,1}$ and does not converge otherwise. Recall that $h_q^{(n+1)}(\sigma)$ is homogeneous of degree q in σ, so $\mid h_q^{(n+1)}(1, \vec{\sigma}) \mid \leq \mid \vec{\sigma} \mid_n^q$ when $\mid \vec{\sigma} \mid_n$ goes to $+\infty$. On the other hand, $h_q^{(n+1)}(1, \vec{\sigma})$ will have the same behaviour as $\mid \vec{\sigma} \mid_n$ at infinity, only if $h_q^{(n+1)}(\sigma)$ contain a term independant of σ_o. In this case the integral in $(IX.6)$ will diverge since the integrand will behave as $\mid \vec{\sigma} \mid_n^{2q-2r} = \mid \vec{\sigma} \mid_n^{-n+1}$, otherwise $\mid h_q^{(n+1)}(1, \vec{\sigma}) \mid \leq \mid \vec{\sigma} \mid_n^{q-1}$ and $(IX.6)$ will converge. Let $\Phi_{qr} \in W_{+,\epsilon}$. Then, there exists a homogeneous polynomial of degree $k \leq q$ with

$$k = r - \frac{n}{2} + \frac{1}{2} - 2s - \epsilon = q - 2s - \epsilon \quad s \in \mathbf{N} \tag{IX.7}$$

such that

$$h_q^{(n+1)}(\sigma) = H(\sigma_o^{q-k} h_k^{(n)}(\vec{\sigma})) \tag{IX.8}$$

where H denotes the operator of harmonisation. H is defined, for all harmonic polynomial P on $\mathbf{R}^{n+1}$, homogeneous of degree l, by (see [19], IX, §2.5) :

$$(H(P))(\sigma) = \sum_{t=0}^{[\frac{l}{2}]} \alpha_t (\sigma_o^2 + \vec{\sigma}^2)^t (\ {}^t_{n+1} P)(\sigma), \tag{IX.9}$$

the α_t's are real numbers less or equal to 1 with an explicit form given in [19], IX, §2.5. Consider the case where $\epsilon = 1$, then $(IX.7)$ implies that $q - k$ is odd. Since $\ _{n+1} = \ _n + \frac{\partial^2}{\partial\sigma_o^2}$, $\ ^t_{n+1}$ will produces on $\sigma_o^{q-k} h_k^{(n)}(\vec{\sigma})$ a polynomial with all terms containing a factor σ_o^p, $p \neq 0$. It follows that $W_{+,1} \subset L^2(X, d\mu)$. Now let $\epsilon = 0$. In this case $(IX.7)$ becomes $q - k = 2s$ so that

$$h_q^{(n+1)}(\sigma) = [\sum_{t=0}^{[\frac{q}{2}]} \alpha_t (\sigma_o^2 + (\vec{\sigma})^2)^t (\frac{\partial^{2t}}{\partial \sigma_o^{2t}} \sigma_o^{q-k})] h_k^{(n)}(\vec{\sigma}). \tag{IX.10}$$

since ${}_{n+1} = {}_n + \frac{\partial^2}{\partial \sigma_o^2}$ and ${}_n h_k^{(n)} = 0$. Moreover we have :

$$h_q^{(n+1)}(\sigma) = \sum_{\substack{t=0 \\ 2t \leq q-k}}^{[\frac{q}{2}]} \alpha_t (\sigma_o^2 + (\vec{\sigma})^2)^t \sigma_o^{q-t-2t} (q-k)(q-k-1) \cdots (q-k-2t+1) h_k^{(n)}(\vec{\sigma}), \tag{IX.11}$$

hence

$$h_q^{(n+1)}(\sigma) = \sum_{t=0}^{s} \alpha_t (\sigma_o^2 + (\vec{\sigma})^2)^t \sigma_o^{q-t-2t} (q-k)(q-k-1) \cdots (q-k-2t+1) h_k^{(n)}(\vec{\sigma}). \tag{IX.12}$$

Finally, we get that

$$h_q^{(n+1)}(1, \vec{\sigma}) \sim \gamma \mid \vec{\sigma} \mid_n^{2s+k} = \gamma \mid \vec{\sigma} \mid_n^q \quad when \ \mid \vec{\sigma} \mid_n \to +\infty \tag{IX.13}$$

and the integral

$$\int_X \mid \Phi_{qr} \mid^2 d\mu \tag{IX.14}$$

diverges. On the other hand, since $W_{+,0}$ is a simple G-module then the G-submodule $W_{+,0} \cap L^2(X, d\mu)$ of $W_{+,0}$ is either reduced to $\{0\}$ or is $W_{+,0}$ itself. But if we choose $\Phi \in W_{+,0}$ non-zero such that $\Phi = \Phi_{qr}$ is given by $(IX.3)$, then we just have seen that $\Phi \notin L^2(X, d\mu)$, so that $W_{+,0} \cap L^2(X, d\mu) = \{0\}$. The argument is analog for $W_{-,1}$ and $W_{-,0}$. Finally, we have :

$$W_{\pm,1} = \Psi(\Pi \mathcal{H}_\pm) \cap L^2(X, d\mu)$$

but

$$W_{\pm,0} \cap L^2(X, d\mu) = \{0\}$$

Since not all of $\Psi(\mathcal{H}_+)$ belongs to $L^2(X, d\mu)$, it is completely natural to look for a nice geometric expression for the inner product, different from the usual L^2-integral. Actually, we shall compare the unitary structure on $\mathcal{H}$ given by $(V.11)$ with the Klein-Gordon scalar product on the hyperboloid X which we briefly recall now.

IX.2 The Klein-Gordon scalar product.

Let (M, g) be a $n+1$-dimensional Lorentz manifold and Σ a Riemannian hypersurface in M. If $f \in C^\infty(M)$, define the vector field ∇f (gradiant of f) by :

$$g(\nabla f, X) = df(X) \ \ \forall X \in TM. \tag{IX.15}$$

If $d\mu$ is the volume form on M, as soon as the integral converges, consider

$$< f_1, f_2 >_{KG,\Sigma} = i \int_\Sigma (\overline{f_1} \nabla f_2 - \overline{\nabla f_1} f_2) \rfloor d\mu \tag{IX.16}$$

where $\rfloor$ denotes the inner product. The interest of this definition is based on the following fact. If Δ_M denotes the Laplace-Beltrami operator on M, one easily checks that, if $\Delta_M f_1 = \alpha f_1$ and $\Delta_M f_2 = \alpha f_2$ ($\alpha \in \mathbf{R}$), then

$$d[(\overline{f_1}\nabla f_2 - \overline{\nabla f_1} f_2)\rfloor d\mu] = 0. \qquad \text{(IX.17)}$$

On the other hand, suppose now that M is globally hyperbolic [25]-[26], and that Σ_1 and Σ_2 are two Cauchy surfaces in M. If the solutions f_1 and f_2 have compact support on Σ_1, then f_1 and f_2 have also a compact support on Σ_2. It follows from Stokes' Theorem that :

$$< f_1, f_2 >_{KG,\Sigma_1} = < f_1, f_2 >_{KG,\Sigma_2} . \qquad \text{(IX.18)}$$

We shall then write $< f_1, f_2 >_{KG}$. Note that this scalar product is automatically invariant under isometries of (M, g), since an isometry maps a Cauchy surface onto another. If the manifold (M, g) does not admit Cauchy surfaces, the situation is not so clear. This is the case for X, which is not globally hyperbolic. However, we shall show in the next section that a Klein-Gordon scalar product properly defined on X induces on $\mathcal{H}$ a scalar product not only G-invariant but also G_c-invariant.

IX.3 The Klein-Gordon scalar product on $\mathcal{H}$.

Let $\Phi \in \mathcal{H}_\pm \subset C^\infty_\epsilon(C_{n+1}, l = -\frac{n}{2} + \frac{1}{2})$. Since X is the hypersurface in C_{n+1} defined by the equation $x_o = 1$, one can realize $\mathcal{H}_\pm$ as a subspace of $C^\infty(X) \bigcap C^\infty_{\frac{n^2-1}{4}}(X)$ via the map Ψ (V.3), where $C^\infty_{\frac{n^2-1}{4}}(X)$ denotes the space of smooth real eigenfunctions of Δ_X with eigenvalue $\frac{n^2-1}{4}$. Here, for convenience, we still denote by $\mathcal{H}_\pm$ its image in $C^\infty(X)$. To define the Klein-Gordon scalar product on $\mathcal{H}_\pm$, we first choose a Riemannian hypersurface in X. Let $\phi \in [0, 2\pi[$ be defined by :

$$\begin{aligned} x_{n+1} &= \sqrt{x_{n+1}^2 + x_{n+2}^2}\cos\phi \\ x_{n+2} &= \sqrt{x_{n+1}^2 + x_{n+2}^2}\sin\phi \end{aligned} \qquad \text{(IX.19)}$$

and put :

$$S_{\phi_o} = \{x \in X \mid \phi = \phi_o\} \text{ and } \hat{S}_{\phi_o} = S_{\phi_o} \cup S_{\phi_o+\pi}. \qquad \text{(IX.20)}$$

Note that all timelike geodesics cross S_{ϕ_o} once, and hence $\hat{S}_{\phi_o}$ twice. However almost all the lightlike geodesics cross either S_{ϕ_o} or $S_{\phi_o+\pi}$. Actually those which do not intersect $\hat{S}_{\phi_o}$ are all contained in $X \cap \{x \in \mathbf{R}^{n,2} \mid x_{n+2} = 1\}$ [15]. Then we define for Φ, $\Phi' \in \mathcal{H}_\pm$ and $\phi \in [0, 2\pi[$.

$$< \Phi, \Phi' >_{KG,\hat{S}_{\phi_o}} = \pm i \int_{\hat{S}_{\phi_o}} (\overline{\Phi}\nabla\Phi' - \overline{\nabla\Phi}\Phi')\rfloor d\mu. \qquad \text{(IX.21)}$$

Proposition IX.2 *The scalar product* $< \cdot, \cdot >_{KG,\hat{S}_{\phi_o}}$ *is well defined on* $\mathcal{H}_\pm$ *and is* G_c*-invariant. It does not depend on* ϕ_o*. More precisely, we have :*

$$< \cdot, \cdot >_{KG,\hat{S}_{\phi_o}} = \frac{1}{2\pi} < \cdot, \cdot >_{BZ} . \qquad \text{(IX.22)}$$

Proof : We shall just consider $\mathcal{H}_+$, since our arguments will still hold for $\mathcal{H}_-$. If $\Phi \in \mathcal{H}_+$, Φ is a finite sum of terms :

$$\Phi_{qr}(x) \equiv h_q^{(n+1)}(\sigma) h_r^{(2)}(\tau) \mid \sigma \mid_{n+1}^{l-q-r} \quad x = (\sigma, \tau) \in \mathbf{R}^{n+1} \times \mathbf{R}^2 \qquad \text{(IX.23)}$$

with $l = -\frac{n}{2} + \frac{1}{2}$, $q = r - \frac{n}{2} + \frac{1}{2}$ and $r \in \mathbf{N}$. Here $h_k^{(m)}$ denotes a harmonic polynomial on $\mathbf{R}^m$ homogeneous of degree k. We have also :

$$h_r^{(2)}(\tau) = (\tau_1 + i\tau_2)^r = \mid \tau \mid_2 e^{ir\phi}, \qquad \text{(IX.24)}$$

where $\tau_1 =\mid \tau \mid_2 \cos\phi$ and $\tau_2 =\mid \tau \mid_2 \sin\phi$. Finally, since $\mid \tau \mid_2 = \mid \sigma \mid_{n+1}$ if $x = (\sigma, \tau) \in C_{n+1}$, we get :

$$\Phi_{qr}(x) \equiv h_q^{(n+1)}(\sigma)(\tau_1 + i\tau_2)^r \mid \tau \mid_2^{-2r} = h_q^{(n+1)}(\sigma) e^{ir\phi} \mid \tau \mid_2^{-r} = h_q^{(n+1)}(\sigma) e^{ir\phi} \mid \sigma \mid_{n+1}^{-r} . \quad \text{(IX.25)}$$

So it suffices to show that :

$$< \Phi_{qr}, \Phi'_{q'r'} >_{KG,\hat{S}_{\phi_o}} = \frac{1}{2\pi} < \Phi_{qr}, \Phi'_{q'r'} >_{BZ} \quad \text{(IX.26)}$$

for all r and $r' \in \mathbf{N}$, and r and $r' \leq \frac{n}{2} - \frac{1}{2}$. First note that from $(V.11)$, one has :

$$< \Phi_{qr}, \Phi'_{q'r'} >_{BZ} = 2r\delta_{rr'} \int_{\Sigma_{n+1}} \overline{\Phi}_{qr} \Phi'_{q'r'} d\sigma \quad \text{(IX.27)}$$

where

$$\Sigma_{n+1} = \{x = (\sigma, \tau) \in \mathbf{R}^{n+1} \times \mathbf{R}^2 \mid \mid \sigma \mid_{n+1} = \mid \tau \mid_2 = 1\} \subset C_{n+1}. \quad \text{(IX.28)}$$

On the other hand, it is not hard to check that :

$$\begin{aligned} & < \Phi_{qr}, \Phi'_{q'r'} >_{KG,\hat{S}_{\phi_o}} \\ = \quad & (r + r') e^{i(r-r')\phi_o} (1 + e^{i(r-r')\pi}) \int_{\mathbf{R}^n} \overline{h_q^{(n+1)}(1, \vec{\sigma})} h_{q'}^{(n+1)'}(1, \vec{\sigma}) (1 + \vec{\sigma}^2)^{-1-\frac{r+r'}{2}} d^n\sigma \end{aligned} \quad \text{(IX.29)}$$

where $d^n\sigma = d\sigma_1 \wedge .... \wedge d\sigma_n$. From the homogeneity of $h_q^{(n+1)}$, we see that this integral converges.

Considering the coordinates $(\vec{\sigma}, \phi)$ on X, a simple calculation shows that [15] :

$$d\mu = d\sigma_1 \wedge \cdots \wedge d\sigma_n \wedge d\phi \quad \text{(IX.30)}$$

and then

$$\int_{\mathbf{R}^n} \overline{h_q^{(n+1)}(1, \vec{\sigma})} h_{q'}^{(n+1)'}(1, \vec{\sigma}) (1+\vec{\sigma}^2)^{-1-\frac{r+r'}{2}} d^n\sigma = \frac{1}{2\pi} \int_X \overline{h_q^{(n+1)}(1, \vec{\sigma})} h_{q'}^{(n+1)'}(1, \vec{\sigma}) (1+\vec{\sigma}^2)^{-1-\frac{r+r'}{2}} d\mu, \quad \text{(IX.31)}$$

where the integrand is the restriction to X of the homogeneous function on C_{n+1} of degree $-n-1$ given by :

$$\overline{h_q^{(n+1)}(\sigma)} h_{q'}^{(n+1)'}(\sigma) \mid \tau \mid^{-2(1+\frac{r+r'}{2})} . \quad \text{(IX.32)}$$

On the other hand, since $\Sigma_{n+1}^{\pm} = \{x = (\sigma, \tau) \in \Sigma_{n+1} \mid \pm\sigma_o > 0\}$, then

$$\Sigma_{n+1}^{+} \to X, \; x = (\sigma, \tau) \mapsto (1, \frac{\sigma}{\sigma_o}, \frac{\tau}{\sigma_o}) \quad \text{(IX.33)}$$

is a bijection and we have $\frac{1}{\sigma_o^{n+1}} d\sigma = d\mu$ [16]. Moreover the change of variables $(VI.18)$ gives :

$$\int_{\mathbf{R}^n} \overline{h_q^{(n+1)}(1, \vec{\sigma})} h_{q'}^{(n+1)'}(1, \vec{\sigma}) (1 + \vec{\sigma}^2)^{-1-\frac{r+r'}{2}} d^n\sigma = \frac{1}{2\pi} \int_{\Sigma_{n+1}^{+}} \overline{h_q^{(n+1)}(\sigma)} h_{q'}^{(n+1)'}(\sigma) d\sigma \quad \text{(IX.34)}$$

hence

$$< \Phi_{qr}, \Phi'_{q'r'} >_{KG,\hat{S}_{\phi_o}} = \frac{1}{2\pi} (r + r') e^{i(r-r')\phi_o} (1 + e^{i(r-r')\pi}) \int_{\sigma_{n+1}^{+}} \overline{h_q^{(n+1)}(\sigma)} h_{q'}^{(n+1)'}(\sigma) d\sigma. \quad \text{(IX.35)}$$

Moreover if $r \equiv r' + 1[2]$, then $< \Phi_{qr}, \Phi'_{q'r'} >_{KG,\hat{S}_{\phi_o}} = 0$. Now assume that $r \equiv r'[2]$, then $q \equiv q'[2]$. In this case we have :

$$\int_{\Sigma^+_{n+1}} \overline{h_q^{(n+1)}(\sigma)} h_{q'}^{(n+1)'}(\sigma) d\sigma = \frac{1}{2}\delta_{qq'} \int_{\Sigma_{n+1}} \overline{h_q^{(n+1)}(\sigma)} h_q^{(n+1)'}(\sigma) d\sigma. \qquad (IX.36)$$

Since $\delta_{qq'} = \delta_{rr'}$ it follows, from $(IX.35)$, that for all r and r', we have :

$$< \Phi_{qr}, \Phi'_{q'r'} >_{KG,\hat{S}_{\phi_o}} = \frac{r}{\pi}\delta_{rr'} \int_{\Sigma_{n+1}} \overline{h_q^{(n+1)}(\sigma)} h_q^{(n+1)'}(\sigma) d\sigma = \frac{1}{2\pi} < \Phi_{qr}, \Phi'_{q'r'} >_{BZ} \qquad (IX.37)$$

where we used $(IX.25)$ and $(IX.27)$.

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