Large Population Limit and Time Behaviour of a Stochastic Particle Model Describing an Age-structured Population

Tran Viet Chi[*]

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Abstract

We study a continuous-time discrete population structured by a vector of ages. Individuals reproduce asexually, age and die. The death rate takes interactions into account. Adapting the approach of Fournier and Méléard, we show that in a large population limit, the microscopic process converges to the measure-valued solution of an equation that generalizes the McKendrick-Von Foerster and Gurtin-McCamy PDEs in demography. The large deviations associated with this convergence are studied. The upper-bound is established via exponential tightness, the difficulty being that the marginals of our measure-valued processes are not of bounded masses. The local minoration is proved by linking the trajectories of the action functional's domain to the solutions of perturbations of the PDE obtained in the large population limit. The use of Girsanov theorem then leads us to regularize these perturbations. As an application, we study the logistic age-structured population. In the super-critical case, the deterministic approximation admits a non trivial stationary stable solution, whereas the stochastic microscopic process gets extinct almost surely. We establish estimates of the time during which the microscopic process stays in the neighborhood of the large population equilibrium by generalizing the works of Freidlin and Ventzell to our measure-valued setting.

Keywords: Age-structured population, interacting measure-valued process, large population approximation, large deviations, exit time estimates, Gurtin-McCamy PDE.
AMS Subject Classification: 60J80, 60K35, 92D25, 60F10.

1 Introduction and motivations

Structured population models describe the dynamics of populations in which individuals differ according to variables that affect their reproductive capacities and survivals. In this article, we are interested in a population structured by ages. Age-structures are important to take into account the changes of behaviour of an individual during its life as well as life histories. It is natural to consider many ages. Examples are the *physical age* (the time since which an individual is born), the *biological age* (the intrinsic maturation stage of the individual), the *age of an illness* (the time since which the individual has been infected), the *stage of the illness* (the clinical stage of the illness). To our knowledge, the literature on the subject mostly considers structuration by the only physical age.

Our purpose is to study a microscopic stochastic population structured by a vector of ages that can grow nonlinearly in time and which models the age-dependence of the birth and death rates as well as possible interactions between individuals (competition or cooperation), including

[*]Université Paris X-Nanterre, Equipe Modal'X, bâtiment G, 200 avenue de la République, 92100 Nanterre Cedex, France. email: *viet_chi.tran@u-paris10.fr*

competition of *logistic* type (for which the death rate is linear in the size of the population).

Continuous time physical age-structured models that generalize the models of Malthus [32] and Verhulst's famous *Logistic Equation* [46], have made the object of an abundant literature based on the theory of partial differential equations (PDEs) (see Sharpe and Lotka [43], McKendrick [34], Gurtin and MacCamy [22], Marcati [33], Busenberg and Iannelli [6], Webb [49]).

Stochastic models generalizing the Galton-Watson process [18] have been studied by Bellman and Harris [3, 23], Athreya and Ney ([2] Chapter IV). These models consider non Markovian processes, called *age-structured branching processes*, structured by the physical age and in which the lifelength of an individual does not follow an exponential law. Each particle, at its death, is replaced by a random number of daughters with a law that is independent of the age of the mother and of the state of the population.

The assumptions of birth at the parent's death and of independence between the reproduction law and the age of the parent are biologically restrictive. Kendall [29], Crump and Mode [9, 10], Jagers [26] and Doney [13] have studied birth and death processes in which a particle can give birth many times during its life, with a rate that depends on its age.

In the preceding models, the particles alive at the same time are independent, which is also a biologically restrictive assumption. Wang [48], Solomon [44] consider birth and death processes in which the lifelength are independent, but where the birth rate of a particle depends on the state of the population. Oelschläger [37] generalizes their works to take interactions in both the birth and the death rate into account. However, in these models, the rates remain bounded, which excludes interactions of *logistic* type.

We present here an individual-centered model which takes age-structure into account. We are inspired by the works of Fournier et Méléard [16] and Champagnat *et al.* [8, 7]. Our paper is inspired from more general models of trait and age-structured population considered in [45] to which we refer for examples, more details and full proofs.

Individuals are characterized by their ages with values in $\mathbb{R}_+^d$. Each component of this vector is an age belonging to $\mathbb{R}_+$, which can increase nonlinearly in time. Let $n \in \mathbb{N}^*$ be a fixed integer (its interpretation is given below). We describe the population by a measure belonging to $\mathcal{M}_P^n(\mathbb{R}_+^d)$ (the set of point measures on $\mathbb{R}_+^d$ with atoms weighted by $1/n$) included in $\mathcal{M}_F(\mathbb{R}_+^d)$ (the set of finite measures on $\mathbb{R}_+^d$):

$$Z_t^n = \frac{1}{n}\sum_{i=1}^{N_t^n} \delta_{a_i(t)}, \qquad \text{where} \qquad N_t^n = n\langle Z_t^n, 1\rangle = n\int_{\mathbb{R}_+^d} Z_t^n(da) \tag{1.1}$$

is the number of individuals living at time t.

An individual of ages $a \in \mathbb{R}_+^d$ in the population $Z \in \mathcal{M}_F(\mathbb{R}_+^d)$ reproduces asexually, ages and dies:

1. It gives birth to a new individual of age zero with rate $b(a) \in \mathbb{R}_+$.

2. Let $U : \mathbb{R}_+^d \times \mathbb{R}_+^d \to \mathbb{R}^{d_c}$ be an interaction kernel, $d_c \in \mathbb{N}^*$ being the number of interactions taken into account. For $a, \alpha \in \mathbb{R}_+^d$, $U(a,\alpha)$ models the action of an individual of ages α on an individual of ages a. The death rate of our individual is modelled by $d(a, ZU(a))$, with $d : \mathbb{R}_+^d \times \mathbb{R}^{d_c} \to \mathbb{R}_+$ and where $ZU(a) = \int_{\mathbb{R}_+^d} U(a,\alpha)Z(d\alpha)$,

3. Our individual ages with the speed $v(a) \in \mathbb{R}_+^d$.

We introduce in Section 2 a pathwise description of our microscopic process, using Poisson point measures and the flow of the equation describing the aging phenomenon.

The parameter $n \in \mathbb{N}^*$ in (1.1) is related to the large population limit that will interest us in this work, and which corresponds to $n \to +\infty$. The underlying idea is to let the number of individuals grow proportionally to n while their masses and the intensity of the interactions are renormalized by $1/n$. This can be understand as a constraints in ressources: if we increase the size of the population, we have to decrease the biomass of individuals to keep the system alive. In Section 3, we prove the convergence in law in $\mathbb{D} := \mathbb{D}(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}_+^d))$ of the sequence $(Z^n)_{n\in\mathbb{N}^*}$ to the solution $\xi \in \mathcal{C}(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}_+^d))$ of: $\forall\,(f\,:\,(a,s)\mapsto f_s(a)) \in \mathcal{C}_b^1(\mathbb{R}_+^d\times\mathbb{R}_+,\mathbb{R}_+)$,

$$\langle \xi_t, f_t\rangle = \langle \xi_0, f_0\rangle + \int_0^t \int_{\mathbb{R}_+^d} \left[v(a)\nabla_a f_s(a) + \frac{\partial f_s}{\partial s}(a) + f_s(0)b(a) - f_s(a)d(a,\xi_s U(x,a))\right] \xi_s(da)\,ds. \quad (1.2)$$

This convergence result is an adaptation of results due to Fournier and Méléard [16] and Champagnat *et al.* [7] for populations without age structure. The densities $m(a,t)$ of the measures $\xi_t \in \mathcal{M}_F(\mathbb{R}_+^d)$, when they exist, correspond to the notion of *number density*, and describe the distribution in age of a population consisting in a "continuum" of individuals. They solve the system: $\forall a \in \mathbb{R}_+^d$, $\forall t\in\mathbb{R}_+$,

$$\frac{\partial m}{\partial t}(a,t) = -\nabla_a\left(v(a)m(a,t)\right) - d\left(a, \int_{\mathbb{R}_+} U(a,\alpha)m(\alpha,t)d\alpha\right)m(a,t) \quad (1.3)$$

$$v(0)m(0,t) = \int_{\mathbb{R}_+} m(a,t)b(a)da, \quad m(a,0)=m_0(a). \quad (1.4)$$

These equations generalize the PDEs introduced by McKendrick-Von Foerster [34, 15] and Gurtin MacCamy [22]. This is considered in Section 3.3. Equations (1.2) and (1.3)-(1.4) are macroscopic deterministic approximations describing the ecology at the scale of the population. Individual trajectories are lost, as well as stochasticity since an averaging phenomenon occurs.

Let $T>0$. We consider the evolution problem on the compact time interval $[0,T]$. We use the notation $\mathbb{D}_T := \mathbb{D}([0,T],\mathcal{M}_F(\mathbb{R}_+^d))$. From the exponential deviations of Section 4 and from the central limit theorem proved in [45], we know that on $[0,T]$, the microscopic process behaves like its deterministic approximation up to a small probability set. Their large time behaviors can however be radically different. We present in the following an example to illustrate this.

In the *logistic age-structured population*, represented by $Z\in\mathcal{M}_F(\mathbb{R}_+)$, individuals are characterized by their *scalar physical age* $a\in\mathbb{R}_+$ growing with speed 1, give birth with rate $b(a)$ (continuous and upper bounded by $\bar{b}$) and die with rate $d(a)+\eta\langle Z,1\rangle$. The term d is the natural death rate (assumed continuous and bounded above and below by positive constants $\bar{d}$ and $\underline{d}$), and $\eta\langle Z,1\rangle$ is the logistic competition term of intensity $\eta>0$. The system (1.3)-(1.4) becomes:

$$\begin{aligned}\frac{\partial m}{\partial t}(a,t) &= -\frac{\partial m}{\partial a}(a,t) - \left(d(a)+\eta M_t\right)m(a,t)\\ m(0,t) &= \int_0^{+\infty} b(a)m(a,t)da, \quad m(a,0)=m_0(a), \quad M_t = \int_0^{+\infty} m(a,t)da. \end{aligned} \quad (1.5)$$

If $R_0 := \int_0^{+\infty} b(a)e^{-\int_0^a d(\alpha)d\alpha}da > 1$, we are in a super-critical case and there exists a globally stable stationary solution to (1.5) that we denote by $\widehat{m}(a)$ (see [22, 6, 49]). The microscopic process Z^n has a different long time behaviour. It follows it deterministic approximation on compact time intervals but leaves almost surely the neighborhood of $\widehat{m}(a)da$ to drive the population to extinction (see Propositions 5.5 and 5.6 in Section 5).

For various applications, including the studies of evolution problems for trait and age structured population, which is a work under progress, it is interesting to establish estimates for the time of exit from a neighborhood of the stable equilibrium of (1.5). This can be obtained by using large deviations techniques and by adapting the results of Freidlin and Ventzell to our measure-valued processes.

Exponential deviations are considered in Section 4 for the general model. The large deviation upper-bound is proved by establishing exponential tightness. The main difficulty lies in the fact

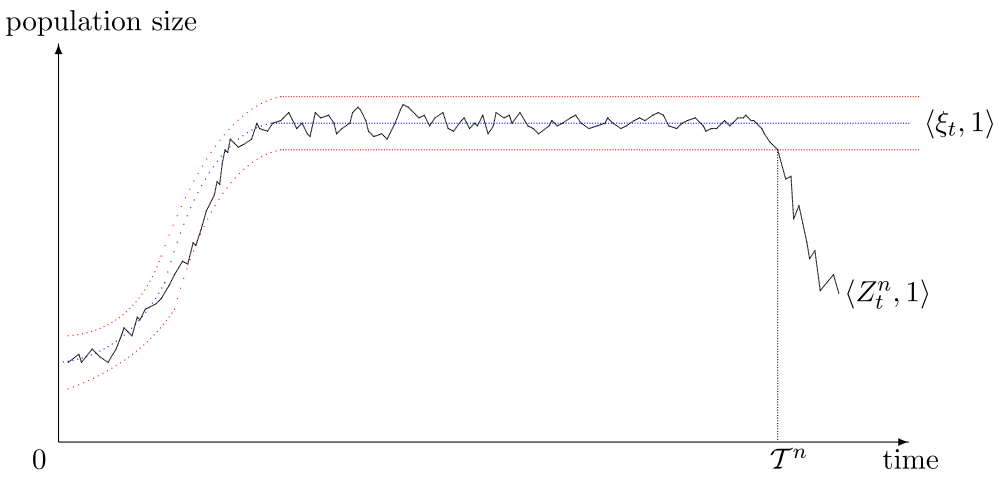

Figure 1: *Time behaviors of the microscopic process and of its large population approximation. For sufficiently large n, the behaviour of the stochastic microscopic population Z^n follows the one of its deterministic approximation ξ on compact time intervals. In the long time, however, Z^n leaves the neighborhood of the stationary stable solution of ξ and the population gets extinct.*

that $\mathcal{M}_F(\mathbb{R}^d_+)$ is not compact. Our proof of the local lower bound relies on the use of a Riesz theorem in Orlicz spaces. This allows us to establish the links between the trajectories in the domain of the rate function and solutions of PDEs obtained by perturbing (1.2). Regularizing the perturbations and using a Girsanov theorem allow us to conclude.

Notation: For two metric spaces E and F, $\mathcal{C}_b(E,F)$ (resp. $\mathbb{D}(E,F)$, $\mathcal{C}_0(E,\mathbb{R})$, $\mathcal{C}^1_b(E,F)$, $\mathcal{C}_K(E,\mathbb{R})$, $\mathcal{B}_b(E,F)$) the set of continuous bounded functions from E to F embedded with the uniform convergence norm (resp. of càdlàg functions from E to F embedded with the Skorohod distance, of real continuous functions with limit 0 at infinity, of differentiable and bounded functions with bounded partial derivatives, of continuous functions with compact support, of bounded measurable functions).

The space of finite measures on $\mathbb{R}^d_+$ is denoted by $\mathcal{M}_F(\mathbb{R}^d_+)$. It can be embedded with the weak or vague convergence topology. By default, we will consider the weak convergence topology. We will write $(\mathcal{M}_F(\mathbb{R}^d_+), w)$ or $(\mathcal{M}_F(\mathbb{R}^d_+), v)$ to precise it when necessary.

We will consider the total variation norm and the L^1-Vaserstein distance on $\mathcal{M}_F(\mathbb{R}^d_+)$:

$$\forall \mu,\, \nu \in \mathcal{M}_F(\mathbb{R}^d_+),\ \|\nu-\mu\|_{TV} = \sup_{\substack{f\ \mathcal{C}_b(\mathbb{R}^d_+,\mathbb{R})\\ \|f\|_\infty \le 1}} \left| \int_{\mathbb{R}^d_+} f d\mu - \int_{\mathbb{R}^d_+} f d\nu \right|, \tag{1.6}$$

$$\mathcal{W}_1(\mu,\nu) = \inf\left\{ \int_{(\mathbb{R}^d_+)^2} (|a-\alpha| \wedge 1)\, d\pi(a,\alpha) \right\} = \sup_{\substack{f\ 1-Lip(\mathbb{R}^d_+)\\ \|f\|_\infty \le 1}} \left| \int_{\mathbb{R}^d_+} f d\mu - \int_{\mathbb{R}^d_+} f d\nu \right|. \tag{1.7}$$

the infimum being taken on the set of measures $\pi \in \mathcal{M}_F((\mathbb{R}^d_+)^2)$ with marginals μ and ν (see Rachev [39], Villani [47] Theorem 7.12 and Remark 7.5).

For $m \in \mathcal{M}_F(\mathbb{R}^d_+)$ and $f \in \mathcal{B}_b(\mathbb{R}^d_+)$, we write $\langle m, f\rangle = \int_E f dm$ and $\langle m, f(a)\rangle = \int_{\mathbb{R}^d_+} f(a) m(da)$.

C is a constant that can change from line to line.

2 Microscopic Process

We now precise the individual dynamic of our model and describe the path of Z^n by a SDE.

2.1 Aging phenomenon

The aging phenomenon is deterministic, and we describe it thanks to the flow of an ordinary differential equation (ODE). The ages of an individual aged $a \in \mathbb{R}_+^d$ at time $t_0 \in \mathbb{R}_+$ satisfy:

$$\forall t \geq t_0,\ \frac{da}{dt} = v(a(t)), \quad a(t_0) = a_0, \tag{2.1}$$

where $v(a) = (v_1(a), \cdots, v_d(a)) \in \mathbb{R}_+^d$. The i^{th} component $v_i(a)$ of $v(a)$ is the speed of aging of the i^{th} age. The constant components correspond to ages which increase linearly in time. Non constant speeds of aging modelize ages which are measured on physiological criteria and which evolve non linearly in time.

Assumption 2.1. *$v \in \mathcal{C}_b^1(\mathbb{R}_+^d, \mathbb{R}_+^d)$ and $\exists \bar{v} > 0$, $\forall i \in [\![1, d]\!]$, $\forall a \in \mathbb{R}_+^d$, $0 < v_i(a) \leq \bar{v}(1 + a_i)$.*

This assumption is a technical assumption under which:

Proposition 2.2. *Under Assumption 2.1.*
(i) The system (2.1) admits for every $t_0 \in \mathbb{R}_+$, $a_0 \in \mathbb{R}_+^d$ a unique solution, defining a $\mathcal{C}^1$-flow:

$$A\,:\,(t, t_0, a_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^d \mapsto A(t, t_0, a_0) \in \mathbb{R}^d. \tag{2.2}$$

Each component of this flow is increasing in t.
(ii) $\forall t_1, t_2 \in \mathbb{R}_+$, $a \in \mathbb{R}_+^d \mapsto A(t_1, t_2, a) \in \mathbb{R}^d$ defines a $\mathcal{C}^1$-diffeomorphism from $\mathbb{R}_+^d$ on its image.
(iii) When $d = 1$, $\forall a_0 \in \mathbb{R}_+$, $\forall t_0 \in \mathbb{R}_+$, $t \in \mathbb{R}_+ \mapsto A(t, t_0, a_0) \in \mathbb{R}$ defines a $\mathcal{C}^1$-diffeomorphism from $\mathbb{R}_+$ on its image.

Proposition 2.2 is a consequence of classical ODE results (see for instance [50], Chapter 10). The following PDE (2.3), which often appears in the sequel, is solved by using the flow (2.2):

Proposition 2.3. *Under Assumption 2.1, for $T > 0$ and $\phi \in \mathcal{C}^1(\mathbb{R}_+^d, \mathbb{R}_+)$, the following transport equation with condition at time T: $\forall a \in \mathbb{R}_+^d$, $\forall t \in \mathbb{R}_+$,*

$$\frac{\partial f}{\partial t}(a, t) + v(a)\nabla_a f(a, t) = 0, \quad f(a, T) = \phi(a), \tag{2.3}$$

admits a unique solution $f \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}_+^d)$ given by $\forall t \in \mathbb{R}_+$, $\forall a \in \mathbb{R}_+^d$, $f(a, t) = \phi(A(T, t, a))$, where A is the flow defined in (2.2).

The proof is given in [19].

The other assumptions, concerning the birth and death rates are the following:

Assumption 2.4. *We assume that b, d and U are continuous and that:*
(i) $\exists \bar{b} > 0$, $\forall a \in \mathbb{R}_+$, $|b(a)| \leq \bar{b}$.
(ii) $\exists \bar{U} > 0$, $\forall a, \alpha \in \mathbb{R}_+^d$, $|U(a, \alpha)| \leq \bar{U}$.
(iii) $\exists L_d > 0$, $\forall u, v \in \mathbb{R}^{d_c}$, $\forall a \in \mathbb{R}_+^d$, $|d(a, u) - d(a, v)| \leq L_d|u - v|$, and $\exists \bar{d} > 0$, $\forall a \in \mathbb{R}_+, \forall u \in \mathbb{R}^{d_c}$, $d(a, u) \leq \bar{d}(1 + |u|)$.
(iv) $\exists \underline{d} \in \mathcal{C}_b(\mathbb{R}_+^d, \mathbb{R}_+)$, $\forall a \in \mathbb{R}_+$, $\forall u \in \mathbb{R}^{d_c}$, $d(a, u) \geq \underline{d}(a)$, and $\forall t_0 \in \mathbb{R}_+$, $\forall a_0 \in \mathbb{R}_+^d$,

$$\int_{t_0}^{+\infty} \underline{d}(A(t, t_0, a_0))dt = +\infty.$$

The function $\underline{d}$ in (iv) can be interpreted as a natural death rate, and Point (iv) can be linked to the *survival probability*: the probability that an individual of ages a_0 at time t_0 is still alive at time t is:

$$\Pi(t_0, a_0, t) = \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} d(A(u, t_0, a_0), Z_u U(x, A(u, t_0, a_0)))du\right)\right].$$

Under Point (iv), $\lim_{t\to+\infty} \Pi(t_0, a_0, t) = 0$.

2.2 Stochastic Differential Equation

We introduce a SDE driven by a Poisson Point Measure, for which existence and uniqueness of the solution are stated. The solution is a Markov process, with a generator that corresponds to the dynamics described previously. We follow in this the approach of [16, 8, 7].

Let us introduce the following map $A = (A_1, \cdots, A_N, \cdots)$ from $\bigcup_{n\in\mathbb{N}^*} \mathcal{M}_P^n(\mathbb{R}_+^d)$ in $(\mathbb{R}_+^d)^{\mathbb{N}}$ that will be useful to extract a particular individual from the population where the particles are ranked in the lexicographical order of $\mathbb{R}_+^d$: $\forall n,\ N \in \mathbb{N}^*$,

$$A\left(\frac{1}{n}\sum_{i=1}^{N} \delta_{a_i}\right) = (a_1, \cdots, a_N, 0, \cdots, 0, \cdots), \tag{2.4}$$

Definition 2.5. *On the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*:*
1. Let $Z_0^n \in \mathcal{M}_P^n(\mathbb{R}_+^d)$ *be a random variable such that* $\sup_{n\in\mathbb{N}^*} \mathbb{E}(\langle Z_0^n, 1\rangle) < +\infty$,
2. Let $Q(ds, di, d\theta)$ *be a Poisson point measure on* $\mathbb{R}_+ \times \mathcal{E}$ *where* $\mathcal{E} := \mathbb{N}^* \times \mathbb{R}_+$ *of intensity* $ds \otimes n(di) \otimes d\theta$ *independent from* Z_0 *(where* ds *and* $d\theta$ *are Lebesgue measures on* $\mathbb{R}_+$ *and* $n(di)$ *is the counting measure on* $\mathbb{N}^*$*).*
We denote by $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ *the canonical filtration associated with* Z_0^n *and* Q*. For* $t \in \mathbb{R}_+$, Z_t^n *is described by:*

$$\begin{aligned} Z_t^n =& \frac{1}{n}\sum_{i=1}^{N_0^n} \delta_{A(t,0,A_i(Z_0^n))} + \frac{1}{n}\int_0^t \int_{\mathcal{E}} \mathbf{1}_{\{i\le N_{s\_}^n\}} \Big[\delta_{A(t,s,0)} \mathbf{1}_{\{0\le\theta<m_1(s,Z_{s\_}^n,i)\}} \\ &- \delta_{A(t,s,A_i(Z_{s\_}^n))} \mathbf{1}_{\{m_1(s,Z_{s\_}^n,i)\le\theta<m_2(s,Z_{s\_}^n,i)\}}\Big] Q(ds, di, d\theta), \end{aligned} \tag{2.5}$$

where N_s^n *is defined in (1.1), where* A *is the flow defined in (2.2), and where:*

$$m_1(s, Z_{s_-}^n, i) = b(A_i(Z_{s_-}^n)), \quad m_2(s, Z_{s_-}^n, i) = m_1(s, Z_{s_-}^n, i) + d(A_i(Z_{s_-}^n), Z_{s\_}^n U(A_i(Z_{s_-}^n))).$$

Let $T_0^n = 0$. Assume that the size N_t^n of the population at time $t \in \mathbb{R}_+$ is finite. Under Assumptions 2.4, there exists a positive constant $\bar{C}$ (say $\bar{b} + \bar{d}$) such that the global jump rate at time $t \in \mathbb{R}_+$ is upper bounded by $\bar{C} N_{t\_}^n (1 + N_{t\_}^n)$, which is finite. Hence, it is possible to define the sequence of event times $(T_k^n)_{k\in\mathbb{N}^*}$ of Z^n almost surely. Since it forms an increasing sequence, $T_\infty^n := \lim_{k\to+\infty} T_k^n$ is well defined. It is proved in [45] (Sections 2.2, 2.3 and 3.1) that:

Theorem 2.6. ***Existence and uniqueness of the solutions of (2.5)***
Under Assumptions 2.1, 2.4 and Point 1 of Definition 2.5, $T_\infty^n = +\infty$ $\mathbb{P}$*-a.s. and SDE (2.5) admits for every* $n \in \mathbb{N}^*$ *a unique pathwise solution* $(Z_t^n)_{t\in\mathbb{R}_+} \in \mathbb{D}$.

Since for $t \in \mathbb{R}_+$, N_t^n and $\langle Z_t^n, |a|\rangle$ are finite but unbounded, we introduce the following stopping times: let $N \in \mathbb{R}_+^*$,

$$\tau_N^n = \inf\{t \ge 0,\ N_t^n \ge N\}, \quad \zeta_N^n = \inf\{t \ge 0,\ N_t^n \ge N \text{ or } \langle Z_t^n, |a|\rangle \ge N\}. \tag{2.6}$$

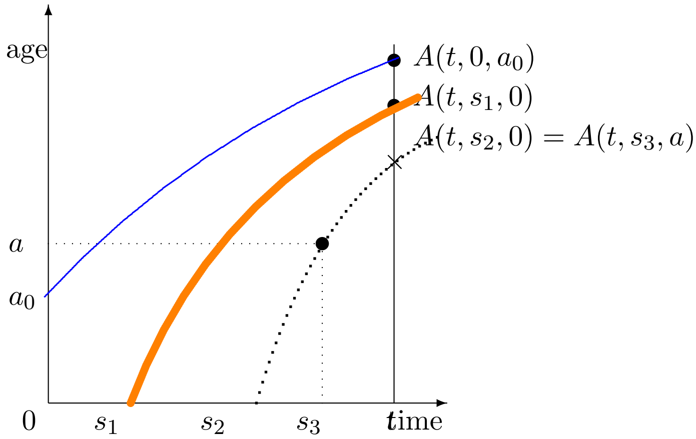

Figure 2: *The interpretation is that the state of the population at time t is obtained by considering the initial particles with their ages at times t (for example, $A(t,0,a_0)$ for the blue thin-line particle), by adding the particles born between $s=0$ and $s=t$ with their age at time t ($A(t,s_1,0)$ and $A(t,s_2,0)$ for the particles in thick orange and black dot lines) and by suppressing the particles which have died before time t ($A(t,s_3,a)$ for the particle in black dot-line).*

2.3 Moment and martingale properties

We give some moments and martingale properties that will be useful in the sequel.

Lemma 2.7. *Let $F\in\mathcal{C}^1(\mathbb{R},\mathbb{R})$, $(f:(a,s)\mapsto f_s(a))\in\mathcal{B}_b(\mathbb{R}_+^d\times\mathbb{R}_+,\mathbb{R})\cap\mathcal{C}_b^1(\mathbb{R}_+^d\times\mathbb{R}_+,\mathbb{R})$. Then, $\forall t\in\mathbb{R}_+$, $\forall n\in\mathbb{N}^*$,*

$$F\left(\langle Z_t^n,\,f_t\rangle\right)=F\left(\langle Z_0^n,\,f_0\rangle\right)+\int_0^t\left\langle Z_s^n,\,v\nabla_a f_s+\frac{\partial f_s}{\partial s}\right\rangle F'\left(\langle Z_s^n,\,f_s\rangle\right)ds \tag{2.7}$$
$$+\int_0^t\int_{\mathcal{E}}\mathbf{1}_{\{i\le N_t^n\}}\left[\left(F\left(\langle Z_{s_-}^n,\,f_s\rangle+\frac{f_s(0)}{n}\right)-F\left(\langle Z_{s_-}^n,\,f_s\rangle\right)\right)\mathbf{1}_{\{0\le\theta<m_1(s,Z_{s\_}^n,i)\}}\right.$$
$$\left.+\left(F\left(\langle Z_{s_-}^n,\,f_s\rangle-f_s(A_i(Z_{s_-}^n))\right)-F\left(\langle Z_{s_-}^n,\,f_s\rangle\right)\right)\mathbf{1}_{\{m_1(s,Z_{s\_}^n,i)\le\theta<m_2(s,Z_{s\_}^n,i)\}}\right]Q(ds,di,d\theta).$$

Proof. Integrating $f_t(a)$ with respect to (2.5) gives:

$$\langle Z_t^n,f_t\rangle=\frac{1}{n}\sum_{i=1}^{N_0^n}f_t\left(A(t,0,A_i(Z_0^n))\right)+\frac{1}{n}\int_0^t\int_{\mathcal{E}}\mathbf{1}_{\{i\le N_{s\_}^n\}}\left[f_t\left(A(t,s,0)\right)\mathbf{1}_{\{0\le\theta<m_1(s,Z_{s\_}^n,i)\}}\right.$$
$$\left.-f_t\left(A(t,s,A_i(Z_{s\_}^n))\right)\mathbf{1}_{\{m_1(s,Z_{s\_}^n,i)\le\theta<m_2(s,Z_{s\_}^n,i)\}}\right]Q(ds,di,d\theta).$$

Since $\forall a\in\mathbb{R}_+^d$, $\forall 0\le s\le t$,

$$f_t(A(t,s,a))=f_s(a)+\int_s^t\left(\frac{\partial f_u}{\partial u}(A(u,s,a))+v(A(u,s,a))\nabla_a f_u(A(u,s,a))\right)du,$$

we deduce:

$$\langle Z_t^n,f_t\rangle=\frac{1}{n}\sum_{i=1}^{N_0^n}f_0\left(A_i(Z_0)\right)+\frac{1}{n}\int_0^t\int_{\mathcal{E}}\mathbf{1}_{\{i\le N_{s\_}^n\}}\left[f_s\left(0\right)\mathbf{1}_{\{0\le\theta<m_1(s,Z_{s\_}^n,i)\}}\right. \tag{2.8}$$
$$\left.-f_s\left(A_i(Z_{s\_}^n)\right)\mathbf{1}_{\{m_1(s,Z_{s\_}^n,i)\le\theta<m_2(s,Z_{s\_}^n,i)\}}\right]Q(ds,di,d\theta)+T_1+T_2+T_3,$$

where:

$$T_1 = \frac{1}{n}\int_0^t \sum_{i=1}^{N_0^n} \left(\frac{\partial f_s}{\partial s}(A(s,0,A_i(Z_0^n))) + v(A(s,0,A_i(Z_0^n)))\,\nabla_a f_s(A(s,0,A_i(Z_0^n)))\right) ds$$

$$T_2 = \frac{1}{n}\int_0^t \int_{\mathcal{E}} \mathbf{1}_{\{i\le N_{s_-}^n\}} \int_s^t \left(\frac{\partial f_u}{\partial u}(A(u,s,0)) + v(A(u,s,0))\,\nabla_a f_u(A(u,s,0))\right) du$$
$$\mathbf{1}_{\{0\le\theta<m_1(s,Z_{s_-}^n,i)\}} Q(ds,di,d\theta)$$

$$T_3 = \frac{1}{n}\int_0^t \int_{\mathcal{E}} \mathbf{1}_{\{i\le N_{s_-}^n\}} \int_s^t \left(\frac{\partial f_u}{\partial u}(A(u,s,A_i(Z_{s_-}^n))) + v(A(u,s,A_i(Z_{s_-}^n)))\,\nabla_a f_u(A(u,s,A_i(Z_{s_-}^n)))\right) du$$
$$\mathbf{1}_{\{m_2(s,Z_{s_-}^n,i)\le\theta<m_3(s,Z_{s_-}^n,i)\}} Q(ds,di,d\theta).$$

It is possible to apply Fubini's theorem to T_2 and T_3 and by (2.5) we recognize:

$$T_1 + T_2 + T_3 = \int_0^t \left[\int_{\mathbb{R}_+^d} \left(\frac{\partial f_u}{\partial u}(a) + v(a)\nabla_a f_u(a)\right) Z_u^n(da)\right] du \tag{2.9}$$

From (2.8), (2.9), and applying Itô's formula with jump terms, we obtain (2.7). ∎

Proposition 2.8. *Under Assumptions 2.1 and 2.4,*
(i) if $\exists q\ge 1$, $\sup_{n\in\mathbb{N}^*}\mathbb{E}(\langle Z_0^n,1\rangle^q) < +\infty$, *then:*

$$\forall n\in\mathbb{N}^*,\ \lim_{N\to+\infty}\tau_N^n = +\infty,\ \mathbb{P}-a.s., \quad \text{and} \quad \forall T>0,\ \sup_{n\in\mathbb{N}^*}\mathbb{E}\left(\sup_{t\in[0,T]}\langle Z_t^n,1\rangle^q\right) < +\infty. \tag{2.10}$$

(ii) if $\exists q\ge 1$, $m\ge 1$, $\sup_{n\in\mathbb{N}^*}\mathbb{E}(\langle Z_0^n,1\rangle^m) < +\infty$ *and* $\sup_{n\in\mathbb{N}^*}\mathbb{E}(\langle Z_0^n,|a|^q\rangle^m) < +\infty$, *then:*

$$\forall T>0,\ \sup_{n\in\mathbb{N}^*}\mathbb{E}\left(\sup_{t\in[0,T]}\langle Z_t^n,|a|^q\rangle^m\right) < +\infty. \tag{2.11}$$

(iii) Let $n\in\mathbb{N}^*$. *If* $\mathbb{E}\left(\langle Z_0^n,1\rangle^2\right) < +\infty$ *and* $\mathbb{E}(\langle Z_0^n,|a|\rangle) < +\infty$ *then:* $\forall f\in\mathcal{C}_b^1(\mathbb{R}_+^d\times\mathbb{R}_+,\mathbb{R})$,

$$\begin{aligned} M_t^{n,f} =& \langle Z_t^n,f_t\rangle - \langle Z_0^n,f_0\rangle \\ &- \int_0^t\int_{\mathbb{R}_+}\left[v(a)\nabla_a f_s(a) + \frac{\partial f_s}{\partial s}(a) + f_s(0)b(a) - f_s(a)d(a,Z_s^nU(a))\right] Z_s^n(da)\,ds, \end{aligned} \tag{2.12}$$

is a square integrable càdlàg martingale starting from 0, with previsible quadratic variation:

$$\langle M^{n,f}\rangle_t = \frac{1}{n}\int_0^t\int_{\mathbb{R}_+}\left[f_s^2(0)b(a) + f_s^2(a)d(a,Z_s^nU(a))\right] Z_s^n(da)\,ds. \tag{2.13}$$

Proof. Point (i) is a direct adaptation of the proof of Lemma 5.2 in [16]. Let us consider Point (ii). Let $N>0$, $n\in\mathbb{N}^*$ and ζ_N^n be the stopping time introduced in (2.6). For this proof, we will consider the norm 1 on $\mathbb{R}^d$: $\forall a=(a_1,\cdots a_d)\in\mathbb{R}_+^d$, $|a| = \sum_{i=1}^d a_i$. This choice is made since it simplifies the calculations. It is not restrictive since the norms in $\mathbb{R}^d$ are equivalent. The function $f:a\in\mathbb{R}_+^d\mapsto |a|^q = \left(\sum_{i=1}^d a_i\right)^q\in\mathbb{R}_+$ is differentiable and $\forall i\in[\![1,d]\!]$, $\partial f/\partial a_i = q|a|^{q-1}$. The map $F:x\in\mathbb{R}_+\mapsto x^m\in\mathbb{R}_+$ is also differentiable and $F'(x) = mx^{m-1}$. Applying Lemma 2.7 and neglecting the non positive terms gives:

$$\mathbb{E}\left(\sup_{u\in[0,t\wedge\zeta_N^n]}\langle Z_u^n,|a|^q\rangle^m\right) \le \mathbb{E}(\langle Z_0^n,|a|^q\rangle^m) + mq\bar{v}\,\mathbb{E}\left(\int_0^{t\wedge\zeta_N^n}\langle Z_s^n,|a|^{q-1}(d+|a|)\rangle\langle Z_s^n,|a|^q\rangle^{m-1}ds\right),$$

by using the fact that $0 \leq \sum_{i=1}^d v_i(x,a) \leq \bar{v}(d+|a|)$ (Assumption 2.1). Since:

$$\langle Z_s^n, |a|^{q-1}\rangle \leq \langle Z_s^n, (|a|\vee 1)^{q-1}\rangle \leq \langle Z_s^n, (|a|\vee 1)^{q}\rangle \leq \langle Z_s^n, 1\rangle + \langle Z_s^n, |a|^q\rangle, \tag{2.14}$$

we have:

$$\langle Z_s^n, |a|^{q-1}\rangle \langle Z_s^n, |a|^q\rangle^{m-1} \leq \langle Z_s^n, 1\rangle \langle Z_s^n, |a|^q\rangle^{m-1} + \langle Z_s^n, |a|^q\rangle^m \leq \langle Z_s^n, 1\rangle^m + 2\langle Z_s^n, |a|^q\rangle^m,$$

and hence by the moment assumptions, (2.10) and by the Fubini's theorem:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{u\in[0,t\wedge\zeta_N^n]} \langle Z_u^n, |a|^q\rangle^m\right) \leq \mathbb{E}\left(\langle Z_0^n, |a|^q\rangle^m\right) + mq\bar{v}dT\mathbb{E}\left(\sup_{u\in[0,T\wedge\zeta_N^n]} \langle Z_u^n, 1\rangle^m\right)\\
&+mq\bar{v}(2d+1)\int_0^t \mathbb{E}\left(\sup_{u\in[0,s\wedge\zeta_N^n]} \langle Z_u^n, |a|^q\rangle^m\right) ds\\
&\leq \left(\sup_{n\in\mathbb{N}^*} \mathbb{E}\left(\langle Z_0^n, |a|^q\rangle^m\right) + mq\bar{v}dT\, C(m,T)\right) e^{mq\bar{v}(2d+1)T} =: D(q,T),
\end{aligned} \tag{2.15}$$

by Gronwall's lemma. As $D(q,T)$ does not depend on N, we deduce that:

$$\lim_{N\to+\infty} \zeta_N^n = +\infty, \ \mathbb{P}-\text{a.s.} \tag{2.16}$$

Assume indeed that (2.16) is not satisfied. For $q=1$, there exists $M>0$ and $A_M \subset \Omega$ with $\mathbb{P}(A_M)>0$ such that $\forall \omega \in A_M$, $\lim_{N\to+\infty} \zeta_N^n(\omega) < M$. Then, for $T>M$, $\mathbb{E}\left(\sup_{t\in[0,T\wedge\zeta_N^n]}\langle Z_t^n, |a|\rangle^m\right) \geq \mathbb{P}(A_M)N^m$, which can not be upper bounded independently of N. Hence, (2.16) is satisfied. Letting N tend to infinity in (2.15) gives (2.11) by Fatou's lemma.

Point (iii) is obtained by using the compensated Poisson point measure $\widetilde{Q}$ of Q. We have:

$$\begin{aligned}
M_t^{n,f} = &-\frac{1}{n}\int_0^t\int_{\mathcal{E}} \mathbf{1}_{\{i\leq N_{s_}^n\}} \Big(f_s(0)\mathbf{1}_{\{0\leq\theta<m_1(s,Z_{s_}^n,i)\}}\\
&- f_s(A_i(Z_{s_}^n))\mathbf{1}_{\{m_1(s,Z_{s_}^n,i)\leq\theta\leq m_2(s,Z_{s_}^n,i)\}}\Big)\widetilde{Q}(ds,di,d\theta).
\end{aligned}$$

Showing that it is a square integrable martingale and computing its quadratic variation is then relatively standard (see [45]). ■

The infinitesimal generator of (2.5) corresponds to the description of the introduction:

Theorem 2.9. *For $n\in\mathbb{N}^*$, $Z^n \in \mathbb{D}$ is a Markov process of infinitesimal generator defined by: $\forall f \in \mathcal{C}_b^1(\mathbb{R}_+^d, \mathbb{R})$, $\forall F\in \mathcal{C}_b^1(\mathbb{R},\mathbb{R})$,*

$$\begin{aligned}
LF_f(Z_0^n) = \frac{\partial}{\partial t}\mathbb{E}\left(F_f(Z_t^n)\right)|_{t=0} = &\int_{\widetilde{\mathcal{X}}} v(a)\nabla_a f(a) F'\left(\langle Z_0^n, f\rangle\right) Z_0^n(da)\\
&+ n\int_{\widetilde{\mathcal{X}}}\left[\left(F_f\left(Z_0^n + \frac{1}{n}\delta_0\right) - F_f\left(Z_0^n\right)\right) b(a) Z_0^n(da)\right.\\
&\left.+\left(F_f\left(Z_0^n - \frac{1}{n}\delta_a\right) - F_f\left(Z_0^n\right)\right) d(a, Z_0^n U(a))\right] Z_0^n(da),
\end{aligned}$$

where $\forall Z \in \mathcal{M}_F(\mathbb{R}_+^d)$, $F_f(Z) = F(\langle Z, f\rangle)$.

Proof. We refer to [45] for the proof, which consists in proving that it is possible to take the derivative under the expectation. ■

We conclude this part with the consideration of exponential moments, which will be useful in the Section dealing with exponential deviations.

Assumption 2.10. *We assume that:*
1. $\forall\lambda > 0$, $\sup_{n\in\mathbb{N}^*} e^{\lambda\langle Z_0^n,1\rangle} < +\infty$, *and* $\exists\eta \in]0,1[$, $\sup_{n\in\mathbb{N}^*}\left[\int_{\mathbb{R}_+^d} |a| Z_0^n(da)\right]^{1+\eta} < +\infty$.
2. $\exists i_0 \in [\![1,d_c]\!]$, $\exists n_0 \in \mathbb{N}^*$, $\exists U_0 \in \mathbb{R}_+^*$, $\forall n \geq n_0$, $\forall u \in \mathbb{R}^{d_c}$, $\forall a \in \mathbb{R}_+^d$, $\forall\lambda > 0$, $[u_{i_0} \geq U_0] \Rightarrow \left[(e^{\lambda/n}-1)b(a) + (e^{-\lambda/n}-1)d(a,u) < 0\right]$, *($d_c$ is the number of interactions).*
3. $\exists\underline{U} > 0$, $\forall a,\alpha \in \mathbb{R}_+$, $U_{i_0}(a,\alpha) > \underline{U}$, *for the index i_0 of Point 2.*

Since $\forall a \in \mathbb{R}_+^d$, $b(a) \leq \bar{b}$, Point 2 is for instance satisfied if $\lim_{|u|\to+\infty} d(a,u) = +\infty$ (this happens for the logistic model).

Proposition 2.11. *Under Assumptions 2.1, 2.4, 2.10 :*
(i) $\forall\lambda > 0$, $\forall t \in [0,T]$,

$$\sup_{n\in\mathbb{N}^*} \mathbb{E}\left(e^{\lambda\langle Z_t^n,1\rangle}\right) \leq \sup_{n\in\mathbb{N}^*} e^{\lambda\langle Z_0^n,1\rangle} + \lambda\bar{b}MTe^{\lambda M} < +\infty, \tag{2.17}$$

where $M := U_0/\underline{U}$ with the constants U_0 and $\underline{U}$ of Assumption 2.10.
(ii) Let $\rho(x) = e^x - x - 1$. We consider the martingale $(M_t^{n,f})_{t\in\mathbb{R}_+}$ defined in (2.12). Then, the process defined for $t \in [0,T]$, $n \in \mathbb{N}^*$ and $f \in \mathcal{C}_b(\mathbb{R}_+^d \times [0,T],\mathbb{R})$ by $\Lambda_t^{n,f} = e^{M_t^{n,f}-\Xi_t^{n,f}}$,

$$\textit{with: } \Xi_t^{n,f} = \int_0^t \int_{\mathbb{R}_+^d} n\left[\rho\left(\frac{f(0,s)}{n}\right)b(a) - \rho\left(-\frac{f(a,s)}{n}\right)d(a,Z_s^n U(a))\right] Z_s^n(da)ds, \tag{2.18}$$

is a martingale.

Sketch of proof. We refer to [45] for a complete proof. For Point (i), the idea is that under Assumptions 2.10, the drift of the semi-martingale $(e^{\lambda\langle Z^n_{t\wedge\tau_N^n},1\rangle})_{t\in[0,T]}$ is nonnegative only for measures that have a mass bounded by M. For Point (ii), applying Itô's formula to $\Lambda_t^{n,f}$ localized by the stopping time τ_N^n (2.6) yields that $\Lambda^{n,f}$ is a local martingale. From Assumptions 2.10 and Point (i) of Proposition 3.3, $\lim_{N\to+\infty}\tau_N^n = +\infty$ $\mathbb{P}$-a.s. Using Fatou's lemma, we show that $\Lambda^{n,f}$ is a real martingale. ■

3 Convergence to the weak measure solution of a PDE in the Large Population Limit

We study the limit of $(Z^n)_{n\in\mathbb{N}^*}$ when $n \to +\infty$, under the following assumptions:

Assumption 3.1. *We assume that $(Z_0^n)_{n\in\mathbb{N}^*} \in \mathcal{M}_F(\mathbb{R}_+^d)$ converges in law in $(\mathcal{M}_F(\mathbb{R}_+^d),w)$ to the deterministic measure $\xi_0 \in \mathcal{M}_F(\mathbb{R}_+^d)$ and that the following moment conditions are satisfied:*

$$\exists\eta \in]0,1[,\quad \sup_{n\in\mathbb{N}^*} \mathbb{E}\left(\langle Z_0^n,1\rangle^{2+\eta}\right) < +\infty, \qquad \textit{and} \qquad \sup_{n\in\mathbb{N}^*} \mathbb{E}\left(\langle Z_0^n,|a|\rangle^{1+\eta}\right) < +\infty.$$

In this section, we prove the convergence of the sequence $(Z^n)_{n\in\mathbb{N}^*}$ to the solution of (1.2) by a tightness-uniqueness argument.

3.1 Tightness of the sequence $(Z^n)_{n\in\mathbb{N}^*}$

Proposition 3.2. *Under Assumptions 2.1, 2.4, 3.1, $(Z^n)_{n\in\mathbb{N}^*}$ is tight on $\mathbb{D}(\mathbb{R}_+,(\mathcal{M}_F(\mathbb{R}_+^d),w))$. Its limit values are continuous measure-valued processes satisfying (1.2).*

Proof. The proof is inspired by the proof of Theorem 5.3 in [16], and a detailed proof stands in [45], Theorem 3.2.2. We begin with establishing the tightness of the laws of $(Z^n)_{n\in\mathbb{N}^*}$ considered as processes of $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_F(\mathbb{R}_+^d), v))$, where $\mathcal{M}_F(\mathbb{R}_+^d)$ is embedded with the topology of vague convergence. For this purpose, we use a criterium due to Roelly ([41], Theorem 2.1): it is sufficient to prove that for $f \in \mathcal{C}_b^1(\mathbb{R}_+^d, \mathbb{R}) \cap \mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R})$, which is dense in $\mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R})$, the sequence $(Z^{n,f} = \langle Z^n, f\rangle)_{n\in\mathbb{N}^*}$ is tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$. Thanks to Proposition 2.8 and Assumptions 3.1, we can prove that Aldous and Rebolledo criteria [1, 28] are satisfied, which is a sufficient condition for $(Z^{n,f})_{n\in\mathbb{N}^*}$ to be tight.

Notice that the particular choice of $f \equiv 1$ implies the tightness of the sequence $(\langle Z^n, 1\rangle)_{n\in\mathbb{N}^*}$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ with the same arguments.

By Prohorov Theorem, it is possible to extract from $(Z^n)_{n\in\mathbb{N}^*}$ a subsequence $(Z^{\phi(n)})_{n\in\mathbb{N}^*}$ that converges in law in $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_F(\mathbb{R}_+^d), v))$ to a limiting process Z. Since we also proved the tightness of $(\langle Z^n, 1\rangle)_{n\in\mathbb{N}^*}$, it is possible to choose this subsequence such that $(\langle Z^{\phi(n)}, 1\rangle)_{n\in\mathbb{N}^*}$ converges in law to $\langle Z, 1\rangle$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$. By construction, the jumps of $(Z^n)_{n\in\mathbb{N}^*}$ are of order $1/n$ and entails that the limit process Z is almost surely continuous. Using Theorem 3 in Méléard and Roelly [35], $(Z^{\phi(n)})_{n\in\mathbb{N}^*}$ also converges in law in $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_F(\mathbb{R}_+^d), w))$, where $\mathcal{M}_F(\mathbb{R}_+^d)$ is embedded with the topology of weak convergence. Applying Prohorov Theorem again, we deduce that the sequence $(Z^n)_{n\in\mathbb{N}^*}$ is tight in $\mathbb{D}(\mathbb{R}_+, (\mathcal{M}_F(\mathbb{R}_+^d), w))$.

To identify the limit, we have to prove that $\mathbb{P}$-a.s., $\forall [f : (a,t) \mapsto f_t(a)] \in \mathcal{C}_b^1(\mathbb{R}_+^d \times \mathbb{R}_+, \mathbb{R})$, $\forall t \in \mathbb{R}_+$, $\Psi_t(Z) = 0$, where:

$$\Psi_t(Z) = \langle Z_t, f_t\rangle - \langle Z_0, f_0\rangle - \int_0^t \int_{\mathbb{R}_+^d} \left[v(a)\nabla_a f_s(a) + \frac{\partial f_s}{\partial s}(a) + f(0,s)b(a) - f(a,s)d(a, Z_s U(a)) \right] Z_s(da)\, ds.$$

This is obtained by a direct adaptation of the proof of [16, 45]. ■

In order to prove that the whole sequence $(Z^n)_{n\in\mathbb{N}^*}$ converges, we will show that it admits a unique limit value. This is related to the uniqueness of the solution of (1.2).

3.2 Existence and uniqueness of the solution of (1.2)

Existence of the solutions of (1.2) is a consequence of Proposition 3.2, since the limit values of $(Z^n)_{n\in\mathbb{N}^*}$ are solutions. Proposition 3.3 deals with the uniqueness problem.

Proposition 3.3. *Let $\xi_0 \in \mathcal{M}_F(\mathbb{R}_+^d)$ be a deterministic initial condition. Let $(\xi_t^1)_{t\in\mathbb{R}_+}$ and $(\xi_t^2)_{t\in\mathbb{R}_+}$ be two solutions of (1.2) in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}_+^d))$, starting from ξ_0. Under Assumptions 2.1, 2.4, 3.1, we have: $\forall t \in \mathbb{R}_+$, $\xi_t^1(dx, da) = \xi_t^2(dx, da)$.*

Proof. The solutions of (1.2) are continuous with bounded masses on finite time intervals: $\forall t \geq 0$, $\langle \xi_t, 1\rangle \leq \langle \xi_0, 1\rangle + \int_0^t \bar{b}\langle \xi_s, 1\rangle\, ds \leq \langle \xi_0, 1\rangle \exp\left(\bar{b}t\right) < +\infty$. For $T > 0$, we can define $A_T := \sup_{t\in[0,T]} \langle \xi_t^1 + \xi_t^2, 1\rangle < +\infty$. Let $\phi \in \mathcal{C}_b^1(\mathbb{R}_+^d, \mathbb{R})$ such that $\|\phi\|_\infty \leq 1$. For $t \in [0,T]$, we can define $\forall a \in \mathbb{R}_+^d$, $\forall s \in \mathbb{R}_+$, $f(a,s) = \phi(A(t,s,a))$. By Proposition 2.3: $\forall i \in \{1,2\}$, $\forall t \in [0,T]$,

$$\langle \xi_t^i, \phi\rangle = \int_{\mathbb{R}_+^d} \phi(A(t,0,a))\xi_0^i(da) + \int_0^t \int_{\mathbb{R}_+^d} \left[\phi(A(t,s,0))b(a) - \phi(A(t,s,a))d(a, \xi_s^i U(a))\right] \xi_s^i(da)\, ds$$

Hence: $\forall t \in [0,T]$,

$$|\langle \xi_t^1 - \xi_t^2, \phi\rangle| \leq \int_0^t \left[\left| \int_{\mathbb{R}_+^d} \left(\phi(A(t,s,0))b(a) - \phi(A(t,s,a))d(a, \xi_s^1 U(a))\right) (\xi_s^1(da) - \xi_s^2(da)) \right| + \left| \int_{\mathbb{R}_+^d} \phi(A(t,s,a)) \left(d(a, \xi_s^1 U(a)) - d(a, \xi_s^2 U(a))\right) \xi_s^2(da) \right| \right] ds \leq C(T) \int_0^t \|\xi_s^1 - \xi_s^2\|_{TV}\, ds, \quad (3.1)$$

where $C(T) = \bar{b} + \bar{d}(1 + A_T) + A_T\, L_d\, \bar{U}$. By taking the sup in ϕ in the left hand side and by noticing that every function $\phi \in \mathcal{C}_b(\mathbb{R}_+, \mathbb{R})$ is the limit for the bounded pointwise convergence of a sequence of functions of $\mathcal{C}_b^1(\mathbb{R}_+, \mathbb{R})$:

$$\forall t \in [0, T],\ ||\xi_t^1 - \xi_t^2||_{TV} \leq C(T) \int_0^t ||\xi_s^1 - \xi_s^2||_{TV}\, ds.$$

The result is given by Gronwall Lemma, and by the fact that T is arbitrary. ■

Corollary 3.4. *Under Assumptions 2.1, 2.4, 3.1 and for every initial condition $\xi_0 \in \mathcal{M}_F(\mathbb{R}_+^d)$, Equation (1.2) admits a unique solution $(\xi_t)_{t \in \mathbb{R}_+} \in \mathcal{C}(\mathbb{R}_+, \mathcal{M}_F(\mathbb{R}_+^d))$ to which the sequence $(Z^n)_{n \in \mathbb{N}^*}$ of $\mathbb{D}$ converges in probability.*

3.3 Absolute continuity of the solutions of (1.2)

Let us now study the existence of densities for the measures $\xi_t(da)$ with respect to the Lebesgue measure of $\mathbb{R}_+^d$. If they exist, they define a weak solution of (1.5) which generalizes the McKendrick and Von Foerster equations that are classical in demography. In the case of a scalar age we have the propagation of the absolute continuity for positive times (Proposition 3.5). However, when many ages are taken into account, densities do not exist any more (Remark 3.6).

Proposition 3.5. *In the case $d = 1$, under Assumptions 2.1, 2.4, 3.1 and if $\xi_0(da) = m_0(a)da$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$, then the time marginals $\xi_t(da)$ also admit densities $m_t(a)$ for every $t \in \mathbb{R}_+$.*

Proof. Let $t \in \mathbb{R}_+$ and let $\phi \in \mathcal{C}_b^1(\mathbb{R}_+, \mathbb{R}_+)$ nonnegative. For f defined by $\forall s \in \mathbb{R}_+, \forall a \in \mathbb{R}_+,\ f(a, s) = \phi(A(t, s, a))$, where A is defined in (2.2), we have:

$$\langle \xi_t, \phi \rangle \quad \leq \quad \int_{\mathbb{R}_+} \phi(A(t, 0, a))\xi_0(da) + \int_0^t \int_{\mathbb{R}_+} \phi(A(t, s, 0))b(a)\xi_s(da)\, ds \tag{3.2}$$

since ϕ is nonnegative. Let us consider the first term. By Point (ii) of Proposition 2.2, the map $a \mapsto A(t, 0, a)$ defines a $\mathcal{C}^1$-diffeomorphism from $\mathbb{R}_+$ into $[A(t, 0, 0), +\infty[$. Let us denote by $A^{-1}(t, \alpha)$ the inverse diffeomorphism and by $JA^{-1}(t, \alpha)$ its jacobian matrix. Using the change of variable associated with $a \mapsto A(t, 0, a)$:

$$\int_{\mathbb{R}_+} \phi(A(t, 0, a))m_0(a)da = \int_{\mathbb{R}_+} \int_{A(t,0,0)}^{+\infty} \phi(\alpha)m_0(A^{-1}(t, \alpha))|JA^{-1}(t, \alpha)|\, d\alpha. \tag{3.3}$$

For the second term, the change of variable given by the $\mathcal{C}^1$-diffeomorphism $s \mapsto A(t, s, 0)$ yields:

$$\int_0^t \int_{\mathbb{R}_+} \phi(A(t, s, 0))b(a)\xi_s(da)\, ds = \int_0^{A(t,0,0)} \int_{\mathbb{R}_+} \phi(\alpha)\, b(a)|J\widetilde{A^{-1}}(\alpha)|\xi_{\widetilde{A^{-1}}(\alpha)}(da)\, d\alpha. \tag{3.4}$$

From (3.3) and (3.4), we deduce that:

$$0 \leq \int_{\mathbb{R}_+} \phi(a)\xi_t(da) \leq \int_0^{+\infty} \phi(\alpha)H(\alpha, t)d\alpha, \text{ where:} \tag{3.5}$$

$$H(\alpha, t) = \mathbf{1}_{A(t,0,0) \leq \alpha}\, n_0(A_x^{-1}(t, \alpha))|JA^{-1}(t, \alpha)| + \mathbf{1}_{\alpha \leq A(t,0,0)} \int_{\mathbb{R}_+} b(a)|J\widetilde{A^{-1}}(\alpha)|\xi_{\widetilde{A^{-1}}(\alpha)}(da)\, d\alpha.$$

The function H is nonnegative and integrable, since the right hand side of (3.5) equals the right hand side of (3.2), which is finite. Since $\mathcal{C}_b^1(\mathbb{R}_+, \mathbb{R}_+)$ is dense in the set of measurable functions that are ξ_t-almost everywhere bounded (see Rudin [42], p. 69, ξ_t being a finite measure), we deduce from (3.5) that the measure $\xi_t(da)$ is dominated by a measure that is absolutely continuous with respect to the Lebesgue measure and is hence itself absolutely continuous. ■

Remark 3.6. *In the case where $d > 1$, the map $t \in \mathbb{R}_+ \mapsto A(t,0,0) \in \mathbb{R}_+^d$ does not define a bijection any more, and the ages of individuals born after $s = 0$ at time t belong to the set $\{A(t,s,0) \in \mathbb{R}_+^d,\ s \in [0,t]\}$, which is a one-dimensional variety of $\mathbb{R}_+^d$. Thus, the measures ξ_t can not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+^d$.*

The following Proposition linking (1.2) and (1.3, 1.4) is proved in Section 3.2.4 of [45]:

Proposition 3.7. *For $d = 1$ and under the Assumptions of Proposition 3.5, the weak function solution $(a,t) \mapsto m(a,t)$ of (1.2) is well defined. If this solution belongs to $\mathcal{C}_b^1(\mathbb{R}_+^2, \mathbb{R}_+)$ and if $\forall t \in \mathbb{R}_+$, $\frac{\partial}{\partial t}\int_{\mathbb{R}_+} m(a,t)da = \int_{\mathbb{R}_+} \frac{\partial m}{\partial t}(a,t)da$, then it is the classical solution of (1.3, 1.4).*

Proof. We replace $\xi_s(da)$ in (1.2) by $m(a,s)da$ and differentiate with respect to time. $\forall t \in \mathbb{R}_+$,

$$\frac{\partial}{\partial t}\int_{\mathbb{R}_+} f(a)m(a,t)da = \int_{\mathbb{R}_+} \Big[v(a)\nabla_a\left(f(a)\right) \\ +f(0)b(a) - f(a)d\left(a, \int_{\mathbb{R}_+} U(a,\alpha)m(\alpha,t)d\alpha\right)\Big] m(a,t)da. \quad (3.6)$$

Let us first consider test functions $f \in \mathcal{C}_{b,K}^1(\mathbb{R}_+, \mathbb{R})$ bounded with compact support in $\mathbb{R}_+^*$. in particular $f(0) = 0$. By Fubini theorem and by an integration by parts formula for the aging term: $\forall t \in \mathbb{R}_+$,

$$\int_{\mathbb{R}_+} \nabla_a\left(f(a)\right)v(a)m(a,t)da = \left[f(a)v(a)m(a,t)\right]_{a=0}^{+\infty} - \int_{\mathbb{R}_+} f(a)\nabla_a\left(v(a)m(a,t)\right)da. \quad (3.7)$$

The bracket in the right hand side being zero since $\lim_{a\to+\infty} m(a,t) = 0$ for every $t \in \mathbb{R}_+$. From (3.6) and (3.7), we obtain that da-almost surely on $\mathbb{R}_+^*$, and $\forall t \in \mathbb{R}_+$,

$$\frac{\partial}{\partial t}m(a,t) + \nabla_a\left(v(a)m(a,t)\right) = -d\left(a, \int_{\mathbb{R}_+} U(a,\alpha)m(\alpha,t)d\alpha\right)m(a,t). \quad (3.8)$$

Let us now consider test functions $f \in \mathcal{C}_b^1(\mathbb{R}_+, \mathbb{R})$ with arbitrary support. From (3.7) and (3.6): $\forall t \in \mathbb{R}_+$,

$$\int_{\mathbb{R}_+}\left[\frac{\partial}{\partial t}m(a,t) + \nabla_a\left(v(a)m(a,t)\right)\right]f(a)da = -f(0)v(0)n(0,t) \\ + \int_{\widetilde{\mathcal{X}}}\left[f(0)b(a) - f(a)d\left(a, \int_{\mathbb{R}_+} U(a,\alpha)m(\alpha,t)d\alpha\right)\right]m(a,t)da. \quad (3.9)$$

We simplify (3.9) with (3.8) and: $\forall t \in \mathbb{R}_+$, $f(0)v(0)n(0,t) = \int_{\mathbb{R}_+} f(0)b(a)m(a,t)da$. By identification of the integrands, we obtain the boundary condition in (1.4). ∎

4 Exponential deviations

In this section, we set $T > 0$ and study the exponential deviations associated with the convergence of $(Z^n)_{n\in\mathbb{N}^*}$ to the solution ξ of (1.2) in $\mathbb{D}_T$ (Corollary 3.4). More precisely, we look for estimates of $(1/n)\log\mathbb{P}(Z^n \in B)$, for any Borel set $B \subset \mathbb{D}_T$, when $n \to +\infty$. Applications will be considered in Section 5.

A first difficulty arise from the fact that $\mathcal{M}_F(\mathbb{R}_+^d)$ is not compact. The mass of the microscopic process is not necessarily bounded. The exponential tightness of $(Z^n_{.\wedge\zeta_N^n})_{n\in\mathbb{N}^*}$ on $\mathbb{D}_T$ is classical and can be established by adapting techniques from Dawson and Gärtner [11] and Graham and Méléard [21]. The relaxation of the localization by ζ_N^n relies on exponential deviation inequalities proved in Lemmas 4.7 and 4.8.

The lower bound issue is treated in Section 4.3. A second difficulty comes from the fact that the events describing the dynamics are of different nature (births, deaths) and that they are associated with nonlinear rates. Following the approach of Léonard [31], Kipnis and Léonard [30], we base our proof on the use of a Riesz theorem on Orlicz spaces, which gives us a representation of the trajectories of the action functional as solutions of perturbed PDEs obtained from (1.2) and on a Girsanov theorem. A key point in the the proof is the use of a density Theorem due to Bishop-Phelps-Israel [5, 24].

Notation: We define $E := \mathbb{R}_+^d \times \{0,1\}$. With each function $f \in \mathcal{B}(\mathbb{R}_+^d \times [0,T], \mathbb{R})$, we can associate a map $\psi(f)$ defined on $E \times [0,T]$ by: $\forall (a,u,t) \in E \times [0,T]$,

$$\psi(f)(a,u,t) = f(0,t)\mathbf{1}_{u=0} - f(a,t)\mathbf{1}_{u=1}. \tag{4.1}$$

For $t \in [0,T]$ and $z \in \mathbb{D}_T$, let us introduce the following positive finite measure on E:

$$m_t^{z,T}(da,du) = [b(a)\delta_0(du) + d(a, z_{t-}U(a))\delta_1(du)]\, z_{t-}(da). \tag{4.2}$$

The Laplace transform ρ of the standard centered Poisson law and its Legendre transform are:

$$\rho(x) = e^x - x - 1, \quad \rho^*(y) = \begin{cases} (y+1)\log(y+1) - y & \text{si } y > -1, \\ 1 & \text{si } y = -1, \\ +\infty & \text{si } y < -1, \end{cases} \tag{4.3}$$

ρ^* is nonnegative and vanishes only at $y = 0$. The functions ρ and ρ^* are convex conjugated. For $\alpha \in \{\rho, \rho^*\}$, $T > 0$ and $z \in \mathbb{D}_T$, we define the following Orlicz norm on the space of real Borel functions on $E \times \mathbb{R}_+$:

$$||g||_{\alpha,z} \quad = \quad \inf\left\{\kappa > 0,\ \int_0^T \int_E \alpha\left(\frac{|g(a,u,s)|}{\kappa}\right) m_s^{z,T}(da,du)ds \leq 1\right\}. \tag{4.4}$$

The set of functions with finite Orlicz norms are denoted by $L^{\rho,z}$ and $L^{\rho^*,z}$. The closure $E^{\rho,z}(E\times \mathbb{R}_+)$ of the space of Borel *bounded* functions for the norm $\|.\|_{\rho,z}$ is strictly included in $L^{\rho,z}$ (see [30] Section 3, for instance) and admits a topological dual which can be identified with $L^{\rho^*,z}$ thanks to the following representation theorem:

Theorem 4.1. ***Riesz Theorem in Orlicz spaces*** *(see Rao et Ren [40], p.93 and following or Kipnis and Léonard [30]) For every continuous linear form ℓ on $E^{\rho,z}(E \times \mathbb{R}_+)$, there exists a function $h \in L^{\rho^*,z}$ such that:*

$$\forall g \in E^{\rho,z}(E \times \mathbb{R}_+),\ \ell(g) = \int_0^T \int_E g(a,u,s)h(a,u,s)dm_s^{z,T}(da,du)ds.$$

4.1 Main Result

The main result of this Section is enounced in Theorem 4.4.

Assumption 4.2. *We assume that Assumptions 2.10 are satisfied and that:*
1. The sequence $(Z_0^n)_{n\in\mathbb{N}^*}$ is deterministic and converges in $(\mathcal{M}_F(\mathbb{R}_+^d), w)$ to a measure ξ_0.
2. $\exists C_0 > 0$, $\langle \xi_0, 1+|a|\rangle < C_0$ and $\exists n_0 \in \mathbb{N}^*$, $\forall n \geq n_0$, $\langle Z_0^n, 1+|a|\rangle < C_0$.

The Assumption of Point 1 is made for simplification. It can be weaken by assuming that $(Z_0^n)_{n\in\mathbb{N}^*}$ converges in law in $(\mathcal{M}_F(\mathbb{R}_+^d), w)$ to ξ_0.

We now define the action functional that appears in our deviation result.

Definition 4.3. *For $\xi_0 \in \mathcal{M}_F(\mathbb{R}_+^d)$ and $z \in \mathbb{D}_T$, let us define $\mathcal{I}_{\xi_0}^T(z)$ by:*

$$\mathcal{I}_{\xi_0}^T(z) = \begin{cases} \sup_{f\in\mathcal{C}_b^1(\mathbb{R}_+^d\times[0,T],\mathbb{R})} \mathcal{I}^{f,T}(z), \text{ if } z_0 = \xi_0 \\ +\infty, \text{ else,} \end{cases} , \quad \text{with: } \mathcal{I}^{f,T}(z) = \ell^T(f,z) - c^T(f,z), \quad (4.5)$$

where $\ell^T(f,z)$ and $c^T(f,z)$ are defined for $(f : (a,s) \mapsto f_s(a)) \in \mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T], \mathbb{R})$ by:

$$\begin{aligned} \ell^T(f,z) =& \langle z_t, f_t\rangle - \langle z_0, f_0\rangle - \int_0^t \left[\int_{\mathbb{R}_+^d} \left(v(a)\nabla_a f_s(a) + \frac{\partial f_s}{\partial s}(a) \right) z_s(da) - \int_E \psi(f) dm_s^{z,T} \right] ds, \\ c^T(f,z) =& \int_0^T \int_E \rho(\psi(f)(a,u,s)) m_s^{z,T}(da,du)\, ds. \end{aligned} \quad (4.6)$$

Theorem 4.4. *Under Assumptions 2.4, 2.1, 4.2.*
(i) Let $z \in \mathbb{D}_T$ such that $\mathcal{I}_{\xi_0}^T(z) < +\infty$. There exists $h^z \in L^{\rho^*,z}$ such that z is the solution of: $\forall (f : (a,s) \mapsto f_s(a)) \in \mathcal{C}_b^1(\mathbb{R}_+^d \times \mathbb{R}_+, \mathbb{R})$, $\forall t \in [0,T]$,

$$\langle z_t, f_t\rangle = \langle z_0, f_0\rangle + \int_0^t \left\langle z_s, v\nabla_a f_s + \frac{\partial f_s}{\partial s} \right\rangle ds + \int_0^T \int_E (1+h^z)\psi(f) dm_s^{z,T}\, ds. \quad (4.7)$$

We have then have the following non-variational representation of the action functional:

$$\mathcal{I}_{\xi_0}^T(z) = \int_0^T \int_E \rho^*\left(h^z(a,u,s)\right) m_s^{z,T}(da,du)\, ds. \quad (4.8)$$

(ii) Let:

$$\mathfrak{G} = \left\{ z \in \mathbb{D}_T \mid \mathcal{I}_{\xi_0}^T(z) < +\infty, \quad \text{and} \quad h^z \in L^\infty(E\times[0,T],\mathbb{R}) \right\} \quad (4.9)$$

The sequence $(Z^n)_{n\in\mathbb{N}^*}$ of $\mathbb{D}_T$ satisfies the following deviation inequalities: $\forall B$ Borel set of $\mathbb{D}_T$,

$$-\inf_{z\in\mathring{B}\cap\mathfrak{G}} \mathcal{I}_{\xi_0}^T(z) \le \liminf_{n\to+\infty} \frac{1}{n}\log\mathbb{P}(Z^n\in B) \le \limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}(Z^n\in B) \le -\inf_{z\in\bar{B}} \mathcal{I}_{\xi_0}^T(z), \quad (4.10)$$

with the convention $\inf\emptyset = +\infty$.

Remark 4.5. *(4.7) corresponds to a perturbation of the birth and death rates of (1.2) by factors $(1+h^z(a,0,s))$ and $(1+h^z(a,1,s))$ which are different (and hence the introduction of the mark $u\in\{0,1\}$). We will prove in Proposition 4.14 (Point 4) that these factors are nonnegative.*

The upper bound of (4.10) is a large deviation upper bound. The lower bound is local, the infimum being taken on $\mathring{B}\cap\mathfrak{G}$ and not $\mathring{B}$. In the proof of the lower bound, we will be lead to regularize h^z by approximating it with a sequence $(h_m)_{m\in\mathbb{N}^*}$ of perturbations with particular forms, continuous and bounded. With each h_m, we can associate the solution z^m of the evolution equation (4.7) perturbed by h_m. The convergence of z^m to z when h^z is bounded is proved in Proposition 4.18, the general case is still open. In the works which have inspired us [31, 30] the similar difficulty is encountered, but under a different form: the problem is a uniqueness problem for the perturbed equation (4.7) and not a convergence problem.

The local minoration that we obtain is still a useful result, that we will use in Section 5. It provides an information as soon as $\mathring{B}\cap\mathfrak{G}\neq\emptyset$, in which case, the lower bound of (4.10) is not $-\infty$. This is the case when $\mathring{B}$ contains the solution of (1.2) which corresponds to a zero perturbation for instance.

4.2 Exponential tightness and large deviation upper bound

We prove the exponential tightness of $(Z^n)_{n\in\mathbb{N}^*}$ on $\mathbb{D}_T$. The large deviation upper bound is then a generalization of Theorem 4.4.2 and Lemma 4.4.5 of Dembo and Zeitouni [12], obtained by replacing the limits with lim sup.

Recall that the continuity modulus for $y\in\mathbb{D}([0,T],\mathbb{R})$ and $\delta\in]0,1[$ is defined by:

$$w'(y,\delta)=\inf_{\{t_l\}}\left(\max_{1<i\leq card\{t_l\}} w(y,[t_{i-1},t_i[)\right)\qquad \text{where } w(y,[t_{i-1},t_i[)=\sup_{s,t\in[t_{i-1},t_i[}|y_s-y_t|,$$

and where the infimum is considered on the subdivisions $0=t_0<t_1\cdots<t_{card\{t_l\}}=T$ such that $\forall i\in[1,card\{t_l\}]$, $t_i-t_{i-1}>\delta$.

Proposition 4.6. *Under Assumptions 2.1, 2.4, 4.2:*
(i) Let $(\varphi_r)_{r\in\mathbb{N}^*}$ be a denumberable dense family of $\mathcal{C}_0^1(\mathbb{R}_+^d,\mathbb{R})$, and let us set $\varphi_0\equiv 1$. $\forall L>0$, $\forall N>0$, $\exists\mathcal{K}^1_{L,N}$ closed subset of $\mathbb{D}_T$, $\exists n_0\in\mathbb{N}^*$, $\forall n\geq n_0$:

$$\mathbb{P}\left(Z^n_{.\wedge\zeta^n_N}\notin\mathcal{K}^1_{L,N}\right)\leq\frac{e^{-nL}(2-e^{-L})}{(1-e^{-L})^2}\ \text{and such that: } \forall r\in\mathbb{N},\ \lim_{\delta\to 0}\sup_{z\in\mathcal{K}^1_{L,N}} w'(\langle z,\varphi_r\rangle,\delta)=0. \tag{4.11}$$

(ii) $\forall L>0$, $\exists\mathcal{K}^2_L$ compact subset of $(\mathcal{M}_F(\mathbb{R}^d_+),w)$,

$$\limsup_{n\to+\infty}\frac{1}{n}\log\left(\mathbb{P}\left(\exists t\in[0,T],\ Z^n_t\notin\mathcal{K}^2_L\right)\right)\leq -L. \tag{4.12}$$

The measure-valued process $Z^n_{.\wedge\zeta^n_N}$ has marginals of mass bounded by $N>0$. The proof of Point (i) is an adaption of the proof in [11] (see [45] for a detailed proof). It relies on the proof of the following estimate, obtained by an adaptation of techniques used in [21, 20]: $\forall\varepsilon>0$, $\forall N>0$, $\forall c>0$, $\forall\delta>0$, $\forall n\in\mathbb{N}$, $\forall\varphi\in\mathcal{C}^1_b(\mathbb{R}^d_+)$,

$$\mathbb{P}\left(\sup_{|t-s|\leq\delta}|\langle Z^n_{t\wedge\zeta^n_N},\varphi\rangle-\langle Z^n_{s\wedge\zeta^n_N},\varphi\rangle|>4\varepsilon\right)\ \leq\ 8\left(\frac{T}{\delta}+1\right)e^{-nc\varepsilon+2\delta nS_c(N,\phi)} \tag{4.13}$$

where $S_c(N,\varphi)=\left[\rho(c\|\varphi\|_\infty)N(\bar b+\bar d(1+N))\right]\vee\left[c(\bar v2N||\nabla_a\varphi||_\infty+N\|\varphi\|_\infty(\bar b+\bar d(1+N)))\right]$. The upper bound in (4.13) depends on N, and we can not take the limit in $N\to+\infty$.

The second point of Proposition 4.6 and the relaxation of the stopping time ζ^n_N make the object of the end of this section. We generalize an argument used by Dawson Gärtner [11] in the case of continuous probability-valued processes. We prove that there exists $N>0$ such that the probability $\mathbb{P}(\zeta^n_N\leq T)$ is exponentially small. Then, the exponential tightness of the laws of $((Z^n_{t\wedge\zeta^n_N})_{t\in[0,T]})_{n\in\mathbb{N}^*}$ implies the exponential tightness of the laws of $((Z^n_t)_{t\in[0,T]})_{n\in\mathbb{N}^*}$.

We set $\phi_0\equiv 1$ and introduce a denumberable family $\mathfrak{K}=(\phi_r)_{r\in\mathbb{N}}$ such that for every $r\in\mathbb{N}^*$, ϕ_r is constant outside a compact set of $\mathbb{R}^d_+$, polynômial on this compact set, and bounded up and below by positive constants, and such that for every $\phi\in\mathcal{C}_0(\mathbb{R}^d_+,\mathbb{R}_+)$ there exists a sub-sequence $(\phi_{\psi(r)})_{r\in\mathbb{N}}$ of $(\phi_r)_{r\in\mathbb{N}}$ that converges uniformly to ϕ.

In order to establish (4.12), we will need the following Lemma proved in the sequel:

Lemma 4.7. *Under Assumptions 2.4, 2.1, 4.2.*
(i) $\forall L>0$, $\exists N=N(T,L,C_0)>0$, $\exists n_0\in\mathbb{N}^*$, $\forall n>n_0$, $\mathbb{P}(\tau^n_N>T)\leq e^{-nL}$, and $\mathbb{P}(\zeta^n_N>T)\leq e^{-nL}$, C_0 being defined in Point 2 of Assumptions 2.10.
(ii) $\forall\phi\in\mathfrak{K}$, $\forall L>0$, $\exists C=C(\phi,T,L,C_0)>0$, $\exists n_0\in\mathbb{N}^*$, $\forall n>n_0$, $\mathbb{P}\left(\sup_{t\in[0,T]}\langle Z^n_t,\phi\rangle>C\right)\leq e^{-nL}$.

Sketch of the proof of Point (ii) of Proposition 4.6. Let us introduce the following sets for $C > 0$ and $\phi \in \mathcal{C}_b(\mathbb{R}_+^d, \mathbb{R}_+)$:

$$\mathcal{K}_\phi(C) = \left\{ z \in \mathcal{M}_F(\mathbb{R}_+^d) \mid \langle z, \phi \rangle \leq C \right\}.$$

Let us prove that the set $\mathcal{K}_L^2 = \text{adh} \left[\bigcap_{r \in \mathbb{N}} \mathcal{K}_{\phi_r}(C_r) \right]$, with $C_r := C(\phi_r, T, rL \vee L, C_0)$ of Lemma 4.7, is compact in $(\mathcal{M}_F(\mathbb{R}_+^d), w)$ and that (4.12) is satisfied.

Let $(z_m)_{m \in \mathbb{N}^*}$ be a sequence of $\bigcap_{r \in \mathbb{N}} \mathcal{K}_{\phi_r}(C_r)$. For every given $r \in \mathbb{N}$, $(\langle z_m, \phi_r \rangle)_{m \in \mathbb{N}^*}$ is a nonnegative real sequence upper bounded by C_r and hence relatively compact.

By a diagonal procedure, we can extract from $(z_m)_{m \in \mathbb{N}^*}$ a subsequence, again denoted by $(z_m)_{m \in \mathbb{N}^*}$, such that for every $r \in \mathbb{N}$, $(\langle z_m, \phi_r \rangle)_{m \in \mathbb{N}^*}$ converges to a limit $\ell(\phi_r)$. Let $g \in \mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R}_+)$. By choice of $\mathfrak{K}$, there exists a subsequence $(\phi_{\psi(r)})_{r \in \mathbb{N}}$ of $(\phi_r)_{r \in \mathbb{N}}$ that converges uniformly to g. The sequence $(\ell(\phi_{\psi(r)}))_{r \in \mathbb{N}}$ is then a Cauchy sequence in $\mathbb{R}$ and we can define the limit $\ell(g) = \lim_{r \to +\infty} \ell(\phi_{\psi(r)})$, which does not depend on the choice of the subsequence $(\phi_{\psi(r)})_{r \in \mathbb{N}}$. With a non negative function $g \in \mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R}_-)$, we associate $\ell(g) := -\ell(-g)$. With $g \in \mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R})$, we associate $\ell(g) = \ell([g]_+) - \ell([g]_-)$, where $[.]_+$ and $[.]_-$ denote the positive and negative parts.

If $\ell(1) \neq 0$, we can prove that the map $g \in \mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R}) \mapsto \ell(g)/\ell(1) \in \mathbb{R}$ is well-defined, linear, continuous, of norm 1 and nonnegative. Thanks to the Riesz representation Theorem (see Rudin [42], Theorem 6.19 p.130), there exists a unique probability measure ν, such that $\forall g \in \mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R})$, $\ell(g) = \ell(1) \int_{\mathbb{R}_+} g(a) \nu(da)$. By construction, for every $g \in \mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R})$, $(\langle z_m, g \rangle)_{m \in \mathbb{N}^*}$ converges to $\langle \ell(1)\nu, g \rangle$. Since $(\langle z_m, 1 \rangle)_{m \in \mathbb{N}^*}$ converges to $\ell(1) = \langle \ell(1)\nu, 1 \rangle$, $(z_m)_{m \in \mathbb{N}^*}$ converges weakly to $\ell(1)\nu$ (see [35]). If $\ell(1) = 0$, then $\forall g \in \mathcal{C}_0(\mathbb{R}_+^d, \mathbb{R})$, $(\langle z_m, g \rangle)_{m \in \mathbb{N}^*}$ converges to 0 and $(\langle z_m, 1 \rangle)_{m \in \mathbb{N}^*}$ also converges to 0. This implies that $(z_m)_{m \in \mathbb{N}^*}$ converges weakly to the null measure.

The set $\bigcap_{r \in \mathbb{N}} \mathcal{K}_{\phi_r}(C_r)$ is hence relatively compact in $(\mathcal{M}_F(\mathbb{R}_+^d), w)$, and its adherence $\mathcal{K}_L^2$ is compact in $(\mathcal{M}_F(\mathbb{R}_+^d), w)$. Finally:

$$\begin{aligned}
\mathbb{P}\left(\exists t \in [0,T],\, Z_t^n \notin \mathcal{K}_L^2\right) &\leq \sum_{r \in \mathbb{N}} \mathbb{P}\left(\exists t \in [0,T],\, Z_t^n \notin \mathcal{K}_{\phi_r}(C(\phi_r, T, L_r, C_0))\right) \\
&\leq \sum_{r \in \mathbb{N}^*} e^{-nrL} + e^{-nL} \leq \frac{e^{-nL}(2 - e^{-nL})}{1 - e^{-nL}}
\end{aligned} \tag{4.14}$$

by Point (ii) of Lemma 4.7. This yields (4.12). ■

The proof of Lemma 4.7 is based of the following Lemma 4.8, where we establish a sufficient condition to obtain the upper bounds of Lemma 4.7.

Lemma 4.8. *Under Assumptions 2.1, 2.4, 4.2, for C_0 and n_0 defined in Assumptions 4.2, for $\phi \in \mathcal{C}_b^1(\mathbb{R}_+^d, \mathbb{R})$ such that $\exists C(\phi) > 0$, $\forall a \in \mathbb{R}_+^d$, $|\phi(a)| \leq C(\phi)(1 + |a|)$, there exists $\gamma = \gamma(\phi, T, C_0) > 0$ such that $\forall t \in [0,T]$, $\forall n > n_0$, $\mathbb{P}$-a.s.:*

$$\int_E \psi(\phi) dm_t^{Z^n,T} + \langle Z_t^n, v\nabla_a \phi \rangle - \gamma \langle Z_t^n, \phi \rangle \leq -\int_E e^{\gamma t} \rho\left(\psi(\phi)\, e^{-\gamma t}\right) dm_t^{Z^n,T}, \tag{4.15}$$

then: $\forall L > 0$, $\exists C = C(\phi, T, L, C_0) > 0$, $\forall n > n_0$,

$$\mathbb{P}\left(\sup_{t \in [0,T]} \langle Z_t^n, \phi \rangle > C\right) \leq e^{-nL}. \tag{4.16}$$

Proof of Lemma 4.8. We are inspired by techniques from [11] for continuous probability-valued processes, and consider the semi-martingale $(e^{-\gamma t} \langle Z_t^n, \phi \rangle)_{t \in [0,T]}$. By Point 2 of Assumptions 4.2,

$2C_0C(\phi) + \langle \xi_0 - Z_0^n, \phi\rangle \geq 0$ for sufficiently large n. Then:

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} \langle Z_t^n, \phi\rangle > e^{\gamma T}\left(L + 2C_0C(\phi) + \langle \xi_0, \phi\rangle\right)\right)\\
=&\mathbb{P}\left(\exists t \in [0,T],\, e^{-\gamma t}\langle Z_t^n, \phi\rangle > e^{\gamma(T-t)}\left(L + 2C_0C(\phi) + \langle \xi_0, \phi\rangle\right)\right)\\
\leq&\mathbb{P}\left(\sup_{t\in[0,T]} e^{-\gamma t}\langle Z_t^n, \phi\rangle - \langle Z_0^n, \phi\rangle > L + 2C_0C(\phi) + \langle \xi_0 - Z_0^n, \phi\rangle\right) \quad \text{since } e^{\gamma(T-t)} \geq 1\\
\leq&\mathbb{P}\left(\sup_{t\in[0,T]} M_t^{n,\gamma,\phi} + \int_0^t e^{-\gamma s}\langle Z_s^n, v\nabla_a \phi - \gamma\phi\rangle ds + \int_0^t \int_E e^{-\gamma s}\psi(\phi) dm_s^{Z^n,T}\, ds > L\right), \qquad (4.17)
\end{aligned}$$

where:

$$M_t^{n,\gamma,\phi} \;:=\; e^{-\gamma t}\langle Z_t^n, \phi\rangle - \langle Z_0^n, \phi\rangle - \int_0^t \langle Z_s^n,\, v\nabla_a\phi - \gamma\phi\rangle e^{-\gamma s} ds - \int_0^t \int_E \psi(\phi) e^{-\gamma s} dm_s^{Z^n,T} ds,$$

is a local martingale starting from 0 (Point (iii) of Proposition 2.8). Let us define $\Xi_t^{n,\gamma,\phi} := \Xi_t^{n,\Psi}$ with the notation of (2.18). By the Assumption (4.15):

$$\begin{aligned}
&\int_0^t \langle Z_s^n,\, v\nabla_a\phi - \gamma\phi\rangle e^{-\gamma s} ds + \int_0^t \int_E \psi(\phi) e^{-\gamma s} dm_s^{Z^n,T} ds\\
\leq& -\int_0^t e^{-\gamma s}\left[\int_E e^{\gamma s}\rho\left(\psi(\phi)\, e^{-\gamma s}\right) dm_s^{Z^n,T}\right] ds = -\int_0^t \int_E \rho\left(\psi(\phi)\, e^{-\gamma s}\right) dm_s^{Z^n,T} ds = -\frac{1}{n}\Xi_t^{n,\gamma,n\phi}.
\end{aligned}$$

Thus:

$$\mathbb{P}\left(\sup_{t\in[0,T]} \langle Z_t^n, \phi\rangle > e^{\gamma T}\left(L + 2C_0C(\phi) + \langle \xi_0, \phi\rangle\right)\right) \leq \mathbb{P}\left(\sup_{t\in[0,T]} M_t^{n,\gamma,n\phi} - \Xi_t^{n,\gamma,n\phi} > nL\right) \leq e^{-nL},$$

by the Doob inequality and Point (iii) of Proposition 3.3. ■

Proof of Lemma 4.7. We begin with establishing a sufficient condition (Inequality (4.19)) for (4.15) to be satisfied. We obtain the results announced in Lemma 4.7 thanks to Lemma 4.8, by proving that this sufficient condition is satisfied for proper choices of functions ϕ.

Let $t \in [0,T]$ and $n \in \mathbb{N}^*$. In the case where $\langle Z_t^n, 1\rangle = 0$, the two members of (4.15) are zero. Assume that $\langle Z_t^n, 1\rangle > 0$. Let $\phi \in \mathcal{C}^1(\mathbb{R}_+^d, \mathbb{R}_+)$ be a positive function and let $\gamma > 0$.

$$\begin{aligned}
e^{\gamma t}\rho\left(\psi(\phi)\, e^{-\gamma t}\right) + \psi(\phi) =& e^{\gamma t}\left(\exp\left(\psi(\phi)\, e^{-\gamma t}\right) - \psi(\phi)\, e^{-\gamma t} - 1\right) + \psi(\phi)\\
=& \sum_{k=1}^{+\infty} [\psi(\phi)]^k \frac{e^{-(k-1)\gamma t}}{k!} = \psi(\phi) \sum_{k=0}^{+\infty} [\psi(\phi)]^k \frac{e^{-k\gamma t}}{k!(k+1)}
\end{aligned}$$

Since for every $k \in \mathbb{N}$, $1/2^k \leq 1/(k+1) \leq 1$, we have:

$$\frac{1}{k!}\left[\frac{e^{-\gamma t}}{2}\psi(\phi)\right]^k \;\leq\; [\psi(\phi)]^k \frac{e^{-k\gamma t}}{k!(k+1)} \leq \frac{1}{k!}\left[e^{-\gamma t}\psi(\phi)\right]^k,$$

summing over k gives:

$$\psi(\phi)\exp\left(\frac{e^{-\gamma t}}{2}\psi(\phi)\right) \leq e^{\gamma t}\rho\left(\psi(\phi)\, e^{-\gamma t}\right) + \psi(\phi) \leq \psi(\phi)\exp\left(e^{-\gamma t}\psi(\phi)\right).$$

Since ϕ is nonnegative:

$$\int_E \left[e^{\gamma t}\rho\left(\psi\left(\phi\right)e^{-\gamma t}\right)+\psi(\phi)\right] dm_t^{Z^n,T} \leq \int_{\mathbb{R}_+} \phi(0)\exp\left(e^{-\gamma t}\phi(0)\right) b(a) Z_t^n(da). \tag{4.18}$$

A sufficient condition for (4.15) to be satisfied is that there exists $\gamma > 0$ such that:

$$\langle Z_t^n, v\nabla_a\phi - \gamma\phi\rangle + \int_{\mathbb{R}_+} \phi(0)\exp\left(e^{-\gamma t}\phi(0)\right) b(a) Z_t^n(da) \leq 0. \tag{4.19}$$

We now prove Point (i) of Lemma 4.7. Noticing that $\mathbb{P}\left(\tau_N^n \leq T\right) = \mathbb{P}\left(\sup_{t\in[0,T]}\langle Z_t^n, 1\rangle \geq N\right)$, it is sufficient, by Lemma 4.8, to prove that (4.15) is satisfied for $\phi \equiv 1$. This is fulfilled since (4.19) is satisfied for $\phi \equiv 1$ and $\gamma > e\bar{b}$.

We proceed in a similar way to establish the deviation inequality for ζ_N^n. For ϕ defined by: $\forall a \in \mathbb{R}_+^d$, $\phi(a) = 1 + |a| = 1 + \sqrt{\sum_{i=1}^d a_i^2}$, (4.19) becomes:

$$\int_{\mathbb{R}_+} \left(\exp\left(e^{-\gamma t}\right) b(a) + \sum_{i=1}^d \frac{2v_i(a)a_i}{|a|} - \gamma - \gamma|a|\right) Z_t^n(da) \leq 0.$$

By Assumptions 2.1 and since $\left(\sum_{i=1}^d a_i\right)^2 \leq C(d)\sum_{i=1}^d a_i^2$, we have $\sum_{i=1}^d \frac{2v_i(a)a_i}{|a|} \leq 2\bar{v}\left(\sqrt{C(d)} + |a|\right)$. Then, (4.19) is satisfied for $\gamma > e\bar{b} + \sqrt{C(d)}2\bar{v}$. By Lemma 4.8:

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(\zeta_N^n \leq T\right) &= \limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[0,T]}\langle Z_t^n, 1\rangle \geq N \quad \text{or} \quad \sup_{t\in[0,T]}\int_{\mathbb{R}_+}|a|Z_t^n(da) \geq N\right)\\ &\leq \limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[0,T]}\langle Z_t^n, 1+|a|\rangle \geq N\right) \leq -L.\end{aligned}$$

This concludes the proof of Point (i).

Let us now consider Point (ii). Let $\phi \in \mathfrak{K}$. The condition (4.19) is satisfied if: $\forall a \in \mathbb{R}_+^d$,

$$v(a)\nabla_a\phi(a) - \gamma\phi(a) + \phi(0)\exp\left(\phi(0)\right) b(a) \leq 0,$$

which is satisfied as soon as:

$$\gamma > \sup_{a\in\mathbb{R}_+^d}\left[\frac{1}{\phi(a)}\left(v(a)\nabla_a\phi(a) + \phi(0)\exp\left(\phi(0)\right) b(a)\right)\right]. \tag{4.20}$$

By the definition of $\mathfrak{K}$, in particular since the function ϕ is polynomial on a compact set and constant outside this set, it is possible to choose γ satisfying (4.20) since the right member is bounded by a constant independent from $a \in \mathbb{R}_+^d$ as:

$$\lim_{|a|\to+\infty}\frac{\sum_{i=1}^d(1+a_i)\frac{\partial\phi}{\partial a_i}(a)}{\phi(a)} \leq C < +\infty.$$

Lemma 4.8 applies and concludes the proof. ∎

Corollary 4.9. *Under Assumptions 2.1, 2.4, 4.2, the sequence of laws of* $(Z^n)_{n\in\mathbb{N}^*}$ *is exponentially tight on* $\mathbb{D}_T$*:* $\forall L > 0$, $\exists K_L \subset \mathbb{D}_T$ *compact,* $\limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(Z^n \notin K_L\right) \leq -L$.

Proof. Let $N = N(T, L, C_0)$ be the positive constant given in Point (i) of Lemma 4.7. Consider the sets $\mathcal{K}^1_{L,N}$ and $\mathcal{K}^2_L$ given by Proposition 4.6. Let us define:

$$\mathcal{K}_L \quad = \quad \mathcal{K}^1_{L,N} \cap \left\{ z \in \mathbb{D}_T \quad \text{such that } \forall t \in [0,T],\, z_t \in \mathcal{K}^2_L \right\} \tag{4.21}$$

Using a criterium from Jakubowski [27] (Theorem 1.7 and Lemma 3.3), the set $\mathcal{K}_L$ is relatively compact in $\mathbb{D}_T$. Since:

$$\mathbb{P}\left(Z^n \notin \mathcal{K}_L\right) \leq \mathbb{P}\left(Z^n_{.\wedge\zeta^n_N} \notin \mathcal{K}^1_{L,N}\right) + \mathbb{P}\left(\exists t \in [0,T],\, Z^n_{t\wedge\zeta^n_N} \notin \mathcal{K}^2_L\right) + \mathbb{P}\left(\zeta^n_N \leq T\right)$$

we have:

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(Z^n \notin \mathcal{K}_L\right) \leq \max\left(\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(Z^n_{.\wedge\zeta^n_N} \notin \mathcal{K}^1_{L,N}\right),\right.$$
$$\left.\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(\exists t \in [0,T],\, Z^n_{t\wedge\zeta^n_N} \notin \mathcal{K}^2_L\right),\, \limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(\zeta^n_N \leq T\right)\right) \leq -L,$$

This concludes the proof. ■

4.3 Lower bound of Theorem 4.4

The proof of the lower bound relies on a change of probability and on the use of Girsanov Theorem. We follow the work of Léonard [31], Kipnis Léonard [30], and thanks to a Riesz theorem, use the links between the trajectories in the domain of the action functional and between some evolution equations obtained by perturbation of (1.2). We prove a local minoration for the neighborhood of trajectories associated to bounded perturbations.

4.3.1 Representation of the paths of the action functional

Proposition 4.10. *Under Assumptions 2.1, 2.4 and for $z \in \mathbb{D}_T$ such that $\mathcal{I}^T_{\xi_0}(z) < +\infty$, there exists a function $h^z \in L^{\rho^*,z}$ satisfying $\forall (f : (a,s) \mapsto f_s(a)) \in \mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})$:*

$$\ell^T(f,z) \quad = \quad \int_0^T \int_E \psi(f)(a,u,s) h^z(a,u,s) m^{z,T}_s(da,du)\, ds, \tag{4.22}$$

where $\ell^T(f,z)$ has been defined in (4.6). Thus:

$$\mathcal{I}^{f,T}(z) = \int_0^T \int_E \left[h^z \psi(f) - \rho(\psi(f))\right] dm^{z,T}_s\, ds, \tag{4.23}$$

and z is the solution of Equation (4.7).

Proof. From the definition of $\mathcal{I}^T_{\xi_0}$, we have for every $f \in \mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})$ and every $\kappa \in \mathbb{R}^*$:

$$\mathcal{I}^{f/\kappa,T}(z) \quad = \quad \frac{1}{\kappa}\ell^T(f,z) - c^T\left(\frac{f}{\kappa}, z\right) \leq \mathcal{I}_{\xi_0}(z) < +\infty.$$

Using the notation $\tau(x) = \max(\rho(x), \rho(-x)) = \rho(|x|)$:

$$\frac{1}{\kappa}\ell^T(f,z) \leq \mathcal{I}^T_{\xi_0}(z) + \int_0^T \int_E \rho\left(\frac{\psi(f)}{\kappa}\right) dm^{z,T}_s ds \leq \mathcal{I}^T_{\xi_0}(z) + \int_0^T \int_E \tau\left(\frac{\psi(f)}{\kappa}\right) dm^{z,T}_s ds.$$

Choosing $\kappa = ||\psi(f)||_{\rho,z}$ et $\kappa = -||\psi(f)||_{\rho,z}$, we obtain $|\ell^T(f,z)| \leq \left(1 + \mathcal{I}^T_{\xi_0}(z)\right) ||\psi(f)||_{\rho,z}$, from (4.4). The map $\psi(f) \mapsto \ell^T(f,z)$ is hence linear and continuous for the norm $||.||_{\rho,z}$ on $\psi\left(\mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})\right) \subset \mathcal{B}_b(E \times [0,T], \mathbb{R})$. By Hahn-Banach's theorem, it is possible to extend this continuous linear form into a continuous linear form on $E^{\rho,z}(E \times [0,T])$. Thanks to Theorem 4.1, there exists a function $h^z \in L^{\rho^*,z}$ satisfying (4.22). From (4.22) and (4.5), we deduce (4.23). (4.7) is a consequence of (4.22) and (4.6). ■

The function h^z given by the previous proposition is not unique, since it is only characterized by the functions of $\psi\left(\mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T],\mathbb{R})\right) \subset \mathcal{B}_b(E \times [0,T],\mathbb{R})$.

Lemma 4.11. *We can choose h^z such that $h^z(a,0,s)$ depends only on $s \in [0,T]$.*

Proof. Let us $h^z \in L^{\rho^*,z}$ be the perturbation given by Proposition 4.10. We are going to modify it on $\mathbb{R}_+^d \times \{0\} \times [0,T]$. $\forall f \in \mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T],\mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}_+^d} f(0,s) h^z(a,0,s) b(a) z_s(da) ds = \int_0^T \int_{\mathbb{R}_+^d} f(0,s) \bar{h}^z(s) b(a) z_s(da) ds,$$

$$\text{with: } \bar{h}^z(s) = \frac{\int_{\mathbb{R}_+^d} h^z(a,0,s) b(a) z_s(da)}{\int_{\mathbb{R}_+^d} b(a) z_s(da)}.$$

(If $\int_{\mathbb{R}_+^d} b(a) z_s(da) = 0$, the choice of h^z on $\mathbb{R}_+^d \times \{0\} \times [0,T]$ is arbitrary). By the convexity of ρ^* and by Jensen's inequality, $\forall \kappa > 0$,

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}_+^d} \rho^*\left(\frac{\bar{h}^z(s)}{\kappa}\right) b(a) z_s(da) ds &= \int_0^T \int_{\mathbb{R}_+^d} \rho^*\left(\frac{\int_{\mathbb{R}_+^d} h^z(\alpha,0,s) b(\alpha) z_s(d\alpha)}{\kappa \int_{\mathbb{R}_+^d} b(\alpha) z_s(d\alpha)}\right) b(a) z_s(da) ds \\
&\leq \int_0^T \int_{\mathbb{R}_+^d} \frac{1}{\int_{\mathbb{R}_+^d} b(\alpha) z_s(d\alpha)} \left[\int_{\mathbb{R}_+^d} \rho^*\left(\frac{h^z(\alpha,0,s)}{\kappa}\right) b(\alpha) z_s(da)\right] b(a) z_s(da) ds \\
&\leq \int_0^T \int_{\mathbb{R}_+^d} \rho^*\left(\frac{h^z(a,0,s)}{\kappa}\right) b(a) z_s(da) ds,
\end{aligned}$$

and the new choice of perturbation obtained by replacing h^z with $\bar{h}^z$ on $\mathbb{R}_+^d \times \{0\} \times [0,T]$ is again a function of $L^{\rho^*,z}$. ∎

4.3.2 Non-variational formula for $\mathcal{I}_{\xi_0}^T$

In order to obtain a non-variational formula for the action functional, we are going to use (4.23) and approximate the perturbations h^z by perturbations of the form $h_m = e^{\psi(f_m)} - 1$ with $(f_m)_{m \in \mathbb{N}^*}$ a sequence of $\mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T],\mathbb{R})$. The result of this section is given in Proposition 4.14. To establish it, we will use the following lemma, which is a particular case of a result due to Israel [24] who generalized a Theorem of Bishop and Phelps [5]:

Lemma 4.12. *Let $\mathfrak{F}$ be a closed subspace of a Banach space $\mathfrak{E}$ and let $\mathfrak{E}'$ the topological dual of $\mathfrak{E}$. Let Λ be a convex continuous function from $\mathfrak{E}$ to $\mathbb{R}$ and let $\partial\Lambda(x)$ be its sub-differential in $x \in \mathfrak{E}$ defined by $\partial\Lambda(x) = \{\ell \in \mathfrak{E}' \mid \forall y \in \mathfrak{E}, \Lambda(x) + \ell(y) \leq \Lambda(x+y)\}$. Let $\ell_0 \in \mathfrak{E}'$ such that:*

$$\exists c \in \mathbb{R}, \forall x \in \mathfrak{E}, \Lambda(x) \geq \ell_0(x) + c, \tag{4.24}$$

then: $\forall \varepsilon > 0$, $\exists x' \in \mathfrak{F}$, $\exists \ell' \in \partial\Lambda(x')$, $\forall x \in \mathfrak{F}$, $|\ell_0(x) - \ell'(x)| \leq \varepsilon \|x\|$.

Definition 4.13. $\forall z \in \mathbb{D}_T$, $\forall g \in \mathcal{B}_b(E \times [0,T],\mathbb{R})$, $\forall h \in \mathcal{B}(E \times [0,T],\mathbb{R})$, $\forall \nu \in (L^\infty(E \times [0,T],\mathbb{R}))'$,

$$\Gamma_{\rho,z}(g) := \int_0^T \int_E \rho(g(a,u,s)) m_s^{z,T}(da,du)\, ds \tag{4.25}$$

$$\Gamma_{\rho,z}^*(h) := \sup_{f \in \mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T],\mathbb{R})} \left\{\int_0^T \int_E h\psi(f)\, dm_s^{z,T}\, ds - \int_0^T \int_E \rho(\psi(f)) dm_s^{z,T}\, ds\right\} \tag{4.26}$$

$$\widetilde{\Gamma}_{\rho,z}^*(\nu) := \sup_{g \in \mathcal{B}_b(E \times [0,T],\mathbb{R})} \{\langle \nu, g \rangle - \Gamma_{\rho,z}(g)\}. \tag{4.27}$$

By (4.5) and (4.23):

$$\mathcal{I}_{\xi_0}^T(z) = \Gamma_{\rho,z}^*(h^z). \tag{4.28}$$

The main result of this Section 4.3.2 provides a non-variational formula for the action functional $\mathcal{I}_{\xi_0}^T(z) = \Gamma_{\rho,z}^*(h^z)$, and an approximation result when we regularize h^z:

Proposition 4.14. *Under Assumptions 2.1, 2.4, for $z \in \mathbb{D}_T$ such that $\mathcal{I}_{\xi_0}^T(z) < +\infty$, and for the associated perturbation $h^z \in L^{\rho^*,z}$ (see Proposition 4.10), there exists a sequence $(h_m)_{m\in\mathbb{N}^*}$ of $\mathcal{C}_b(E \times [0,T], \mathbb{R})$ such that:*
(i) $\forall m \in \mathbb{N}^*$, $\exists f_m \in \mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T], \mathbb{R})$, $h_m = e^{\psi(f_m)} - 1$,
(ii) the sequence $(h_m)_{m\in\mathbb{N}^*}$ converges to h^z $dm_s^{z,T}ds$-a.e.
(iii) $\forall m \in \mathbb{N}^*$, $\Gamma_{\rho,z}^*(h_m) = \Gamma_{\rho^*,z}(h_m)$.
(iv) $h^z \in [-1, +\infty[$ $dm_s^{z,T}ds$-a.e. and there exists $f^z \in \mathcal{B}(\mathbb{R}_+^d \times [0,T], \mathbb{R})$ such that $dm_s^{z,T}ds$-a.e. on $\{h^z > -1\}$, $h^z(a,u,s) = e^{\psi(f^z)(a,u,s)} - 1$.
(v) We have the following non-variational representation for the action functional:

$$\begin{aligned}\mathcal{I}_{\xi_0}^T(z) = \Gamma_{\rho^*,z}(h^z) = \int_0^T \int_E \rho^*(h^z) dm_s^{z,T}\, ds \\ = \int_0^T \int_E \mathbf{1}_{\{h^z>-1\}} \left(h^z\psi(f^z) - \rho(\psi(f^z))\right) dm_s^{z,T} ds + \int_0^T \int_E \mathbf{1}_{\{h^z=-1\}} dm_s^{z,T} ds.\end{aligned} \tag{4.29}$$

A difficulty lies in the fact that $\Gamma_{\rho,z}^*(h^z)$ and $\widetilde{\Gamma}_{\rho,z}^*(h^z\, dm_s^{z,T} ds)$ do not coincide *a priori*, since the sup in (4.26) and (4.27) are taken on different sets. This is due to the particular formulation of our problem which leads us to deal with births and deaths.

Before enoucing Proposition 4.14, we give a lemma for the functions of Definition 4.13.

Lemma 4.15. *Under Assumptions 2.1, 2.4, for $z \in \mathbb{D}_T$,*
(i) $\Gamma_{\rho,z}$ is convex continuous,
(ii) Let $f \in \mathcal{C}_b(\mathbb{R}_+^d \times [0,T], \mathbb{R})$. $\partial\Gamma_{\rho,z}(\psi(f))$ is a singleton $\{\nu\}$ characterized by:

$$\forall g \in \mathcal{B}_b(E \times [0,T], \mathbb{R}),\ \langle \nu, g\rangle = \int_0^T \int_E g\left(e^{\psi(f)} - 1\right) dm_s^{z,T}\, ds. \tag{4.30}$$

(iii) We have: $\forall h \in \mathcal{B}(E \times [0,T], \mathbb{R})$, $h \geq -1$ $dm_s^{z,T}ds$-a.e, and:

$$\Gamma_{\rho,z}^*(h) \leq \widetilde{\Gamma}_{\rho,z}^*(h\, dm_s^{z,T} ds) \leq \Gamma_{\rho^*,z}(h). \tag{4.31}$$

Proof. The convexity of $\Gamma_{\rho,z}$ is a direct consequence of the convexity of ρ. For the continuity, we can see by dominated convergence that if $g \in \mathcal{B}_b(E \times [0,T], \mathbb{R})$ and if $(g_q)_{q\in\mathbb{N}^*}$ is a sequence that converges uniformly to g in $\mathcal{B}_b(E \times [0,T], \mathbb{R})$, we have $\lim_{q\to+\infty} \Gamma_{\rho,z}(g_q) = \Gamma_{\rho,z}(g)$.

To characterize $\partial\Gamma_{\rho,z}(\psi(f))$, let $g \in \mathcal{B}_b(E \times [0,T], \mathbb{R})$,

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\Gamma_{\rho,z}(\psi(f) + \varepsilon g) - \Gamma_{\rho,z}(\psi(f))}{\varepsilon} &= \lim_{\varepsilon\to 0} \int_0^T \int_E \left(\frac{\rho(\psi(f) + \varepsilon g) - \rho(\psi(f))}{\varepsilon}\right) dm_s^{z,T}\, ds \\ &= \int_0^T \int_E g\left[e^{\psi(f)} - 1\right] dm_s^{z,T}\, ds.\end{aligned} \tag{4.32}$$

This is obtained by taking the derivative under the integral. $\Gamma_{\rho,z}$ is hence Gâteau-differentiable in $\psi(f)$ and its sub-differential in $\psi(f)$ exists and is a singleton characterized by (4.30) (Ekeland and Temam [14], Proposition 5.3 Chapitre I).

The first inequality of (4.31) is a consequence of the inclusion $\psi(\mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T], \mathbb{R})) \subset \mathcal{B}_b(E \times [0,T], \mathbb{R})$. The second inequality is a consequence of the fact that $\forall (a,u,s) \in E \times [0,T]$, $\forall g \in \mathcal{B}_b(E \times [0,T], \mathbb{R})$, $h(a,u,s)g(a,u,s) - \rho(g(a,u,s)) \leq \rho^*(h(a,u,s))$. ■

We can now prove Proposition 4.14.

Proof of Proposition 4.14. We have $\mathcal{I}^T_{\xi_0}(z) = \Gamma^*_{\rho,z}(h^z) < +\infty$ by assumption, but we do not know whether $\widetilde{\Gamma}^*_{\rho,z}(h^z dm^{z,T}_s ds) < +\infty$. Since $h^z \in L^{\rho^*,z}$, we however know by (4.31) and by definition of the Orlicz norm $\|.\|_{\rho^*,z}$ (4.4) that:

$$\widetilde{\Gamma}^*_{\rho,z}\left(\frac{h^z}{\|h^z\|_{\rho^*,z}} dm^{z,T}_s ds\right) \leq \Gamma_{\rho^*,z}\left(\frac{h^z}{\|h^z\|_{\rho^*,z}}\right) \leq 1. \tag{4.33}$$

By definition of $\widetilde{\Gamma}^*_{\rho,z}$, and by (4.33), it is possible to write: $\forall g \in \mathcal{B}_b(E \times [0,T], \mathbb{R})$,

$$\Gamma_{\rho,z}(g) \geq \int_0^T \int_E \frac{h^z}{\|h^z\|_{\rho^*,z}} g\, dm^{z,T}_s ds - \widetilde{\Gamma}^*_{\rho,z}\left(\frac{h^z}{\|h^z\|_{\rho^*,z}} dm^{z,T}_s ds\right),$$

and the condition (4.24) is satisfied. Applying Lemma 4.12 with the closed sub-space $\mathfrak{F} = \mathrm{adh}\left(\psi\left(\mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})\right)\right)$ of $L^\infty(E \times [0,T], \mathbb{R})$ gives: $\forall m \in \mathbb{N}^*$, $\exists \widetilde{f}_m \in \mathrm{adh}\left(\mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})\right)$, $\exists \widetilde{\nu}_m \in \partial\Gamma_{\rho,z}(\psi(\widetilde{f}_m))$, $\forall g \in \mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})$,

$$\left|\int_0^T \int_E \psi(g) \frac{h^z}{\|h^z\|_{\rho^*,z}}\, dm^{z,T}_s\, ds - \langle \widetilde{\nu}_m, \psi(g)\rangle\right| \leq \frac{\|g\|_\infty}{2m\|h^z\|_{\rho^*,z}} \tag{4.34}$$

By Point (ii) of Lemma 4.15 and setting $\widetilde{h}_m := \|h^z\|_{\rho^*,z}(e^{\psi(\widetilde{f}_m)} - 1)$:

$$\left|\int_0^T \int_E \psi(g)\left(\frac{h^z}{\|h^z\|_{\rho^*,z}} - \frac{\widetilde{h}_m}{\|h^z\|_{\rho^*,z}}\right) dm^{z,T}_s\, ds\right| \leq \frac{\|g\|_\infty}{2m\|h^z\|_{\rho^*,z}}. \tag{4.35}$$

Since $\widetilde{f}_m$ is bounded and since $dm^{z,T}_s ds$ is a finite measure, $\Gamma_{\rho^*,z}(\widetilde{h}_m) < +\infty$, implying by (4.31), that $\widetilde{\Gamma}^*_{\rho,z}(\widetilde{h}_m dm^{z,T}_s ds) < +\infty$. From the definition of $\widetilde{\Gamma}^*_{\rho,z}$: $\forall g \in \mathcal{B}_b(E \times [0,T], \mathbb{R})$,

$$\Gamma_{\rho,z}(g) \geq \int_0^T \int_E \widetilde{h}_m g\, dm^{z,T}_s ds - \widetilde{\Gamma}^*_{\rho,z}\left(\widetilde{h}_m dm^{z,T}_s ds\right).$$

Applying the Lemma 4.12 and Point (ii) of Lemma 4.15, there exists a sequence $(f_m)_{m\in\mathbb{N}^*}$ of $\mathrm{adh}(\mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R}))$, and a sequence of continuous and bounded perturbations $(h_m)_{m\in\mathbb{N}^*}$ defined by $h_m := e^{\psi(f_m)} - 1$ such that: $\forall m \in \mathbb{N}^*$, $\forall g \in \mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})$,

$$\left|\int_0^T \int_E \psi(g)\left(h_m - \widetilde{h}_m\right) dm^{z,T}_s\, ds\right| \leq \frac{\|g\|_\infty}{4m}.$$

Since each function f_m is the uniform limit of a sequence $(f_{m,p})_{p\in\mathbb{N}^*}$ of $\mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})$, h_m is the limit, for the uniform norm and in $L^1(E \times [0,T],\, dm^{z,T}_s ds)$ (since the measure $dm^{z,T}_s ds$ is finite) of the sequence $h_{m,p} = e^{\psi(f_{m,p})} - 1$. Replacing h_m with $h_{m,p}$ for sufficiently large p, we obtain the existence of a sequence $(f_m)_{m\in\mathbb{N}^*}$ of $\mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})$ satisfying:

$$\left|\int_0^T \int_E \psi(g)\left(h_m - \widetilde{h}_m\right) dm^{z,T}_s\, ds\right| \leq \frac{\|g\|_\infty}{2m}. \tag{4.36}$$

From (4.35) and (4.36): $\forall m \in \mathbb{N}^*$, $\forall g \in \mathcal{C}^1_b(\mathbb{R}^d_+ \times [0,T], \mathbb{R})$,

$$\left|\int_0^T \int_E \psi(g)\left(h_m - h^z\right) dm^{z,T}_s\, ds\right| \leq \frac{\|g\|_\infty}{m}, \tag{4.37}$$

with:

$$\int_0^T \int_E \psi(g)\,(h_m - h^z)\, dm_s^{z,T}\, ds = \int_0^T \int_{\mathbb{R}_+^d} g(0,s)\left(e^{f_m(0,s)} - 1 - h^z(0,0,s)\right) b(a) z_s(da) ds$$
$$- \int_0^T \int_{\mathbb{R}_+^d} g(a,s)\left(e^{-f_m(a,s)} - 1 - h^z(a,1,s)\right) d(a, z_s U(a)) z_s(da) ds. \quad (4.38)$$

For $g \in \mathcal{C}_K^1((\mathbb{R}_+^*)^d \times [0,T], \mathbb{R})$, which satisfies $\forall s \in [0,T]$, $g(0,s) = 0$, we obtain the vague convergence of the sequence of measures $(h_m\, dm_s^{z,T} ds)_{m\in\mathbb{N}^*}$ to $h^z\, dm_s^{z,T} ds$ on $(\mathbb{R}_+^*)^d \times \{1\} \times [0,T]$. Hence, there exists a sub-sequence of $(e^{-f_m} - 1)_{m\in\mathbb{N}^*}$ that converges to $h^z(.,1,.)\ z_s(da)ds$-a.e. on $(\mathbb{R}_+^*)^d \times [0,T]$. For $g \in \mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T], \mathbb{R})$, and for the previous sub-sequence of $(f_m)_{m\in\mathbb{N}^*}$:

$$\left| \int_0^T g(0,s) \left[\int_{\mathbb{R}_+^d} \left(e^{f_m(0,s)} - 1 - h^z(0,0,s)\right) b(a) z_s(da) \right.\right.$$
$$\left.\left. - \int_{\mathbb{R}_+^d} \mathbf{1}_{a=0} \left(e^{-f_m(a,s)} - 1 - h^z(a,1,s)\right) d(a, z_s U(a)) z_s(da) \right] ds \right| \le \frac{C\|g\|_\infty}{m} \quad (4.39)$$

If we prove that:

$$\forall t \in [0,T],\ \langle z_t(da),\ \mathbf{1}_{a=0}\rangle = 0, \quad (4.40)$$

then, $h^z(0,0,s)$ will be the ds-a.e. limit of a sub-sequence of $(e^{f_m(0,s)} - 1)_{m\in\mathbb{N}^*}$, which will entail that h^z is the $dm_s^{z,T} ds$-a.e. limit of a sub-sequence of $(e^{\psi(f_m)} - 1)_{m\in\mathbb{N}^*}$. Let us prove (4.40). For $\varepsilon > 0$, let $\phi_\varepsilon \in \mathcal{C}_b^1(\mathbb{R}_+^d, \mathbb{R})$ be a non-negative function, bounded by 1, such that $\phi_\varepsilon(0) = 1$ and $\phi_\varepsilon(a) = 0$ for $|a| \ge \varepsilon$. For $t \in [0,T]$ and for the function $f_\varepsilon(a,s) = \phi_\varepsilon(A(t,s,a))$:

$$\langle z_t, \phi_\varepsilon\rangle = \langle z_t, f_t\rangle = \langle \xi_0, f_0\rangle + \int_0^t \int_{\mathbb{R}_+^d} \phi_\varepsilon(A(t,s,0))(1 + h^z(a,0,s)) b(a) z_s(da) ds$$
$$- \int_0^t \int_{\mathbb{R}_+^d} \phi_\varepsilon(A(t,s,a))(1 + h^z(a,1,s)) d(a, z_s U(a)) z_s(da) ds. \quad (4.41)$$

For $s < t$ and $a \in \mathbb{R}_+^d$, $\phi_\varepsilon(A(t,s,a)) \to 0$ when $\varepsilon \to 0$. Since $\phi_\varepsilon(A(t,s,0))(1 + h^z(a,0,s))b(a)$ and $\phi_\varepsilon(A(t,s,a))(1 + h^z(a,1,s))d(a, z_sU(a))$ are dominated by $(1 + h^z(a,0,s))b(a)$ and $(1 + h^z(a,1,s))d(a, z_sU(a))$ which are integrable with respect to $z_s(da)\, ds$, the right hand side of (4.41) converges to 0. The left hand side converges to $\langle z_t, \mathbf{1}_{a=0}\rangle$. This proves (4.40) and thus Points (i) and (ii).

Since $h_m = e^{\psi(f_m)} - 1$, $\rho^*(h_m) = e^{\psi(f_m)}\psi(f_m) - e^{\psi(f_m)} + 1 = h_m\psi(f_m) - \left(e^{\psi(f_m)} - \psi(f_m) - 1\right)$:

$$\Gamma_{\rho^*,z}(h_m) = \int_0^T \int_E \left[h_m \psi(f_m) - \rho\left(\psi(f_m)\right)\right] dm_s^{z,T}\, ds$$
$$\le \sup_{f\in\mathcal{C}_b^1(\mathbb{R}_+^d\times[0,T])} \left[\int_0^T \int_E h_m \psi(f)\, dm_s^{z,T}\, ds - \Gamma_{\rho,z}(\psi(f)) \right] = \Gamma_{\rho,z}^*(h_m). \quad (4.42)$$

From (4.31) and (4.42), $\Gamma_{\rho,z}^*(h_m) = \widetilde{\Gamma}_{\rho^*,z}(h_m dm_s^{z,T} ds) = \Gamma_{\rho^*,z}(h_m)$. This proves Point (iii).

Since $(h_m)_{m\in\mathbb{N}^*}$ converges $dm_s^{z,T} ds$-a.e. to h^z and since $\forall m \in \mathbb{N}^*$, $h_m > -1\ dm_s^{z,T} ds$-a.e., then $h^z \ge -1\ dm_s^{z,T} ds$-a.e. As $h^z \in L^{\rho^*,z} \subset L^1(E \times [0,T], dm_s^{z,T} ds)$, $h^z < +\infty\ dm_s^{z,T} ds$-a.e. In the sequel, we have to separate the cases $h^z = -1$ and $h^z > -1$ to take the logarithm of $1 + h^z$.

On $\{h^z > -1\}$, $\psi(f_m)(a,u,s) = \log(h_m(a,u,s)+1)$ converges $dm_s^{z,T}ds$-a.e. to $\psi(f^z)(a,u,s) := \log(h^z(a,u,s)+1)$. The sequence $h^z\psi(f_m)-\rho(\psi(f_m))$ thus converge to $h^z\psi(f^z)-\rho(\psi(f^z))$ when $m\to+\infty$, and the latter function is nonnegative:

$$\begin{aligned}0\le\rho^*(h^z) =& e^{\psi(f^z)}\psi(f^z) - e^{\psi(f^z)} + 1\\ &= \left(e^{\psi(f^z)}+1\right)\psi(f^z) - \left(e^{\psi(f^z)}-\psi(f^z)-1\right) = h^z\psi(f^z)-\rho(\psi(f^z)).\end{aligned} \tag{4.43}$$

On $\{h^z = -1\}$, the sequence $(\psi(f_m))_{m\in\mathbb{N}^*}$ diverges to $-\infty$. Then, $h^z\psi(f_m)-\rho(\psi(f_m)) = -e^{\psi(f_m)}+1$ converges to 1.

Let us define the sequence $(\breve{f}_m)_{m\in\mathbb{N}^*}$ by:

$$\psi(\breve{f}_m) = \psi(f_m)\mathbf{1}_{\{h^z\,\psi(f_m)-\rho(\psi(f_m))\ge 0,\, h^z>-1\}} + \psi(f_m)\mathbf{1}_{\{\psi(f_m)\le 0,\, h^z=-1\}}. \tag{4.44}$$

The nonnegative sequence $(h^z\psi(\breve{f}_m)-\rho(\psi(\breve{f}_m)))_{m\in\mathbb{N}^*}$ converges to $[h^z\psi(f^z)-\rho(\psi(f^z))]\,\mathbf{1}_{\{h^z>-1\}}+\mathbf{1}_{\{h^z=-1\}} = \rho^*(h^z)$. Thus, $\Gamma_{\rho^*,z}(h^z)<+\infty$. Else, by Fatou Lemma and (4.43):

$$\begin{aligned}\Gamma^*_{\rho,z}(h^z) &= \sup_{f\in\mathcal{C}_b^1(\mathbb{R}_+^d\times[0,T],\mathbb{R})}\left\{\int_0^T\int_E\left(h^z\psi(f)-\rho(\psi(f))\right)dm_s^{z,T}ds\right\}\\ &\ge \liminf_{m\to+\infty}\int_0^T\int_E\left(h^z\psi(\breve{f}_m)-\rho(\psi(\breve{f}_m))\right)dm_s^{z,T}ds \ge \int_0^T\int_E\liminf_{m\to+\infty}\left(h^z\psi(\breve{f}_m)-\rho(\psi(\breve{f}_m))\right)dm_s^{z,T}ds\\ &= \int_0^T\int_E\rho^*(h^z)dm_s^{z,T}ds = \Gamma_{\rho^*,z}(h^z) = +\infty,\end{aligned} \tag{4.45}$$

which contradicts the assumption $\Gamma^*_{\rho,z}(h^z)<+\infty$. Integrating (4.43) gives the last equality of (4.29). Moreover Point (iii) of Lemma 4.15 and (4.45) imply the first equality of (4.29). ■

Remark 4.16. *When h^z is bounded, we can choose the functions h_m bounded by $\|h^z\|_\infty$. Indeed, if we regularize the functions f_m in the neighborhood of the points where $\psi(f_m)\ge\log(\|h^z\|_\infty+1)$, we obtain a new sequence $(\widehat{h}_m = e^{\psi(\widehat{f}_m)}-1)_{m\in\mathbb{N}^*}$ bounded by $\|h^z\|_\infty$ and satisfying $|h^z-\widehat{h}_m| = \left|e^{\psi(f^z)}-e^{\psi(\widehat{f}_m)}\right| \le \left|e^{\psi(f^z)}-e^{\psi(f_m)}\right| = |h^z-h_m|$.*

4.3.3 Local lower bound

The proof of the local lower bound enounced in Theorem 4.4 relies on a change of probability, and on the use of Girsanov Theorem. Let $P^n = \mathcal{L}(Z^n)$. It is a probability law on $\mathbb{D}_T$. Let $z\in\mathbb{D}_T$ be such that $\mathcal{I}^T_{\xi_0}(z)<+\infty$. Our purpose is to construct a sequence of probability measures $(P'^n)_{n\in\mathbb{N}^*}$ on $\mathbb{D}_T$ that are absolutely continuous with respect to $(P^n)_{n\in\mathbb{N}^*}$ and which can be interpreted as a recentering on z. To achieve this, we use the perturbed equation (4.7) satisfied by z. The use of the Girsanov Theorem is allowed if h^z has a particular form and proper regularities. Since $h^z\in L^{\rho^*,z}$ is not *a priori* bounded or differentiable, we use regularized perturbations h_m of the form given in Point 1 of Proposition 4.14. It is hence natural to introduced the sequence $(z^m)_{m\in\mathbb{N}^*}$ of the solutions of (4.7) perturbed by the h_m and to apply a recentering associated with these z^m. Unfortunately, we can prove the convergence of $(z^m)_{m\in\mathbb{N}^*}$ to z only in the case were h^z is bounded. The case where h^z is unbounded remains open. Thus, we will be lead to introduce the additional assumption $h^z\in L^\infty(E\times[0,T],\mathbb{R})$.

The following result implies the lower bound of Theorem 4.4:

Proposition 4.17. *Under Assumptions 2.1, 2.4, 4.2, for $z\in\mathbb{D}_T$ such that $\mathcal{I}^T_{\xi_0}(z)<+\infty$ and $h^z\in L^\infty(E\times[0,T],\mathbb{R})$, we have for any open subset O of $\mathbb{D}_T$ containing z:*

$$\liminf_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(Z^n\in O\right)\ge -\mathcal{I}^T_{\xi_0}(z). \tag{4.46}$$

(in the case where $\mathcal{I}^T_{\xi_0}(z)=+\infty$, the result is automatically satisfied).

First, we complete the approximation result of Proposition 4.14 by showing that every path z of the domain of the action functional $\mathcal{I}_{\xi_0}^T$ associated to a bounded h^z can be approximated by a sequence of paths $(z^m)_{m\in\mathbb{N}^*}$ associated to perturbations h_m of the form $h_m = e^{f_m} - 1$ with $f_m \in \mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T], \mathbb{R})$.

Proposition 4.18. *Under Assumptions 2.1, 2.4, for $z \in \mathbb{D}_T$ such that $\mathcal{I}_{\xi_0}^T(z) < +\infty$ and $h^z \in L^{\rho^*,z} \cap L^\infty(E \times [0,T], \mathbb{R})$, there exists a sequence $(f_m)_{m\in\mathbb{N}^*}$ of $\mathcal{C}_b^1(\mathbb{R}_+^d \times [0,T], \mathbb{R})$ such that the sequence defined by $h_m = e^{\psi(f_m)} - 1$ satisfies:*
(i) $(h_m)_{m\in\mathbb{N}^*}$ converges to h^z $dm_s^{z,T}ds$-a.e. and (4.37) is fulfilled,
(ii) For every $m \in \mathbb{N}^*$, the evolution equation: $\forall f \in \mathcal{C}_b^1(\mathbb{R}_+^d \times \mathbb{R}_+)$,

$$\langle y_t, f_t\rangle = \quad \langle \xi_0, f_0\rangle + \int_0^t \left\langle y_s, v\nabla_a f_s + \frac{\partial f_s}{\partial s}\right\rangle ds + \int_0^T \int_E \exp\left(\psi(f_m)\right)\psi(f) dm_s^{y,T}\, ds \qquad (4.47)$$

admits a unique solution in y, which is denoted by $z^m \in \mathcal{C}([0,T], \mathcal{M}_F(\mathbb{R}_+^d))$.
(iii) The sequence $(z^m)_{m\in\mathbb{N}^*}$ converges to $z \in (\mathcal{C}([0,T], (\mathcal{M}_F(\mathbb{R}_+^d), \|.\|_{TV})), \|.\|_\infty)$.
(iv) We have $\mathcal{I}_{\xi_0}^T(z^m) = \int_0^T \int_E \left(h_m\psi(f_m) - \rho(\psi(f_m))\right) dm_s^{z^m,T} ds = \int_0^T \int_E \rho^*(h_m) dm_s^{z^m,T} ds$ and:

$$\lim_{m\to+\infty} \mathcal{I}_{\xi_0}^T(z^m) = \mathcal{I}_{\xi_0}^T(z). \qquad (4.48)$$

Proof. Point (i) is a consequence of Proposition 4.14. By Remark 4.16:

$$\langle z_t^m, 1\rangle \leq \langle \xi_0, 1\rangle + \int_0^t \int_{\mathbb{R}_+^d} e^{f_m(0,s)} b(a) z_s^m(da)\, ds \leq \langle \xi_0, 1\rangle e^{(\|h^z\|_\infty+1)\bar{b}t},$$

we can define $A_T = \max\left(\sup_{t\in[0,T]}\langle z_t, 1\rangle,\, \sup_{t\in[0,T]}\langle z_t^m, 1\rangle,\, m \in N^*\right)$ which exists and is finite and independent from m.

Existence and uniqueness of the solution $z^m \in \mathcal{C}([0,T], \mathcal{M}_F(\mathbb{R}_+^d))$ of (4.47) is an adaptation of the results of Section 3.2 (the birth and death rates are perturbed by continuous bounded factors and Assumptions 2.4 are still satisfied).

Let $t \in [0,T]$ and $\phi \in \mathcal{C}_b^1(\mathbb{R}_+^d, \mathbb{R})$ such that $\|\phi\|_\infty \leq 1$. Let $f : (a,s) \in \mathbb{R}_+^d \times [0,T] \mapsto f(a,s) = \phi(A(t,s,a)) \in \mathbb{R}$.

$$|\langle z_t - z_t^m, \phi\rangle| = \left|\int_0^t \int_E (1 + h^z)\,\psi(f) dm_s^{z,T}\, ds - \int_0^t \int_E (1 + h_m)\,\psi(f) dm_s^{z^m,T}\, ds\right| \leq A + B, \qquad (4.49)$$

where by (4.37):

$$A = \left|\int_0^t \int_E (h^z - h_m)\,\psi(f) dm_s^{z,T}\, ds\right| \leq \frac{1}{m}, \qquad (4.50)$$

and, by Remark 4.16 and computations similar to (3.1):

$$B = \left|\int_0^t \int_E (1 + h_m)\,\psi(f)\left[dm_s^{z,T} - dm_s^{z^m,T}\right]\, ds\right| \leq C(\|h^z\|_\infty, T)\int_0^t \|z_s - z_s^m\|_{TV} ds, \qquad (4.51)$$

We deduce from (4.49), (4.50), (4.51) that:

$$|\langle z_t - z_t^m, \phi\rangle| \leq \frac{1}{m} + C(\|h^z\|_\infty, T)\int_0^t \|z_s - z_s^m\|_{TV} ds,$$

Since every function $\phi \in \mathcal{C}_b(\mathbb{R}_+^d, \mathbb{R}_+)$ bounded by 1 can be written as the pointwise limit of a sequence of $\mathcal{C}_b^1(\mathbb{R}_+^d, \mathbb{R}_+)$ bounded by 1, we obtain by taking the supremum in the left hand side:

$$\sup_{u\in[0,t]} \|z_u - z_u^m\|_{TV} \leq \frac{1}{m} + C(\|h^z\|_\infty, T) \int_0^t \sup_{u\in[0,s]} \|z_u - z_u^m\|_{TV} ds, \tag{4.52}$$

and by Gronwall Lemma: $\sup_{t\in[0,T]} \|z_t - z_t^m\|_{TV} \leq C(\|h^z\|_\infty, T)/m$. Hence:

$$\forall \varepsilon > 0,\ \exists m_0 = m_0(\varepsilon, T) \in \mathbb{N}^*,\ \forall m \geq m_0,\ \sup_{t\in[0,T]} \|z_t - z_t^m\|_{TV} \leq \varepsilon, \tag{4.53}$$

and $(z^m)_{m\in\mathbb{N}^*}$ converges to z in $\mathcal{C}([0,T], (\mathcal{M}_F(\mathbb{R}_+^d), \|.\|_{TV}))$ uniformly in $t \in [0,T]$.

By an adaptation of the proof of Point 3 of Proposition 4.14:

$$\Gamma_{\rho^*, z^m}(h_m) = \Gamma^*_{\rho, z^m}(h_m). \tag{4.54}$$

By (4.5), (4.23), (4.31) and (4.54) $\mathcal{I}_{\xi_0}^T(z^m) = \Gamma^*_{\rho,z^m}(h_m) = \Gamma_{\rho^*,z^m}(h_m)$, which proves the first part of (iv). Finally let us prove (4.48).

$$\begin{aligned} 0 &\leq \left|\mathcal{I}_{\xi_0}^T(z^m) - \mathcal{I}_{\xi_0}^T(z)\right| \leq A + B \\ A &= \left| \int_0^T \int_E \rho^*(h_m) dm_s^{z,T} ds - \int_0^T \int_E \rho^*(h^z) dm_s^{z,T} ds \right| \\ B &= \left| \int_0^T \int_E \rho^*(h_m) dm_s^{z^m,T} ds - \int_0^T \int_E \rho^*(h_m) dm_s^{z,T} ds \right| \end{aligned} \tag{4.55}$$

By (4.53): $\forall m \geq m_0$, $B \leq \int_0^T \rho^*(\|h_m\|_\infty)(\bar{b} + \bar{d}(1+A_T) + L_d \bar{U} A_T) \|z_s - z_s^m\|_{TV} ds \leq C(\|h^z\|_\infty, T)\varepsilon$. Since $(h_m)_{m\in\mathbb{N}^*}$ converges $dm_s^{z,T} ds$-a.e. to h^z, and since these functions are bounded, we have by dominated convergence that $\lim_{m\to+\infty} \Gamma_{\rho^*,z}(h_m) = \Gamma_{\rho^*,z}(h^z) = \mathcal{I}_{\xi_0}^T(z)$, and $\lim_{m\to+\infty} A = 0$. This concludes the proof of (4.48). ■

Thanks to Propositions 4.14 and 4.18 and under the Assumptions of Proposition 4.17, we can restrict the study of the local minoration to "regular" paths:

Assumption 4.19. *We consider a path z belonging to the domain of the action functional $\mathcal{I}_{\xi_0}^T$ and such that $h^z = e^{f^z} - 1$ with $f^z \in \mathcal{C}_b^1(\mathbb{R}_+^d \times \mathbb{R}_+)$. Then:*

$$\mathcal{I}_{\xi_0}^T(z) = \int_0^T \int_E \left(h^z \psi(f^z) - \rho(\psi(f^z))\right) dm_s^{z,T}\, ds.$$

We construct the change of probability corresponding to a centering around a such path z.

Proposition 4.20. *For a path z as in Assumption 4.19, the following SDE on $[0,T]$:*

$$\begin{aligned} D_t^n = 1 + \int_0^t \int_{\mathcal{E}} D_{s\_}^n \mathbf{1}_{\{i \leq N_t^n\}} \Big[&\left(e^{f^z(0,s)} - 1\right) \mathbf{1}_{\{0 \leq \theta \leq m_1(s, Z_{s\_}^n, i)\}} \\ &+ \left(e^{f^z(A_i(Z_{s\_}^n), s)} - 1\right) \mathbf{1}_{\{m_1(s,Z_{s\_}^n,i) < \theta \leq m_2(s, Z_{s\_}^n, i)\}} \Big] \widetilde{Q}(ds, di, d\theta) \end{aligned} \tag{4.56}$$

admits a unique solution $(D_t^n)_{t\in[0,T]}$ which is a P^n-exponential martingale:

$$\begin{aligned} D_t^n &= \exp\Bigg(n \Bigg[\langle Z_t^n, f_t^z \rangle - \langle Z_0^n, f_0^z \rangle - \int_0^t \int_{\mathbb{R}_+^d} \left(v(a) \nabla_a f_s^z(a) + \frac{\partial f_s^z}{\partial s}(a) \right) Z_s^n(da)\, ds \\ &- \int_0^t \int_E \psi(f^z)\, dm_s^{Z^n,T}\, ds - \int_0^t \int_E \rho\left(\psi(f^z)\right) dm_s^{Z^n,T}\, ds \Bigg] \Bigg), \end{aligned} \tag{4.57}$$

Under the probability $P'^n := D^n_T \cdot P^n$, the compensator of the random measure Q is:

$$\widehat{q}(ds, di, d\theta) = 1_{\{i \leq N^n_t\}} \Big[e^{f^z(0,s)} \mathbf{1}_{\{0 \leq \theta \leq m_1(s, Z^n_{s\_}, i)\}} \\ + e^{f^z(A_i(Z^n_{s\_}), s)} \mathbf{1}_{\{m_1(s, Z^n_{s\_}, i) < \theta \leq m_2(s, Z^n_{s\_}, i)\}} \Big] \, ds \, n(di) \, d\theta. \quad (4.58)$$

Proof. We wish to perturb the birth and death rate of the particles, so that the new rates are multiplied by $1 + h^z(a, u, s) = e^{\psi(f^z)(a,u,s)}$, $u \in \{0, 1\}$. This is why we introduce SDE (4.56). For the proofs, we refer to Jacod and Shiryaev [25], Theorems 3.24 p.159 and 5.19 p.181. ■

Proposition 4.21. *Under Assumptions 2.1, 2.4, 4.2, for $z \in \mathbb{D}_T$ satisfying Assumption 4.19 and under $(P'^n)_{n \in \mathbb{N}^*}$ of Proposition 4.20, the sequence $(Z^n)_{n \in \mathbb{N}^*}$ converges in probability to z.*

Proof. Birth and death rates are perturbed by continuous bounded functions:

$$b(a) \leftrightarrow e^{f^z(0,s)} b(a), \quad d(a, Z^n_s U(a)) \leftrightarrow e^{-f^z(a,s)} d(a, Z^n_s U(a)). \quad (4.59)$$

Assumptions 2.4 are still satisfied. We can then prove, with proofs similar to the ones of Section 3 (Corollary 3.4), that Z^n converges weakly to z in P'^n-probability. ■

We are now able to prove the lower bound announced in (4.46).

Proof of Proposition 4.17. We have $P^n(O) = P'^n(O) \mathbb{E}^{P'^n} \left(\mathbf{1}_O \frac{dP^n}{dP'^n} / P'^n(O) \right)$ (these terms may all be equal to zero). By Jensen's inequality and the definition of D^n_T:

$$\liminf_{n \to \infty} \frac{1}{n} \log P^n(O) \geq \liminf_{n \to \infty} \frac{1}{n} \log P'^n(O) + \liminf_{n \to \infty} \mathbb{E}^{P'^n} \left(- \frac{\mathbf{1}_O}{P'^n(O)} \frac{1}{n} \log D^n_T \right).$$

By Proposition 4.21, $\lim_{n \to +\infty} \log P'^n(O) \to 0$, which implies: $\liminf_{n \to \infty} \frac{1}{n} \log P'^n(O) = 0$. From (4.57), from Proposition 4.21, from the fact that z is continuous and that h^z and f^z are continuous and bounded, we obtain the following convergence in probability:

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n} \log D^n_T &= \lim_{n \to \infty} \Bigg[\langle Z^n_t, f^z_t \rangle - \langle Z^n_0, f^z_0 \rangle - \int_0^t \int_{\mathbb{R}^d_+} \left(v(a) \nabla_a f^z_s(a) + \frac{\partial f^z_s}{\partial s} \right) Z^n_s(da) ds \\ &\quad - \int_0^t \int_E \psi(f^z) dm^{Z^n,T}_s \, ds - \int_0^t \int_E \rho\left(\psi\left(f^z\right)\right) dm^{Z^n,T}_s \, ds \Bigg] \\ &= \int_0^t \int_E h^z \, \psi(f^z) dm^{z,T}_s \, ds - \int_0^t \int_E \rho\left(\psi\left(f^z\right)\right) dm^{z,T}_s \, ds = \mathcal{I}^T_{\xi_0}(z), \end{aligned}$$

by (4.22) and (4.29). Since: $\frac{1}{n} \log D^n_T \leq C(T, \|f^z\|_\infty) \left(1 + \sup_{t \in [0,T]} \langle Z^n_t, 1 \rangle^2 \right)$, we obtain, by Point 2 of Assumptions 2.10 and by Proposition 2.8:

$$\limsup_{n \to \infty} \mathbb{E}^{P'^n} \left(\frac{\mathbf{1}_O}{P'^n(O)} \frac{1}{n} \log D^n_T \right) \leq \int_0^t \int_E \left[h^z \psi(f^z) - \rho\left(\psi(f^z)\right) \right] dm^{z,T} \, ds = \mathcal{I}^T_{\xi_0}(z).$$

This concludes the proof of Theorem 4.4. ■

5 Application to problems of exit of domains

We deduce from Theorem 4.4 some estimates on times of exit of domains. The computations that we present are inspired by results obtained by Freidlin and Ventzell [17], Dembo and Zeitouni [12] (Section 5) in finite dimension, and Da Prato and Zabczyk [38] (Chapter 12). We adapt these works to the case of our measure-valued processes.

This allows us to control the probability of exit of a tube around the deterministic solution (Section 5.1) and to give estimates for the exit time of a neighborhood of an attractive domain of the deterministic limit solution in the particular case of logistic age-structured populations.

In the proofs, the distance $\mathcal{W}_1$ appears naturally since the consideration of an age-structure leads us to introduce functions f of $\mathcal{C}^1_b(\mathbb{R}^d_+,\mathbb{R})$ such that $\nabla_a f$ is bounded. These functions are Lipschitz continuous. The total variation norm arise since we do not know if the perturbations h^z are Lipschitz continuous. However, under the constraint that $z\in\mathfrak{G}$, these perturbations remain bounded. The difficulty will be to deal with these two distances. We have:

$$\forall\mu,\,\nu\in\mathcal{M}_F(\mathbb{R}^d_+),\ \mathcal{W}_1(\mu,\nu)\leq\|\mu-\nu\|_{TV}. \tag{5.1}$$

We denote by $B_{TV}(\mu,r)$ (resp. $B_{\mathcal{W}_1}(\mu,r)$) the $\|.\|_{TV}$-ball (resp. the $\mathcal{W}_1$-ball) of radius $r>0$, centered in $\mu\in\mathcal{M}_F(\widetilde{\mathcal{X}})$. $d_{\mathbb{D},T}$ is the Skorohod distance on $\mathbb{D}_T$ (see. [4, 25]).

5.1 Exit times from tubes and pits

Proposition 5.1. *Let $R>0$, $\delta>0$ and let $K(R)$ and $B_{\mathcal{W}_1}(K(R),\delta)$ be the sets defined by:*

$$K(R)=\left\{z\in\mathbb{D}_T,\ \ \mathcal{I}^T_{\xi_0}(z)\leq R\right\}\qquad\textit{and}\qquad K(R)_\delta=\left\{z\in\mathbb{D}_T,\ \ \exists z'\in K(R),\,d_{\mathbb{D},T}(z,z')<\delta\right\}.$$

Under Assumptions 2.1, 2.4, 4.2:
(i) $\forall R>0,\,\forall\delta>0,\,\forall\varepsilon>0,\,\exists n_0\in\mathbb{N}^*,\,\forall n\geq n_0,\ \mathbb{P}\left(Z^n\in K(R)_\delta\right)\geq 1-e^{-n(R-\varepsilon)}$.
(ii) $\forall z\in\mathfrak{G},\,\forall\delta>0,\,\forall\varepsilon>0,\,\exists n_0\in\mathbb{N}^*,\,\forall n\geq n_0,\ \mathbb{P}\left(d_{\mathbb{D},T}(Z^n,z)<\delta\right)\geq e^{-n\left[\mathcal{I}^T_{\xi_0}(z)+\varepsilon\right]}$.

Proof. For $R>0$, $\delta>0$ and $z\in\mathbb{D}_T$, $K(R)^c_\delta$ is closed in $\mathbb{D}_T$ and $\{y\in\mathbb{D}_T\ \mid\ d_{\mathbb{D},T}(z,y)<\delta\}$ is open. The results are then consequences of Theorem 4.4. ■

Remark 5.2. *Since $\{y\,|\,d_{\mathbb{D},T}(y,\xi)<\delta\}\cap\mathfrak{G}\neq\emptyset$, the lower bound of (4.10) provides information for the times of exit from tubes centered on the solution ξ of (1.2).*

5.2 Exit times from the neighborhood of a stationary state of the Logistic age-structured population

Recall the example of the logistic population structured by a scalar age presented in the introduction. Let us introduce the probability $\Pi(a_1,a_2)$ that an individual of age a_1 lives until age a_2 and the *net reproduction rate* R_0.

$$\Pi(a_1,a_2)=\exp\left(-\int_{a_1}^{a_2}d(\alpha)d\alpha\right)\qquad\text{and}\qquad R_0=\int_0^{+\infty}b(a)\Pi(0,a)da. \tag{5.2}$$

Assumption 5.3. *In the following, we assume that $R_0>1$, implying that $\bar{b}>\underline{d}$ since: $\bar{b}/\underline{d}=\int_0^{+\infty}\bar{b}e^{-\underline{d}a}da\geq R_0>1$.*

There exists a unique classical solution to PDE (1.5) that describe the large population limit of the microscopic process and we have an explicit expression (Proposition 5.4). Under Assumption 5.3, the behaviour of the solution of (1.5) is moreover known: there exists a unique nontrivial stable stationary solution to which every solution starting from a nonzero initial condition converge (Proposition 5.5). These are results due to [6, 22, 33, 49].

The estimates given by Section 5.1 tell us that the microscopic stochastic process behaves as its deterministic approximation on compact time intervals, and we prove in Proposition 5.6 that for sufficiently large $T > 0$ and n, Z^n_T finds itself with "large probability" in a neighborhood of the stationary solution $\widehat{m}(a)da$ of (1.5).

However, the same Proposition 5.6 indicates that in long time, $(Z^n)_{n\in\mathbb{N}^*}$ does not converge to $\widehat{m}(a)da$, but to the null measure. The realizations $(Z^n_t)_{t\in\mathbb{R}_+}$ almost surely leave the neighborhood of $\widehat{m}(a)da$ to drive the population to extinction.

Proposition 5.4. *Under Assumptions 2.1, 2.4, 3.1, if* $\exists m_0 \in \mathcal{C}^1_b(\mathbb{R}_+, \mathbb{R}_+)$, $\xi_0(da) = m_0(a)da$:
(i) $(Z^n)_{n\in\mathbb{N}^*}$ *converges in probability in* $\mathbb{D}$ *to the weak measure solution* $(\xi_t)_{t\in\mathbb{R}_+}$ *of (1.5).*
(ii) For every $t \in \mathbb{R}_+$, ξ_t *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}_+$. *The family of the densities* $(m(.,t))_{t\in\mathbb{R}_+}$ *is the unique classical solution of (1.5). It belongs to* $\mathcal{C}^1(\mathbb{R}^2_+, \mathbb{R}_+) \cap L^1(\mathbb{R}^2_+, \mathbb{R}_+)$ *and its explicit expression is given by:* $\forall a \in \mathbb{R}_+$, $\forall t \in \mathbb{R}_+$,

$$m(a,t) = \frac{M_0\, \nu(a,t)}{1 + M_0 \int_0^t \int_0^{+\infty} \nu(\alpha,s)d\alpha\, ds}, \quad \textit{with: } \nu(a,t) = \begin{cases} \frac{m_0(a-t)}{M_0}\Pi(a-t,a) & si \quad a \geq t, \\ B(t-a)\Pi(a,0) & si \quad a < t, \end{cases}$$

Π *being defined in (5.2) and for* g^{*n} *the* n^{th}*-convolution of the function* g *with itself:*

$$B(t) = B_0 * \left(\sum_{n=0}^{+\infty} g^{*n}(t)\right),\ B_0(t) = \mathbf{1}_{t\geq 0}\int_0^{+\infty} b(a+t)\frac{n_0(a)}{N_0}\Pi(a,a+t)da,\ g(a) = b(a)\Pi(0,a)\mathbf{1}_{a\geq 0}.$$

Proof. Existence and uniqueness of a weak function solution are particular cases of Corollary 3.4 and Proposition 3.5. The computations of [6, 22, 33, 49] tell us that (1.5) admits a classical solution. This provides regularity information and an explicit expression of the function solution of (1.2) in this case. ■

Proposition 5.5. *Under Assumption 5.3:*
(i) The following equation: $1 = \int_0^{+\infty} e^{-\lambda_1 a} b(a)\Pi(0,a)da$, *admits a unique solution* $\lambda_1 > 0$.
(ii) The solution $m(a,t)$ *of (1.5) has the following long-time behaviour:*

$$\lim_{t\to+\infty} m(a,t) = \frac{\lambda_1 e^{-\lambda_1 a}\Pi(0,a)}{\eta \int_0^{+\infty} e^{-\lambda_1 \alpha}\Pi(0,\alpha)d\alpha} =: \widehat{m}(a), \tag{5.3}$$

the following limit being uniformly in age on the bounded interval of $\mathbb{R}_+$.
(iii) $\widehat{m}$ *is exponentially asymptotically stable and* $\forall \xi_0 \in \mathcal{M}_F(\mathbb{R}^d_+)$, $\forall R > 0$, $\exists T_1 = T_1(R,\xi_0) \geq 0$, $\forall t \geq T_1$, $\mathcal{W}_1(\xi_t, \widehat{\xi}) < R$, *where* $\widehat{\xi}(da) = \widehat{m}(a)da$.

Proof. These assertions are proved in [6, 33, 49] and Proposition 5.6.4 of [45]. ■

Proposition 5.6. *If* $\exists m_0 \in \mathcal{C}^1_b(\mathbb{R}_+, \mathbb{R}_+)$, $\xi_0(da) = m_0(a)da$:
(i) For $n \in \mathbb{N}^*$. $\mathbb{P}$*-a.s.* $\lim_{t\to+\infty}\langle Z^n_t, 1\rangle = 0$.
(ii) $\forall \varepsilon > 0$, $\forall \varsigma > 0$, $\exists T_1 = T_1(\varsigma) > 0$, $\exists n_1 = n_1(\varepsilon, T_1(\varsigma)) \in \mathbb{N}^*$, $\forall n \geq n_1$, $\mathbb{P}(\mathcal{W}_1(Z^n_{T_1}, \widehat{\xi}) > \varsigma) \leq \varepsilon$.

Proof. Let $n \in \mathbb{N}^*$ be fixed. The idea in the proof of Point (i) is to dominate stochastically $(\langle Z^n_t, 1\rangle)_{t\in\mathbb{R}_+}$ by the following process $(Y^n_t)_{t\in\mathbb{R}_+}$ with values in $(1/n)\mathbb{N}$:

$$\begin{aligned} Y^n_t =& \langle Z^n_0, 1\rangle + \int_0^t \int_{\mathcal{E}} \mathbf{1}_{\{i\leq nY^n_{s\_}\}}\frac{1}{n}\left(\mathbf{1}_{0\leq\theta<b(A_i(Z^n_{s\_}))} - \mathbf{1}_{b(A_i(Z^n_{s\_}))\leq\theta<b(A_i(Z^n_{s\_}))+\underline{d}+\eta Y^n_{s\_}}\right) Q(ds,di,d\theta) \\ &+ \int_0^t \int_{\mathcal{E}} \frac{1}{n}\left[\mathbf{1}_{\{Y^n_{s\_}=0,\, i\in[\![1,n]\!],\, \theta\in[0,1]\}} + \mathbf{1}_{\{i\leq nY^n_{s\_}\}}\mathbf{1}_{0\leq\theta<\bar{b}-b(A_i(Z^n_{s\_}))}\right] Q'(ds,di,d\theta). \end{aligned} \tag{5.4}$$

Q is the Poisson point measure introduced in Definition 2.5 and Q' is an independent copy. The second term of (5.4) implies that 0 is not an absorbing state for Y^n and, when $Y^n_t > 0$, completes the births events so that the birth rate is $\bar{b}$. Let us prove that Y^n is recurrent positive. This process is irreducible. Its generator $\mathcal{A}$ is defined by: $\forall F \in \mathcal{C}((1/n)\mathbb{N}, \mathbb{R}_+)$, $\forall i \in \mathbb{N}$,

$$\begin{aligned}\mathcal{A}F(i/n) =& \mathbf{1}_{i>0}\left\{\left(F\left(\frac{i+1}{n}\right) - F\left(\frac{i}{n}\right)\right)\bar{b}\frac{i}{n} + \left(F\left(\frac{i-1}{n}\right) - F\left(\frac{i}{n}\right)\right)\left(\underline{d} + \eta\frac{i}{n}\right)\frac{i}{n}\right\} \\ &+ \left(F\left(\frac{1}{n}\right) - F(0)\right)\mathbf{1}_{i=0} \end{aligned} \tag{5.5}$$

If we choose for F the Lyapounov function $V^n : i/n \mapsto i/n$, then $\forall i \in \mathbb{N}$,

$$\mathcal{A}V^n\left(\frac{i}{n}\right) = \frac{i}{n^2}\left[\bar{b} - \underline{d} - \eta\frac{i}{n}\right]\mathbf{1}_{i>0} + \frac{1}{n}\mathbf{1}_{i=0} \leq -\frac{1}{n}V^n\left(\frac{i}{n}\right) + \frac{(\bar{b}-\underline{d})^2}{4n^3\eta} \vee \frac{1}{n}, \tag{5.6}$$

since $\bar{b} - \underline{d} - \eta i/n < -1$ for $i > n\left(\bar{b} + 1 - \underline{d}\right)/\eta$, and

$$\sup_{0<i\leq \frac{n(\bar{b}+1-\underline{d})}{\eta}} \left\{\frac{i}{n^2}\left[\bar{b} - \underline{d} - \eta\frac{i}{n}\right]\right\} \leq \frac{(\bar{b}-\underline{d})^2}{4n^3\eta}.$$

The inequality (5.6) is of the form $\forall i \in \mathbb{N}$, $\mathcal{A}V^n\left(\frac{i}{n}\right) \leq -cV^n\left(\frac{i}{n}\right) + c'$. A criterium of Meyn et Tweedie ([36], Theorems 6.1 and 7.1) then entails that Y^n is recurrent positive.

A sufficient condition for Point (i) to be proved is that $\mathbb{P}\left(\exists t \in \mathbb{R}_+,\, Y^n_t = 0\right) = 1$. Let $C > 1$ and $\tau_C = \inf\left\{t > 0,\, Y^n_t = 0 \text{ or } Y^n_t = C/n\right\}$. Since $(Y^n_t)_{t\in\mathbb{R}_+}$ is recurrent positive, $\mathbb{P}_{C/n}\left(Y^n_{\tau_C} = 0\right) > 0$ (under $\mathbb{P}_{C/n}$, Y^n starts from C/n).

$$\begin{aligned}&\mathbb{P}\left(\exists t \in \mathbb{R}_+,\, Y^n_t = 0\right) = \mathbb{P}\left(Y^n_{\tau_C} = 0\right) + \mathbb{P}\left(Y^n_{\tau_C} = C/n \text{ et } \exists t \in \mathbb{R}_+,\, Y^n_{t+\tau_C} = 0\right) \\ =& \mathbb{P}\left(Y^n_{\tau_C} = 0\right) + \mathbb{P}\left(Y^n_{\tau_C} = C/n\right)\mathbb{P}_{C/n}\left(\exists t \in \mathbb{R}_+,\, Y^n_t = 0\right), \text{ by the strong Markov property} \\ =& \mathbb{P}\left(Y^n_{\tau_C} = 0\right) + \mathbb{P}\left(Y^n_{\tau_C} = C/n\right)\left[\mathbb{P}_{C/n}\left(Y^n_{\tau_C} = 0\right) + \mathbb{P}_{C/n}\left(Y^n_{\tau_C} = C/n \text{ et } \exists t \in \mathbb{R}_+,\, Y^n_{t+\tau_C} = 0\right)\right] \\ =& \mathbb{P}\left(Y^n_{\tau_C} = 0\right) + \mathbb{P}\left(Y^n_{\tau_C} = C/n\right)\left[\sum_{\ell=0}^{+\infty}\mathbb{P}_{C/n}\left(Y^n_{\tau_C} = 0\right)\mathbb{P}_{C/n}\left(Y^n_{\tau_C} = C/n\right)^{\ell}\right], \text{ by recursion} \\ =& \mathbb{P}\left(Y^n_{\tau_C} = 0\right) + \mathbb{P}\left(Y^n_{\tau_C} = C/n\right)\frac{\mathbb{P}_{C/n}\left(Y^n_{\tau_C} = 0\right)}{1 - \mathbb{P}_{C/n}\left(Y^n_{\tau_C} = C/n\right)} = \mathbb{P}\left(Y^n_{\tau_C} = 0\right) + \mathbb{P}\left(Y^n_{\tau_C} = C/n\right) = 1.\end{aligned}$$

This concludes the proof. Point (ii) is a consequence of Proposition 5.1. ■

Assumption 5.7. *Assume that the functions $a \mapsto b(a)$ and $a \mapsto d(a)$ are Lipschitz continuous from $\mathbb{R}_+$ to $\mathbb{R}_+$, with constants L_b and L_d.*
We assume that the limit ξ_0 of $(Z^n_0)_{n\in\mathbb{N}^*}$ in $(\mathcal{M}_F(\mathbb{R}_+),\ w)$ is absolutely continuous with respect to the Lebesgue measure, with a density $m_0 \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+) \cap L^1(\mathbb{R}_+, \mathbb{R}_+)$.

For numerous applications (see [45], Chapter 6 for instance), it is of great interest to evaluate the time during which the microscopic process stays in the neighborhood of the stationary stable solution $\widehat{m}(a)da$ of its deterministic approximation.

Let $0 < \gamma < \langle\widehat{\xi}, 1\rangle$. We study the problem of exit from the domain $B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$ using the results of Theorem 4.4. Results are stated in Propositions 5.8 and 5.13. Notice that for this choice of γ, this neighborhood does not contain the null measure. The exit time of $B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$ is:

$$\mathcal{T}^n = \inf\left\{t \geq 0, \ \mid \ Z^n_t \notin B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right\}. \tag{5.7}$$

For $R>0$ and $\mathfrak{G}$ defined in (4.9):

$$\bar{V} = \inf\left\{\mathcal{I}^T_{\xi_0}(z) \mid T>0,\ z\in\mathbb{D}_T\cap\mathfrak{G},\ z_0=\widehat{\xi},\ z_T\notin B_{\mathcal{W}_1}(\widehat{\xi},\gamma)\right\}, \tag{5.8}$$

$$\underline{V}(R) = \inf\left\{\mathcal{I}^T_{z_0}(z) \mid T>0,\ z\in\mathbb{D}_T,\ \mathcal{W}_1(z_0,\widehat{\xi})<R,\ z_T\notin B_{\mathcal{W}_1}(\widehat{\xi},\gamma)\right\}, \tag{5.9}$$

$$\underline{V} = \lim_{R\to 0}\underline{V}(R). \tag{5.10}$$

When the set in (5.8) (resp. (5.9)) is empty, $\bar{V}$ (resp. $\underline{V}(R)$) is infinite. The estimates given in the following sections (Propositions 5.8 and 5.13) are then automatically satisfied. Notice that $\underline{V}$ is well defined in $\mathbb{R}_+\cup\{+\infty\}$: $(\underline{V}(R))_{R>0}$ is an increasing sequence when $R\downarrow 0$ and:

$$\underline{V} := \lim_{R\to 0}\underline{V}(R) \le \bar{V}. \tag{5.11}$$

Notation:

Recall that ξ is the solution of (1.2) starting from $\xi_0\in\mathcal{M}_F(\mathbb{R}_+)$, which is the limit of Z^n_0 for the weak convergence. Until the end of this section, we will consider $\xi_0\in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$. We denote by $\xi^{\mu_0}\in\mathcal{C}(\mathbb{R}_+,\mathcal{M}_F(\mathbb{R}_+))$ the solution of (1.2) starting from the initial condition $\mu_0\in\mathcal{M}_F(\mathbb{R}_+)$.

For $z\in\mathbb{D}_T$ such that $\mathcal{I}^T_{\xi_0}(z)<+\infty$ with $T>0$, there exists $h^z\in L^{\rho^*,z}$ such that z is the solution of (4.7) perturbed by h^z. We denote by z^{μ_0} the solution of this equation perturbed by h^z starting from $\mu_0\in\mathcal{M}_F(\mathbb{R}_+)$.

5.2.1 Upper bound of the exit time

Proposition 5.8. *Let $\xi_0\in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$. Under Assumptions 2.4, 4.2, 4.19, 5.7:*

$$\limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{E}\left(\mathcal{T}^n\right)\le\bar{V}. \tag{5.12}$$

The following corollary, obtained by the Markov inequality, give us an interpretation of this result: the exit time $\mathcal{T}^n$ is upper bounded by $e^{n\bar{V}}$ up to an exponentially small probability.

Corollary 5.9. *Let $\xi_0\in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$. Under Assumptions 2.4, 4.2, 4.19, 5.7 : $\forall\delta>0$, $\exists C>0$,*

$$\limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(\mathcal{T}^n\ge e^{n(\bar{V}+\delta)}\right)\le -C. \tag{5.13}$$

The proof of Proposition 5.8 relies on the following lemmas. Lemma 5.10, proved in the end of this section, establishes a result of continuity with respect to the initial condition. Lemma 5.11 shows that $z\mapsto\mathcal{I}^T_{\xi_0}(z)$ wanishes only at ξ. Its proof stands in [20] page 490. Notice that the results given by these lemmas use the total variation norm. Indeed, we do not know if the perturbations h^z are Lipschitz continuous. However, under the constraint that $z\in\mathfrak{G}$ in the definition of $\bar{V}$ and in the lower bound of Theorem 4.4, these perturbations remain bounded.

Lemma 5.10. *Let $T>0$, let $z\in\mathbb{D}_T\cap\mathfrak{G}$ with initial condition $z_0\in\mathcal{M}_F(\mathbb{R}_+)$.*
(i) $\forall\varepsilon>0$, $\exists\delta>0$, $\forall\mu_0\in B_{TV}(z_0,\delta)$, $\sup_{t\in[0,T]}\|z^{\mu_0}_t-z_t\|_{TV}<\varepsilon$,
(ii) The map $\mu_0\mapsto\mathcal{I}^T_{\mu_0}(z^{\mu_0})$ from $(\mathcal{M}_F(\mathbb{R}_+),\|.\|_{TV})$ into $\mathbb{R}_+$ is continuous at point $\mu_0=z_0$.

Lemma 5.11. *Let $T>0$. We have $\mathcal{I}^T_{\xi_0}(z)=0$ is and only if $z=\xi$, the unique weak measure solution of (1.4)-(1.3) starting from ξ_0.*

Proof Proposition 5.8. Let $z\in\mathbb{D}(\mathbb{R}_+,\mathcal{M}_F(\mathbb{R}_+))$ such that $z_0=\widehat{\xi}$, $\exists T>0$, $z_T\notin B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$, $(z_t)_{t\in[0,T]}\in\mathfrak{G}$. We do this assumption since the upper bound (5.12) is based on the lower bound in (4.10). In order to establish the Proposition 5.8, it is sufficient to prove that:

$$\limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{E}\left(\mathcal{T}^n\right)\le\mathcal{I}^T_{\widehat{\xi}}(z). \tag{5.14}$$

The left hand side does not depend on z, and we can take the infimum in z in the right hand side, which gives (5.12) by (5.8).

In order to obtain (5.14), we establish an upper bound of the form:

$$\forall n \in \mathbb{N}^*,\ \forall k \in \mathbb{N}^*,\ \mathbb{P}\left(\mathcal{T}^n > kC(z,T)\right) \leq (p(z,T))^k, \tag{5.15}$$

where $C(z,T)$ is a nonnegative constant. For this purpose, we lower bound the probability $\mathbb{P}\left(\mathcal{T}^n \leq kC(z,T)\right)$ by the probability that $(Z^n_t)_{t\in[0,kC(z,T)]}$ goes through the neighborhood of $z_T \in \mathcal{M}_F(\mathbb{R}_+)$ situated outside of $B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$.

Step 1 : We begin with bounding below the exit probability of $B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$ when the initial condition is "near" $\widehat{\xi}$, by using the parth z.

Let $\delta := \inf\{\mathcal{W}_1(z_T,y),\ y \in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)\}$. Since $z_T \notin B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$, $\delta > 0$. Let us consider the perturbed evolution equation (4.7) satisfied by z starting from $\widehat{\xi}$ and let $\mu_0 \in \mathcal{M}_F(\mathbb{R}_+)$.

By Point (i) of Lemma 5.10, $\forall \delta_1 \in]0,\delta[$, $\exists \delta_2 > 0$, $\left[\|\mu_0 - \widehat{\xi}\|_{TV} < \delta_2\right] \Rightarrow \left[\|z^{\mu_0}_T - z_T\|_{TV} < \delta - \delta_1\right]$. We deduce by the triangular inequality: $\forall \mu_0 \in B_{TV}(\widehat{\xi},\delta_2)$,

$$\inf_{y\in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)} \mathcal{W}_1(z^{\mu_0}_T,y) \geq \inf_{y\in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)} \mathcal{W}_1(z_T,y) - \|z^{\mu_0}_T - z_T\|_{TV} \geq \delta - (\delta-\delta_1) = \delta_1 > 0,$$

and thus $z^{\mu_0}_T \notin B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$.

By Point (ii) of Lemma 5.10:

$$\forall \varepsilon > 0,\ \exists \delta_0 > 0,\ \forall \mu_0 \in B_{TV}(\widehat{\xi},\delta_0),\ |\mathcal{I}^T_{\mu_0}(z^{\mu_0}) - \mathcal{I}^T_{\widehat{\xi}}(z)| \leq \varepsilon. \tag{5.16}$$

Assume that $\xi_0 \in B_{TV}(\widehat{\xi},\delta_0 \wedge \delta_2)$. We can lower bound the probability of exit of $B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$ by the probability that the path stays in a tube of radius δ_1 around z^{ξ_0}: $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}^*$, $\forall n \geq n_0$,

$$\mathbb{P}\left(\mathcal{T}^n \leq T\right) \geq \mathbb{P}\left(d_{\mathbb{D},T}(Z^n, z^{\xi_0}) < \delta_1\right) \geq e^{-n\ \mathcal{I}^T_{\xi_0}(z^{\xi_0})+\varepsilon} \geq e^{-n\ \mathcal{I}^T_{\widehat{\xi}}(z)+2\varepsilon}\ , \tag{5.17}$$

by Proposition 5.1 and by (5.16).

Step 2 : We now consider an initial condition $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$ satisfying Assumption 5.7. By Proposition 3.5, the marginals ξ_t of the solution of (1.2) are absolutely continuous with respect to the Lebesgue measure. By Point (iii) of Proposition 5.5,

$$\exists T_1 = T_1(\delta_0 \wedge \delta_2, \widehat{\xi}, \gamma) \geq 0,\ \|\xi_{T_1} - \widehat{\xi}\|_{TV} < \delta_0 \wedge \delta_2. \tag{5.18}$$

Let $\delta_3 = \inf_{t\in[0,T_1]}\left(\gamma - \mathcal{W}_1(\xi_t,\widehat{\xi})\right) > 0$ (the deterministic path ξ remains in $B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$). Since $(Z^n_{T_1})_{n\in\mathbb{N}^*}$ converges in probability to $\xi_{T_1} \in B_{TV}(\widehat{\xi},\delta_0\wedge\delta_2)$, we obtain from (5.17), and from the strong Markov property:

$$\begin{aligned}\mathbb{P}\left(\mathcal{T}^n \leq T + T_1\right) \geq & \mathbb{E}\left(\mathbf{1}_{\mathcal{T}^n > T_1}\mathbb{P}_{Z^n_{T_1}}\left(\mathcal{T}^n \leq T\right)\right)\\ \geq & \mathbb{P}\left(d_{\mathbb{D},T_1}(Z^n,\xi) \leq \delta_3\right) e^{-n(\mathcal{I}^T_{\widehat{\xi}}(z)+2\varepsilon)} \geq e^{-n(\mathcal{I}^T_{\widehat{\xi}}(z)+3\varepsilon)},\end{aligned} \tag{5.19}$$

where the lower bound for $\mathbb{P}\left(d_{\mathbb{D},T_1}(Z^n,\xi) \leq \delta_3\right)$ is obtained by Proposition 5.1 and Lemma 5.11.

Step 3: We now establish (5.15). Let $p := 1 - e^{-n(\mathcal{I}^T_{\widehat{\xi}}(z)+3\varepsilon)}$. Assume that for $k \in \mathbb{N}^*$, we have shown that for $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$:

$$\mathbb{P}\left(\mathcal{T}^n > k(T+T_1)\right) \leq p^k. \tag{5.20}$$

By the strong Markov property and since on $\{\mathcal{T}^n > k(T+T_1)\}$, $Z^n_{k(T+T_1)} \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$ converges to $\xi_{k(T+T_1)} \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$:

$$\begin{aligned}\mathbb{P}\left(\mathcal{T}^n > (k+1)(T+T_1)\right) \leq & \mathbb{E}\left(\mathbb{P}_{Z^n_{k(T+T_1)}}\left(\mathcal{T}^n > T+T_1\right)\mathbf{1}_{\mathcal{T}^n > k(T+T_1)}\right) \\ \leq & p\,\mathbb{P}\left(\mathcal{T}^n \geq k(T+T_1)\right) \leq p^{k+1}.\end{aligned} \tag{5.21}$$

By recursion, we obtain (5.20) for every $k \in \mathbb{N}^*$.

Step 4: We deduce for $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$:

$$\begin{aligned}\mathbb{E}\left(\frac{\mathcal{T}^n}{T+T_1}\right) = & \sum_{k=0}^{+\infty}\int_k^{k+1}\mathbb{P}\left(\frac{\mathcal{T}^n}{T+T_1} > t\right)dt \\ \leq & \sum_{k=0}^{+\infty}\mathbb{P}\left(\mathcal{T}^n > k(T+T_1)\right) \leq \frac{1}{1-p} = e^{n(\mathcal{I}^T_{\widehat{\xi}}(z)+3\varepsilon)},\end{aligned} \tag{5.22}$$

by definition of p. Then $\limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{E}\left(\mathcal{T}^n\right) \leq \mathcal{I}^T_{\widehat{\xi}}(z) + 3\varepsilon$. The choice of $\varepsilon > 0$ is arbitrary and the Proposition is proved. ■

Remark 5.12. *In order to obtain the inequalities (5.17) and (5.19), the fact that we only have a local minoration in (4.10) is not restrictive, since we consider deviations with respect to the path ξ, which belongs to $\mathfrak{G}$ (4.9).*

Proof of Lemma 5.10. Let $\varepsilon > 0$, $\delta \in]0,1]$ and $\mu_0 \in B_{TV}(\xi_0, \delta)$. We have $\sup_{t\in[0,T]}\langle z_t + z^{\mu_0}_t, 1\rangle \leq (2\langle\xi_0, 1\rangle + 1)e^{\bar{b}T} =: A_T$ independent of μ_0 and δ.

Let $\phi \in \mathcal{C}^1_b(\mathbb{R}_+, \mathbb{R}_+)$ such that $\|\phi\|_\infty \leq 1$. Let $t \in [0,T]$ and $(f : (a,s) \mapsto \phi(a+t-s)) \in \mathcal{C}^1_b(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$. With a computation similar to (3.1):

$$\begin{aligned}|\langle z_t - z^{\mu_0}_t, \phi\rangle| \leq & \left|\int_{\mathbb{R}_+}\phi(a+t)z_0(da) - \int_{\mathbb{R}_+}\phi(a+t)\mu_0(da)\right| \\ & + \left|\int_0^t\int_E\psi(f)(h^z+1)dm^{z,T}_s ds - \int_0^t\int_E\psi(f)(h^z+1)dm^{z^{\mu_0},T}_s ds\right| \\ \leq & \|z_0 - \mu_0\|_{TV} + (\|h^z\|_\infty + 1)\left[\bar{b} + \bar{d} + 2\eta A_T\right]\int_0^t\|z_s - z^{\mu_0}_s\|_{TV}\,ds\end{aligned} \tag{5.23}$$

Taking the supremum in $\phi \in \mathcal{C}^1_b(\mathbb{R}_+, \mathbb{R}_+)$ bounded by 1 gives:

$$\|z_t - z^{\mu_0}_t\|_{TV} \leq \delta\exp\left((\|h^z\|_\infty + 1)\left[\bar{b} + \bar{d} + 2\eta A_T\right]T\right), \tag{5.24}$$

by Gronwall's Lemma and by choice of μ_0. Choosing $\delta = \min\left(1, \varepsilon e^{-(\|h^z\|_\infty+1)\left[\bar{b}+\bar{d}+2\eta A_T\right]T}\right)$, we have $\delta \in]0,1]$ and can upper bound the right hand side of (5.24) by ε, which proves Point (i).

Now let us consider Point (ii). For f^z defined in Proposition 4.14 and using (4.29):

$$\left|\mathcal{I}^T_{\mu_0}(z^{\mu_0}) - \mathcal{I}^T_{z_0}(z)\right| \leq \|h^z\psi(f^z) - \rho(\psi(f^z))\|_\infty T\left(\bar{b} + \bar{d} + 2\eta A_T\right)\sup_{t\in[0,T]}\|z_t - z^{\mu_0}_t\|_{TV}. \tag{5.25}$$

Thanks to Point (i), there exists $\delta > 0$ such that for all $\mu_0 \in B(z_0, \delta)$, the right hand side of (5.25) is upper bounded by ε. ■

5.2.2 Lower bound of the exit time

We now consider the probability $\mathbb{P}(\mathcal{T}^n > e^{n(\underline{V}-\delta)})$ for $\delta > 0$. We rely on the large deviation upper bound (4.10). The skeleton of the proof looks like its counterpart in finite dimension (see [12], Section 5), but the proofs of all the technical lemmas have to be changed, when dealing with our measure-valued processes. Another difficulty is that $\underline{V}$ is possibly infinite.

Proposition 5.13. *Let $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$. Under Assumptions 2.4, 4.2, 5.7 :*
1. If $\underline{V} < +\infty$, then $\forall \delta > 0$, $\lim_{n\to+\infty} \mathbb{P}\left(\mathcal{T}^n > e^{n(\underline{V}-\delta)}\right) = 1$.
2. If $\underline{V} = +\infty$, then $\forall V > 0$, $\lim_{n\to+\infty} \mathbb{P}\left(\mathcal{T}^n > e^{nV}\right) = 1$.

Let $\rho \in]0, \gamma/2[$. For such ρ, we define the following stopping times:

$$\sigma_\rho = \inf\left\{t \geq 0,\ \ Z_t^n \in B_{\mathcal{W}_1}(\widehat{\xi}, \rho) \cup B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right\}. \tag{5.26}$$

σ_ρ is the first time when $(Z_t^n)_{t\in\mathbb{R}_+}$ enters $B_{\mathcal{W}_1}(\widehat{\xi}, \rho)$ or leaves $B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$. The proof of Proposition 5.13 is based on the following lemmas, proved at the end of the section. The interesting vase is when $\rho \leq \mathcal{W}_1(\xi_0, \widehat{\xi})$, $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma) \setminus B_{\mathcal{W}_1}(\widehat{\xi}, \rho)$. The estimates of Lemmas 5.14, 5.15 and 5.16 are automatically satisfied if $\rho > \mathcal{W}_1(\xi_0, \widehat{\xi})$.

Lemma 5.14. *Let $\rho \in]0, \gamma/2[$, $\lim_{t\to+\infty} \limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}(\sigma_\rho > t) = -\infty$.*

Lemma 5.15. *(i) $\lim_{\rho\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(Z^n_{\sigma_\rho} \in B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right) \leq -\underline{V}$ (where $\underline{V}$ is possibly equal to $+\infty$).*
(ii) We deduce that for $\rho \in]0, \gamma/2[$, $\lim_{n\to+\infty} \mathbb{P}\left(Z^n_{\sigma_\rho} \in B_{\mathcal{W}_1}(\widehat{\xi}, \rho)\right) = 1$.

Lemma 5.16. *Let $\rho \in]0, \gamma/2[$. $\forall c > 0$, $\exists T(c, \rho) > 0$,*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(\sup_{t\in[0,T(c,\rho)]} \mathcal{W}_1(Z_t^n, Z_0^n) \geq \rho\right) < -c. \tag{5.27}$$

Lemmas 5.14, 5.15 and 5.16 tell us that the following probabilities are exponentially small:
1. the probability that the path $(Z_s^n)_{s\in[0,t]}$ remains in the set $B_{\mathcal{W}_1}(\widehat{\xi}, \gamma) \backslash B_{\mathcal{W}_1}(\widehat{\xi}, \rho)$ when $t \to +\infty$,
2. the probability that Z^n leaves this set by leaving $B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$,
3. the probability that Z^n "strongly" deviates from its initial condition Z_0^n in short time.
Notice that the use of $\mathcal{W}_1$ is fundamental in the proof of Lemma 5.16.

Proof of Proposition 5.13. Let $\delta > 0$ and let:

$$\widetilde{V} = \begin{cases} \underline{V} & \text{if } \underline{V} < +\infty, \\ V + \delta & \text{if } \underline{V} = +\infty, \end{cases} \tag{5.28}$$

with an arbitrary $V > 0$ as in the terms of the Proposition. By Lemma 5.15, $\exists \rho \in]0, \gamma/2[$, $\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(Z^n_{\sigma_\rho} \in B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right) < -\widetilde{V} + \delta/4$, and: $\exists n_0 \in \mathbb{N}^*$, $\forall n \geq n_0$,

$$\mathbb{P}\left(Z^n_{\sigma_\rho} \in B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right) < e^{-n(\widetilde{V}-\delta/2)}. \tag{5.29}$$

We define the following stopping times: $\forall k \in \mathbb{N}$,

$$\theta_0 = 0 \tag{5.30}$$

$$\tau_k = \inf\left\{t \geq \theta_k,\ \ Z_t^n \in B_{\mathcal{W}_1}(\widehat{\xi}, \rho) \cup B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right\}, \tag{5.31}$$

$$\theta_{k+1} = \inf\left\{t \geq \tau_k,\ \ Z_t^n \in B^c_{\mathcal{W}_1}(\widehat{\xi}, 2\rho)\right\}, \tag{5.32}$$

with the convention $\theta_{k+1} = +\infty$ si $Z^n_{\tau_k} \in B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$. τ_k is the first time of exit from $B_{\mathcal{W}_1}(\widehat{\xi}, \gamma) \setminus B_{\mathcal{W}_1}(\widehat{\xi}, \rho)$ after θ_k, and if $Z^n_{\tau_k} \in B_{\mathcal{W}_1}(\widehat{\xi}, \rho)$, θ_{k+1} is the first time of exit of $B_{\mathcal{W}_1}(\widehat{\xi}, 2\rho)$. Notice that the exit time $\mathcal{T}^n$ is one of the τ_k, $k \in \mathbb{N}$.

Let $q \in \mathbb{N}^*$ and $T_0 = T(\widetilde{V}, \rho)$ as given by Lemma 5.16. The event $\{\mathcal{T}^n \leq qT_0\}$ implies that there exists $k \in [\![0, q]\!]$ such that either $\mathcal{T}^n = \tau_k$ or one of the q first "excursions" $[\tau_{k-1}, \tau_k]$ for $k \in [\![1, q]\!]$ is of length upper bounded by T_0. In this last case, one of the trajectories $[\tau_{k-1}, \theta_k[$ is covered in less than T_0. Thus:

$$\mathbb{P}\left(\mathcal{T}^n \leq qT_0\right) \leq \sum_{k=1}^{q} \left[\mathbb{P}\left(\mathcal{T}^n = \tau_k\right) + \mathbb{P}\left(\min_{1 \leq k \leq q} (\theta_k - \tau_{k-1}) \leq T_0\right)\right] + \mathbb{P}\left(\mathcal{T}^n = \tau_0\right) \tag{5.33}$$

By Markov strong property and by (5.29), $\exists n_0 > 0$, $\forall n \geq n_0$, $\forall k \in \mathbb{N}^*$,

$$\begin{aligned} \mathbb{P}\left(\mathcal{T}^n = \tau_k\right) =& \mathbb{P}\left(\mathbb{P}_{Z^n_{\theta_k}}\left(\mathcal{T}^n = \tau_0\right) \mathbf{1}_{\mathcal{T}^n > \tau_{k-1}}\right) \\ =& \mathbb{P}\left(\mathbb{P}_{Z^n_{\theta_k}}\left(Z^n_{\sigma_\rho} \in B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right) \mathbf{1}_{\mathcal{T}^n > \tau_{k-1}}\right) \leq e^{-n(\widetilde{V} - \delta/2)}, \end{aligned} \tag{5.34}$$

by noticing that for sufficiently large n, $Z^n_{\theta_k}$ belongs to $B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$.

By choice of T_0, by the strong Markov property: $\forall n \geq n_0$, $\forall k \in \mathbb{N}^*$,

$$\mathbb{P}\left(\theta_k - \tau_{k-1} \leq T_0\right) \leq \mathbb{P}\left(\sup_{t \in [0, T_0]} \mathcal{W}_1(Z^n_t, Z^n_0) \geq \rho\right) \leq e^{-n(\widetilde{V} - \delta/2)}. \tag{5.35}$$

By (5.33), (5.34), (5.35) : $\forall q \in \mathbb{N}^*$,

$$\mathbb{P}\left(\mathcal{T}^n \leq qT_0\right) \leq \mathbb{P}\left(\mathcal{T}^n = \tau_0\right) + 2qe^{-n(\widetilde{V} - \delta/2)}. \tag{5.36}$$

Since $\{\mathcal{T}^n = \tau_0\} = \{Z^n_{\sigma_\rho} \in B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\}$, then, for the choice of $q = [e^{n(\widetilde{V} - \delta)}/T_0] + 1$:

$$\mathbb{P}\left(\mathcal{T}^n \leq e^{n(\widetilde{V} - \delta)}\right) \leq \mathbb{P}\left(\mathcal{T}^n \leq qT_0\right) \leq \mathbb{P}\left(Z^n_{\sigma_\rho} \in B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right) + \frac{4}{T_0} e^{-n\delta/2}. \tag{5.37}$$

The right hand side of (5.37) converges to 0 when $n \to +\infty$, as:

$$\lim_{n \to +\infty} \mathbb{P}\left(Z^n_{\sigma_\rho} \in B^c_{\mathcal{W}_1}(\widehat{\xi}, \gamma)\right) = 0, \tag{5.38}$$

by Lemma 5.15. ■

Proof of Lemma 5.14. If $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi}, \rho)$, Lemma 5.14 is true. Assume that $\rho \leq \mathcal{W}_1(\widehat{\xi}, \xi_0)$. Then, $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma) \setminus B_{\mathcal{W}_1}(\widehat{\xi}, \rho)$. Let $L > 0$. It is sufficient to show that: $\exists T = T(L) > 0$,

$$\limsup_{n \to +\infty} \frac{1}{n} \log \mathbb{P}\left(\sigma_\rho > T\right) < -L.$$

By Point (iii) of Proposition 5.5: $\forall \xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma)$, $\exists T_1(\rho, \widehat{\xi}, \gamma) > 0$, $\forall t \geq T_1$, $\mathcal{W}_1(\xi^{\xi_0}_t, \widehat{\xi}) < \rho/2$.

Let $T \geq T_1$. If the path $(Z^n_t)_{t \in [0,T]}$ remains in $B_{\mathcal{W}_1}(\widehat{\xi}, \gamma) \setminus B_{\mathcal{W}_1}(\widehat{\xi}, \rho)$, we have necessarily $\mathcal{W}_1(Z^n_T, \xi^{\xi_0}_T) > \rho/2$.

The sequence $(\mathcal{L}(Z^n))_{n \in \mathbb{N}^*}$ being exponentially tight, $\exists K(L, T) \subset \mathbb{D}_T$ compact, $\exists n_0 \in \mathbb{N}^*$, $\forall n \geq n_0$, $\mathbb{P}\left(Z^n \notin K(L, T)\right) \leq e^{-nL}$.

The following set:

$$\mathcal{A}(T) = \text{adh}\left\{z \in \mathbb{D}_T,\ \forall t \in [0, T],\ z_t \in B_{\mathcal{W}_1}(\widehat{\xi}, \gamma) \setminus B_{\mathcal{W}_1}(\widehat{\xi}, \rho)\right\} \cap K(L, T). \tag{5.39}$$

is compact, does not contain ξ by choice of T and $\forall z \in \mathcal{A}(T)$, $\mathcal{I}^T_{\xi_0}(z) > 0$ (Lemma 5.11). Then:

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(\sigma_\rho > T\right) \leq \max\left(\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(Z^n \in \mathcal{A}(T)\right), \limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(Z^n \notin K(L,T)\right)\right)$$
$$\leq \max\left(-\inf_{z\in\mathcal{A}(T)} \mathcal{I}^T_{\xi_0}(z), -L\right) < 0.$$

It remains to show that for sufficiently large T,

$$\inf_{z\in\mathcal{A}(T)} \mathcal{I}^T_{\xi_0}(z) > L. \tag{5.40}$$

Let $z \in \mathcal{A}(T)$ starting from $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi},\gamma) \setminus B_{\mathcal{W}_1}(\widehat{\xi},\rho)$ with $T \geq T_1$. Let $\phi \in \mathcal{C}^1(\mathbb{R}_+,\mathbb{R})$ be bounded by 1, $t \in [0,T]$ and $f : (a,s) \in \mathbb{R}^2_+ \mapsto f_s(a) = \phi(a+t-s) \in \mathbb{R}$. We have:

$$\langle z_t - \xi_t, \phi\rangle = \int_0^t \int_{\mathbb{R}_+} \left(b(a)\phi(t-s) - \phi(a+t-s)\left(d(a) - \eta\langle z_s, 1\rangle\right)\right)(z_s - \xi_s)(da)ds$$
$$- \int_0^t \int_{\mathbb{R}_+} \phi(a+t-s)\eta\left(\langle z_s - \xi_s, 1\rangle\right)\xi_s(da)\, ds + \int_0^t \int_E \psi(f) h^z dm^{z,T}_s\, ds.$$

For $z \in \mathcal{A}(T)$ and by choice of γ, $\sup_{t\in[0,T]}(\langle \xi_t, 1\rangle + \langle z_t, 1\rangle) \leq 4\langle\widehat{\xi}, 1\rangle e^{\bar{b}T} := A_T$, which is finite and does not depend on $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)$ nor on $z \in \mathcal{A}(T)$. We then deduce that:

$$|\langle z_t - \xi_t, \phi\rangle| \leq \left(\bar{b} + \bar{d} + 2\eta A_T\right) \int_0^t \sup_{u\in[0,s]} \|z_u - \xi_u\|_{TV}\, ds + \int_0^t \int_E |h^z| dm^{z,T}_s\, ds$$

Taking the supremum in ϕ in the left hand side, by Gronwall's inequality and by (5.1):

$$\sup_{t\in[0,T]} \mathcal{W}_1(z_t, \xi_t) \leq \sup_{t\in[0,T]} \|z_t - \xi_t\|_{TV} \leq \left(\int_0^T \int_E |h^z| dm^{z,T}_s\, ds\right) e^{(\bar{b}+\bar{d}+2\eta A_T)T}. \tag{5.41}$$

Since $\sup_{t\in[0,T]} \mathcal{W}_1(z_t, \xi_t) \geq \rho/2$:

$$\int_0^T \int_E |h^z| \frac{dm^{z,T}_s\, ds}{\int_0^T \int_E dm^{z,T}_s\, ds} \geq \frac{\rho}{2} \frac{1}{\int_0^T \int_E dm^{z,T}_s\, ds} e^{-(\bar{b}+\bar{d}+2\eta A_T)T}$$

and since ρ^* is an increasing function on $\mathbb{R}_+$:

$$\rho^*\left(\int_0^T \int_E |h^z| \frac{dm^{z,T}_s\, ds}{\int_0^T \int_E dm^{z,T}_s\, ds}\right) \geq \rho^*\left(\frac{\rho}{2} \frac{1}{(\bar{b}+\bar{d}+\eta A_T)A_T\, T} e^{-(\bar{b}+\bar{d}+2\eta A_T)T}\right), \tag{5.42}$$

On the other hand, by Jensen's inequality:

$$\rho^*\left(\int_0^T \int_E |h^z| \frac{dm^{z,T}_s ds}{\int_0^T \int_E dm^{z,T}_s ds}\right) \leq \int_0^T \int_E \rho^*\left(|h^z|\right) \frac{dm^{z,T}_s ds}{\int_0^T \int_E dm^{z,T}_s ds} = \frac{\mathcal{I}^T_{\xi_0}(z)}{\int_0^T \int_E dm^{z,T}_s ds}. \tag{5.43}$$

By (5.42) and (5.43), and for $T = T_1$:

$$\int_0^{T_1} \int_E \rho^*(h^z) dm^{z,T_1}_s ds = \mathcal{I}^{T_1}_{\xi_0}(z)$$
$$\geq \left(\int_0^{T_1} \int_E dm^{z,T_1}_s\, ds\right) \rho^*\left(\frac{\rho}{2} \frac{1}{(\bar{b}+\bar{d}+\eta A_{T_1})A_{T_1}\, T_1} e^{-(\bar{b}+\bar{d}+2\eta A_{T_1})T_1}\right)$$
$$\geq \left(\inf_{z\in\mathcal{A}(T_1)} \int_0^{T_1} \int_E dm^{z,T_1}_s ds\right) \rho^*\left(\frac{\rho}{2} \frac{e^{-(\bar{b}+\bar{d}+2\eta A_{T_1})T_1}}{(\bar{b}+\bar{d}+\eta A_{T_1})A_{T_1}\, T_1}\right) := C_2 > 0, \tag{5.44}$$

by compactness of $\mathcal{A}(T_1)$ with paths that do not vanish (by choice of γ and since they remain in $adh(B_{\mathcal{W}_1}(\widehat{\xi},\gamma))$). Notice that the constant C_2 does not depend on the particular choice of the initial condition $\xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi},\gamma) \setminus B_{\mathcal{W}_1}(\widehat{\xi},\rho)$ nor on the path $z \in \mathcal{A}(T)$.

Let now $T > 0$ be such that $T > J\,T_1$ with $J \in \mathbb{N}^*$. We have:

$$\mathcal{I}_{\xi_0}^T(z) \geq \sum_{j=1}^J \int_{(j-1)T_1}^{jT_1} \int_E \rho^*(h^z) dm_s^{z,T}\, ds = \sum_{j=1}^J \int_0^{T_1} \int_E \rho^*(h^{z^j}) dm_s^{z^j,T}\, ds, \tag{5.45}$$

where $\forall t \in [0,T_1]$, $z_t^j = z_{(j-1)T_1+t}$ is a path in $\mathcal{A}(T_1)$ which solves: $\forall\, (f : (a,s) \mapsto f_s(a)) \in \mathcal{C}_b^1(\mathbb{R}_+ \times [0,T_1], \mathbb{R})$,

$$\langle z_t^j,\, f_t\rangle = \langle z_{(j-1)T_1},\, f_0\rangle + \int_0^t \left\langle z_s^j,\, \frac{\partial}{\partial a} f_s + \frac{\partial f_s}{\partial s}\right\rangle ds + \int_0^t \int_E (1+h^{z^j})\psi(f) dm_s^{z,T}\, ds,$$

which is an equation obtained by the perturbation of (1.2) with $h^{z^j}(a,u,s) = h^z(a,u,(j-1)T_1+s)$ for $s \in [0,T_1]$. By (5.44) and by recursion $\mathcal{I}_{\xi_0}^T(z) \geq JC_2$. This proves (5.40). ■

Proof of Lemma 5.15. Let us prove Point (i). Let $\rho \in]0,\gamma/2[$. Let $R > 0$ be such that $\underline{V}(R) < +\infty$. By Lemma 5.14 :

$$\exists T_2 > 0,\, \exists n_0 \in \mathbb{N}^*,\, \forall n \geq n_0,\, \mathbb{P}\left(\sigma_\rho > T_2\right) < e^{-n\underline{V}(R)}, \tag{5.46}$$

where $\underline{V}(R)$ is defined in (5.9). For $T > 0$, we define:

$$A(T) := \left\{ z \in \mathbb{D}([0,T], \mathcal{M}_F(\mathbb{R}_+)) \;\mid\; \exists t \in [0,T],\, z_t \notin B_{\mathcal{W}_1}(\widehat{\xi},\gamma) \right\} \tag{5.47}$$

which is closed since $z \mapsto \sup_{t\in[0,T]} \mathcal{W}_1(z_t,\widehat{\xi})$ is continuous. From (5.9): $\forall \xi_0 \in B_{\mathcal{W}_1}(\widehat{\xi},2\rho)$, $\inf_{z\in A(T_2)} \mathcal{I}_{\xi_0}^{T_2}(z) \geq \underline{V}(2\rho)$. Using the large deviation upper bound (Theorem 4.4) :

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}_{Z_0^n \in B_{\mathcal{W}_1}(\widehat{\xi},\gamma)\setminus B_{\mathcal{W}_1}(\widehat{\xi},\rho)} \left(Z^n \in A(T_2)\right) \leq -\inf_{z\in A(T_2)} \mathcal{I}_{\xi_0}^{T_2}(z) \leq -\underline{V}(2\rho). \tag{5.48}$$

By (5.46) and (5.48) :

$$\begin{aligned}
&\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(Z_{\sigma_\rho}^n \in B_{\mathcal{W}_1}^c(\widehat{\xi},\gamma)\right) \\
&\leq \max\left(\limsup_{n\to+\infty} \frac{1}{n}\log \mathbb{P}\left(\sigma_\rho > T_2\right), \limsup_{n\to+\infty} \frac{1}{n}\log \mathbb{P}\left(Z^n \in A(T_2)\right)\right) \leq \max\left(-\underline{V}(R), -\underline{V}(2\rho)\right).
\end{aligned}$$

By (5.10), $\lim_{\rho\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(Z_{\sigma_\rho}^n \in B_{\mathcal{W}_1}^c(\widehat{\xi},\gamma)\right) \leq -\underline{V}(R)$, and letting R tend to 0, we obtain the result of Point (i), since the left hand side does not depend on R.

Let us consider Point (ii). Let $\varepsilon > 0$, $\rho \in]0,\gamma/2[$ and $R > 0$ satisfying $\underline{V}(R) < +\infty$ and:

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}(Z_{\sigma_\rho}^n \in B_{\mathcal{W}_1}^c(\widehat{\xi},\gamma)) \leq -\underline{V}(R) + \frac{\varepsilon}{2}.$$

(the existence of such R is given by (i)) Then $\exists n_0 \in \mathbb{N}^*, \forall n \geq n_0$, $\mathbb{P}\left(Z_{\sigma_\rho}^n \in B_{\mathcal{W}_1}^c(\widehat{\xi},\gamma)\right) \leq e^{-n\underline{V}(R)+n\varepsilon}$, and:

$$\mathbb{P}\left(Z_{\sigma_\rho}^n \in B_{\mathcal{W}_1}(\widehat{\xi},\rho)\right) = 1 - \mathbb{P}\left(Z_{\sigma_\rho}^n \in B_{\mathcal{W}_1}^c(\widehat{\xi},\gamma)\right) \geq 1 - e^{-n(\underline{V}(R)-\varepsilon)}.$$

Choosing $\varepsilon < \underline{V}(R)$ and letting n tend to $+\infty$ completes the proof. ■

Proof of Lemma 5.16. Let $T > 0$ and $c > 0$. By Point (i) of Lemma 4.7, there exists $N = N(c,T) > 0$ such that:

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbb{P}\left(\zeta_N^n < T\right) < -c. \tag{5.49}$$

Let $\phi \in \mathcal{C}_b^1(\mathbb{R}_+,\mathbb{R})$ $1-$Lipschitz continuous bounded by 1, $t \in [0,T]$, and $f(a,s) = \phi(a+t-s) \in \mathcal{C}_b^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$. With the notation of Definition 2.5:

$$\begin{aligned}\Big|\langle Z^n_{t\wedge\zeta_N^n}, \phi\rangle - \langle Z_0^n, \phi\rangle\Big| \leq \left|\int_{\mathbb{R}_+} (\phi(a+t)-\phi(a))\, Z_0^n(da)\right| + \left|\frac{1}{n}\int_0^{t\wedge\zeta_N^n}\int_{\mathcal{E}} \mathbf{1}_{\{i\leq N^n_s\}} \Big[f(0,s)\mathbf{1}_{0\leq m_1(s,Z^n_{s\_},i)}\right.\\ \left. -\ f(A_i(Z^n_{s\_}),s)\mathbf{1}_{m_1(s,Z^n_{s\_},i)\leq\theta<m_2(s,Z^n_{s\_},i)}\Big]\, Q(ds,di,d\theta,dx')\right| \leq t\langle Z_0^n,1\rangle + \frac{\mathcal{N}(t)}{n},\end{aligned}$$

where $\mathcal{N}(t)$ is a Poisson random variable with parameter $tN(\bar{b}+\bar{d}+\eta N)$. We deduce:

$$\sup_{s\in[0,t]} \mathcal{W}_1(Z^n_{s\wedge\zeta_N^n}, Z_0^n) \leq t\langle Z_0^n,1\rangle + \frac{\mathcal{N}(t)}{n}$$

$$\text{and: } \mathbb{P}\left(\sup_{s\in[0,t]} \mathcal{W}_1(Z^n_{s\wedge\zeta_N^n}, Z_0^n) \geq \rho\right) \leq \mathbb{P}\left(t\langle Z_0^n,1\rangle \geq \frac{\rho}{2}\right) + \mathbb{P}\left(\frac{\mathcal{N}(t)}{n} \geq \frac{\rho}{2}\right).$$

Since:

$$\limsup_{n\to+\infty} \frac{1}{n}\log\mathbb{P}\left(\mathcal{N}(t) \geq \frac{n\rho}{2}\right) \leq \frac{\rho}{2}\log(tN(\bar{b}+\bar{d}+\eta N)),$$

which tends to $-\infty$ when $t \to 0$. Then: $\exists T_3 \in]0,T[$, $\forall t \in [0,T_3]$,

$$\limsup_{n\to+\infty} \frac{1}{n}\log\mathbb{P}\left(\mathcal{N}(t) \geq \frac{n\rho}{2}\right) < -c. \tag{5.50}$$

By Point 2 of Assumptions 4.2, $\exists n_0 \in \mathbb{N}^*$, $\forall n \geq n_0$, $\langle Z_0^n,1\rangle < C_0 + \langle \xi_0,1\rangle$. Then, for $T_4 < \rho/(2(C_0+\langle\xi_0,1\rangle))$: $\forall t \in [0,T_4]$, $\forall n \geq n_0$, $\mathbb{P}\left(t\langle Z_0^n,1\rangle > \frac{\rho}{2}\right) = 0$.

Hence, for $t \in [0, T_3\wedge T_4 \wedge T]$:

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{s\in[0,t]}\mathcal{W}_1(Z_t^n,Z_0^n)\geq\rho\right) \leq \max\Bigg[\limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}(\zeta_N^n<T),\\ \limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(t\langle Z_0^n,1\rangle\geq\frac{\rho}{2}\right), \limsup_{n\to+\infty}\frac{1}{n}\log\mathbb{P}\left(\mathcal{N}(t)\geq\frac{n\rho}{2}\right)\Bigg] < -c.\end{aligned} \tag{5.51}$$

∎

This concludes the proof of Proposition 5.13.

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