Solvability of fully nonlinear elliptic equations with natural growth and unbounded coefficients

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Abstract. We prove several results of existence, uniqueness and non-uniqueness of viscosity solutions of uniformly elliptic fully nonlinear equations with unbounded measurable ingredients and quadratic growth in the gradient.

1 Introduction and Main Results

In this paper we prove results on existence and uniqueness of solutions of the Dirichlet problem for uniformly elliptic fully nonlinear equations in non-divergence form

$$F(D^2u, Du, u, x) = f(x) \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. Our model equation is

$$\mathcal{M}(D^2u) + \mu(x)|Du|^2 + b(x)|Du| + c(x)u = f(x), \tag{2}$$

where $\mathcal{M}$ is a Pucci extremal operator (see below), and

$$\mu \in L^\infty(\Omega), \qquad b \in L^p(\Omega),\ p > N, \qquad c, f \in L^N(\Omega). \tag{3}$$

The quadratic dependence in the gradient is usually referred to in the literature as natural growth, while the condition on b is required for solvability of the Dirichlet problem even for linear equations in non-divergence form. We have new results in both particular cases when $\mu \equiv 0$ or $b \equiv 0$.

Note (2) is only a model case, we shall not suppose that the operator is convex or concave in any of its parts. For instance, our results apply to general Bellman-Isaacs equations (these equations appear frequently in control theory, large deviations problems, game theory)

$$\sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} F^{\alpha,\beta}(u, x) = 0, \tag{4}$$

where $\mathcal{A}$, $\mathcal{B}$ are arbitrary index sets and $F^{\alpha,\beta}(u, x)$ denotes the operator

$$\mathrm{tr}\left(A^{\alpha,\beta}(x)D^2u\right) + <Q^{\alpha,\beta}(x)Du, Du> + <b^{\alpha,\beta}(x), Du> + c^{\alpha,\beta}(x)u - f^{\alpha,\beta}(x).$$

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For the general equation (1), we suppose that $F(0,0,0,x) \equiv 0$, and for some null set $\mathcal{N} \subset \Omega$ and all $M, N \in \mathcal{S}_n(\mathbb{R}), p, q \in \mathbb{R}^N, u, v \in \mathbb{R}, x \in \Omega \setminus \mathcal{N}$,

$$(S) \left\{ \begin{array}{lcl} F(M,p,u,x) - F(N,q,v,x) & \leq & \mathcal{M}^+_{\lambda,\Lambda}(M-N) + \mu(|p|+|q|)|p-q| \\ & & +b(x)|p-q| + d(x)\overline{h}(u,v) \\ & & \\ F(M,p,u,x) - F(N,q,v,x) & \geq & \mathcal{M}^-_{\lambda,\Lambda}(M-N) - \mu(|p|+|q|)|p-q| \\ & & -b(x)|p-q| - d(x)\underline{h}(u,v), \end{array} \right.$$

where $0 < \lambda \leq \Lambda$, $\mu \in \mathbb{R}$, $b \in L^p(\Omega)$ for some $p > N$, $d \in L^N(\Omega)$, $\mu, b, d \geq 0$, and $\overline{h}, \underline{h} \in C(\mathbb{R}^2)$. We recall that Pucci's operators are defined by $\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}} \mathrm{tr}(AM)$, $\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf\limits_{A\in\mathcal{A}} \mathrm{tr}(AM)$, where $\mathcal{A} \subset \mathcal{S}_N$ denotes the set of matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$. Note $\mathcal{M}_{1,1}(D^2u) = \Delta u$.

When we speak of solutions of (1) we shall mean L^N-viscosity solutions – we refer to [CCKS] for a general review of these (we recall definitions and results that we need in the next section). Note that viscosity solutions are continuous and that any function in $W^{2,N}_{\mathrm{loc}}(\Omega)$ satisfies (1) almost everywhere – such a solution is called strong – if and only if it is a L^N-viscosity solution.

Here is our main result. As usual, we set $a^\pm = \max\{\pm a, 0\}$.

Theorem 1 *Suppose* (S) *holds with* $\underline{h} = h((u-v)^+)$, $\overline{h} = h((v-u)^+)$, *for some continuous function* h, *with* $h(0) = 0$. *Let* $c \in L^N(\Omega)$. *Then*

(i) if $c(x) \leq -c_0$ *a.e. in* Ω, *for some* $c_0 > 0$, *then for any* $f \in L^N(\Omega)$ *there exists a solution* $u \in C(\overline{\Omega})$ *of*

$$\left\{ \begin{array}{rcll} F(D^2u, Du, u, x) + c(x)u & = & f(x) & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{5}$$

(ii) there exist positive constants δ_0, c_0, *depending on* $\lambda, \Lambda, p, N, \Omega, \|b\|_{L^p(\Omega)}$, *such that if*

$$\|\mu f\|_{L^N(\Omega)} \leq \delta_0 \qquad \text{and} \qquad \|c^+\|_{L^\infty(\Omega)} < c_0, \tag{6}$$

then there exists a solution $u \in C(\overline{\Omega})$ *of (5).*

(iii) If problem (5) with $c^+ \equiv 0$ *or with* $\mu = 0$ *and* $\|c^+\|_{L^\infty(\Omega)} < c_0$ *has a strong solution then this solution is the unique viscosity solution of (5).*

(iv) The strong solutions of (5) are not unique if $\mu > 0$ *and* $c(x) \equiv c > 0$ *arbitrarily small, even for* $b = d = f \equiv 0$ *(see Theorem 6 in Section 4).*

Remark 1. The choice of $\underline{h}, \overline{h}$ in Theorem 1 means F is nonincreasing in u. Of course the term $c(x)u$ in (5) can be incorporated into F, with corresponding modifications of the hypotheses. We have avoided this, for clarity.
Remark 2. Gradient dependence with precise growth between 1 and 2 can be dealt with through the (trivial) inequality $x^m \le (2-m)x + (m-1)x^2$, valid for all $x \ge 0,\ m \in [1,2]$.
Remark 3. We recall that, even for the simplest equations satisfying our hypotheses, for $\mu = 0$ and $c(x) \equiv c_0$ large, or for μf large and $c \equiv 0$, there may not exist solutions of (5) (see the end of the paper).

Existence of strong solutions of quasilinear uniformly elliptic equations was studied in the classical works of Ladizhenskaya-Uraltseva and Krylov-Safonov, [LU1], [LU2], [KSa], [K]. For viscosity solutions and fully nonlinear equations, in the linear growth case ($\mu = 0$), results for solvability of (5) can be found in [CCKS] and in [CKLS], for equations with bounded coefficients. An existence result for equations where F is convex in D^2u, $c \equiv 0$, and $b \in L^{2N}$ close to the boundary is stated in the thesis work [F1]. An existence result for $\mu > 0$, b bounded and $c = 0$ is contained in the recent paper [KS1], where the authors use results from [F1]. In all these works it is supposed that $c^+ \equiv 0$. The results in (i) and (ii) above unify and extend the results in the fully nonlinear setting, to cover the cases of $b \in L^p, p > N, c, d \in L^N$, and changing-sign zero-order term. Actually, in the fully nonlinear case $c^+ \not\equiv 0$ has only been considered in the recent work [QS2], where convex positively homogeneous operators (like (2) with $\mu \equiv 0$, Bellman equations) were studied and the constant c_0 from (6) is related to eigenvalues of F.

It is important to note that solvability of elliptic equations with natural growth has been studied very extensively for divergence-form equations, where weak solutions can be searched for in Sobolev spaces. A typical example in this case is the equation

$$\mathrm{div}(A(x)Du) + \mu|Du|^2 + b(x).Du + c_0 u = f(x),$$

for $b \in L^N(\Omega), f \in L^q(\Omega)$. Note that in this situation $q = N/2$ is the dividing number for the solutions to be bounded and continuous. We shall quote here [BMP1], [BMP2], [AGP] for the case $c_0 < 0$, and [MPS], [FM], [GMP] for the case $c_0 = 0$ (see also the references in these works). Corresponding uniqueness results in the natural Sobolev spaces were proved in [BM], [BBGK], [BP]. Some results on uniqueness of viscosity solutions of quasilinear equations with Lipschitz ingredients are obtained in [BR].

Our Theorem 1 can be seen as a counterpart of these results for fully nonlinear equations in non-divergence form, where the natural weak notion of a solution is the viscosity one. Of course the methods in the two frameworks are completely different.

Further, the existence hypothesis (6) appears to be new, even in the divergence case. All previous works seem to have concerned either $c^+ \equiv 0$ or $\mu = 0$. Unifying these results leads to qualitatively new phenomena, as the fact that uniqueness breaks down attests.

We remark that in the framework of non-divergence form operators with measurable ingredients uniqueness of viscosity solutions does not hold in general, even for linear equations (see [N] and [Sa] for counter-examples). So, to have a uniqueness result like in Theorem 1 (iii) some additional hypothesis is needed. Generally, the existence of a strong solution is such a hypothesis, verified for example by operators which are convex or concave in D^2u and $F(M, 0, 0, x)$ is continuous.

The existence result in Theorem 1 (i) is based on the method of sub- and supersolutions, together with approximation techniques (see Theorem 4 in Section 4). For the results in (ii) we use in addition some recent results on existence and properties of eigenvalues of convex fully nonlinear operators, obtained in [QS2]. The proof of the result in (iv) is based on results on solvability of superlinear equations, which admit a priori bounds in the uniform norm for their positive solutions. The uniqueness result in (iii) is a consequence of the ABP inequality, see below.

An important point in the existence proofs is a result on global Hölder continuity of solutions of equations of type (1), satisfying (S). This result, which is of clear independent interest, permits to approximate the equations we consider by equations with regular ingredients.

Theorem 2 *Suppose* (S) *holds for* $N = q = v = 0$ *and* $u \in C(\Omega)$ *is a solution of (1). Then*

1. (interior estimate) *there exists* $\alpha \in (0,1)$ *depending only on* N, λ, Λ, p, $\|b\|_{L^p(\Omega)}$, *such that* $u \in C^\alpha_{\text{loc}}(\Omega)$, *and for any subdomain* $\Omega' \subset\subset \Omega$ *we have*

$$\|u\|_{C^\alpha(\Omega')} \le K,$$

where K *depends on* $N, \lambda, \Lambda, \mu, p, \|b\|_{L^p(\Omega)}, \|c\|_{L^N(\Omega)}, \|f\|_{L^N(\Omega)}$, $\operatorname{dist}(\Omega', \partial\Omega)$, *and* $\sup_{\Omega'} u$.

2. (global estimate) *if, in addition,* $u \in C(\overline{\Omega})$, $u|_{\partial\Omega} \in C^\beta(\partial\Omega)$ *for some* $\beta \in (0,1)$, *and* Ω *satisfies an uniform exterior cone condition (with size* L*), then there exists some* $\alpha \in (0,1)$ *depending only on* $N, \lambda, \Lambda, \beta, L, p$, $\|b\|_{L^p(\Omega)}$, *such that* $u \in C^\alpha(\overline{\Omega})$, *and*

$$\|u\|_{C^\alpha(\Omega)} \le K,$$

where K *depends on* $N, \lambda, \Lambda, \mu, p, \|b\|_{L^p(\Omega)}, \|c\|_{L^N(\Omega)}, \|f\|_{L^N(\Omega)}, L$, $\sup_{\Omega} u$, *and* $\operatorname{diam}(\Omega)$.

Interior Hölder estimates for the viscosity solutions of Pucci equations ($\mu = b \equiv 0$ in (2)) were obtained by Caffarelli in his founding work [C], see also [CC]. These estimates were extended to operators with bounded coefficients ($\mu, b, c, f \in L^\infty(\Omega)$) in [W]. The proofs in [W] are somewhat difficult to read, an alternative approach can be found in [F1], see Theorem 1.3 there. We do not know of any works on Hölder estimates for viscosity solutions of equations with unbounded coefficients.

In the above works the Hölder estimate is obtained as a consequence of a Harnack inequality for the corresponding equation. Here we will make use of another idea, whose essence is that one does not need to prove a Harnack inequality if only Hölder estimates are aimed at ; actually, it is enough to have some comparison between the measures of level sets of the solution, which can be achieved by use of simple barriers and the so-called ABP inequality (this exhibits, as in so many other situations, the strength of this splendid result), which we discuss in the next section. The idea of using level sets to prove Harnack inequalities and Hölder regularity is essentially due to Krylov and Safonov. Another adaptation of their methods to viscosity solutions was used in the proof of the interior Harnack inequality for Pucci equations in [CC].

In the next section we give some preliminary results and recall definitions and results which we need, in particular the ABP inequality. The proof of Theorem 2 is given in Section 3, while Section 4 is devoted to the proof of Theorem 1.

2 Preliminaries

Let us start by recalling the definition of a viscosity solution of (1).

Definition 2.1 *We say that the function $u \in C(\Omega)$ is a L^p-viscosity subsolution (supersolution) of (1), provided for any $\varepsilon > 0$, any open subset $\mathcal{O} \subset \Omega$, and any $\varphi \in W^{2,p}(\mathcal{O})$ – we call φ a test function – such that*

$$\begin{array}{rcll} F(D^2\varphi(x), D\varphi(x), u(x), x) & \leq & f(x) - \varepsilon & \\ (F(D^2\varphi(x), D\varphi(x), u(x), x) & \geq & f(x) + \varepsilon) & \quad a.e.\ in\ \mathcal{O}, \end{array}$$

the function $u - \varphi$ cannot achieve a local maximum (minimum) in $\mathcal{O}$. In this case we say that the function w satisfies $F(D^2u, Du, u, x) \geq (\leq) f$ in the L^p-viscosity sense in Ω.

If both F and f are continuous in x and the above definition holds for all $\varphi \in C^2(\mathcal{O})$, then we speak of C-viscosity subsolution (supersolution).

We say that u is a solution of (1) if u is at the same time a subsolution and a supersolution of (1).

We recall that strong and C-viscosity solutions are L^p-viscosity solutions, see [CCKS], and that whenever a function in $W^{2,N}_{\rm loc}(\Omega)$ satisfies an inequality $F(D^2u, Du, u, x) \geq (\leq) f$ a.e. then it is a viscosity solution.

We shall make essential use of the Generalized Maximum Principle for elliptic equations, commonly known as the Alexandrov-Bakelman-Pucci (ABP) inequality. It was obtained by Bakelman and Alexandrov, see [A1], [A2], [Ba], and independently by Pucci [P]. It states that for any measurable matrix $A(x)$, $\lambda I \leq A(x) \leq \Lambda I$, $x \in \Omega$, any $b \in L^N(\Omega)^N, f \in L^N(\Omega)$, and any $u \in W^{2,N}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ such that

$$\mathrm{tr}(A(x)D^2u) + b(x).Du \geq f(x)$$

we have

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C\|f\|_{L^N(\Gamma)}$$

where C depends on $N, \lambda, \Lambda, \|b\|_{L^N(\Omega)}$, $\mathrm{diam}(\Omega)$, and Γ is the upper contact set of u.

A breakthrough in the theory of viscosity solutions of uniformly elliptic equations was the extension of this inequality to viscosity solutions of $\mathcal{M}(D^2u) \geq f$, obtained by Caffarelli in [C]. The result was subsequently extended to equations with bounded measurable coefficients in [W], [CCKS], and to unbounded coefficients in [F2]. A simple self-contained proof of the result from [F2] can be found in [KS2]. We give the statement next.

Theorem 3 ([F2], [KS2]) *Suppose $u \in C(\overline{\Omega})$ is a L^q-viscosity solution of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + b(x)|Du| \geq f(x),$$

where $b \in L^p(\Omega)$ for some $p > N$, and $f \in L^q(\Omega)$, for some $q \geq N$. Then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + (\mathrm{diam}(\Omega))^{2-\frac{N}{q}}\, C_1\|f\|_{L^q(\Omega^+)}, \tag{7}$$

where $\Omega^+ = \{x \in \Omega \,:\, u(x) > 0\}$ and C_1 is a constant which depends on $N, \lambda, \Lambda, p, q, \|b\|_{L^p(\Omega)}$, diam$(\Omega)$, and C_1 remains bounded when all these quantities are bounded.

Remark 1. Note that the norm of f in (7) has to be taken over the whole domain Ω. Then, reapplying the result in the set Ω^+ one sees that it is enough to consider the norm over Ω^+. We are going to use this in the proof of the Hölder regularity.

Remark 2. Theorem 3 is a scaled version (with respect to $\mathrm{diam}(\Omega)$) of either Theorem 1.2 of [F2] or Proposition 2.8 in [KS2]. Actually, the result is based on an upgrade to unbounded coefficients of the basic Lemma 3.1 in [CCKS], where the correct scaling is given.

We recall some easy properties of Pucci operators (see for instance [CC]).

Lemma 2.1 *Let* $M, N \in \mathcal{S}_N$, $\phi(x) \in C(\overline{\Omega})$ *be such that* $0 < a \le \phi(x) \le A$. *Then*

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = -\mathcal{M}^+_{\lambda,\Lambda}(-M),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{\{\nu_i>0\}} \nu_i + \Lambda \sum_{\{\nu_i<0\}} \nu_i, \qquad \text{where } \{\nu_1,\ldots,\nu_N\} = \mathrm{spec}(M),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^-_{\lambda,\Lambda}(N) \le \mathcal{M}^-_{\lambda,\Lambda}(M+N) \le \mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N),$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N) \le \mathcal{M}^+_{\lambda,\Lambda}(M+N) \le \mathcal{M}^+_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N),$$

$$\mathcal{M}^-_{\lambda a,\Lambda A}(M) \le \mathcal{M}^-_{\lambda,\Lambda}(\phi M) \le \mathcal{M}^-_{\lambda A,\Lambda a}(M),$$

We will also use the following simple fact.

Lemma 2.2 *Suppose* $u \in C^2(B)$ *is a radial function, say* $u(x) = g(|x|)$, *defined on a ball* $B \subset \mathbb{R}^N$. *Then* $g''(|x|)$ *is an eigenvalue of the matrix* $D^2u(x)$, *and* $|x|^{-1}g'(|x|)$ *is an eigenvalue of multiplicity* $N-1$.

The following lemma will help us deal with the quadratic dependence in the gradient.

Lemma 2.3 *Let* $u \in W^{2,N}_{\mathrm{loc}}(\Omega)$. *For any* $m > 0$ *set*

$$v = \frac{e^{mu}-1}{m}, \qquad w = \frac{1-e^{-mu}}{m}.$$

Then a.e. in Ω $Dv = (1+mv)Du$, $Dw = (1-mw)Du$,

$$m\lambda|Du|^2 + \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) \;\le\; \frac{\mathcal{M}^\pm_{\lambda,\Lambda}(D^2v)}{1+mv} \;\le\; m\Lambda|Du|^2 + \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u),$$

$$-m\Lambda|Du|^2 + \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u) \;\le\; \frac{\mathcal{M}^\pm_{\lambda,\Lambda}(D^2w)}{1-mw} \;\le\; -m\lambda|Du|^2 + \mathcal{M}^\pm_{\lambda,\Lambda}(D^2u).$$

and, clearly, $u = 0$ *(or* $u > 0$*) is equivalent to* $v = 0$ *(or* $v > 0$*).*

The same inequalities hold in the L^N*-viscosity sense, that is, if, for example,* $u \in C(\Omega)$ *is a viscosity solution of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu|Du|^2 + b(x)|Du| \ge f(x) \tag{8}$$

then v *is a viscosity solution of*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| - (\mu/\lambda)f(x)v \ge f(x), \quad \textit{etc.} \tag{9}$$

Proof. This is a matter of an easy computation and use of Lemma 2.1 and Definition 2.1. Suppose first that $u \in W^{2,N}_{\rm loc}(\Omega)$. Then a.e. in Ω

$$Dv = e^{mu}Du, \qquad D^2v = e^{mu}D^2v + mDu \otimes Du,$$

$$Dw = e^{-mu}Du, \qquad D^2w = e^{-mu}D^2w - mDu \otimes Du,$$

and the inequalities follow by Lemma 2.1, since

$$\mathrm{spec}\,(Du \otimes Du) = \{0, \dots, 0, |Du|^2\}.$$

If u is only continuous and we suppose v does not satisfy (9), then there exists a function $\psi \in W^{2,p}_{\rm loc}(\Omega)$ and $\varepsilon > 0$ such that $\psi - v$ attains a minimum in some open set $\mathcal{O}$, while for $m = \mu/\lambda$

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\psi) + b(x)|D\psi| \leq f(x)e^{mu} - \varepsilon \qquad \text{in } \ \mathcal{O}.$$

By setting $\phi = (1/m)\log(1+m\psi)$ we get a contradiction with the fact that u satisfies (8) in the sense of Definition 2.1, since then $\phi - u$ attains a minimum in $\mathcal{O}$. □

3 Proof of Theorem 2

For any measurable subset A in $\mathbb{R}^N$ we denote the measure of A by $|A|$ or meas(A). As usual, constants denoted by C may change from line to line, and depend only on the appropriate quantities.

For clarity, we are going to start by giving the proof of the interior estimate in the case of the model equation (2), with $\mu = 0, c = f \equiv 0$. The following statements appear, in somewhat different form and for strong solutions of quasilinear equations with bounded ingredients, in the works of Krylov and Safonov (see for instance [K]).

The next proposition states that, for any given subdomain $\Omega' \subset\subset \Omega$, if a level set in Ω' of a positive supersolution has sufficiently small measure with respect to $|\Omega'|$, then this supersolution is uniformly positive in Ω'.

Proposition 3.1 *There exist numbers $\delta, \kappa, \rho_0 > 0$ depending only on N, λ, Λ, $\|b\|_{L^p}$, $p > N$, such that if for some $\rho \in (0, \rho_0)$ the ball $B_{2\rho} \subset \Omega$, and $b \in L^p(B_{2\rho})$, $b \geq 0$, $u \in C(B_{2\rho})$ satisfy*

$$\begin{aligned} G[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| \ &\leq \ 0 \qquad \text{in } \ B_{2\rho} \\ u \ &\geq \ 0 \qquad \text{in } \ B_{2\rho}, \end{aligned}$$

then for any $a > 0$

$$\mathrm{meas}\,\{x \in B_\rho \ : \ u(x) < a\} \ \leq \delta\,|B_\rho| \qquad \text{implies} \qquad u \geq \kappa a \quad \text{in } \ B_\rho.$$

Proof. Without restricting the generality we can suppose $a = 1$ (replace u by u/a). Set

$$v = 1 - \frac{|x|^2}{\rho^2}.$$

Then, by Lemmas 2.1 and 2.2, for any $x \in B_\rho$

$$\begin{aligned}\mathcal{M}^+_{\lambda,\Lambda}(D^2(v-u)) + b(x)|D(v-u)| &\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2v) - b(x)|Dv| - G[u] \\ &\geq -\frac{2}{\rho^2}\left(N\Lambda + b(x)|x|\right) \\ &\geq -\frac{C}{\rho^2}\left(1 + \rho b(x)\right),\end{aligned}$$

provided $u \in W^{2,N}(B_{2\rho})$. Extending this inequality to u only continuous is then easy (and very standard, since $v \in C^2$), by using Definition 2.1 and test functions.

Since $v - u \leq 0$ on ∂B_ρ, by applying Theorem 3 (with $q = N$) to this inequality we obtain

$$\sup_{B_\rho}(v-u) \leq C\rho^{-1}\|1 + \rho b(x)\|_{L^N(B_\rho \cap \{v-u>0\})}.$$

Note that $\{v - u > 0\} \subset \{u < 1\}$, so $\mathrm{meas}(B_\rho \cap \{v - u > 0\}) \leq \delta\, C\rho^N$, by hypothesis. Then the triangle and Hölder inequalities imply

$$\sup_{B_\rho}(v-u) \leq C\delta + \rho^{\varepsilon_1}\|b\|_{L^p(B_\rho)},$$

where $\varepsilon_1 = (p - N)/Np$. By choosing δ and ρ_0 sufficiently small we get

$$\frac{3}{4} - \inf_{B_{\frac{\rho}{2}}} u = \inf_{B_{\frac{\rho}{2}}} v - \inf_{B_{\frac{\rho}{2}}} u \leq \sup_{B_{\frac{\rho}{2}}}(v-u) \leq \sup_{B_\rho}(v-u) \leq \frac{1}{4}$$

for $\rho \leq \rho_0$, so

$$u \geq \frac{1}{2} \qquad \text{in } B_{\frac{\rho}{2}}. \tag{10}$$

Now set, for $s > 0$ and $x \in B_{2\rho} \setminus B_{\frac{\rho}{2}}$,

$$w = \frac{1}{4}\,\frac{|x|^{-s} - (2\rho)^{-s}}{(\rho/2)^{-s} - (2\rho)^{-s}}.$$

It is easy to compute, with the help of Lemma 2.2, that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(|x|^{-s})) - b(x)|D(|x|^{-s})| = s(\lambda(s+1) - \Lambda(N-1) - b(x)|x|)|x|^{-s-2},$$

and hence, fixing s such that $\lambda(s+1)=\Lambda(N-1)$,

$$\begin{array}{rcl}\mathcal{M}^+_{\lambda,\Lambda}(D^2(w-u))+b(x)|D(w-u)| & \geq & \mathcal{M}^-_{\lambda,\Lambda}(D^2w)-b(x)|Dw|-G[u]\\ & \geq & -C\rho^s|x|^{-s-2}b(x)|x|\\ & \geq & -C\rho^{-1}b(x)\end{array}$$

in the set $B_{2\rho}\setminus B_{\frac{\rho}{2}}$. Since $w-u\leq 0$ on $\partial(B_{2\rho}\setminus B_{\frac{\rho}{2}})$, Theorem 3 yields

$$\sup_{B_\rho\setminus B_{\frac{\rho}{2}}}(w-u)\leq \sup_{B_{2\rho}\setminus B_{\frac{\rho}{2}}}(w-u)\leq C\|b\|_{L^N(B_{2\rho})}\leq C\rho^{\varepsilon_1}\|b\|_{L^p(B_\rho)}$$

so, by taking ρ_0 sufficiently small, we have, for $\rho\leq\rho_0$,

$$u(x)\geq \inf_{B_\rho\setminus B_{\frac{\rho}{2}}} w-C\rho^{\varepsilon_1}\geq 2^{-s-3}-C\rho^{\varepsilon_1}\geq 2^{-s-4}\,,\qquad \text{for } x\in B_\rho\setminus B_{\frac{\rho}{2}}$$

which finishes the proof of Proposition 3.1. □

We shall use the following well-known measure theoretic result (Krylov's "propagating ink spots" lemma).

Lemma 3.1 *Let G be a ball and K be some measurable subset of G, such that $|K|\leq\eta|G|$, for some $\eta\in(0,1)$. Let $\mathcal{F}$ be the set of all balls B contained in G, and such that $|B\cap K|\geq\eta|B|$. Then there exists $\zeta>0$ depending only on N,η, such that*

$$\text{meas}(\cup_{B\in\mathcal{F}}B)\geq(1+\zeta)\text{meas}(K).$$

Proof. This is essentially inequality (9.20) from [GT], setting f to be the indicator function of K in the reasoning there. □

Wit the help of this lemma we can prove the result from Proposition (3.1) for any $\delta\in(0,1)$.

Proposition 3.2 *If for some $\rho\in(0,\rho_0)$ (ρ_0 is the number from Proposition 3.1) the ball $B_{2\rho}\subset\Omega$, and $b\in L^p(B_{2\rho})$, $p>N$, $b\geq 0$, $u\in C(B_{2\rho})$ satisfy*

$$\begin{array}{rcll}G[u]:=\mathcal{M}^-_{\lambda,\Lambda}(D^2u)-b(x)|Du| & \leq & 0 & \text{in } B_{2\rho}\\ u & \geq & 0 & \text{in } B_{2\rho},\end{array}$$

then for any $\nu,a>0$ there exists $\overline{\kappa}>0$ depending on $\nu,N,\lambda,\Lambda,\|b\|_{L^p}$, $p>N$, such that

$$\text{meas}\,\{x\in B_\rho\,:\,u(x)\geq a\}\geq\nu\,|B_\rho|\qquad\text{implies}\qquad u\geq\overline{\kappa}a\quad\text{in } B_\rho.$$

Remark. Proposition 3.2 can be viewed as a "very weak" Hölder inequality. Note that the usual weak Hölder inequality $\inf_{B_\rho} u \geq C|B_\rho|^{-1/q}\|u\|_{L^q(B_\rho)}$ contains a much stronger statement.

Proof of Proposition 3.2. Set $K_a = \{x \in B_\rho : u(x) \geq a\}$. We know that

$$|K_a| \geq \nu\, |B_\rho|.$$

If $|K_a| \geq (1-\delta)|B_\rho|$, where δ is the number from Proposition 3.1 then we conclude, by that Proposition.

If, on the other hand, $|K_a| < (1-\delta)|B_\rho|$, we apply Lemma 3.1, with $\eta = 1-\delta$. By Proposition 3.1 we have $u \geq \kappa a$ in each ball in $\mathcal{F}$ (defined in Lemma 3.1), for some $\kappa > 0$, depending on the appropriate quantities. Hence, by Lemma 3.1,

$$|K_{\kappa a}| \geq (1+\zeta)|K_a| \geq \nu(1+\zeta)\,|B_\rho|.$$

We repeat the same reasoning and get either Proposition 3.2 or

$$|K_{\kappa^2 a}| \geq \nu(1+\zeta)^2\,|B_\rho|.$$

This process stops after at most n iterations, where n is a number such that $\nu(1+\zeta)^n \geq 1$. □

Proof of the interior C^α-estimate for (2), $\mu = c = f = 0$. We recall we have a solution $u \in C(\Omega)$ of

$$\mathcal{M}(D^2u) + b(x)|Du| = 0.$$

Then for any ρ such that $B_{2\rho} \subset \Omega$ the functions

$$u_1 := u - \inf_{B_{2\rho}} u, \qquad u_2 := \sup_{B_{2\rho}} u - u,$$

satisfy the hypotheses of Proposition 3.2. In addition,

$$\omega(2\rho) := \underset{B_{2\rho}}{osc}\, u = u_1 + u_2,$$

so at each point of $B_{2\rho}$ one of u_1, u_2 is larger than $\frac{1}{2}\omega(2\rho)$. This implies

$$\text{meas}\left\{x \in B_\rho \,:\, u_i(x) \geq \frac{1}{2}\omega(2\rho)\right\} \geq \frac{1}{2}\,\text{meas}(B_\rho),$$

for one i, say for $i = 1$. Then we can apply Proposition 3.2 to u_1 and infer

$$u - \inf_{B_{2\rho}} u = u_1 \geq \kappa\omega(2\rho) \qquad \text{in } \ B_\rho,$$

which implies

$$\inf_{B_\rho} u \geq \kappa \sup_{B_{2\rho}} u + (1-\kappa) \inf_{B_{2\rho}} u,$$

and hence

$$\omega(\rho) \leq (1-\kappa)\omega(2\rho),$$

for all $\rho \in (0, \rho_0)$.

The proof is now standardly finished, with the help of Lemma 8.23 in [GT], which implies

$$\omega(\rho) \leq C\rho^\alpha \rho_0^{-\alpha} \omega(\rho_0) = C(\sup_{B_{2\rho}} u)\rho^\alpha,$$

for some α depending on $N, \lambda, \Lambda, \|b\|_{L^p(B_{2\rho})}, p > N$. □.

Next, we give the changes in the proofs of Propositions 3.1 and 3.2, which we have to make in order to deal with a nontrivial right-hand side.

Proposition 3.3 *There exist numbers $\delta, \overline{\kappa}, \rho_0, C_0 > 0$ depending only on $N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, such that if for some $\rho \in (0, \rho_0)$ the ball $B_{2\rho} \subset \Omega$ and $f \in L^N(\Omega)$, $b \in L^p(B_{2\rho})$, $b, f \geq 0$, $u \in C(B_{2\rho})$ satisfy*

$$\begin{array}{rcll} G[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| & \leq & f(x) & \text{in } B_{2\rho} \\ u & \geq & 0 & \text{in } B_{2\rho}, \end{array}$$

then, for any $a > 0$, $\operatorname{meas}\{x \in B_\rho \,:\, u(x) < a\} \leq \delta \operatorname{meas}(B_\rho)$ *implies*

$$\inf_{B_\rho} u \geq \overline{\kappa} a - C_0 \rho \|f\|_{L^N(B_{2\rho})}. \tag{11}$$

Proof. The proof goes the same way as the proof of Proposition 3.1, by adding $-f$ to the right-hand sides of the inequalities to which we apply Theorem 3. We can suppose $\operatorname{meas}\{x \in B_\rho \,:\, u(x) < 1\} \leq \delta \operatorname{meas}(B_\rho)$, with f replaced by f/a. Then inequality (10) reads

$$u \geq \frac{1}{2} - C_1 a^{-1} \rho \|f\|_{L^N(B_{2\rho})} \qquad \text{in } B_{\frac{\rho}{2}}\,, \tag{12}$$

where C_1 is the constant from the ABP inequality. We distinguish two cases. First, if $a < 4C_1\rho\|f\|_{L^N(B_\rho)}$ then the conclusion of Proposition 3.3 trivially holds, with $\overline{\kappa} = 1, C_0 = 4C_1$ (so that the right-hand side of (11) be negative). If not, we have $u \geq \frac{1}{4}$ in $B_{\frac{\rho}{2}}$, and we finish the proof as in Proposition 3.1. □

Proposition 3.4 *If for some $\rho \in (0, \rho_0)$ (ρ_0 is the number from Proposition 3.3) the ball $B_{2\rho} \subset \Omega$ and $f \in L^N(\Omega)$, $b \in L^p(B_{2\rho})$, $b, f \geq 0$, $u \in C(B_{2\rho})$ satisfy*

$$\begin{array}{rcll} G[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| & \leq & f(x) & \text{in } B_{2\rho} \\ u & \geq & 0 & \text{in } B_{2\rho}, \end{array}$$

then for any $\nu, a > 0$ there exist $\kappa, C > 0$ depending on $N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, such that meas$\{x \in B_\rho \,:\, u(x) \geq a\} \geq \nu\,\mathrm{meas}(B_\rho)$ implies

$$\inf_{B_\rho} u \geq \kappa a - C\rho \|f\|_{L^N(B_{2\rho})}. \tag{13}$$

Proof. We have to modify the proof of Proposition 3.2 as in the proof of the previous proposition. Suppose meas$(K_a) < (1-\delta)$meas(B_ρ). Then if $a < (2\overline{\kappa})C_0\rho\|f\|_{L^N(B_{2\rho})}$ (the constants $\overline{\kappa}, C_0$ are defined in the Proposition 3.3) then inequality (13) is trivially true, by choosing κ, C such that its right-hand side is negative. If not, then $u \geq \frac{\overline{\kappa}}{2} a$ in each ball in $\mathcal{F}$ (defined in Lemma 3.1), by Proposition 3.3. Then we repeat the same reasoning as in the proof of Proposition 3.2, distinguishing at each step the cases when a is smaller or larger than $(2\overline{\kappa})^l C_0\rho\|f\|_{L^N(B_{2\rho})}$, $l = 1, \ldots, n$. □

Proof of the interior C^α-estimate for (2), $\mu = c = 0, f \neq 0$. We reason in exactly the same way as in the case $f = 0$, only at the end we get

$$\omega(\rho) \leq (1-\kappa)\omega(2\rho) + \left(C\|f\|_{L^N(B_{2\rho})}\right)\rho,$$

to which Lemma 8.23 of [GT] applies as well : for any $\gamma \in (0,1)$ there exists α depending on $\gamma, N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, such that

$$\omega(\rho) \leq C(\sup_{B_{2\rho}} u)\rho^\alpha + C\|f\|_{L^N(B_{2\rho})}\rho^\gamma.$$

Remark. Note that in order to carry out all the above arguments it is actually sufficient to know that

$$\begin{array}{rcl} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + b(x)|Du| & \geq & -f(x), \\ \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| & \leq & f(x), \end{array}$$

The next proposition deals with the extension of the result to the boundary. It uses the well-known idea of extending the function u as a constant outside the domain (like for example in Theorem 8.26 or 9.27 in [GT]).

Proposition 3.5 *There exist constants κ, C, ρ_0 depending on $N, \lambda, \Lambda, \|b\|_{L^p}$, $p > N$, such that if, for some ball $B \subset \mathbb{R}^N$, and $f \in L^N(B)$, $b \in L^p(B)$, $b, f \geq 0$, $u \in C(\overline{\Omega})$ we have*

$$\begin{array}{rcll} G[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b(x)|Du| & \leq & f(x) & \text{in } \Omega \cap B, \\ u & \geq & 0 & \text{in } \Omega \cap B, \\ u & \geq & 2m & \text{on } \partial\Omega \cap B, \end{array}$$

for some $m > 0$, then for any ball $B_{2\rho} \subset \Omega \cup B$, $\rho \leq \rho_0$, and any $\nu, a > 0$

$$\text{meas}\,\{x \in B_\rho \,:\, \overline{u}(x) \geq a\} \geq \nu\,\text{meas}(B_\rho) \tag{14}$$

implies

$$\inf_{B_\rho} \overline{u} \geq \kappa a - C\rho\|f\|_{L^N(\Omega)},$$

where $\overline{u} \in C(B)$ is defined by

$$\overline{u}(x) = \left\{ \begin{array}{cl} m & \text{if } \; x \in \Omega \setminus B \\ \min\{u(x), m\} & \text{if } \; x \in \overline{\Omega} \cap B\,. \end{array} \right.$$

Proof. The function $\overline{u}$ satisfies the hypotheses of Proposition 3.4 in the ball B – since the minimum of two viscosity supersolutions is a viscosity supersolution, and $G[m] \equiv 0 \leq f(x)$. □

Proof of the boundary C^α-estimate for (2), $\mu = c = 0, f \neq 0$. Let $x_0 \in \partial\Omega$. Then by the uniform cone condition, for some $\overline{\rho}$ and some $\xi > 0$ (depending on L), the balls B with center x_0 and radii 2ρ, $\rho \leq \overline{\rho}$, satisfy $\text{meas}(B \setminus \Omega) \geq \xi \text{meas}(B)$.

We want to show that for each ball B_ρ with center x_0 and sufficiently small radius ρ we have

$$\operatorname*{osc}_{\Omega\cap B_{2\rho}} u \leq C\rho^\alpha. \tag{15}$$

First, if

$$\omega(2\rho) := \operatorname*{osc}_{\Omega\cap B_{2\rho}} u \leq 2 \operatorname*{osc}_{\partial\Omega\cap B_{2\rho}} u, \tag{16}$$

inequality (15) follows with $\alpha = \beta$, since $u|_{\partial\Omega}$ is in $C^\beta(\partial\Omega)$. If (16) doesn't hold, then either

$$\inf_{\partial\Omega\cap B_{2\rho}} u - \inf_{\Omega\cap B_{2\rho}} u \geq \frac{1}{4}\omega(2\rho) \qquad \text{or} \qquad \sup_{\Omega\cap B_{2\rho}} u - \sup_{\partial\Omega\cap B_{2\rho}} u \geq \frac{1}{4}\omega(2\rho).$$

Let's say the first of these holds. Then the function

$$u_1 = u - \inf_{\Omega\cap B_{2\rho}} u$$

satisfies the conditions of Proposition 3.5, with $a = m = \omega(2\rho)/8$ – note (14) is automatically satisfied thanks to the exterior cone condition. So

$$u(x) - \inf_{\Omega\cap B_{2\rho}} u \geq \kappa\omega(2\rho) - C\|f\|_{L^N(B_{2\rho})}\rho \qquad \text{for each } \ x \in B_\rho.$$

Hence again

$$\omega(\rho) \leq (1-\kappa)\omega(2\rho) + C\|f\|_{L^N(B_{2\rho})}\, \rho,$$

and

$$\omega(\rho) \leq C(\sup_{B_{2\rho}} u)\rho^\alpha + C\|f\|_{L^p(B_{2\rho})}\rho^\gamma.$$

Proof of the global C^α-estimate for (2), $\mu = c = 0, f \neq 0$. Putting together the interior and the boundary estimates we already proved is standard, see for example the proof of Theorem 8.29 in [GT] or the proof of Proposition 4.13 in [CC].

Proof of Theorem 2. To get the full strength of Theorem 2 we first use (S), transferring the terms $d(x)h(u,0)$ to the right-hand side of the inequalities, which permits to suppose $d \equiv 0$.

Let us prove the interior estimate. By (S) and Lemma 2.1 both u and $-u$ are solutions of

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \mu|Du|^2 - b(x)|Du| \leq |f(x)|,$$

hence, by Lemma 2.3, the functions

$$w_1 = \frac{1-e^{-m(u-\inf_{B_{2\rho}} u)}}{m} \qquad w_2 = \frac{1-e^{-m(\sup_{B_{2\rho}} u - u)}}{m}$$

(with $m = \mu/\lambda$) satisfy

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w_i) - b(x)|Dw_i| \leq |f|(1-mw_i) =: \overline{f}. \tag{17}$$

Since at each point $x \in B_{2\rho}$

$$w_j(x) \geq \frac{1-e^{-m(\omega(2\rho)/2)}}{m}$$

for one j, say $j=1$, reasoning as before we get

$$w_1 \geq \kappa\frac{1-e^{-m(\omega(2\rho)/2)}}{m} - \left(C\|\overline{f}\|_{L^N}\right)\, \rho,$$

for $\rho \leq \rho_0$, the number from Proposition 3.4. Note that for each t_0 there exists $\xi = \xi(t_0, m)$ such that

$$t \geq \frac{1-e^{-mt}}{m} \geq \xi t \qquad \text{for } \ t \in [0, t_0]$$

We apply this with $t_0 = \omega(2\rho_0)/2$ and get

$$u_1 \geq \kappa\xi\omega(2\rho) - C\|\overline{f}\|_{L^N}\,\rho. \qquad \text{in } \; B_\rho,$$

so again

$$\omega(\rho) \leq C(\sup_{B_{2\rho}} u)\rho^\alpha + C\|f\|_{L^N(B_{2\rho})}\rho^\gamma. \tag{18}$$

for $\rho \in (0, \rho_0)$. Note that here α depends on μ and $\sup u$, because of the choice of ξ, but this dependence can easily be transferred to C, by choosing another α, if necessary. Indeed, ρ_0 is independent of μ and $\sup u$ (ρ_0 comes from the applications of the ABP inequality to (17), as in the proofs of Propositions 3.1 and 3.2). Then we can choose $\rho_1 \leq \rho_0$ so small that, by (18), $\operatorname{osc}_{B_{2\rho_1}} u$ is so small that if we repeat the above argument with ρ_0 replaced by ρ_1, we get $\xi \geq 1/2$ – since obviously $\xi \to 1$ as $t_0 \to 0$. This implies (18) holds for $\rho \leq \rho_1$, with a different α_1, which is independent of μ and $\sup u$, the dependence of these now being in the constants C and ρ_1. But since $\rho^\alpha \leq C\rho^{\alpha_1}$ for $\rho \in [\rho_1, \rho_0]$, $C = C(\rho_1, \rho_0)$, we see that we have (18) for α replaced by α_1, and all $\rho \leq \rho_0$.

The boundary estimate is proved similarly. □

4 Proof of Theorem 1

The first lemma concerns existence of sub- and supersolutions in the continuous setting.

Lemma 4.1 *Suppose $\partial\Omega$ is of class C^2. For any positive constants μ, b, c, k, there exist C-viscosity solutions u_1, u_2, such that $u_1 \leq 0 \leq u_2$ in Ω, $u_i = 0$ on $\partial\Omega$, of*

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2u_2) + \mu|Du_2|^2 + b|Du_2| - cu_2 &\leq -k, \\ \mathcal{M}^-_{\lambda,\Lambda}(D^2u_1) - \mu|Du_1|^2 - b|Du_1| - cu_1 &\geq k. \end{aligned}$$

Proof. The proof of this lemma is based on techniques described in [CIL]. Let us prove there exists a solution of the first inequality (note the second inequality is obtained from the first by the change $u \to -u$). In view of Lemma 2.3, it is enough to construct a solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b|Dv| - (c/m)(1+mv)\log(1+mv) \leq -k(1+mv), \tag{19}$$

such that $v = 0$ on $\partial\Omega$, with $m = \mu/\lambda, v$ defined in Lemma 2.3.

To avoid writing constants, suppose for simplicity $c = m = k = 1$. Then $v_1 \equiv e - 1 =: A$ is a solution of (19) in Ω. We search for a neighbourhood of

$\partial\Omega$, denoted by $\Omega_\alpha = \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) < \frac{1}{\alpha}\}$, and a function v_2 which satisfies (19) in Ω_α, such that

$$v_2 = \begin{cases} A+1 & \text{on} \quad \partial\Omega_\alpha \\ 0 & \text{on} \quad \partial\Omega \end{cases}$$

Then the function

$$v = \begin{cases} A & \text{in} \quad \Omega \setminus \Omega_\alpha \\ \min\{v_2, A\} & \text{in} \quad \Omega_\alpha \end{cases}$$

is a solution of (19), again using the fact that the minimum of two viscosity supersolutions is a viscosity supersolution.

So we set

$$v_2 = (A+1)(1+e^{-1})^{-1}\left(1 - e^{-\alpha d(x)}\right) \qquad \text{in} \ \ \Omega_\alpha,$$

where $d(x)$ is the distance function to the boundary and α is chosen sufficiently large so that d is C^2 in Ω_α.

Let

$$B = \max_{t \in [0, A+1]} (1+t)(1 - \log(1+t)).$$

Then, computing Dv_2 and D^2v_2, and using the fact that $|Dd| = 1$ and $D^2 d$ is bounded in Ω_α (see for instance Chapter 14.16 in [GT]) we get, by Lemma 2.1,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 v_2) + b|Dv_2| \leq -C_1\alpha^2 + C_2\alpha < -B,$$

if α is large enough ; here C_1, C_2 depend on the right quantities and $\partial\Omega$. $\square$

Corollary 4.1 *Under the conditions of Theorem 1 (i), problem (5) has a C-viscosity solution provided F is continuous in $x \in \overline{\Omega}$ (so the functions in x appearing in the bounds in (S) can be considered continuous), and $f \in C(\overline{\Omega})$.*

Proof. When $\partial\Omega$ is smooth this follows from (S), the previous lemma and Theorem 3.3 in [CIL]. To extend the result to a domain Ω which only satisfies the uniform exterior cone condition with size L, we approximate Ω by smooth domains Ω_n, which admit exterior cones with size $L/2$, such that $\Omega \subset \Omega_n$, and take solutions u_n of (5) in Ω_n. Then, by our Theorem 2, u_n is uniformly bounded in $C^\alpha(\Omega_n)$, so, by the compact embedding $C^\alpha \hookrightarrow C^0$, a subsequence of u_n converges uniformly in $\overline{\Omega}$ to a function u, which is then a solution of (5) in Ω, by the convergence results in [CIL]. $\square$

Proposition 4.1 *For any $\mu \geq 0$, $c > 0$, $b \in L^p(\Omega), p > N$, $f \in L^N(\Omega)$, $b, f \geq 0$ there exist solutions u_1, u_2 of*

$$\begin{array}{rcl} \mathcal{M}^+_{\lambda,\Lambda}(D^2u_2) + \mu|Du_2|^2 + b(x)|Du_2| - cu_2 & \leq & -f(x), \\ \mathcal{M}^-_{\lambda,\Lambda}(D^2u_1) - \mu|Du_1|^2 - b(x)|Du_1| - cu_1 & \geq & f(x), \end{array}$$

such that $u_1 \leq 0 \leq u_2$ in Ω, $u_i = 0$ on $\partial\Omega$.

Proof. We take sequences of continuous functions b_n, f_n, which approximate b, f in L^p, L^N respectively. By Corollary 4.1 we know that the problem

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu|Du|^2 + b_n(x)|Du| - cu = (\Lambda/\lambda)f_n(x)$$

has a solution u_n, $u_n = 0$ on $\partial\Omega$. By the uniform C^α-estimate (Theorem 2) u_n is bounded in C^α and so a subsequence of u_n converges uniformly on $\overline{\Omega}$ to a function u. Then by Lemma 2.3

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v_n) + b_n(x)|Dv_n| \leq ((\Lambda/\lambda)f_n(x) + cu_n)(1 + mv_n),$$

where $m = \mu/\Lambda$, $v_n = (1/m)(e^{mu_n} - 1) \rightrightarrows v = (1/m)(e^{mu} - 1)$, as in Lemma 2.3. The right-hand side of this inequality converges in $L^N(\Omega)$, while for each $\mathcal{O}$ in Ω and each $\phi \in W^{2,N}(\mathcal{O})$

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + b_n(x)|D\phi| \longrightarrow \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + b(x)|D\phi| \quad \text{in } L^N(\mathcal{O}).$$

Note $W^{2,N}$ is embedded in $W^{1,q}$ for all $q < \infty$, so the convergence of the term $b_n(x)|D\phi|$ in L^N is a simple consequence of the Hölder inequality and $p > N$.

Hence we are in a position to apply Theorem 3.8 in [CCKS], which shows that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| \leq ((\Lambda/\lambda)f(x) + cu)(1 + mv)$$

so, again by Lemma 2.3,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu(\lambda/\Lambda)|Du|^2 + b(x)|Du| - cu \leq (\Lambda/\lambda)f(x),$$

and we conclude by replacing u by $(\Lambda/\lambda)u$.

Remark. Strictly speaking, the operator $\mathcal{M}^+_{\lambda,\Lambda}(D^2\cdot) + b(x)|D\cdot|$ does not satisfy the hypothesis of Theorem 3.8 in [CCKS], since $b \notin L^\infty(\Omega)$. However the proof of this theorem can be repeated without modifications for this operator, only at its end we have to note that solutions of

$$\left\{ \begin{array}{rcll} \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi_n) + b_n(x)|D\phi_n| & = & f_n & \text{in} \quad \Omega \\ \phi_n & = & 0 & \text{on} \quad \partial\Omega \end{array} \right.$$

where $b_n \in L^\infty(\Omega)$ is bounded in $L^p(\Omega)$ (such solutions exist, by the results in [CKLS]) are such that $\phi_n \rightrightarrows 0$ in $\overline{\Omega}$ if $f_n \to 0$ in $L^N(\Omega)$, by the ABP inequality (Theorem 3). □

One of the consequences of this result is a general approximation theorem for operators of our type. It extends Theorem 3.8 in [CCKS] to operators with unbounded coefficients and natural growth in the gradient.

Theorem 4 *Suppose F_n, F are operators which satisfy (S) with $\underline{h}, \overline{h}$ as in Theorem 1. Suppose $f_n, f \in L^N(\Omega)$ and $u_n, u \in C(\Omega)$ are such that u_n is a supersolution (subsolution) of*

$$F_n(D^2u_n, Du_n, u_n, x) = f_n \qquad \text{in } \Omega, \quad \text{for each } n,$$

and u_n converges to u locally uniformly in Ω. If for any ball $B \subset \Omega$ and any $\phi \in W^{2,N}(B)$, setting

$$g_n = F_n(D^2\phi, D\phi, u_n, x) - f_n, \quad g = F(D^2\phi, D\phi, u, x) - f(x),$$

we have

$$\|(g - g_n)^+\|_{L^N(B)} \longrightarrow 0 \qquad \left(\|(g - g_n)^-\|_{L^N(B)} \longrightarrow 0\right),$$

then u is a supersolution (subsolution) of $F(D^2u, Du, u, x) = f(x)$ in Ω.

Proof. The proof is identical to the proof of Theorem 3.8 in [CCKS], using (S) and Proposition 4.1 at the end. □

Proof of Theorem 1 (i). With the previous results at hand, the result is obtained through a standard smoothing argument. For instance, if we have to solve the Dirichlet problem for the model equation (2), we take continuous functions μ_n, b_n, c_n, f_n which converge to $\mu, b, c \le -c_0, f$ respectively in $L^q, q < \infty$, L^p, L^N, and $\|\mu_n\|_{L^\infty} \le \|\mu\|_{L^\infty} + 1$, $c_n \le -c_0/2$. Then by Corollary 4.1 the approximate problems

$$\mathcal{M}(D^2u) + \mu_n(x)|Du|^2 + b_n(x)|Du| + c_n(x)u = f_n(x),$$

have solutions u_n, with $u_n = 0$ on $\partial\Omega$. By the uniform C^α-estimate u_n is bounded in C^α and hence converges (up to a subsequence) uniformly in $\overline{\Omega}$. Then the solvability of (2) follows from Theorem 4, noting again that $b_n|D\phi| \to b|D\phi|$ and $\mu_n|D\phi|^2 \to \mu|D\phi|^2$ for any $\phi \in W^{2,N} \hookrightarrow W^{1,q}, q < \infty$.

For general F we can use the same argument as in the proof of Theorem 4.1 in [CKLS]. Let f_n be a sequence of continuous functions which converges to f in $L^N(\Omega)$, and set

$$F_\varepsilon(M, p, u, x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^N} \eta\left(\frac{x - y}{\varepsilon}\right) F(M, p, u, y)\, dy,$$

where $\eta \geq 0, \eta \in C^\infty$, has compact support and mass 1. Now, for fixed ε, the operator F_ε satisfy the conditions of Corollary 4.1, so the problem $F_\varepsilon = f_\varepsilon$ has a solution u_ε. Then we conclude again with the help of Theorem 2 and the approximation Theorem 4, noting that

$$F_\varepsilon(D^2\phi, D\phi, u_\varepsilon, x) \to F(D^2\phi, D\phi, u, x)$$

is a consequence of the Lebesgue dominated convergence theorem. Part (i) of Theorem 1 is proved. $\square$

Now we turn to the proof of Theorem 1 (ii). We shall use the notion of first eigenvalues for fully nonlinear elliptic operators, recently developed in [FQ], [BEQ] (for the Pucci operator) and in [QS2] for general convex or concave operators. We shall need the following particular case of the results in [QS2].

Theorem 5 ([QS2]) *Given λ, Λ, and $b, c \in L^\infty(\Omega)$, $b \geq 0$, there exist numbers $\lambda_1^+ \leq \lambda_1^-$, and functions $\varphi_1^+, \varphi_1^- \in W^{2,p}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ for each $p < \infty$, such that*

$$\left\{ \begin{array}{rcll} \mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi_1^+) + b(x)|D\varphi_1^+| + c(x)\varphi_1^+ &=& -\lambda_1^+\varphi_1^+ & in \;\; \Omega \\ \varphi_1^+ > 0 \;\; in \;\; \Omega, \qquad \varphi_1^+ &=& 0 & on \;\; \partial\Omega, \end{array} \right.$$

$$\left\{ \begin{array}{rcll} \mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi_1^-) + b(x)|D\varphi_1^-| + c(x)\varphi_1^- &=& -\lambda_1^-\varphi_1^- & in \;\; \Omega \\ \varphi_1^- < 0 \;\; in \;\; \Omega, \qquad \varphi_1^- &=& 0 & on \;\; \partial\Omega. \end{array} \right.$$

In addition, $\lambda_1^+ > 0$ is a sufficient condition for the Dirichlet problem

$$\left\{ \begin{array}{rcll} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + b(x)|Du| + c(x)u &=& f & in \;\; \Omega \\ u &=& 0 & on \;\; \partial\Omega, \end{array} \right.$$

to have a solution in $W^{2,N}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$, for any $f \in L^N(\Omega)$.

The following proposition gives a bound on the eigenvalues in terms of Lebesgue norms of the coefficients.

Proposition 4.2 *Given λ, Λ, and $b, c \in L^\infty(\Omega)$, $b \geq 0$, and $p > N$, there exists a constant C_0 depending only on $\lambda, \Lambda, p, N, \|b\|_{L^p(\Omega)}$, $diam(\Omega)$, such that the number λ_1^+ defined in Theorem 5 satisfies*

$$\lambda_1^+ \geq C_0 - |\Omega|^{-1/N}\|c^+\|_{L^N(\Omega)}.$$

Proof. We apply the ABP inequality (Theorem 3) to

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\varphi_1^+) + b(x)|D\varphi_1^+| = -(\lambda_1^+ + c(x))\varphi_1^+ \geq -(\lambda_1^+ + c^+(x))\varphi_1^+,$$

which yields

$$\sup_\Omega \varphi_1^+ \leq C_1\|\lambda_1^+ + c^+(x)\|_{L^N(\Omega)} \sup_\Omega \varphi_1^+,$$

so

$$\|c^+\|_{L^N(\Omega)} + \lambda_1^+|\Omega|^{1/N} \geq \|\lambda_1^+ + c^+(x)\|_{L^N(\Omega)} \geq 1/C_1,$$

and the result follows. □

We can now deduce a first result on solvability for non-proper equations with unbounded coefficients.

Proposition 4.3 *Given* $\lambda, \Lambda,$ *and* $b \in L^p(\Omega), p > N,\ c \in L^N(\Omega),\ b \geq 0,$ *there exists a constant* δ_0 *depending only on* $\lambda, \Lambda, p, N, \|b\|_{L^p(\Omega)},$ *diam*(Ω), *such that*

$$\|c^+\|_{L^N(\Omega)} < \delta_0$$

is a sufficient condition for the problem

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + b(x)|Du| + c(x)u &= f \quad in \quad \Omega \\ u &= 0 \quad on \quad \partial\Omega, \end{cases}$$

to have a solution in $W^{2,N}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$, *for any* $f \in L^N(\Omega)$. *In addition,* $f \leq (\geq) 0$ *implies* $u \geq (\leq) 0$ *in* Ω, *and*

$$\sup_\Omega |u| \leq \overline{C}\|f\|_{L^N(\Omega)},$$

where $\overline{C}$ *depends on* $\lambda, \Lambda, p, N, \delta_0, \|c^+\|_{L^N(\Omega)}, \|b\|_{L^p(\Omega)},$ *diam*(Ω).

Proof. We approximate b, c in L^p, L^N by sequences of continuous functions b_n, c_n with compact support in Ω, such that $\|c_n^+\|_{L^N(\Omega)} \leq \delta_0$, and δ_0 is fixed so small that the first eigenvalues of the operators

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2\cdot) + b_n|D\cdot| + c_n\cdot$$

be uniformly positive for all n – this is possible by the previous lemma. Hence, by Theorem 5, there exist strong solutions of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + b_n|Du_n| + c_nu_n = f$$

with $u_n = 0$ on $\partial\Omega$. Note that $f \leq (\geq)0$ implies $u_n \geq (\leq)0$ in Ω by the maximum principle, which was shown in [QS2] to hold for operators with positive eigenvalues.

Next, we apply the ABP inequality (Theorem 3) to

$$\begin{aligned}
\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + b_n|Du_n| &= -c_nu_n + f \geq -c_n^+u_n + f \qquad \text{on } \{u_n > 0\} \\
\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + b_n|Du_n| &= -c_nu_n + f \leq -c_n^+u_n + f \qquad \text{on } \{u_n < 0\}
\end{aligned}$$

to get

$$\sup_\Omega |u_n| \leq C_1 \left(\|c_n^+\|_{L^N(\Omega)} \sup_\Omega |u_n| + \|f\|_{L^N(\Omega)} \right),$$

so, if δ_0 is sufficiently small (say $\delta_0 = 1/(2C_1)$),

$$\sup_\Omega |u_n| \leq 2C_1 \|f\|_{L^N(\Omega)},$$

that is, u_n is bounded in $L^\infty(\Omega)$. Then, by Theorem 2, u_n is bounded in $C^\alpha(\Omega)$ so a subsequence of u_n converges uniformly in $\overline{\Omega}$ to a solution u of our problem.

Finally,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + b_n|Du_n| = -c_nu_n + f \qquad \text{in } \Omega \tag{20}$$

implies that u_n is bounded in $W^{2,N}_{\text{loc}}(\Omega)$, hence converges weakly in that space, so u is a strong solution. The boundedness in $W^{2,N}_{\text{loc}}$ is proved through the same cut-off argument as in the proof of Lemma 3.1 in [CCKS]. A precise upgrade of this result to coefficients in $C^0_c(\Omega)$, and bounded in $L^p(\Omega), p > N$, is given in Proposition 2.6 in [KS2] ; actually, (20) can be treated exactly like equation (2.8) in [KS2]. □

Next, we prove a result on existence of strong subsolutions and supersolutions of (5).

Proposition 4.4 *Given λ, Λ, and $b \in L^p(\Omega), p > N, b \geq 0$, $c \in L^N(\Omega)$, $c^+ \in L^\infty(\Omega)$, and an operator F satisfying the hypotheses of Theorem 1 (ii), there exists constants δ_0, c_0 depending only on $\lambda, \Lambda, p, N, \|b\|_{L^p(\Omega)}$, diam$(\Omega)$, such that*

$$\|\mu f\|_{L^N(\Omega)} \leq \delta_0 \qquad \textit{and} \qquad \|c^+\|_{L^\infty(\Omega)} < c_0$$

is a sufficient condition for the existence of functions $v, w \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$, such that $v \leq 0 \leq w$ in Ω,

$$F(D^2v, Dv, v, x) + cv \geq f, \qquad F(D^2w, Dw, w, x) + cw \leq f \quad \textit{in } \Omega,$$

and $v = w = 0$ on $\partial\Omega$.

Proof. Let us prove the existence of a positive supersolution. In view of (S) it is enough to find $u \geq 0$ such that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu|Du|^2 + b(x)|Du| + c(x)u \leq -f^-(x) \qquad \text{in } \ \Omega.$$

We set $v = (1/m)(e^{mu} - 1)$ like in Lemma 2.3, $m = \mu/\lambda$, and we see that it is enough to find a solution $v \geq 0$ of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) + b(x)|Dv| + mf^-(x)v \leq -f^-(x) - \frac{1}{m}\, c^+(x)(1+mv)\log(1+mv).$$

Let $e \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$, $e \geq 0$, be a solution of the problem

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2e) + b(x)|De| + mf^-(x)e \; = \; -\dfrac{f^-(x)}{\|f^-\|_{L^N(\Omega)}} - 1 & \text{in } \ \Omega \\ \\ e \; = \; 0 & \text{on } \ \partial\Omega, \end{cases}$$

This problem is solvable, by Proposition 4.3, provided

$$\|mf^-(x)\|_{L^N(\Omega)} < \delta_0,$$

where δ_0 is the number from Proposition 4.3. Then

$$\|e\|_{L^\infty(\Omega)} \leq \overline{C} \left\| \frac{f^-(x)}{\|f^-\|_{L^N(\Omega)}} + 1 \right\|_{L^N(\Omega)} = \overline{C}\left(1 + |\Omega|^{1/N}\right) =: \overline{C}_1,$$

by the same proposition.

Set $v = \|f^-\|_{L^N(\Omega)} e$. This function is a solution of the inequality we aim to solve provided

$$\|f^-\|_{L^N(\Omega)} \geq \frac{1}{m}\, c^+(x)(1 + mv(x))\log(1 + mv(x)),$$

for all $x \in \Omega$, which is implied by

$$\|mf^-\|_{L^N} \geq \frac{1}{m}\, \|c^+\|_{L^\infty}(1 + m\|f^-\|_{L^N}\|e\|_{L^\infty})\log(1 + m\|f^-\|_{L^N}\|e\|_{L^\infty}).$$

The last inequality holds if we choose c_0 such that $\|c^+\|_{L^\infty(\Omega)} \leq c_0$, and

$$t \geq c_0(1 + \overline{C}_1 t)\log(1 + \overline{C}_1 t), \qquad \text{for all } \ t \in [0, \delta_0].$$

This holds if we take for example

$$c_0 = \frac{1}{1 + \log(1 + \overline{C}_1\delta_0)}.$$

The existence of a subsolution is proved analogously. □

Proof of Theorem 1 (ii). Let δ_0, c_0 be the numbers from the previous proposition and let $u_0 = v \leq 0 \leq w$ be the subsolution and supersolution obtained in that proposition. We solve the hierarchy of problems in Ω

$$\begin{cases} F(D^2u_n, Du_n, u_n, x) + c(x)u_n - (c_0+1)u_n &= f(x) - (c_0+1)u_{n-1} \\ u &= 0 \quad \text{on } \partial\Omega, \quad n \geq 1. \end{cases}$$

Each of these problems is solvable, by Theorem 1 (i), which we already proved. Since u_0, v_0 are *strong* solutions, it is easily seen, by induction, that

$$u_0 \leq u_n \leq v_0 \qquad \text{for all } n \geq 1. \tag{21}$$

For instance, for any n, if we know that $u_{n-1} \geq u_0$ then we have

$$F(D^2u_n, Du_n, u_n, x) - F(D^2u_0, Du_0, u_0, x) + (c - c_0 - 1)(u_n - u_0) \leq 0,$$

so, using (S) we see that the function $w = u_n - u_0$, if we know that $u_{n-1} \geq u_0$, satisfies

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w) - (\mu|Dw| + 2\mu|Du_0| + b)|Dw| - d(x)h(w^+) + (c - c_0 - 1)w \leq 0$$

which gives a contradiction in case w attains a negative minimum in Ω (or, alternatively, $w \geq 0$ is implied by Lemma 2.3 and Theorem 3, applied in the set $\{w < 0\}$). Note that, since u_0 is a strong solution, we can use (S) as if both u_n and u_0 were strong - this is trivially seen with the help of test functions.

By (21) u_n is bounded in $L^\infty(\Omega)$, so it is bounded in C^α, for some $\alpha > 0$, by Theorem 2. Hence a subsequence of u_n converges uniformly in $\overline{\Omega}$ to a function u, which is then a solution to (5), by the approximation Theorem 4. Theorem 1 (ii) is proved. □

Proof of Theorem 1 (iii). Suppose u_1 and u_2 are solutions of (5) and $u_2 \in W^{2,N}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$. Set $u = u_1 - u_2$. Then, as above, by (S),

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - (\mu|Du| + 2\mu|Du_2| + b)|Du| - d(x)h(u^+) + c(x)u \leq 0.$$

First, if $c \leq 0$ in Ω then, by Lemma 2.3,

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w) - \widetilde{b}(x)|Dw| \leq 0 \qquad \text{in } \{u < 0\} = \{w < 0\},$$

where $w = 1/m(1 - e^{-mu})$, $m = \mu/\Lambda$, $\widetilde{b} = 2\mu|Du_2| + b$. Hence we have $w^- \equiv 0$, by Theorem 3.

Second, if $\mu = 0$ then we apply Theorem 3 directly to

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b|Du| \leq c^+(x)u \qquad \text{in } \{u < 0\},$$

and conclude $u^- \equiv 0$ for $\|c^+\|$ small, like in the proof of Proposition 4.3.

The fact that $u^+ \equiv 0$ is proved analogously. □

Proof of Theorem 1 (iv). For clarity, let us first consider the problem

$$\begin{array}{rcll} \Delta u + \mu|Du|^2 + c_0 u &=& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega. \end{array} \tag{22}$$

We want to show that, for all c_0 small, this problem has a solution different from the trivial one $u \equiv 0$. We shall in fact show that (22) has a classical positive solution provided

$$0 < c_0 < \lambda_1, \tag{23}$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian in Ω.

Setting, as before, $v = (1/\mu)\,(e^{\mu u} - 1)$, we see that (22) transforms into

$$\begin{array}{rcll} -\Delta v &=& (c_0/\mu)(1+\mu v)\log(1+\mu v) =: f(v) & \text{in } \Omega \\ u &>& 0 \quad \text{in } \Omega & \\ u &=& 0 \quad \text{on } \partial\Omega. & \end{array} \tag{24}$$

Since $f(0) = 0$, $f(v) > 0$ for $v > 0$,

$$f'(0) = c_0, \qquad \lim_{t\to\infty} \frac{f(t)}{t} = \infty,$$

the nonlinearity $f(u)$ is *superlinear* provided (23) holds.

It is well known that superlinear problems of this type possess a positive solution provided they admit *a priori bounds*, that is, if we able to show that all (eventual) positive solutions of (24) are uniformly bounded in the L^∞-norm by a constant which, in this case, depends only on Ω, μ, and an upper bound for c_0.

So let us show (24) admits a priori bounds. We can use the well-known "blow-up" method of Gidas and Spruck [GS]. Suppose for contradiction that there exists a sequence v_n of solutions of (24) such that $\|v_n\|_{L^\infty(\Omega)} \to \infty$. Set

$$s_n = \log \|v_n\|_{L^\infty(\Omega)}\,,$$

and make the change of unknowns

$$v_n(x) = e^{s_n} w_n(y), \qquad y = \sqrt{s_n}(x - x_n),$$

where $x_n \in \Omega$ is a point where v_n attains its maximum. Then $0 \le w_n \le 1$, $w_n(0) = 1$, and

$$-\Delta w_n(y) = s_n^{-1} e^{-s_n} f(e^{s_n} w_n(y)) \qquad \text{for } \ y \in \Omega_n,$$

where $\Omega_n := \sqrt{s_n}(\Omega - x_n)$ is a domain which converges either to $\mathbb{R}^N$ or to a half-space in $\mathbb{R}^N$.

It is trivial to see that the right-hand side of the last equation remains bounded (recall $0 \le w_n \le 1$), hence by elliptic estimates w_n converges in $W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ to a function w, such that $0 \le w \le 1$ and $w(0) = 1$. Further, we have pointwise, and hence in $L^p_{\text{loc}}(\mathbb{R}^N)$ (by Lebesgue dominated convergence)

$$s_n^{-1} e^{-s_n} f(e^{s_n} w_n) \to c_0 w$$

so w is a positive (by the strong maximum principle) solution of

$$-\Delta w = c_0 w$$

in $\mathbb{R}^N$ or in a half-space, which is a contradiction with $c_0 > 0$, since the spectrum of the Laplacian in these spaces does not meet the positive half-line.

Remark. When the elliptic operator is in divergence form (as in the above particular case), the problem can also be tackled via variational methods. For nonlinearities similar to $f(v)$ in (24), some related problems have been studied in [J], [JT]. We would like also to remark that the nonlinearity

$$(1+u)\log(1+u) = \log(1+u) + u\log(1+u)$$

is a sort of "concave-convex" nonlinearity, but the derivatives of its concave and convex part have different behaviour than in the case $u^q + u^p, q < 1 < p$, which has been studied very extensively in the last years, starting with the work [ABC] (see also the survey [AAP]). It seems that neither this type of nonlinearities, nor their connection with problems with natural growth in the gradient have been studied before. More developments on this topic will be given in [Sn].

Let us now state a more general result.

Theorem 6 *The problem*

$$\begin{cases} F(D^2u, Du, x) + c(x)u &= 0 \quad in \quad \Omega \\ u &> 0 \quad in \quad \Omega \\ u &= 0 \quad on \quad \partial\Omega. \end{cases} \tag{25}$$

has a solution if F satisfies the hypotheses of Theorem 1 (in (S) we suppose $b(x) \in L^\infty(\Omega)$, $d \equiv 0$), $0 < c_1 \le c(x) \le c_2$, and c_2 is sufficiently small.

Proof. By (S), positive solutions u_1, u_2 of the problems

$$\begin{aligned} \mathcal{M}^+_{\lambda,\Lambda}(D^2u_1) + \mu|Du_1|^2 + b(x)|Du_1| + c_2u_1 &= 0 \quad \text{in } \Omega \quad (26)\\ \mathcal{M}^-_{\lambda,\Lambda}(D^2u_2) - \mu|Du_2|^2 - b(x)|Du_2| + c_1u_2 &= 0 \quad \text{in } \Omega \quad (27)\\ u_i &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

are respectively a supersolution and a subsolution of problem (25).

The existence of u_1, u_2 is proved through by using the results in the recent paper [QS1], where fully nonlinear superlinear problems were studied, on the following problems (obtained from (26), (27) by a change of unknowns)

$$\begin{aligned} -\mathcal{M}^+_{\lambda,\Lambda}(D^2v_1) - b(x)|Dv_1| &= (c_2/m)(1+mv_1)\log(1+mv_1) \quad \text{in } \Omega \quad (28)\\ -\mathcal{M}^-_{\lambda,\Lambda}(D^2v_2) + b(x)|Dv_2| &= -(c_1/m)(1-mv_2)\log(1-mv_2) \quad \text{in } \Omega \quad (29)\\ u_i &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Uniform a priori bounds for the positive solutions of (28), (29) are proved through the same argument that we employed above for the Laplacian. Note that at the end of the blow-up argument the nonexistence of a positive solution of

$$\mathcal{M}(D^2u) = -c_iu \qquad \text{in } G = \mathbb{R}^N \text{ or } \mathbb{R}^N_+ \qquad (30)$$

follows from the results in [QS2] or [BEQ], where it is shown that the existence of a positive solution of (30) implies that for any ball $B_R \subset G$ we have $\lambda_1^+(\mathcal{M}, B_R) \geq c_i$ (by the definition of λ_1^+), and that $\lambda_1^+(\mathcal{L}, B_R) \leq CR^{-2}$, a contradiction for large R.

Finally, to infer the existence of a solution of (25) we need to know that $u_1 \geq u_2$ in Ω (then we can use the same iteration argument as the one used to prove Theorem 1 (ii)). Note that, by the a priori L^∞-bounds we proved for (28), (29), and by $C^{1,\alpha}$-estimates for classical or strong solutions of elliptic equations (see for example [GT]), we get from (28), (29) that

$$\|Du_i\|_{L^\infty(\Omega)} \leq C(\lambda, \Lambda, \mu, b, c_i, \Omega).$$

By (S) the function $u = u_1 - u_2$ is a solution of

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \tilde{b}|Du| \leq c_2u^-,$$

so the ABP inequality implies $u \geq 0$ in Ω, provided c_2 is small enough, like in the proof of Proposition 4.3. Here

$$\tilde{b} = \|b\|_{L^\infty(\Omega)} + \mu\|Du_1\|_{L^\infty(\Omega)} + \mu\|Du_2\|_{L^\infty(\Omega)}.$$

This finishes the proof of Theorem 1. □

Some simple equations without solutions. Here we recall some well-known results about existence and non-existence of solutions of

$$\begin{cases} \Delta u + \mu|Du|^2 + c_0 u &= -A \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{31}$$

where μ, c_0, A are nonnegative constants. First, if $\mu = 0$ then it is well-known that this problem, for any $A > 0$, has a positive solution if $c_0 < \lambda_1$, and has no solutions for $c_0 = \lambda_1$ (multiply by the first eigenfunction of the Laplacian). Second, if $c_0 = 0$, then problem (31) is equivalent to

$$\begin{cases} \Delta v + A\mu v &= -A \quad \text{in} \quad \Omega \\ v &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{32}$$

by the change $v = (1/\mu)(e^{\mu u} - 1)$. Again, for any $A > 0$, problem (32) has a positive solution if $A\mu < \lambda_1$ and has no solutions if $A\mu = \lambda_1$.

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