

# Some results of error evaluation for a non-Gaussian simulation method

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## ABSTRACT

In a first part is recalled a simulation method for a strictly stationary non-Gaussian process given its one-dimensional marginal distribution (or its  $N$ -first statistical moments) and its autocorrelation function. This method was already widely treated in the articles [14] and [13]. The objective of the present paper is twofold : firstly simplify this method when by Mehler formula it is possible to find an autocorrelation function yielding the target autocorrelation function, secondly analyze the error made between the given autocorrelation function and the model one.

**Keywords:** Monte-Carlo simulation, non-Gaussian process, Hermite polynomials, maximum entropy principle.

## 1 Introduction

The simulation of Gaussian processes is a well-posed problem, since these processes are totally characterized only by the data of their mean function and their autocorrelation function. The Gaussian simulation methods are numerous and effective (cf. [1, 3, 7, 8, 9, 11, 17]). On the contrary,

in the general case, the simulation of (non-Gaussian) stochastic processes is an ill-posed problem. To characterize a (non-Gaussian) stochastic process  $(X_t, t \in \mathbb{R}^d)$ , it is necessary (and sufficient) to know the entire (infinite) family of joint probability distributions  $\{\mathcal{L}(X_{t_1}, \dots, X_{t_n}), n \geq 1, t_i \in \mathbb{R}\}$ . Since this information is not accessible for the real-life processes, only a partial description of the non-Gaussian process is given. In this paper, the objective is to simulate the paths of a strictly stationary scalar process  $(X_t, t \in \mathbb{R})$  which is known only by its one-dimensional marginal distribution (which may be only known by its  $N$ -first statistical moments ( $N > 1$ )) and its autocorrelation function. Various methods have already been proposed to simulate paths of non-Gaussian processes (cf. [4, 5, 7, 10, 12, 15, 16, 18]). Many simulation techniques for strictly stationary non-Gaussian processes are based on the idea which consists in trying to write the strictly stationary non-Gaussian process as a translation process, that is a memoryless non-linear transformation  $Y_t = f(G_t)$  of a stationary Gaussian process  $(G_t, t \in \mathbb{R})$ . In this problem, the memoryless transformation  $f$  is determined thanks to the one-dimensional marginal distribution which itself may be determined by its first  $N$  moments if these are the data of the problem. In [5, 12, 18], an iterative algorithm is used to find an approximate spectral density function of an underlying Gaussian process in order to match the desired target spectral density, but as written in [6], "This iterative procedure does not guarantee convergence for all classes of non-linear processes", and there does not exist any theoretical study for the convergence of the algorithm. The method of [4] is more analytical. Once the memoryless transform has been determined, the author uses a (rather complicated) formula giving the autocorrelation function of  $(f(G_t), t \in \mathbb{R})$  as a function of the autocorrelation function of  $(G_t, t \in \mathbb{R})$ . If this formula with the target covariance function as datum can be inverted and if one obtains a solution which is nonnegative definite, then the problem is solved. Unfortunately, as written in [4], "it is not always possible to find a covariance function [of the underlying Gaussian process] yielding the target covariance function".

In [14] and [13], a general method has been proposed to simulate strictly stationary non-Gaussian processes based on the same ideas. After having determined a one-dimensional marginal distribution (if the given data are the  $N$ -first statistical moments), and since "it is not always possible to find a covariance function [of the underlying Gaussian process] yielding the target covariance function", [14] proposes to *find a translation process  $(f(G_t), t \in \mathbb{R})$  which is the best approximation of the strictly stationary non-Gaussian process  $(X_t, t \in \mathbb{R})$* , in the sense that the quadratic norm of the difference between the autocorrelation function of the strictly stationary non-Gaussian

process  $(X_t, t \in \mathbb{R})$  and the autocorrelation function of the translation process  $(f(G_t), t \in \mathbb{R})$  is minimized. However, from the practical point of view, the memoryless non-linear transform  $f$  is developed as a Hermite polynomials series and approximated by a finite truncated sum (at the order  $M$ )  $f^M$ . In [13], with some regularity hypotheses on the function  $f$  (which are verified if the one-dimensional marginal distribution is determined by its first  $N$  moments and the maximum entropy principle), a precise error bound of the error made in replacing the function  $f$  by the function  $f^M$  was established (cf. proposition 2.2).

The new contribution of the present paper is twofold. Firstly by Mehler formula, there is a very simple formula giving the autocorrelation function of  $(f(G_t), t \in \mathbb{R})$  as a function of the autocorrelation function of  $(G_t, t \in \mathbb{R})$  (the autocorrelation function of  $(f(G_t), t \in \mathbb{R})$  is an analytical function of the autocorrelation function of  $(G_t, t \in \mathbb{R})$ , cf. equation (16)), this analytical function being approximated by a polynomial function with a few terms, cf. equation (15)). If, as in Grigoriu's method ([4]), this formula with the target covariance function as datum can be inverted and gives a solution which is nonnegative definite, then the problem is simply solved, with a simpler formula than in Grigoriu's method. The second contribution of the present paper is to give an evaluation of the quadratic error (minoration and majoration) between the autocorrelation function of the strictly stationary non-Gaussian process  $(X_t, t \in \mathbb{R})$  and the autocorrelation function of the translation process  $(f(G_t), t \in \mathbb{R})$  found by the minimization procedure (cf. propositions 4.1 and 5.1). These two new results allow to improve the non-Gaussian simulation method and to control the quality of the simulation. Finally the method is illustrated by some examples.

## 2 Modelling and simulation of non-Gaussian processes

The considered non-Gaussian simulation method is recalled in this section. Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be the probability space implicitly used in what follows.

### 2.1 Simulation background

The objective is to simulate the paths of a strictly stationary non-Gaussian process  $(X_t, t \in \mathbb{R})$  with the following statistical data:

- i. for all  $t \in \mathbb{R}$ ,  $X_t$  has the same law that a random variable  $Y$  whose one-dimensional marginal

distribution is given by its cumulative distribution function  $F_Y$  (we suppose in this case that  $F_Y$  is a strictly increasing function) or is only known by its  $N$ -first statistical moments  $\mu_1, \mu_2, \dots, \mu_N$  ( $N > 1$ ) (let us assume without loss of generality that in the two cases  $\mu_1 = 0$  and  $\mu_2 = 1$ ),

- ii.  $R : \mathbb{R} \rightarrow \mathbb{R}$  is a given autocorrelation function (it is an even, non-negative definite function and we suppose that  $R$  is continuous and  $R(0) = 1$ ). Moreover let us assume that  $R \in L^2(\mathbb{R}, dt)$  and that the spectral measure obtained with Bochner theorem admits a density  $S$  (therefore we have  $S \in L^1(\mathbb{R}, d\omega) \cap L^2(\mathbb{R}, d\omega)$ ).

## 2.2 Simulation method

### 2.2.1 Choice of the marginal distribution : maximum entropy principle

If only the  $N$ -first moments are given, there can be an infinity of distributions whose first  $N$  moments are the same. Without further information on the one-dimensional marginal distribution, a criterion of model choice is used. The criterion of maximum entropy principle is well-adapted for the present case. It is described here such as used in our problem.

Let  $Y$  be a random variable, whose distribution is to be determined, such that

$$\mu_k = \mathbb{E}(Y^k), \quad k = 1, \dots, N. \quad (1)$$

$N$  is assumed here to be even. Two conditions are imposed to our criterion of model choice: the support of the wanted distribution is  $\mathbb{R}$  and this distribution is absolutely continuous with respect to the Lebesgue-measure. The maximum entropy principle consists in an optimization problem under constraints. Actually, we seek a density function  $p$  with support  $\mathbb{R}$  which maximizes the entropy function  $H$ :

$$H[p] = - \int_{\mathbb{R}} p(y) \ln(p(y)) dy, \quad (2)$$

under the constraints

$$\mu_k = \int_{\mathbb{R}} y^k p(y) dy, \quad k = 0, \dots, N \quad (\text{where } \mu_0 \stackrel{\text{def}}{=} 1). \quad (3)$$

The form of  $p$  is then

$$p(y) = e^{-\lambda_0 - 1} \exp \left( - \sum_{k=1}^N \lambda_k y^k \right), \quad (4)$$

where the  $(\lambda_k)$  are the Lagrange multipliers of the optimization problem.

### 2.2.2 Inversion method and use of Hermite polynomials (cf. [14])

Denote  $F_Y^{-1}$  the reciprocal function of  $F_Y$ , the cumulative distribution function of the one-dimensional distribution function which is given or which is obtained by the maximum entropy principle. If  $G$  is a Gaussian random variable with the standard Gaussian distribution  $\mathcal{N}(0, 1)$  and if  $F_G$  is its cumulative distribution function, it is known that  $F_G(G)$  is a random variable of uniform distribution on  $[0, 1]$  and then  $F_Y^{-1} \circ F_G(G)$  is a random variable which has the same distribution as the random variable  $Y$ . In particular the random variable  $F_Y^{-1} \circ F_G(G)$  has the required statistical moments.

Then the following model is considered

$$Y_t = F_Y^{-1} \circ F_G(G_t), \quad (5)$$

where  $(G_t, t \in \mathbb{R})$  is a standard stationary Gaussian process (that is  $(G_t, t \in \mathbb{R})$  is a stationary Gaussian process such that  $\forall t \in \mathbb{R}, \mathbb{E}(G_t) = 0, \mathbb{E}(G_t^2) = 1$ ) whose autocorrelation function is to be determined. We shall set

$$f = F_Y^{-1} \circ F_G \quad (6)$$

#### Lemma 2.1

The function  $f = F_Y^{-1} \circ F_G \in L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)$  and  $\|f\|_{L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)} = 1$ .

**Proof.** By assumption  $Y$  has a finite second-order statistical moment and  $\mathbb{E}(Y^2) = 1$ . Then

$$\mathbb{E}(Y^2) = \mathbb{E}((F_Y^{-1} \circ F_G(G))^2) \quad (7)$$

$$= \int_{\mathbb{R}} (F_Y^{-1} \circ F_G(x))^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = 1. \quad (8)$$

□

For all  $x \in \mathbb{R}$ , the Hermite polynomials are defined by

$$H_0(x) = 1, \quad H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad n \in \mathbb{N}^*, \quad (9)$$

and the family  $((\sqrt{n!})^{-1} H_n)_{n \in \mathbb{N}}$  is an orthonormal base of  $L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)$ .

Then the function  $f$  can be projected on the base  $((\sqrt{n!})^{-1} H_n)_{n \in \mathbb{N}}$  :

$$f = \sum_{n=0}^{\infty} f_n H_n \quad (10)$$

where

$$f_n = \frac{1}{n!} \int_{\mathbb{R}} F_Y^{-1} \circ F_G(x) H_n(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx, \quad (11)$$

the series being convergent in  $L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)$ . Since  $\mathbb{E}(Y) = 0$ , we have  $f_0 = 0$ , and since  $\|f\|_{L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)} = 1$ , we have

$$\sum_{n=1}^{+\infty} n! f_n^2 = 1, \quad (12)$$

We shall set for all  $M \in \mathbb{N}^*$ ,

$$f^M = \sum_{n=1}^M f_n H_n \quad (13)$$

### 2.2.3 Determination of the autocorrelation function of the underlying Gaussian process

$$(G_t, t \in \mathbb{R})$$

If  $X = (X_t, t \in \mathbb{R})$  is a strictly stationary stochastic process, let us denote  $R_X$  its autocorrelation function, that is

$$R_X(t) = E(X_s X_{t+s}), \forall s, t \in \mathbb{R} \quad (14)$$

If  $X = (X_t, t \in \mathbb{R})$  is a strictly stationary stochastic process, and if  $g$  is any regular (measurable) function, we shall denote  $g(X)$  the strictly stationary stochastic process  $(g(X_t), t \in \mathbb{R})$ . If  $G = (G_t, t \in \mathbb{R})$  is a standard stationary Gaussian process, by Mehler formula we have

$$R_{f^M(G)} = \sum_{n=1}^M n! f_n^2 (R_G)^n, \forall M \in \mathbb{N}^* \quad (15)$$

and

$$R_{f(G)} = \sum_{n=1}^{+\infty} n! f_n^2 (R_G)^n \quad (16)$$

the series being absolutely and uniformly convergent on  $\mathbb{R}$  because of (12) and since  $\forall t \in \mathbb{R}$ ,  $|R_G(t)|^2 \leq \mathbb{E}(G_t^2) \mathbb{E}(G_0^2) = 1$ . If moreover, we suppose that  $R_G \in L^2(\mathbb{R}, dt)$ , then the series (16) is convergent in  $L^2(\mathbb{R}, dt)$ , because of (12) and since  $\forall n \in \mathbb{N}^*$ ,  $|(R_G)^n| \leq |R_G| |(R_G)^{n-1}| \leq |R_G|$ .

The idea of the method is to find an autocorrelation function  $R_G \in L^2(\mathbb{R}, dt)$  of a standard stationary Gaussian process which minimizes the quantity

$$\|R - R_{f(G)}\|_{L^2(\mathbb{R}, dt)} = \|R - \sum_{n=1}^{+\infty} n! f_n^2 (R_G)^n\|_{L^2(\mathbb{R}, dt)} \quad (17)$$

but from a numerical point of view, it is only possible to deal with a finite sum, so for  $M \in \mathbb{N}^*$ , we shall solve the problem : find an autocorrelation function  $R_G \in L^2(\mathbb{R}, dt)$  of a standard stationary Gaussian process which minimizes the quantity

$$\|R - R_{f^M(G)}\|_{L^2(\mathbb{R}, dt)} = \|R - \sum_{n=1}^M n! f_n^2(R_G)^n\|_{L^2(\mathbb{R}, dt)} \quad (18)$$

Denoting  $S_G$  the spectral density function of  $(G_t, t \in \mathbb{R})$  (assuming such density exists), the problem becomes:

minimize the quantity

$$\|R - R_{f^M(G)}\|_{L^2(\mathbb{R}, dt)} = \|R - \sum_{n=1}^M n! f_n^2 \left( \int_{\mathbb{R}} S_G(\omega) e^{i\omega \cdot} d\omega \right)^n\|_{L^2(\mathbb{R}, dt)}, \quad (19)$$

under the constraint that  $S_G$  belongs to the space :

$$E = \{g \in L^1(\mathbb{R}, d\omega) \cap L^2(\mathbb{R}, d\omega), g \text{ even}, g \geq 0, \int_{\mathbb{R}} g(\omega) d\omega = 1\} \quad (20)$$

By the Plancherel theorem, we have

$$\|R - R_{f^M(G)}\|_{L^2(\mathbb{R}, dt)} = \sqrt{2\pi} \|S - \Phi^M(S_G)\|_{L^2(\mathbb{R}, d\omega)}, \quad (21)$$

$$\|R - R_{f(G)}\|_{L^2(\mathbb{R}, dt)} = \sqrt{2\pi} \|S - \Phi(S_G)\|_{L^2(\mathbb{R}, d\omega)}, \quad (22)$$

with

$$\Phi^M(S_G) = \sum_{n=1}^M n! f_n^2 S_G^{*n} \quad (23)$$

$$\Phi(S_G) = \sum_{n=1}^{+\infty} n! f_n^2 S_G^{*n} \quad (24)$$

where  $S_G^{*n} = \underbrace{S_G * \cdots * S_G}_{n \text{ times}}$  (\* denotes the convolution product of two functions), the series (24) being convergent in  $L^2(\mathbb{R}, d\omega)$ .

Set

$$I = \inf_{S_G \in E} \|R - R_{f(G)}\|_{L^2(\mathbb{R}, dt)} = \sqrt{2\pi} \inf_{S_G \in E} \|S - \Phi(S_G)\|_{L^2(\mathbb{R}, d\omega)} \quad (25)$$

$$I^M = \inf_{S_G \in E} \|R - R_{f^M(G)}\|_{L^2(\mathbb{R}, dt)} = \sqrt{2\pi} \inf_{S_G \in E} \|S - \Phi^M(S_G)\|_{L^2(\mathbb{R}, d\omega)} \quad (26)$$

Then for  $M \in \mathbb{N}^*$  and  $\alpha^M \in \mathbb{R}_+^*$  (small), let us find a standard stationary Gaussian process  $G^M = (G_t^M, t \in \mathbb{R})$  with the spectral density function  $S_{G^M} \in E$  such that

$$I^M \leq \|R - R_{f^M(G^M)}\|_{L^2(\mathbb{R}, dt)} \leq I^M + \alpha^M \quad (27)$$

Thus for  $M \in \mathbb{N}^*$ , the spectral method or the markovianization method is used to generate the paths of the underlying standard Gaussian process  $(G_t^M, t \in \mathbb{R})$  with the spectral density function  $S_{G^M}$  (cf. [14] for the different formulations of the minimization problem) and the simulation is achieved by constructing the truncated sum  $f^M(G_t^M) = \sum_{n=1}^M f_n H_n(G_t^M)$ .

### 2.2.4 Some convergence results

If  $g \in L^2\left(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx\right)$ , we shall denote by  $(c_n(g))_{n \in \mathbb{N}}$  the coefficients of its Hermite expansion. In [13] the following convergence result was shown

#### Proposition 2.2

Let for all  $M \in \mathbb{N}^*$   $(G_t^M, t \in \mathbb{R})$  be the stationary Gaussian process given by (27), with notations (6), (25) and if moreover  $f \in C^k(\mathbb{R})$  and  $f, f', \dots, f^{(k)}$  belong to  $L^2\left(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx\right)$  (that is the case when  $F_Y$  is determined by the use of the maximum entropy principle), then there exists a constant  $C_1(f)$  depending on  $f$  and a constant  $C_2(S, I, f)$  depending on  $S, I$  and  $f$  such that  $\forall k \in \mathbb{N}^*, \forall M \in \mathbb{N}^*$ ,

$$\begin{aligned} I &\leq \|R - R_{f(G^M)}\|_{L^2(\mathbb{R}, dt)} \\ &\leq I + C_1(f)\alpha^M + C_2(S, I, f) \left(\frac{(M+1-k)!}{(M+1)!}\right) \left(\sum_{n=M+1-k}^{\infty} n! c_n(f^{(k)})^2\right) \\ &= I + C_1(f)\alpha^M + o_{M \rightarrow +\infty}\left(\frac{1}{M^k}\right) \end{aligned} \quad (28)$$

## 3 Supplementary results and hypotheses

The following power series is considered

$$\Psi(Z) = \sum_{n=1}^{\infty} n! f_n^2 Z^n. \quad (29)$$

From (12), it follows that the radius of convergence of this series is  $\geq 1$ .



**Lemma 3.1**

We have  $R(\mathbb{R}) \subset [-1, 1]$ . Since  $R$  is continuous and  $R(0) = 1$ , it follows that there exists  $b \in [-1, 1]$  such that  $R(\mathbb{R}) = [b, 1]$  or  $]b, 1]$ .

We have  $\Psi([-1, 1]) \subset [-1, 1]$ . Suppose there exists  $a \leq 1$  such that  $\Psi$  is defined and strictly increasing on the interval  $[a, 1]$  (if  $a < -1$ , it is supposed that the radius of convergence of the series (29) is  $> |a|$ ) and such that  $\Psi([a, 1]) = [b, 1]$ . Then there is an unique function  $R_1$  defined on  $\mathbb{R}$  such that

$$R = \Psi(R_1) \tag{30}$$

and  $R_1$  verifies

$$\begin{cases} R_1(0) & = 1 \\ |R_1(t)| & \leq \sup(|a|, 1), \forall t \in \mathbb{R} \\ R_1 & \text{is continuous on } \mathbb{R} \\ R_1 & \text{is even.} \end{cases} \tag{31}$$

**Proof.**

We have

$$\forall t \in \mathbb{R}, |R(t)| \leq \int_{\mathbb{R}} S(\omega) d\omega = R(0) = 1. \tag{32}$$

Therefore

$$R(\mathbb{R}) \subset [-1, 1]. \tag{33}$$

Since  $R$  is continuous, and  $R(0) = 1$ , then  $R(\mathbb{R})$  is an interval of the form  $[b, 1]$  or  $]b, 1]$  with  $-1 \leq b \leq 1$ .

Moreover,

$$\forall Z \in [-1, 1], |\Psi(Z)| \leq \sum_{n=0}^{\infty} n! f_n^2 = 1, \tag{34}$$

then

$$\Psi([-1, 1]) \subset [-1, 1]. \tag{35}$$

With the lemma assumptions,  $\Psi^{-1}$  is well defined and continuous on  $[b, 1]$  and then  $R_1 = \Psi^{-1}(R)$  is well defined and continuous on  $\mathbb{R}$ . The other properties of  $R_1$  can be simply obtained.  $\square$

**Lemma 3.2**

Under the assumptions of lemma 3.1 and if  $f_1 \neq 0$ , then

$$R_1 \in L^2(\mathbb{R}, dt). \tag{36}$$

**Proof.** The mean-value theorem gives:

$$\forall x \in [-1, 1] \setminus \{0\}, \exists c \in ]0, x[ \text{ or } ]x, 0[ / \Psi(x) - \Psi(0) = \Psi'(c)(x - 0), \quad (37)$$

We have  $\Psi(0) = 0$  and since  $f_1 \neq 0$ , it follows that  $\Psi'(0) \neq 0$ . Then

$$\exists \alpha, 0 < \alpha < 1, \exists M > 0 / |x| \leq \alpha \Rightarrow |\Psi(x)| \geq M|x|. \quad (38)$$

But

- i.  $\lim_{|t| \rightarrow \infty} R(t) = 0$  (because  $R$  is the Fourier transform of a  $L^1$ -function),
- ii.  $\Psi^{-1}$  is continuous at 0 (because  $\Psi'(0) \neq 0$ ) then

$$\exists \beta > 0 / |x| \leq \beta \Rightarrow |\Psi^{-1}(x)| \leq \alpha. \quad (39)$$

Thus

$$\exists A > 0 / |t| \geq A \Rightarrow |R(t)| \leq \beta \quad (40)$$

$$\Rightarrow |\Psi^{-1}(R(t))| \leq \alpha \quad (41)$$

$$\Rightarrow |R(t)| \geq M|R_1(t)|. \quad (42)$$

Then

$$R \in L^2(\mathbb{R}, dt) \Rightarrow R_1 \in L^2(\mathbb{R}, dt). \quad (43)$$

□

Under the assumptions of lemmas 3.1 and 3.2, we denote  $S_1 \in L^2(\mathbb{R}, d\omega)$  the inverse Fourier transform of  $R_1$  :

$$S_1(\omega) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\omega t} R_1(t) dt, \forall \omega \in \mathbb{R}. \quad (44)$$

### Lemma 3.3

Under the assumptions of lemmas 3.1 and 3.2, suppose moreover that  $\omega \mapsto \omega S(\omega) \in L^2(\mathbb{R}, d\omega)$  and that  $\Psi'(x) \neq 0$  for all  $x \in [a, 1]$ . Then  $S_1$ , the inverse Fourier transform of  $R_1$ , verifies

$$S_1 \in L^1(\mathbb{R}, d\omega). \quad (45)$$

**Proof.** Since  $\omega \mapsto \omega S(\omega) \in L^2(\mathbb{R}, d\omega)$ , then  $R \in H^1(\mathbb{R}, dt)$ , the Sobolev space defined by  $H^1(\mathbb{R}, dt) = \{g \in L^2(\mathbb{R}, dt) / g' \text{ (in the distribution sense)} \in L^2(\mathbb{R}, dt)\}$ . Since  $\Psi'(x) \neq 0$  for all  $x \in [a, 1]$ , it follows that  $\Psi^{-1} \in \mathcal{C}^1([b, 1])$ . By an adaptation of a result in [2], p.131, Corollaire VIII.10, it is shown that  $R_1 = \Psi^{-1}(R) \in H^1(\mathbb{R}, dt)$ , then  $\omega \mapsto \omega S_1(\omega) \in L^2(\mathbb{R}, d\omega)$ . Now since we can write  $S_1(\omega) = ((1 + |\omega|)S_1(\omega)) \frac{1}{1+|\omega|}$  ( $\forall \omega \in \mathbb{R}$ ) and since  $\omega \mapsto ((1 + |\omega|)S_1(\omega)) \in L^2(\mathbb{R}, d\omega)$  and  $\omega \mapsto \frac{1}{1+|\omega|} \in L^2(\mathbb{R}, d\omega)$ , it follows that  $S_1 \in L^1(\mathbb{R}, d\omega)$ .  $\square$

In the sequel we shall do the assumptions of lemmas 3.1, 3.2 and 3.3. Then by lemma 3.3, we have  $S_1 \in L^1(\mathbb{R}, d\omega)$  and by lemma 3.1, we have  $\int_{\mathbb{R}} S_1(\omega) d\omega = R_1(0) = 1$ . By lemma 3.1,  $R_1$  is an even function, then  $S_1$  too. Moreover,

$$\overline{S_1(\omega)} = (2\pi)^{-1} \int_{\mathbb{R}} e^{i\omega t} R_1(t) dt \quad (46)$$

$$= (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\omega t} R_1(-t) dt \quad (47)$$

$$= (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\omega t} R_1(t) dt \quad (48)$$

$$= S_1(\omega), \forall \omega \in \mathbb{R}. \quad (49)$$

so  $S_1$  has real values.

The function  $R_1$  is not necessarily non-negative definite in general. This is true when  $S_1 \geq 0$ , and then this function must be chosen for the spectral density of the underlying Gaussian process  $(G_t, t \in \mathbb{R})$  (in this case  $S_1 \in E$  (defined by (20)) and the minimization problem is resolved and has for minimum 0 :  $I$  defined by (25) is equal to 0). In the opposite case, the minimization procedure described in the previous section must be used. Then, in the following sections, an error evaluation on the autocorrelation function of the model will be exposed.

## 4 Majoration result

For all  $g \in E$  (defined by (20)), let  $\hat{g} = \int_{\mathbb{R}} e^{i\omega \cdot} g(\omega) d\omega$  be the Fourier transform of  $g$ . The central proposition of this section is

**Proposition 4.1**

If  $a < -1$ , then

$$I = \inf_{S_G \in E} \|R - R_{f(G)}\|_{L^2(\mathbb{R}, dt)} \leq \Psi'(|a|) \sqrt{2\pi} \inf_{g \in E} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)} \quad (50)$$

If  $a \geq -1$ , if  $f \in \mathcal{C}^1(\mathbb{R})$  and if  $f' \in L^2(\frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)$ , then

$$I = \inf_{S_G \in E} \|R - R_{f(G)}\|_{L^2(\mathbb{R}, dt)} \leq \|f'\|_{L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)}^2 \sqrt{2\pi} \inf_{g \in E} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)} \quad (51)$$

**Proof.** We have

$$R(t) = \Psi(R_1(t)) = \sum_{n=0}^{\infty} n! f_n^2 (R_1(t))^n, \forall t \in \mathbb{R}. \quad (52)$$

Since  $|R_1| \leq \sup(|a|, 1)$ , we have

$$\|R_1^n\|_{L^2(\mathbb{R}, dt)} \leq (\sup(|a|, 1))^{n-1} \|R_1\|_{L^2(\mathbb{R}, dt)}, \forall n \in \mathbb{N}^* \quad (53)$$

therefore, since the radius of convergence of the power series  $\Psi(Z)$  is  $> |a|$  when  $a < -1$ , it follows that

$$\sum_{n=0}^{\infty} n! f_n^2 \|R_1^n\|_{L^2(\mathbb{R}, dt)} \leq \sum_{n=0}^{\infty} n! f_n^2 (\sup(|a|, 1))^{n-1} \|R_1\|_{L^2(\mathbb{R}, dt)} \quad (54)$$

$$< \infty. \quad (55)$$

Thus, by uniqueness of the limit we deduce that

$$\sum_{n=0}^{\infty} n! f_n^2 R_1^n = R \quad (56)$$

the series being convergent in  $L^2(\mathbb{R}, dt)$ . The Fourier transform and the Plancherel theorem can be used to obtain:

$$\int_{\mathbb{R}} e^{-i \cdot t} R(t) dt = \sum_{n=0}^{\infty} n! f_n^2 \int_{\mathbb{R}} e^{-i \cdot t} R_1^n(t) dt \text{ in } L^2(\mathbb{R}, d\omega) \quad (57)$$

and then

$$S = \sum_{n=0}^{\infty} n! f_n^2 S_1^{*n} \text{ in } L^2(\mathbb{R}, d\omega). \quad (58)$$

It is easily seen that

$$I = \inf_{S_G \in E} \|R - R_{f(G)}\|_{L^2(\mathbb{R}, dt)} = \inf_{g \in E} \|\Psi(R_1) - \Psi(\hat{g})\|_{L^2(\mathbb{R}, dt)} \quad (59)$$

If  $f \in C^1(\mathbb{R})$  and  $f' \in L^2(\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx)$ , then  $\Psi$  is differentiable at  $1^-$  and  $\Psi'(1^-)$  noted  $\Psi'(1)$  verifies

$$\Psi'(1) = \sum_{n=1}^{\infty} n! f_n^2 n = \|f'\|_{L^2(\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx)}^2 \quad (60)$$

Then in the two cases considered in the proposition,  $\Psi'(\sup(|a|, 1))$  exists. For all  $g \in E$ , we have

$$|\hat{g}(t)| \leq \int_{\mathbb{R}} |e^{i\omega t} g(\omega)| d\omega \leq \int_{\mathbb{R}} g(\omega) d\omega = 1, \quad \forall t \in \mathbb{R} \quad (61)$$

and, owing to the inequalities  $|R_1| \leq \sup(|a|, 1)$ ,  $|\hat{g}| \leq 1$  and to the mean-value theorem, we can write

$$\|\Psi(R_1) - \Psi(\hat{g})\|_{L^2(\mathbb{R}, dt)} \leq \Psi'(\sup(|a|, 1)) \|R_1 - \hat{g}\|_{L^2(\mathbb{R}, dt)} \quad (62)$$

$$= \sum_{n=1}^{\infty} n! f_n^2 n (\sup(|a|, 1))^{n-1} \|R_1 - \hat{g}\|_{L^2(\mathbb{R}, dt)} \quad (63)$$

Since

$$\|R_1 - \hat{g}\|_{L^2(\mathbb{R}, dt)} = \sqrt{2\pi} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)} \quad (64)$$

the proposition is proved.  $\square$

The following lemma completes the preceding proposition.

**Lemma 4.2**

$\inf_{g \in E} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)}$  is reached at a single point  $g_0$  of  $E$ .

**Proof.** Let the set

$$E' = \{g \in L^1(\mathbb{R}, d\omega) \cap L^2(\mathbb{R}, d\omega), g \text{ even}, g \geq 0, \int_{\mathbb{R}} g(\omega) d\omega \leq 1\} \quad (65)$$

$E \subset E'$ ,  $E'$  is convex, and we shall show that it is closed in  $L^2(\mathbb{R}, d\omega)$ .

Let  $g_n \in E'$  be a sequence such that  $g_n \rightarrow g$  in  $L^2(\mathbb{R}, d\omega)$ . Clearly  $g \geq 0$  and  $g$  is even. But Fatou's lemma gives

$$0 \leq \int_{\mathbb{R}} \liminf_{n \rightarrow \infty} g_n(\omega) d\omega \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} g_n(\omega) d\omega, \quad (66)$$

whence

$$\int_{\mathbb{R}} g(\omega) d\omega \leq 1 \quad (67)$$

and  $g \in L^1(\mathbb{R}, d\omega)$ .

The theorem of projection on a closed convex set in a Hilbert space can be applied :

$$\exists! g_0 \in E' / \inf_{g \in E'} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)} = \|S_1 - g_0\|_{L^2(\mathbb{R}, d\omega)}. \quad (68)$$

We want then to show that  $g_0 \in E$  i.e.  $\int_{\mathbb{R}} g_0(\omega) d\omega = 1$ .

We write

$$S_1 = S_1^+ - S_1^-. \quad (69)$$

$$R_1(0) = \int_{\mathbb{R}} S_1(\omega) d\omega = 1 \Rightarrow \int_{\mathbb{R}} S_1^+(\omega) d\omega = \int_{\mathbb{R}} S_1^-(\omega) d\omega + 1. \quad (70)$$

Let the decomposition  $A^+ = \{\omega \in \mathbb{R} / S_1(\omega) > 0\}$ ,  $A^- = \{\omega \in \mathbb{R} / S_1(\omega) \leq 0\}$ .

Then  $\lambda(A^+) > 0$  because  $\int_{A^+} S_1^+(\omega) d\omega = \int_{A^-} S_1^-(\omega) d\omega + 1 > 0$ .

**1st case:**  $\int_{A^-} S_1^-(\omega) d\omega = 0$ .

Consequently  $S_1 = S_1^+ \geq 0$  and  $S_1 \in E \subset E'$ , therefore  $g_0 = S_1 \in E$ .

**2nd case:**  $\int_{A^-} S_1^-(\omega) d\omega > 0$ .

Then we have  $\lambda(A^-) > 0$ . We want to show that  $g_0 = 0$  on  $A^-$  (a.s.). Suppose the contrary : in this case, if  $A_1^- = \{\omega \in A^- / g_0(\omega) > 0\}$ , then  $\lambda(A_1^-) > 0$ .

Consider the function  $g'_0$  defined by

$$\begin{cases} g'_0 = 0 \text{ on } A_1^- \\ g'_0 = g_0 \text{ elsewhere.} \end{cases} \quad (71)$$

We have

$$\begin{cases} g'_0 \in L^1(\mathbb{R}, d\omega) \cap L^2(\mathbb{R}, d\omega) \\ g'_0 \geq 0 \\ \int_{\mathbb{R}} g'_0(\omega) d\omega \leq \int_{\mathbb{R}} g_0(\omega) d\omega \leq 1. \end{cases} \quad (72)$$

On the other hand

$$\begin{cases} S_1 \text{ even} \Rightarrow A^- \text{ symmetric with respect to } 0 \\ g_0 \text{ even} \Rightarrow A_1^- \text{ symmetric with respect to } 0. \end{cases}$$

Then  $g'_0$  is even and  $g'_0 \in E'$ . Moreover

$$\|S_1 - g'_0\|_{L^2(\mathbb{R}, d\omega)}^2 = \|S_1 - g'_0\|_{L^2(A^-, d\omega)}^2 + \|S_1 - g'_0\|_{L^2(A^+, d\omega)}^2 \quad (73)$$

$$= \|S_1 - g'_0\|_{L^2(A_1^-, d\omega)}^2 + \|S_1 - g_0\|_{L^2(A^- \setminus A_1^-, d\omega)}^2 + \|S_1 - g_0\|_{L^2(A^+, d\omega)}^2 \quad (74)$$

On  $A_1^-$ , we have:

$$0 \leq g'_0 - S_1 < g_0 - S_1, \quad (75)$$

then

$$\|S_1 - g'_0\|_{L^2(\mathbb{R}, d\omega)}^2 < \|S_1 - g_0\|_{L^2(\mathbb{R}, d\omega)}^2. \quad (76)$$

This leads to a contradiction and thus we conclude that

$$g_0 = 0 \text{ on } A^-. \quad (77)$$

Let us show that  $g_0 \leq S_1$  on  $A^+$  (a.s.). Suppose the contrary : in this case, if  $A_1^+ = \{\omega \in A^+ / g_0(\omega) > S_1(\omega)\}$ , then  $\lambda(A_1^+) > 0$ .

Let  $g'_0$  be the function defined by

$$\begin{cases} g'_0 = S_1 \text{ on } A_1^+ \\ g'_0 = g_0 \text{ elsewhere.} \end{cases} \quad (78)$$

As previously, we have

$$\begin{cases} g'_0 \in L^1(\mathbb{R}, d\omega) \cap L^2(\mathbb{R}, d\omega) \\ g'_0 \geq 0 \\ \int_{\mathbb{R}} g'_0(\omega) d\omega \leq \int_{\mathbb{R}} g_0(\omega) d\omega \leq 1, \end{cases} \quad (79)$$

and

$$\begin{cases} S_1 \text{ even} \Rightarrow A^+ \text{ symmetric with respect to } 0 \\ g_0 \text{ even} \Rightarrow A_1^+ \text{ symmetric with respect to } 0. \end{cases}$$

Then  $g'_0$  is even and  $g'_0 \in E'$ . Moreover

$$\|S_1 - g'_0\|_{L^2(\mathbb{R}, d\omega)}^2 = \|S_1 - g'_0\|_{L^2(A^-, d\omega)}^2 + \|S_1 - g'_0\|_{L^2(A^+, d\omega)}^2 \quad (80)$$

$$= \|S_1 - g_0\|_{L^2(A^-, d\omega)}^2 + \|S_1 - g'_0\|_{L^2(A_1^+, d\omega)}^2 + \|S_1 - g_0\|_{L^2(A^+ \setminus A_1^+, d\omega)}^2 \quad (81)$$

On  $A_1^+$ , we have:

$$0 = g'_0 - S_1 < g_0 - S_1 \quad (82)$$

then

$$\|S_1 - g'_0\|_{L^2(\mathbb{R}, d\omega)}^2 < \|S_1 - g_0\|_{L^2(\mathbb{R}, d\omega)}^2. \quad (83)$$

This leads to a contradiction, from which we obtain

$$g_0 \leq S_1 \text{ on } A^+. \quad (84)$$

Finally let us show that  $\int_{\mathbb{R}} g_0(\omega) d\omega = 1$ . On the contrary suppose that  $\int_{\mathbb{R}} g_0(\omega) d\omega < 1$ . Let  $g'_0$  be the function defined by

$$\begin{cases} g'_0 = g_0 + \alpha(S_1 - g_0) & \text{on } A^+ \\ g'_0 = g_0 = 0 & \text{on } A^-, \end{cases} \quad (85)$$

where  $\alpha$  is such that  $\int_{A^+} g_0(\omega) d\omega + \alpha \int_{A^+} (S_1(\omega) - g_0(\omega)) d\omega = 1$ .

It is possible to define such an  $\alpha$  because

$$\int_{\mathbb{R}} g_0(\omega) d\omega < 1 \Leftrightarrow \int_{A^+} g_0(\omega) d\omega < 1,$$

and

$$\int_{\mathbb{R}} S_1(\omega) d\omega = 1 \Rightarrow \int_{A^+} S_1^+(\omega) d\omega = \int_{A^-} S_1^-(\omega) d\omega + 1 > 1,$$

implies

$$\int_{A^+} (S_1(\omega) - g_0(\omega)) d\omega > 0.$$

We have  $\alpha > 0$  and  $\alpha < 1$ . Actually, the function  $\alpha \mapsto \int_{A^+} g_0(\omega) d\omega + \alpha \int_{A^+} (S_1(\omega) - g_0(\omega)) d\omega$  is affine, strictly increasing and

$$\begin{cases} \text{for } \alpha = 0, \text{ this function equals } & \int_{A^+} g_0(\omega) d\omega < 1, \\ \text{for } \alpha = 1, \text{ this function equals } & \int_{A^+} S_1(\omega) d\omega > 1. \end{cases}$$

Consequently  $g'_0 \in E$  and moreover

$$\|S_1 - g'_0\|_{L^2(\mathbb{R}, d\omega)}^2 = \|S_1 - g'_0\|_{L^2(A^-, d\omega)}^2 + \|S_1 - g'_0\|_{L^2(A^+, d\omega)}^2 \quad (86)$$

$$= \|S_1 - g_0\|_{L^2(A^-, d\omega)}^2 + \|S_1 - (g_0 + \alpha(S_1 - g_0))\|_{L^2(A^+, d\omega)}^2 \quad (87)$$

$$= \|S_1 - g_0\|_{L^2(A^-, d\omega)}^2 + \|(1 - \alpha)(S_1 - g_0)\|_{L^2(A^+, d\omega)}^2 \quad (88)$$

$$< \|S_1 - g_0\|_{L^2(\mathbb{R}, d\omega)}^2. \quad (89)$$

We obtain a contradiction, hence we conclude that  $\int_{\mathbb{R}} g_0(\omega) d\omega = 1$  and

$$g_0 \in E. \quad (90)$$

Let us now end the proof.

As  $E \subset E'$ , we get

$$\inf_{g \in E'} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)} \leq \inf_{g \in E} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)}. \quad (91)$$



But

$$\inf_{g \in E'} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)} = \|S_1 - g_0\|_{L^2(\mathbb{R}, d\omega)}, \quad (92)$$

with  $g_0 \in E$ , then

$$\inf_{g \in E'} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)} = \inf_{g \in E} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)} = \|S_1 - g_0\|_{L^2(\mathbb{R}, d\omega)}, \quad (93)$$

i.e.  $\inf_{g \in E} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)}$  is reached at the single point  $g_0 \in E$ .  $\square$

## 5 Minoration result

If the coefficient  $f_1$  is "large", the error  $I$  can be evaluated more precisely, because of the following minoration result :

### Proposition 5.1

If

$$f_1^2 > \sum_{n \geq 2} f_n^2 (n!) n (\sup(|a|, 1))^{n-1},$$

then

$$I \geq \left( f_1^2 - \sum_{n \geq 2} n! f_n^2 n (\sup(|a|, 1))^{n-1} \right) \sqrt{2\pi} \inf_{g \in E} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)}. \quad (94)$$

**Proof.** Writing for all  $g \in E$

$$\|R - \Psi(\hat{g})\|_{L^2(\mathbb{R}, dt)} = \|\Psi(R_1) - \Psi(\hat{g})\|_{L^2(\mathbb{R}, dt)} \quad (95)$$

$$\geq f_1^2 \|R_1 - \hat{g}\|_{L^2(\mathbb{R}, dt)} - \left\| \sum_{n \geq 2} n! f_n^2 (R_1^n - \hat{g}^n) \right\|_{L^2(\mathbb{R}, dt)} \quad (96)$$

$$\geq f_1^2 \|R_1 - \hat{g}\|_{L^2(\mathbb{R}, dt)} - \quad (97)$$

$$\sum_{n \geq 2} n! f_n^2 n (\sup(|a|, 1))^{n-1} \|R_1 - \hat{g}\|_{L^2(\mathbb{R}, dt)} \quad (98)$$

$$\geq \left( f_1^2 - \sum_{n \geq 2} n! f_n^2 n (\sup(|a|, 1))^{n-1} \right) \sqrt{2\pi} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)}. \quad (99)$$

we get

$$\inf_{g \in E} \|R - \Psi(\hat{g})\|_{L^2(\mathbb{R}, dt)} \geq \left( f_1^2 - \sum_{n \geq 2} n! f_n^2 n (\sup(|a|, 1))^{n-1} \right) \sqrt{2\pi} \inf_{g \in E} \|S_1 - g\|_{L^2(\mathbb{R}, d\omega)}. \quad (100)$$

□

## 6 Numerical experiments

We present here some numerical examples where the hypotheses of section 3 are verified.

### 6.1 First example of spectral density

The spectral density is given by:

$$S(\omega) = \frac{1}{2\pi} \frac{100}{270} \frac{1 + 0.6558\omega^2}{(1 + 0.2459\omega^2)^{11/6}}.$$

(We have  $\omega \mapsto \omega S(\omega) \in L^2(\mathbb{R}, d\omega)$ )

This is a Von Karman spectrum, often used as a model in atmospheric turbulence.

Four statistical moments are given

$$\mu_1 = 0, \quad \mu_2 = 1, \quad \mu_3 = 2, \quad \mu_4 = 9.$$

This choice of moments obviously assures the non-Gaussian nature of the process.

The obtained probability density function is

$$p(y) = e^{-\lambda_0 - 1} \exp\left(-\sum_{k=1}^4 \lambda_k y^k\right),$$

with the values

$\lambda_0$	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$
-0.3179	0.840332	0.915891	-0.393295	0.0424441

We consider the model  $f^5(G_t^5) = \sum_{n=1}^5 f_n H_n(G_t^5)$  and the function  $\Psi^5$  as an approximation of  $\Psi$  (where  $\Psi^M(Z) = \sum_{n=1}^M n! f_n^2 Z^n, \forall M \in \mathbb{N}^*$ ). The hypotheses of lemma 3.1 are verified with  $a \approx -0.0280$  and  $b \approx -0.0236$ .

The chosen spectral domain is  $\Omega = [-50, 50]$ rad/s, discretized in 1024 points. The temporal domain is chosen and discretized in order to verify the hypotheses of the Shannon sampling theorem. We can see on figure 1 the function  $S_1$  obtained as inverse Fourier transform of  $(\Psi^5)^{-1}(R)$ .

This function being non-negative, it must be chosen for spectral density of  $(G_t, t \in \mathbb{R})$ . We obtain the same function as by the minimization procedure but the present method is simpler and faster. If instead of fixing the four-first moments, the given one-dimensional marginal distribution is the Pareto distribution with parameters (1,4), i.e. the distribution with density function  $f(x) = \frac{1}{x^5} \mathbb{I}_{[1,+\infty[}(x)$  which is centered and normalized, we obtain the reciprocal cumulative distribution function:

$$F_Y^{-1}(y) = \frac{3(1-y)^{-1/4} - 4}{\sqrt{2}}. \quad (101)$$

The hypotheses of lemma 3.1 are verified with  $a \approx -0.0622$  and  $b \approx -0.0352$ . The function  $S_1$  obtained in figure 2 is still non-negative.

## 6.2 Second example of spectral density

The spectral density function is given by:

$$S(\omega) = \frac{e^{5/4}}{|\omega|^5} \exp\left(-\frac{5}{4\omega^4}\right).$$

(We have  $\omega \mapsto \omega S(\omega) \in L^2(\mathbb{R}, d\omega)$ )

This kind of spectrum is used in ocean engineering (Pierson-Moskowitz spectrum). Four moments are given

$$\mu_1 = 0, \quad \mu_2 = 1, \quad \mu_3 = 1, \quad \mu_4 = 4.$$

The coefficients values  $\lambda_k$  involved in the maximum entropy distribution are

$\lambda_0$	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$
-0.1081	0.744447	0.519957	-0.32643	0.058711

The Hermite development and the entire series  $\Psi$  are truncated at the order 4. The hypotheses of lemma 3.1 are verified with  $a \approx -0.8074$  and  $b \approx -0.7240$ . The function  $S_1$  resulting of the inverse Fourier transform of  $(\Psi^4)^{-1}(R)$  is not always non-negative (figure 3).

Let's consider now as one-dimensional distribution the centered and normalized Rayleigh distribution with parameter 1. The reciprocal cumulative distribution function is

$$F_Y^{-1}(y) = \frac{\sqrt{-2 \ln(1-y)} - \sqrt{\pi/2}}{\sqrt{2 - \pi/2}}.$$

The Hermite development is truncated at the order 5. The hypotheses of lemma 3.1 are verified with  $a \approx -0.7165$  and  $b \approx -0.6836$ . The obtained function  $S_1$  is still not always non-negative (figure 4).

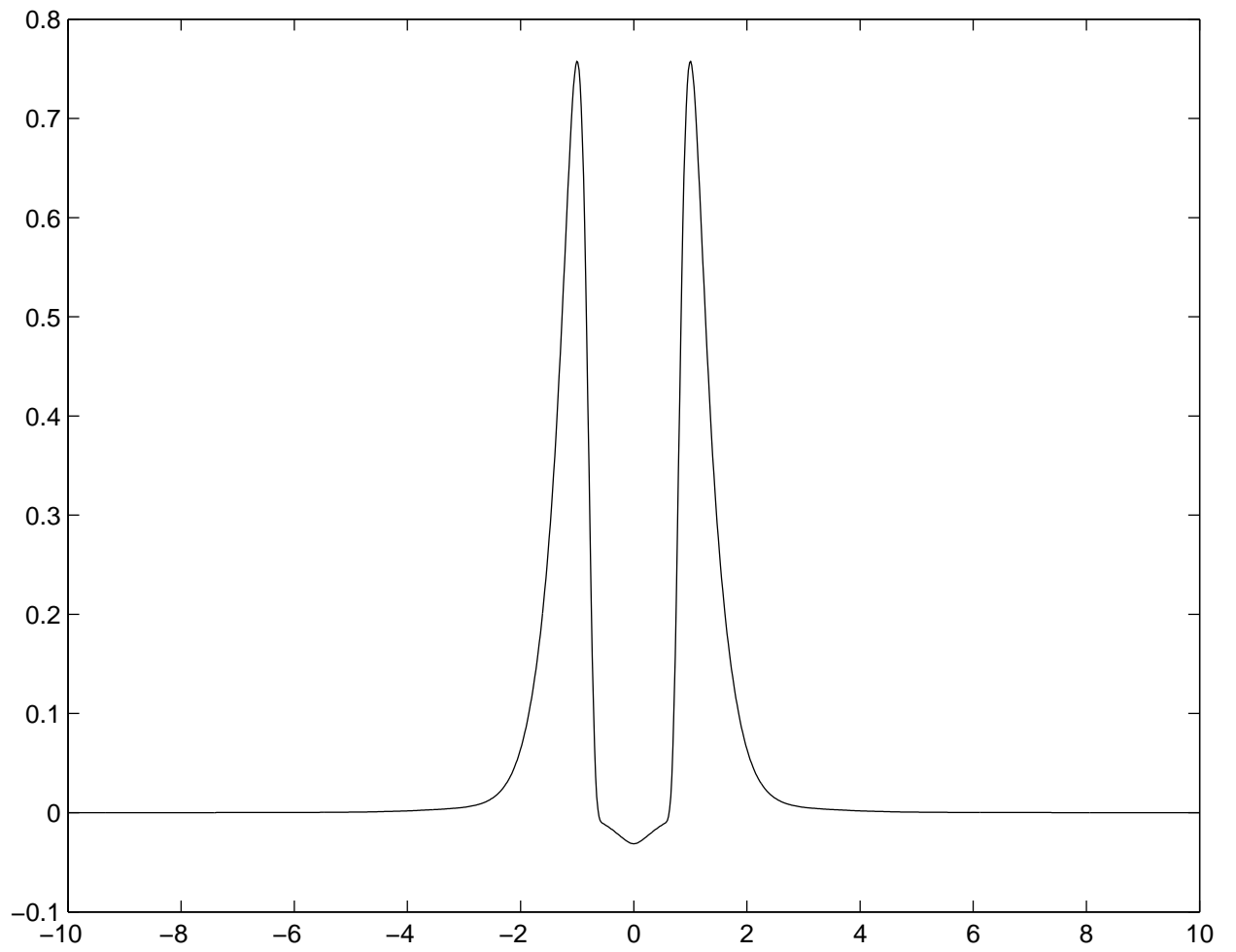


FIGURE 1:

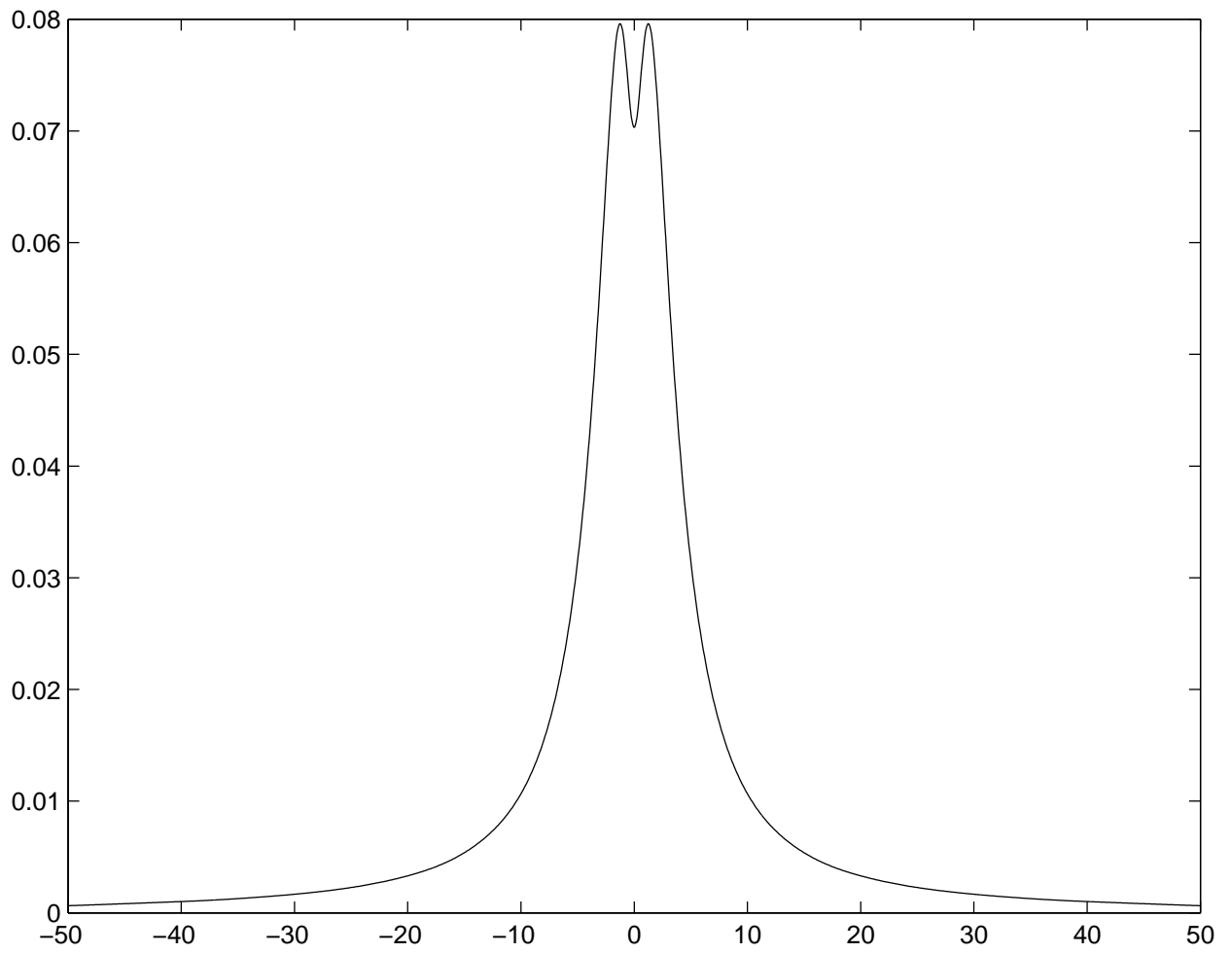


FIGURE 2:

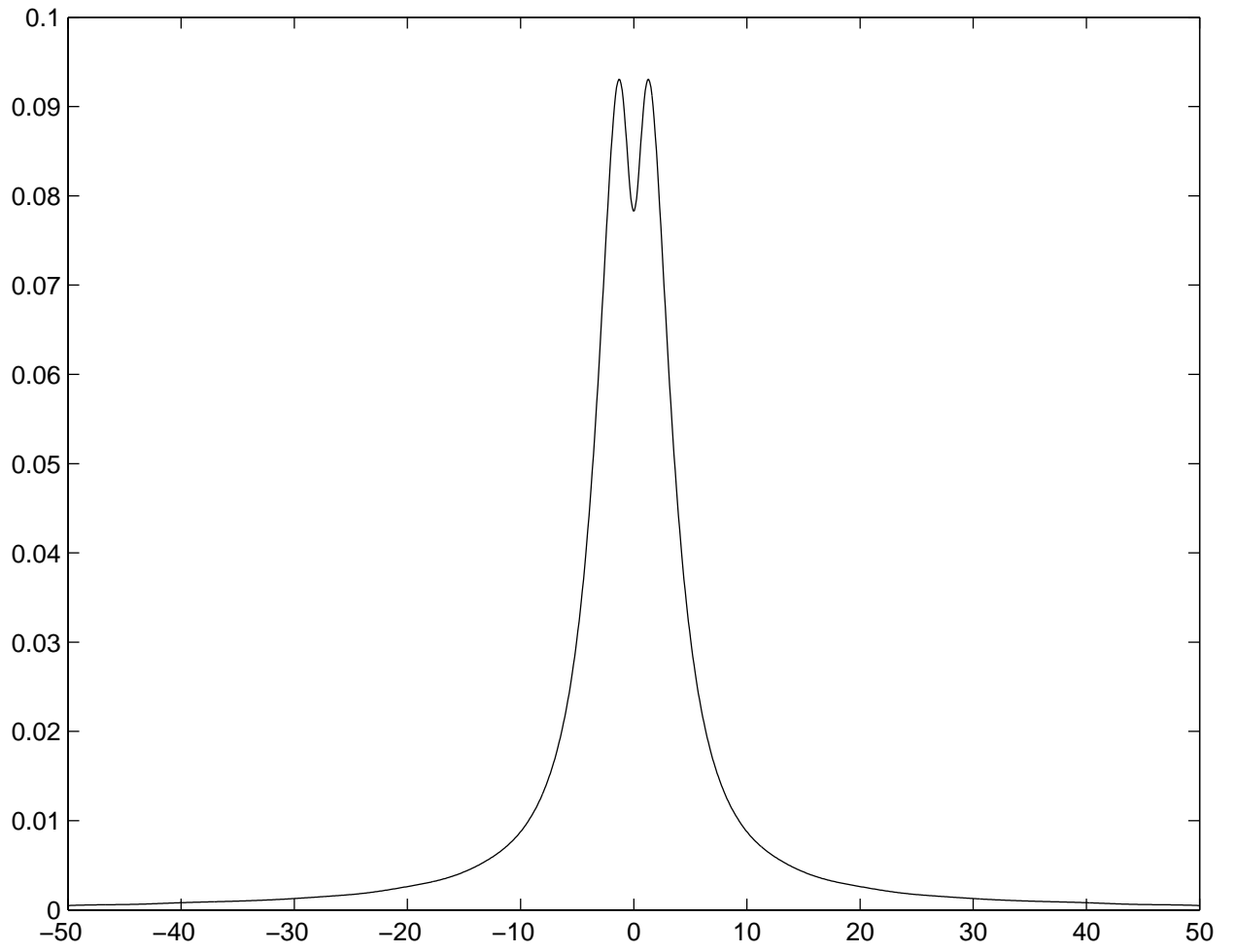


FIGURE 3:

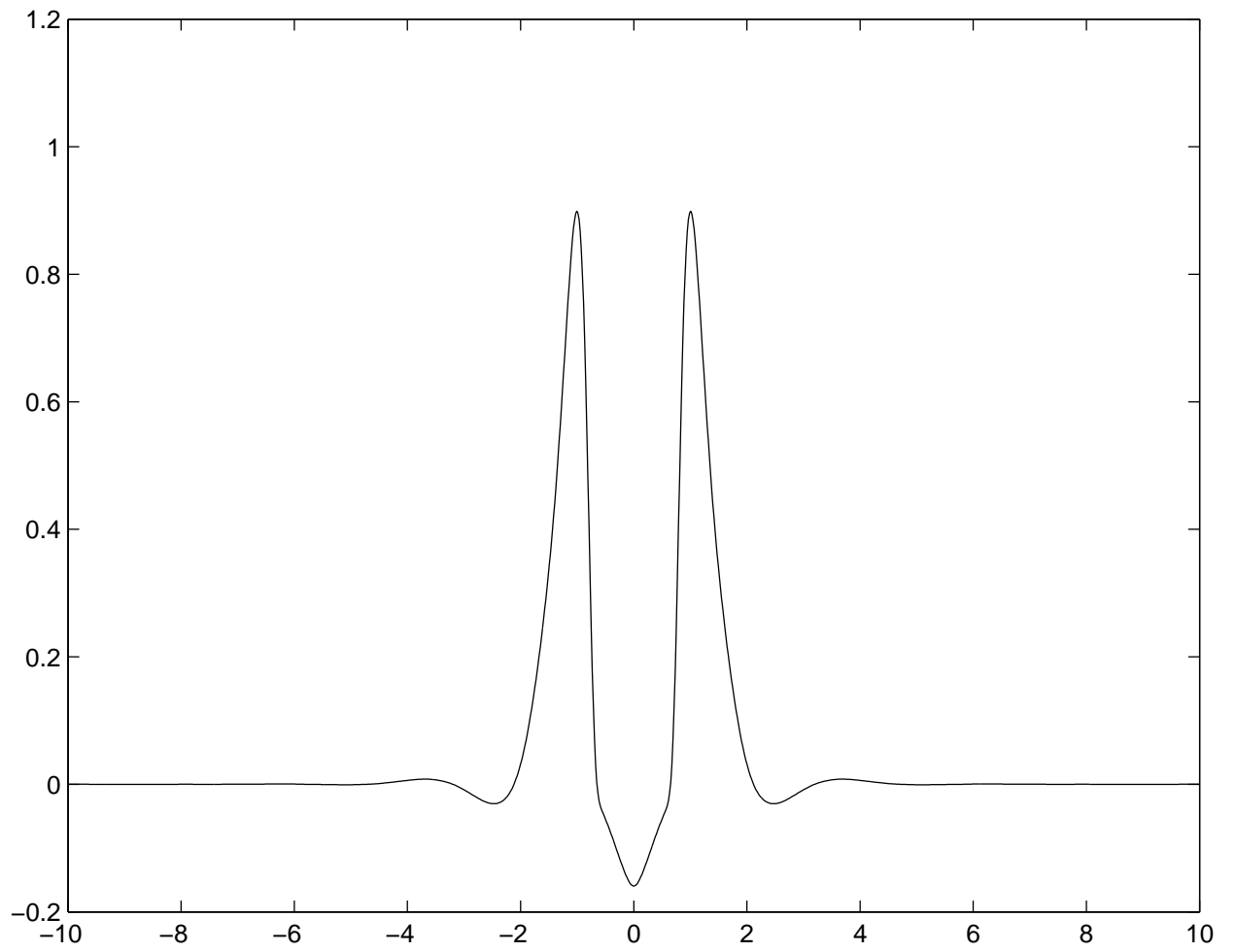


FIGURE 4:



**Figure 1** : Von Karman spectrum/moments fixed

**Figure 2** : Von Karman spectrum/Pareto distribution

**Figure 3** : Pierson-Moskowitz spectrum/moments fixed

**Figure 4** : Pierson-Moskowitz spectrum/Rayleigh distribution

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