A probabilistic interpretation and stochastic particle approximations of the 3-dimensional Navier-Stokes equations

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Abstract

We develop a probabilistic interpretation of local mild solutions of the three dimensional Navier-Stokes equation in the L^p spaces, when the initial vorticity field is integrable. This is done by associating a generalized nonlinear diffusion of the McKean-Vlasov type with the solution of the corresponding vortex equation. We then construct trajectorial (chaotic) stochastic particle approximations of this nonlinear process. These results provide the first complete proof of convergence of a stochastic vortex method for the Navier-Stokes equation in three dimensions, and rectify the algorithm conjectured by Esposito and Pulvirenti in 1989. Our techniques rely on a fine regularity study of the vortex equation in the supercritical L^p spaces, and on an extension of the classic McKean-Vlasov model, which incorporates the derivative of the stochastic flow of the nonlinear process to explain the vortex stretching phenomenon proper to dimension three.

1 Introduction

The Navier-Stokes equation for an homogeneous and incompressible fluid in the whole space or plane, is given by

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla\mathbf{p}; \\ &div\ \mathbf{u}(t,x) = 0; \quad \mathbf{u}(t,x)\to 0 \text{ as } |x|\to\infty, \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the velocity field, $\mathbf{p}$ is the pressure function and $\nu > 0$ is the viscosity coefficient assumed to be constant.
In this work we develop a probabilistic interpretation of the Navier-Stokes equation (1) in three dimensions. More precisely, we will consider the vortex equation satisfied by the vorticity field *curl* $\mathbf{u}$, and we will show in a general functional framework that it can be viewed as a generalized McKean-Vlasov equation associated with a nonlinear diffusion process. As a consequence, we will construct and prove the convergence of a stochastic particle method for the solution of (1) in that functional setting.

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Thirty years ago, Chorin [9] proposed an heuristical probabilistic algorithm to numerically simulate the solution of the Navier Stokes equation in two dimensions, by approximating the (scalar) vorticity function by random interacting "point vortices". The convergence of Chorin's vortex method was first mathematically proved in 1982 by Marchioro and Pulvirenti [21], who interpreted the vortex equation in two dimensions with bounded and integrable initial condition as a generalized McKean-Vlasov equation (with a singular interacting kernel) associated with a nonlinear diffusion. (For general expositions on the McKean-Vlasov model and nonlinear processes, we refer the reader to Sznitman [31] or Méléard [23].) Following the pioneering ideas of McKean [20], Marchioro and Pulvirenti defined then some stochastic systems of particles interacting weakly through cutoffed kernels, and for which the empirical measure converges at each time (when the number of particles tends to ∞) to the solution of the vortex equation. The results of [21] were improved by Méléard [24], [25], who showed the convergence in the path space of the empirical measures of the interacting particle systems or, equivalently, the propagation of chaos for the system of particles. (Propagation of chaos for a system of particles without cutoff was proved by Osada [27], but only for large viscosities and initial conditions is a bounded probability density).
A rigorous probabilistic interpretation and a stochastic vortex method for the Navier-Stokes equation in three dimensions have been open problems since the paper [21] appeared. An attempt to extend those results to the three dimensional case was done by Esposito and Pulvirenti [12], but this authors did not furnish rigorous mathematical proofs of crucial facts.

In three dimensions, the vorticity field $\mathbf{w} = curl\ \mathbf{u}$ is a solution of the nonlinear equation

$$\begin{aligned} &\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{w} = (\mathbf{w} \cdot \nabla)\mathbf{u} + \nu \Delta \mathbf{w}, \\ &div\ w_0 = 0, \end{aligned} \tag{2}$$

where, thanks to the condition of incompressibility, $div\ \mathbf{u} = 0$, and by the Biot-Savart law, the velocity field $\mathbf{u}$ is equal to

$$\mathbf{u}(t,x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \wedge \mathbf{w}(t,y) dy. \tag{3}$$

Here, $\wedge$ stands for the vectorial product in $\mathbb{R}^3$ and, with the notation $K(x) := -\frac{1}{4\pi}\frac{x}{|x|^3}$, the vectorial kernel $K(x) \wedge \cdot$ is the so-called *Biot-Savart kernel* in three dimensions. See for instance Chorin and Marsden [10] Ch. 1, Chorin [9] Ch.1 and Marchioro and Pulvirenti [22] for this facts and for background on vorticity.
The vectorial equation (2) is not conservative due to the *vortex stretching* term $(\mathbf{w} \cdot \nabla)\mathbf{u} = \sum_j \mathbf{w}_j \frac{\partial \mathbf{u}}{\partial x_j}$. In fact, vortex stretching lies in the heart of complex three dimensional phenomena such as transfer of energy and turbulence (see [9] Ch. 5), and is also related to the emergence of singularities (see Beale, Kato, Majda [1]).

In this work, we consider the vortex equation (2) with initial condition w_0 in L^p_3 (see the notation below). By adapting to the vortex equation the techniques for equation (1) in the so-called supercritical spaces (see e.g. Cannone [7] Ch.1), we shall first of all provide a local existence and global uniqueness result for the mild version of equation (2) in suitable L^p_3 spaces. These will be, when $\frac{3}{2} < p < 3$.

Then, we will assume that w_0 also belongs to L^1_3, and we will consider a probability density ρ_0 on $\mathbb{R}^3$ and a vectorial "weight function" $h_0 : \mathbb{R}^3 \to \mathbb{R}^3$, respectively given by

$$\rho_0(x) = \frac{|w_0(x)|}{\|w_0\|_1}, \quad \text{and} \quad h_0(x) = \frac{w_0}{\rho_0}(x).$$

Denote by $\mathcal{M}_{3\times3}$ the space of real 3×3 matrices and let $Id \in \mathcal{M}_{3\times3}$ be the identity matrix. Write also $\mathcal{C}_T$ for the space of continuous trajectories $\mathcal{C}_T := C([0,T], \mathbb{R}^3 \times \mathcal{M}_{3\times3})$. Our goal is to study the following nonlinear stochastic differential equation with values in $\mathbb{R}^3 \times \mathcal{M}_{3\times3}$:

$$\begin{aligned} X_t &= X_0 + \sqrt{2\nu}B_t + \int_0^t \int_{\mathcal{C}_T} [K(X_s - x(s)) \wedge \phi_s h_0(x(0))]\, P(dx, d\phi) ds \\ \Phi_t &= Id + \int_0^t \int_{\mathcal{C}_T} [\nabla K(X_s - x(s)) \wedge \phi_s h_0(x(0))]\, P(dx, d\phi) \Phi_s \, ds \end{aligned} \qquad t \in [0,T], \tag{4}$$

under the condition

$$law(X, \Phi) = P \text{ and } law(X_0) = \rho_0(x)dx.$$

(we are using here the notation $\nabla K(y) \wedge z = \nabla_y (K(y) \wedge z)$ for $y, z \in \mathbb{R}^3, y \neq 0$.)
We shall prove uniqueness in law for (4) in certain class of probability measures P on $C([0,T], \mathbb{R}^3 \times \mathcal{M}_{3\times3})$, and establish an equivalence between weak solutions of (4) on $[0,T]$ in that class, and mild solutions $\mathbf{w}$ of (2) in $L^\infty([0,T], L^p_3)$ satisfying $w_0 \in L^1_3$. This correspondence will be given by the relation

$$\int_{\mathbb{R}^3} \mathbf{f}(y)\mathbf{w}(t,y)dy = E^P(\mathbf{f}(X_t)\Phi_t h_0(X_0)) \tag{5}$$

for functions $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$. In particular, we will obtain local existence (in time) for (4).

By the Biot-Savart law (3) and by (5), the (nonlinear) drift terms in (4) are indeed equal to $\mathbf{u}(s, X_s)$ and to $\nabla \mathbf{u}(s, X_s)\Phi_s$. The study of the nonlinear process (4) will thus require precise regularity estimates for the velocity field $\mathbf{u}(t)$ associated to $\mathbf{w}(t)$. To that end, we will follow a similar strategy as in [13], by proving suitable Sobolev regularity for $\mathbf{w}$ and using then some continuity properties of the Biot-Savart operator (3) and classic embeddings of Sobolev spaces. Under the assumption that $w_0 \in L^p_3 \cap L^1_3$, the functions $\mathbf{u}(t)$ and $\nabla \mathbf{u}(t)$ turn out to be continuous and bounded for each $t \in]0,T]$, but with singularities at $t = 0$. Thus, we will also need to use and extend the techniques of [25] and [13] for singular drift terms to study the martingale problem associated with (4). Here, it will be crucial that the "vortex stretching process", Φ_t, associated with a mild solution $\mathbf{w}$ in $L^\infty([0,T], L^p_3)$ (with $\frac{3}{2} < p < 3$) is *a priori* bounded independently of the randomness.
Our second goal is to construct stochastic particle approximations of $\mathbf{w}$ and $\mathbf{u}$. We will follow a trajectorial approach in the same line of Bossy and Talay [3], Méléard [24] and [25], or Fontbona [13]. Namely, we will prove a propagation of chaos result for a system of particles $(X^{i,n,\varepsilon,R}, \Phi^{i,n,\varepsilon,R})_{i=1}^n, n \in \mathbb{N}$, where ε is a mollifying parameter of the kernel K and R is a cutoff threshold of the approximating vortex stretching processes $\Phi^{i,n,\varepsilon,R}$. Then, we will assume that the conditions ensuring existence of a local solution $\mathbf{w}$ of (2) hold, and we will prove that for suitable sequences $\varepsilon_n \to 0$ and $R > 0$ large enough the system $(X^{i,n,\varepsilon_n,R}, \Phi^{i,n,\varepsilon_n,R})$ is chaotic with limiting law P given by (4). From this, we will deduce the convergence to $\mathbf{w}$ of some "weighted" empirical process of the system (with time dependent vectorial weights), and the convergence of an "approximate velocity field"

to $\mathbf{u} = \mathbf{K}(\mathbf{w})$. This result is the first complete mathematical proof of convergence of a stochastic vortex method for the Navier-Stokes equation in three dimensions, and rectifies the method conjectured by Esposito and Pulvirenti in [12].
We are not able to provide an explicit convergence rate here, mainly due to the loss of regularity of $\mathbf{u}$ at $t = 0$. Such result could be obtained under additional regularity assumptions on the initial condition (as it is done by Méléard [24] in two dimensions). However, the convergence rate one can expect to obtain by this approach is far from being optimal. (This problem remains untreated even for the two-dimensional vortex method.)
We also point out that under the assumption that $w_0 \in L^p_3$ with $\frac{3}{2} < p < 3$, the SDE

$$\xi_t(x) = x + \sqrt{2\nu} B_t + \int_0^t \mathbf{u}(t, \xi_s(x)) ds \tag{6}$$

(with $\mathbf{u}$ given by (3)) will define a C^1 stochastic flow $\xi : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$, and the identity $(X, \Phi) = (\xi(X_0), \nabla_x \xi(X_0))$ will hold in a trajectorial sense. Thus, formula (5) is the fact that the vorticity is transported by the stochastic flow and stretched by its gradient, and generalizes the representation of the vorticity field in the inviscid case $\nu = 0$, in terms of the (deterministic) flow of the solution of the Euler equation (see [10] Ch. 1). A representation formula equivalent to (5) was partially established in [12], under the more restrictive assumption that w_0 and its Fourier transform are in L^1_3. Our interpretation of the vorticity in terms of the weight function h_0 is simpler than the one in [12]. We are inspired here in the approach of Méléard [24] and [25] in two dimensions, where vorticity was represented using a scalar weight, that is simply "transported" by a nonlinear diffusion process. We are also extending in this way the techniques of Jourdain [16] for dealing with signed measures in the McKean-Vlasov context. A representation formula in terms of stochastic flow was also proved in Esposito, Marra, Pulvirenti and Sciaretta [11], but these authors needed to restrict themselves to the equation (1) on the torus in order to define the underlying probability space.

A probabilistic interpretation of the three dimensional Navier-Stokes equation is also developed in Giet [15], in the case of a bounded domain and non-slip boundary condition. This author extends the ideas of Benachour, Roynette and Vallois [2] in two dimensions, by using a diffusion process with jumps to interpret the coupled system (2) with zero-order term, and a branching process to treat the boundary condition. At an advanced stage of this work, we also became aware of the work of Busnello, Frandoli and Romito [6], who also interpret the vorticity in terms of a stochastic flow and its gradient. These authors use a Bismut-Elworthy formula to recover the velocity field (extending the approach of Busnello in two dimensions [5]) and provide a local existence statement. In these works, the approaches are in some sense "dual" to ours: they are based on Feynman-Kac type formulae for the vorticity (in terms of the *linear* SDE (6) reversed in time) and aim to represent classical solutions of (2) by means of probabilistic objects. Due to this fact, they need to assume more regularity of the initial conditions. Furthermore, non of the aforementioned works [11], [6] or [15] relate the nonlinearity to a mean field interaction limit, and they do not lead to stochastic approximations of the solutions of the Navier-Stokes or the vortex equation.

1.1 Notation

- By $\mathcal{M}es^T$ we denote the space of measurable real valued functions on $[0,T] \times \mathbb{R}^3$.

- $C^{1,2}$ is the set of real valued functions on $[0,T]\times\mathbb{R}^3$ with continuous derivatives up to the first order in $t\in[0,T]$ and up to the second order in $x\in\mathbb{R}$. $C_b^{1,2}$ is the subspace of bounded functions in $C^{1,2}$ with bounded derivatives.

- $\mathcal{S}$ is the Schwartz space of infinitely differentiable functions on $\mathbb{R}^3$ all of whose derivatives remain bounded when multiplied by polynomials. $\mathcal{D}$ is the subspace of functions with compact support.

- For all $1\le p\le\infty$ we denote by L^p the space $L^p(\mathbb{R}^3)$ of real valued functions on $\mathbb{R}^3$. By $\|\cdot\|_p$ we denote the corresponding norm and p^* stands for the Hölder conjugate of p. We write $W^{i,p}=W^{i,p}(\mathbb{R}^3)$ for the Sobolev space of functions in L^p with partial derivatives up to the i-th order in L^p.

- If E is a space of real valued functions (defined on $\mathbb{R}^3$ or on $[0,T]\times\mathbb{R}^3$), then the notation E_3 is used for the space of $\mathbb{R}^3$-valued functions whose scalar components belong to E. If E is has a norm, then the norm on E_3 is denoted in the same way.

- For simplicity, if $\mathbf{f}:\mathbb{R}^3\to\mathbb{R}^3$ is a vector field and $Z:\mathbb{R}^3\to\mathcal{M}_{3\times3}$ is a matrix function, we will write $\mathbf{f}Z$ for the product vector $(\mathbf{f}^tZ)_i=\sum_{j=1}^3\mathbf{f}_jZ_{j,i}$. By $\nabla\mathbf{f}$ we denote the gradient of $\mathbf{f}$, that is the matrix $(\nabla\mathbf{f})_{i,j}=\frac{\partial\mathbf{f}_i}{\partial x_j}$.

- $\mathcal{F}(g)$ denotes the Fourier transform of $g:\mathbb{R}^3\to\mathbb{R}$, that is $\mathcal{F}(g)(\xi)=\int_{\mathbb{R}^3}e^{-2\pi i\xi\cdot x}g(x)dx$.

- C and $C(T)$ are finite positive constants that may change from line to line.

2 Preliminaries

Throughout this work, we assume that

- w_0 is a function $w_0:\mathbb{R}^3\to\mathbb{R}^3$ and a distribution in $\mathcal{D}_3'$.

Let G be the fundamental solution of the Laplace operator on $\mathbb{R}^3$. We will denote by K the kernel $K(x)=\nabla G(x)$, that is, the singular kernel given by

$$K(x):=-\frac{1}{4\pi}\frac{x}{|x|^3}\ \text{ for all } x\in\mathbb{R}^3\backslash\{0\}.$$

For functions $w:\mathbb{R}^3\to\mathbb{R}^3$, the *Biot-Savart operator* $\mathbf{K}$ is formally given by

$$w(x)\mapsto\mathbf{K}(w)(x):=\int_{\mathbb{R}^3}K(x-y)\wedge w(y)dy=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)}{|x-y|^3}\wedge w(y)dy. \tag{7}$$

Recall that a function $w\in\mathcal{D}_3'$ is said to have null divergence if $\int_{\mathbb{R}^3}\nabla f(x)w(x)dx=0$ for all $f\in\mathcal{D}$, and this is written $div\ w=0$.

Remark 2.1 *For all $w:\mathbb{R}^3\to\mathbb{R}^3$ that belongs to $\mathcal{D}_3'$, one has $div\ \mathbf{K}(w)=0$.*

2.1 Mild and weak forms of the 3-dimensional vortex equation

We introduce the notation $G^\nu : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$, for the heat kernel

$$G_t^\nu(x) = (4\pi\nu t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{4\nu t}\right), \tag{8}$$

where $\nu > 0$ is a fixed but arbitrary constant.

We will next define the two different notions of solution of the vortex equation (2) we shall work with. Here we denote by $\mathcal{D}^T$ the set of real functions of class C^∞ on $[0,T]\times\mathbb{R}^3$ having compact support.

Definition 2.1 *We say that* $\mathbf{w} \in \mathcal{M}es_3^T$ *is a mild solution of the vortex equation with initial condition* w_0 *(or "mild solution" for short), if the following conditions are satisfied:*

mildV0: *div* $w_0 = 0$.

mildV1: *For each* $j = 1,2,3$ *and* $t \in [0,T]\times\mathbb{R}^3$, *the distribution* $\mathbf{K}(\mathbf{w})_j(t,x) := \mathbf{K}(\mathbf{w}(t,\cdot))_j(x)$ *belongs to* $L^1_{loc}(dx)$.

mildV2: *For all* $i,j = 1,2,3$, *and every function* $f \in \mathcal{D}^T$

$$\int_0^T \int_{\mathbb{R}^3} |\mathbf{K}(\mathbf{w})_j(s,x)||\mathbf{w}_i(s,x)||f(s,x)|dx\ ds < \infty. \tag{9}$$

mildV3: *For almost every* (t,x) *in* $[0,T]\times\mathbb{R}^3$, *one has*

$$\mathbf{w}(t,x) = G_t^\nu * w_0(x) + \int_0^t \nabla G_{t-s}^\nu * \left[\mathbf{K}(\mathbf{w})\otimes\mathbf{w} - \mathbf{w}\otimes\mathbf{K}(\mathbf{w})\right](s,x)ds,$$

and the r.h.s. converges absolutely and belongs to $L^1_{loc}(dx)$.

Explicitly, **mildV3** is written as

$$\begin{aligned}\mathbf{w}(t,x) = &\int_{\mathbb{R}^3} G_t^\nu(x-y)w_0(y)dy \\ &+ \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y)\left[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{w}(s,y) - \mathbf{w}_j(s,y)\mathbf{K}(\mathbf{w})(s,y)\right]dy\ ds.\end{aligned} \tag{10}$$

It is implicitly assumed that a mild solution $\mathbf{w}(t)$ has null divergence for every t (as can be seen from the right hand side of (10)). We shall deal with this form for analytical purposes. In turn, the following form of the equation will appear more naturally in a probabilistic framework:

Definition 2.2 *A function* $\mathbf{w} \in \mathcal{M}es_3^T$ *is a weak solution of the vortex equation with initial condition* w_0 *(or "weak solution"), if the following conditions hold:*

weakV0: *div* $w_0 = 0$.

weakV1: *The integral* $\mathbf{K}(w)(t,x) := \mathbf{K}(w(t,\cdot))(x)$ *exists* $dxdt-a.e$ *on* $[0,T]\times\mathbb{R}^3$*, and* $\mathbf{K}(\mathbf{w})(t,\cdot)$ *and its gradient* $\nabla\mathbf{K}(w)(t,\cdot)$ *are distributions with components in* $L^1_{loc}(dx)$*.*

weakV2: *For all* $f\in\mathcal{D}^T$ *and* $i,j=1,2,3$ *and all* $t\in[0,T]$ *on has*

$$\int_0^T\int_{\mathbb{R}^3}|f(s,x)|\left[|\mathbf{K}(\mathbf{w})_j(s,x)|+\left|\frac{\partial\mathbf{K}(\mathbf{w})_j}{\partial x_k}(s,x)\right|\right]|\mathbf{w}_i(s,x)|dx\;ds<\infty. \tag{11}$$

weakV3: *For all* $\mathbf{f}\in\mathcal{D}_3^T$*, we have*

$$\int_{\mathbb{R}^3}\mathbf{f}(t,y)\mathbf{w}(t,y)dy=\int_{\mathbb{R}^3}\mathbf{f}(0,y)w_0(y)dy+\int_0^t\int_{\mathbb{R}^3}\left[\frac{\partial\mathbf{f}}{\partial s}(s,y)+\nu\triangle\mathbf{f}(s,y)\right.$$
$$\left.+\mathbf{K}(\mathbf{w})(s,y)\nabla\mathbf{f}(s,y)+\mathbf{f}(s,y)\nabla\mathbf{K}(\mathbf{w})(s,y)\right]\mathbf{w}(s,y)\;dy\;ds. \tag{12}$$

We will refer to (10) (resp. (12)) as the *mild equation* (resp. the *weak equation*). These two forms are not equivalent in general. By the moment, we can assert that under additional integrability assumptions, a weak solution satisfies an "intermediate mild form":

Lemma 2.1 *Assume that* $\mathbf{w}\in\mathcal{M}es_3^T$ *is a weak solution, and that*

$$\int_0^t\int_{(\mathbb{R}^3)^2}\sum_{i,j=1}^3\left|\frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\right||\mathbf{K}(\mathbf{w})_j(s,y)||\psi_i(x)||\mathbf{w}_i(s,y)|dx\;dy\;ds$$

and

$$\int_0^t\int_{(\mathbb{R}^3)^2}\sum_{i,j=1}^3|G^\nu_{t-s}(x-y)|\left|\frac{\partial\mathbf{K}(\mathbf{w})_i}{\partial y_j}(s,y)\right||\psi_i(x)||\mathbf{w}_j(s,y)|dx\;dy\;ds$$

are finite for all $i,j=1,2,3$ *and* $\psi\in\mathcal{D}_3$*. Then,* $\mathbf{w}$ *satisfies*

$$\mathbf{w}(t,x)=G^\nu_t*w_0(x)+\int_0^t\sum_{j=1}^3\int_{\mathbb{R}^3}\left[\frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{w}(s,y)]\right.$$
$$\left.+G^\nu_{t-s}(x-y)[\mathbf{w}_j(s,y)\frac{\partial\mathbf{K}(\mathbf{w})}{\partial y_j}(s,y)]\right]dy\;ds. \tag{13}$$

Proof: Take fixed $\psi\in\mathcal{D}_3$ and $t\in[0,T]$ and define $\mathbf{f}_t:[0,t]\times\mathbb{R}^3\to\mathbb{R}^3$ by $\mathbf{f}_t(s,y)=G^\nu_{t-s}*\psi(y)$; this function is of class $(C_b^{1,2})_3$ and solves the backward heat equation on $[0,t]\times\mathbb{R}^3$ with final condition $\mathbf{f}(t,y)=\psi(y)$. If $\mathbf{w}$ is a solution of the weak vortex equation satisfying the hypothesis of the Lemma, by a density argument it also satisfies the weak equation **weakV3** with the function $\mathbf{f}_t(s,y)$ just defined. By Fubini's theorem, we deduce that (13) holds since $\psi\in\mathcal{D}_3$ is arbitrary.

□

Remark 2.2 *Formally, by integrating by parts the last term in the l.h.s. of (13), one can check that a weak solution* $\mathbf{w}(s)$ *as in the previous lemma is also a mild solution if its divergence is null for all* s*. Of course, to make this reasoning rigorous we must ensure that* $\mathbf{w}$ *and* $\mathbf{K}(\mathbf{w})$ *belong to suitable functional spaces . The passage from weak to mild solutions will be important to establish an equivalence between probabilistic and analytic objects.*

2.2 Continuity of the Biot-Savart operator

In order to study the vortex equation in some Lebesgue and Sobolev spaces, we will state here some fundamental continuity results for the operators $\mathbf{K}$ and $\nabla\mathbf{K}$ acting is this type of spaces. These results will also be used to deduce from the properties of the vorticity field some regularity properties of the velocity field. The proofs are given in Section **3.3**.

Lemma 2.2 *Let $1 < p < 3$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$.*

i) For every $w \in L_3^p$, the integral (7) is absolutely convergent for almost every x and one has $\mathbf{K}(w) \in L_3^q$. There exists further a positive constant $\widetilde{C}_{p,q}$ such that

$$\|\mathbf{K}(w)\|_q \leq \widetilde{C}_{p,q} \|w\|_p \tag{14}$$

for all $w \in L_3^p$.

ii) If moreover $w \in W^{1,p}$, then we have $\mathbf{K}(w) \in W_3^{1,q}$, with $\frac{\partial}{\partial x_k}\mathbf{K}(w) = \mathbf{K}\left(\frac{\partial w}{\partial x_k}\right)$, and

$$\left\|\frac{\partial \mathbf{K}(w)}{\partial x_k}\right\|_q \leq \widetilde{C}_{p,q} \left\|\frac{\partial w}{\partial x_k}\right\|_p \tag{15}$$

for all $k = 1, 2, 3$.

The proof of Lemma 2.2 will use some elements from Riesz potentials. Intuitively, part *ii)* results from taking the derivatives of w when differentiating the convolution $\mathbf{K}(w)$. It will ensure that the velocity field at time t, given by $\mathbf{K}(\mathbf{w})(t)$, belongs to $W_3^{1,q}$ (with $q = \frac{3p}{3-p}$) if $\mathbf{w}(t) \in W_3^{1,p}$ and $p \in]1, 3[$. It is however natural to expect the gradient of the velocity field to have the same regularity as the vorticity field. This fact will follow from next lemma, which states the continuity in $W_3^{1,p}$ of the operator $w \mapsto \nabla\mathbf{K}(w) = \nabla(\mathbf{K}(w))$:

Lemma 2.3 *Let $1 < p < \infty$. If $w \in L_3^p$, then each component of the derivative (in distribution sense) $\frac{\partial}{\partial x_k}\mathbf{K}(w)$ is a linear combination of singular integrals of the components of w (in the precise sense given in Theorem 3.4 of Section* **3.3***). We deduce the following continuity estimates:*

i) For all $w \in L_3^p$, we have $\frac{\partial}{\partial x_k}\mathbf{K}(w) \in L_3^p$ for $k = 1, 2, 3$. There exists further a positive constant $\widetilde{C}_p$ depending only on p such that

$$\left\|\frac{\partial \mathbf{K}(w)_j}{\partial x_k}\right\|_p \leq \widetilde{C}_p \|w\|_p \tag{16}$$

for all $j = 1, 2, 3$, where $\mathbf{K}(w)_j$ is the j-th component of $\mathbf{K}(w)$.

ii) If moreover $w \in W_3^{1,p}$, then we have $\frac{\partial}{\partial x_k}\mathbf{K}(w) \in W_3^{1,p}$, and

$$\left\|\frac{\partial^2 \mathbf{K}(w)_j}{\partial x_l \partial x_k}\right\|_p \leq \widetilde{C}_p \left\|\frac{\partial w}{\partial x_l}\right\|_p \tag{17}$$

for all $l = 1, 2, 3$.

Roughly, here we shall use the facts that one can take also derivatives to K, when differentiating $\mathbf{K}(w)$, and that the singular kernel ∇K has the properties required to define a singular integral operator by (principal value) convolution.

3 The vortex equation in the supercritical L^p spaces

We will now establish a general functional framework to study the vortex equation and prove all analytic results we need later on for probabilistic purposes. We assume in the sequel that

- the initial condition w_0 belongs to L^p_3 for some $p \in [1, \infty]$.

First, we will prove local existence and global uniqueness for the mild vortex equation (10), by adapting to the vortex setting general techniques for the usual (velocity field) Navier-Stokes equation in the so-called super-critical spaces (see Cannone [7], Ch.1). These particular results could be deduced from the analogous statements in [7]; our aim however is to establish precise regularity estimates of the velocity field and its gradient (in Hölder and L^∞ norms), in connection with properties of the vorticity field. This type of result is easier to obtain by studying directly the vortex equation. Furthermore, our probabilistic statements (in particular the construction of stochastic particle approximations) will require that existence and regularity statements hold simultaneously for the vortex equation (10) and for a family of approximating equations involving mollified kernels K_ε. We thus need to make explicit the role played by K.

We start with the following well known estimates:

Lemma 3.1 *Let G^ν be the heat kernel defined in (8) and $m \in [1, \infty]$. There exist two positive constants $c(m)$ and $c'(m)$ such that for all $t > 0$,*

$$\|G^\nu_t\|_m \leq c(m)(\nu t)^{-\frac{3}{2}+\frac{3}{2m}} \quad \text{and} \tag{18}$$

$$\|\nabla G^\nu_t\|_m \leq c'(m)(\nu t)^{-2+\frac{3}{2m}}. \tag{19}$$

We shall also need the general version of Young's inequality: if $f \in L^m$ and $g \in L^k$, with $1 \leq m, k \leq \infty$, and $\frac{1}{r} = \frac{1}{m} + \frac{1}{k} - 1 \geq 0$, then,

$$f * g \in L^r \text{ and } \|f * g\|_r \leq \|f\|_m \|g\|_k. \tag{20}$$

We easily deduce the following

Lemma 3.2 *Let $p \in [1, \infty]$, $r \geq p$ and $w_0 \in L^p_3$. There exist positive constants $\overline{C}_0(p), \overline{C}_1(p)$, $\overline{C}_0(p;r)$ and $\overline{C}_1(p;r)$ such that for all $t > 0$,*

i) $\|G^\nu_t * w_0\|_p \leq \overline{C}_0(p)\|w_0\|_p$, *ii)* $\|\nabla G^\nu_t * w_0\|_p \leq \overline{C}_1(p)t^{-\frac{1}{2}}\|w_0\|_p$,

iii) $\|G^\nu_t * w_0\|_r \leq \overline{C}_0(p;r)t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\|w_0\|_p$, *iv)* $\|\nabla G^\nu_t * w_0\|_r \leq \overline{C}_1(p;r)t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\|w_0\|_p$.

According to Lemma 3.2, we define for $\mathbf{w} \in \mathcal{M}es^T_3$ and $p \in [1, \infty]$ the norms:

- $\|\mathbf{w}\|_{0,p,T} = \sup_{0 \leq t \leq T} \|\mathbf{w}(t)\|_p$,
- $\|\mathbf{w}\|_{1,p,T} = \sup_{0 \leq t \leq T} \left\{ \|\mathbf{w}(t)\|_p + t^{\frac{1}{2}} \sum_{k=1}^{3} \left\| \frac{\partial \mathbf{w}(t)}{\partial x_k} \right\|_p \right\}$,

and, for $r \geq p$,

- $|||\mathbf{w}|||_{0,r,(T;p)} = \sup_{0 \leq t \leq T} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|\mathbf{w}(t)\|_r$,

- $|||\mathbf{w}|||_{1,r,(T;p)} = \sup_{0 \leq t \leq T} \left\{ t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|\mathbf{w}(t)\|_r + t^{\frac{1}{2}+\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \sum_{k=1}^{3} \left\| \frac{\partial \mathbf{w}(t)}{\partial x_k} \right\|_r \right\}$.

The following is the notation for associated Banach spaces:

- $\mathbf{F}_{0,p,T} = \{\mathbf{w} \in \mathcal{M}es_3^T : |||\mathbf{w}|||_{0,p,T} < \infty\}$,
- $\mathbf{F}_{1,p,T} = \{\mathbf{w} \in \mathcal{M}es_3^T : |||\mathbf{w}|||_{1,p,T} < \infty\}$,
- $\mathbf{F}_{0,r,(T;p)} = \{\mathbf{w} \in \mathcal{M}es_3^T : |||\mathbf{w}|||_{0,r,(T;p)} < \infty\}$ and
- $\mathbf{F}_{1,r,(T;p)} = \{\mathbf{w} \in \mathcal{M}es_3^T : |||\mathbf{w}|||_{1,r,(T;p)} < \infty\}$

Observe that by Lemma 2.2 *i)* and Lemma 2.3 *i)*, a function $\mathbf{w} \in \mathbf{F}_{0,p,T}$ satisfies conditions **mildV1** and **weakV1** if $p \in]1,3[$, and condition **mildV2** if $p \in [\frac{3}{2},3[$.
It is worth noting also that the L_3^p-spaces, with $p \in]\frac{3}{2},3[$, are in correspondence *via* the operator $\mathbf{K}$ with the supercritical L^q-spaces for the velocity field, that is, the L_3^q-spaces with $q = \frac{3p}{3-p} \in]3,\infty[$. We shall prove existence and uniqueness results for the mild equation in $\mathbf{F}_{0,p,T}$ for these values of p. Then we will show that the solution belongs to $\mathbf{F}_{1,r,(T;p)}$ for all $r \in [p,\infty[$. Finer regularity results for $\mathbf{K}(\mathbf{w})$ will follow as a consequence of the continuity properties of $\mathbf{K}$ and $\nabla\mathbf{K}$ stated in Section **2.2**, and of classic Sobolev embbedings.

A key point to establish these facts is the continuity property of the bilinear term in (10), as an operator acting in some of the spaces previously defined. More precisely, given functions $\mathbf{w},\mathbf{v} \in \mathcal{M}es_3^T$, consider a function $\mathbf{B}(\mathbf{w},\mathbf{v}) : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ formally defined by

$$\mathbf{B}(\mathbf{w},\mathbf{v})(t,x) = \int_0^t \sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^{\nu}}{\partial y_j}(x-y)\left[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{v}(s,y) - \mathbf{v}_j(s,y)\mathbf{K}(\mathbf{w})(s,y)\right] dy\ ds. \tag{21}$$

Proposition 3.1 *Let $p,p' \in [1,\infty]$. Then, $\mathbf{B} : \mathbf{F}^2 \to \mathbf{F}'$ is well defined and continuous whenever*

i) $\frac{3}{2} \leq p < 3,\ \frac{3p}{6-p} \leq p' < \frac{3p}{6-2p}, \quad \mathbf{F} = \mathbf{F}_{0,p,T}$ *and* $\mathbf{F}' = \mathbf{F}_{0,p',T}$.

ii) $\frac{3}{2} \leq p < 3,\ p \leq l < 3,\ \frac{3l}{6-l} \leq l' < \frac{3l}{6-2l}, \quad \mathbf{F} = \mathbf{F}_{0,l,(T;p)}$ *and* $\mathbf{F}' = \mathbf{F}_{0,l',(T;p)}$.

iii) $\frac{3}{2} \leq p < 3,\ \frac{3p}{6-p} \leq p' < \frac{3p}{6-2p}, \quad \mathbf{F} = \mathbf{F}_{1,p,T}$ *and* $\mathbf{F}' = \mathbf{F}_{1,p',T}$.

iv) $\frac{3}{2} \leq p < 3,\ p \leq l < 3,\ \frac{3l}{6-l} \leq l' < \frac{3l}{6-2l}, \quad \mathbf{F} = \mathbf{F}_{1,l,(T;p)}$ *and* $\mathbf{F}' = \mathbf{F}_{1,l',(T;p)}$.

Proof: The following formula will be useful: if $\beta(\varepsilon,\theta) = \int_0^1 x^{\varepsilon-1}(1-x)^{\theta-1}dx$ is the Beta function of real parameters $\varepsilon,\theta > 0$, then

$$\int_0^t s^{\varepsilon-1}(t-s)^{\theta-1}\ ds = t^{\varepsilon+\theta-1}\beta(\varepsilon,\theta),\ \forall t > 0. \tag{22}$$

i) Let $\mathbf{w}, \mathbf{v} \in \mathbf{F}_{0,p,T}$ with $\frac{3}{2} \le p < 3$. We take the $L^{p'}$ norm to the $i-$th component of (21), and apply Young's inequality (20), with $r = p'$, $m = (\frac{4}{3} + \frac{1}{p'} - \frac{2}{p})^{-1}$ and $k = \frac{3p}{6-p}$ (notice that the constraint on p and p' ensures that $1 \le m < \frac{3}{2}$). This yields

$$\begin{aligned}\|\mathbf{B}(\mathbf{w},\mathbf{v})_i(t)\|_{p'} &\le C \sum_{j=1}^{3} \int_0^t \|\nabla G^{\nu}_{t-s}\|_m \left(\|\mathbf{v}_j(s)\ \mathbf{K}(\mathbf{w})_i(s)\|_k + \|\mathbf{v}_i(s)\ \mathbf{K}(\mathbf{w})_j(s)\|_k\right)\ ds \\ &\le C \int_0^t (t-s)^{\frac{3}{2m}-2} \|\mathbf{w}(s)\|_p\ \|\mathbf{v}(s)\|_p\ ds\end{aligned}$$

by using also estimate (19), Hölder's inequality, and inequality (14) for the Biot-Savart operator. Therefore,

$$\|\mathbf{B}(\mathbf{w},\mathbf{v})_i(t)\|_{p'} \le C t^{1-3(\frac{1}{p}-\frac{1}{2p'})} \|\!|\mathbf{w}|\!\|_{0,p,T} \|\!|\mathbf{v}|\!\|_{0,p,T},\ \forall t \in [0,T], \tag{23}$$

and we conclude that

$$\|\!|\mathbf{B}(\mathbf{w},\mathbf{v})|\!\|_{0,p',T} \le C_0(p,p') T^{1-3(\frac{1}{p}-\frac{1}{2p'})} \|\!|\mathbf{w}|\!\|_{0,p,T}\ \|\!|\mathbf{v}|\!\|_{0,p,T},\ \forall \mathbf{w},\mathbf{v} \in \mathbf{F}_{0,p,T},$$

with $C_0(p,p') > 0$ a constant independent of T and $1 - 3(\frac{1}{p} - \frac{1}{2p'}) > 0$.

ii) We proceed as in *i)*, taking now in Young's inequality (20) $r = l'$, $m = (\frac{4}{3} + \frac{1}{l'} - \frac{2}{l})^{-1}$ and $k = \frac{3l}{6-l}$. By similar steps we obtain

$$\begin{aligned}\|\mathbf{B}(\mathbf{w},\mathbf{v})_i(t)\|_{l'} &\le C \int_0^t (t-s)^{\frac{3}{2l'}-\frac{3}{l}} \|\mathbf{w}(s)\|_l\ \|\mathbf{v}(s)\|_l\ ds \\ &\le C \int_0^t (t-s)^{\frac{3}{2l'}-\frac{3}{l}} s^{\frac{3}{l}-\frac{3}{p}} ds \|\!|\mathbf{w}|\!\|_{0,l,(T;p)} \|\!|\mathbf{v}|\!\|_{0,l,(T;p)} \\ &\le C t^{1+\frac{3}{2l'}-\frac{3}{p}} \|\!|\mathbf{w}|\!\|_{0,l,(T;p)} \|\!|\mathbf{v}|\!\|_{0,l,(T;p)}.\end{aligned} \tag{24}$$

The relation between p, l and l' ensured us that $m, k \in [1, \infty[$, and allowed us to use formula (22) here. We conclude as in *i)*.

iii) Assume now $\mathbf{w}, \mathbf{v} \in \mathbf{F}_{1,p,T}$. We have $\frac{3p}{6-p} \in]1, \infty[$ and $\mathbf{K}(\mathbf{w})_j(t)\mathbf{v}(t)_i \in W^{1,\frac{3p}{6-p}}$. Since also $G^{\nu}_t \in W^{1,\frac{3p}{4p-6}}$, we can integrate by parts for each $t \in]0,T]$ and obtain

$$\begin{aligned}&\mathbf{B}(\mathbf{w},\mathbf{v})(t,x) \\ &\quad = -\sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}^3} G^{\nu}_{t-s}(x-y) \frac{\partial}{\partial y_j} \left[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{v}(s,y) - \mathbf{v}_j(s,y)\mathbf{K}(\mathbf{w})(s,y)\right] dy\ ds\end{aligned}$$

.

Take $f \in \mathcal{D}$ and write

$$\overline{\mathbf{B}}^f_i(\mathbf{w},\mathbf{v})(t) := \int_0^t \int \int G^{\nu}_{t-s}(x-y) |f(x)| \left|\frac{\partial}{\partial y_j} \left[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{v}_i(s,y)\right]\right| dx\ dy\ ds.$$

By Hölder's and Young's inequalities applied as in *i)*, we deduce, with $\frac{1}{q}=\frac{1}{p}-\frac{1}{3}$, that

$$\begin{aligned}\overline{\mathbf{B}}_i^f(\mathbf{w},\mathbf{v})(t) \leq & C\|f\|_{(p')^*}\sum_{j=1}^3\int_0^t (t-s)^{\frac{3}{2m}-\frac{3}{2}}\left[\left\|\frac{\partial \mathbf{K}(\mathbf{w})_j(s)}{\partial y_j}\right\|_q \|\mathbf{v}_i(s)\|_p + \left\|\frac{\partial \mathbf{v}_i(s)}{\partial y_j}\right\|_p \|\mathbf{K}(\mathbf{w})_j\|_q\right] ds \\ \leq & C\|f\|_{(p')^*}\int_0^t (t-s)^{\frac{3}{2m}-\frac{3}{2}} s^{-\frac{1}{2}} ds \|\mathbf{w}\|_{1,p,T}\|\mathbf{v}\|_{1,p,T} \\ \leq & C\|f\|_{(p')^*} T^{1-3(\frac{1}{p}-\frac{1}{2p'})}\|\mathbf{w}\|_{1,p,T}\|\mathbf{v}\|_{1,p,T} < \infty. \end{aligned} \tag{25}$$

We have used here (18), (14), the definition of $\|\cdot\|_{1,p,T}$ and formula (22). A similar estimate as (25) holds for the term involving the product $\mathbf{v}_j(s)\mathbf{K}_i(\mathbf{w})(s)$. Therefore, we can apply Fubini's theorem and integrate by parts to deduce that

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathbf{B}(\mathbf{w},\mathbf{v})_i(t,x)\frac{\partial f(x)}{\partial x_k}dx = \\ &\sum_{j=1}^3\int_0^t\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{\partial G^\nu_{t-s}}{\partial x_k}(x-y)f(x)dx\right)\frac{\partial}{\partial y_j}\left[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{v}_i(s,y)-\mathbf{v}_j(s,y)\mathbf{K}(\mathbf{w})_i(s,y)\right] dy\, ds\end{aligned} \tag{26}$$

for all $f\in\mathcal{D}$. Proceeding as before in (25), we deduce now from (26) that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \mathbf{B}(\mathbf{w},\mathbf{v})_i(t,x)\frac{\partial f(x)}{\partial x_k}dx\right| \leq & C\|f\|_{(p')^*}\sum_{j=1}^3\int_0^t (t-s)^{\frac{3}{2m}-2}s^{-\frac{1}{2}}ds\|\mathbf{w}\|_{1,p,T}\|\mathbf{v}\|_{1,p,T} \\ \leq & C\|f\|_{(p')^*} t^{\frac{1}{2}-3(\frac{1}{p}-\frac{1}{2p'})}\|\mathbf{w}\|_{1,p,T}\|\mathbf{v}\|_{1,p,T} < \infty.\end{aligned}$$

From this and (23), we conclude that $\|\mathbf{B}(\mathbf{w},\mathbf{v})\|_{1,p',T}\leq C_1(p,p')T^{1-3(\frac{1}{p}-\frac{1}{2p'})}\|\mathbf{w}\|_{1,p,T}\ \|\mathbf{v}\|_{1,p,T}$, where $C_1(p,p')>0$ does not depend on T.

iv) By similar arguments as in *iii)* (and Young's inequality as in *ii)*) we get similar estimates as (24) for the derivatives, with $t^{\frac{1}{2}+\frac{3}{2l'}-\frac{3}{p}}$ on the right hand side. The statement follows. □

Let us write

$$\mathbf{w}_0(t,x) := G^\nu_t * w_0(x).$$

Remark 3.1 *If* $p\in]\frac{3}{2},3[$ *then we have* $\frac{3p}{6-p}<p<\frac{3p}{6-2p}$.

Thus, Lemma 3.2 and Proposition 3.1 *i)* and *iii)* give sense to the abstract equation

$$\mathbf{w}=\mathbf{w}_0+\mathbf{B}(\mathbf{w},\mathbf{w}),\ \mathbf{w}\in\mathbf{F}, \tag{27}$$

in the spaces $\mathbf{F}=\mathbf{F}_{0,p,T}$ and $\mathbf{F}=\mathbf{F}_{1,p,T}$ when $\frac{3}{2}<p<3$ and $w_0\in L^p_3$. Hence, the mild equation (10) and equation (27) are equivalent in these spaces.

3.1 Local existence and global uniqueness

We assume from now on that

- w_0 is in L^p_3 and $\frac{3}{2} < p < 3$.

Theorem 3.1 *Let $\frac{3}{2} < p < 3$ and $w_0 \in L^p_3$ be given. We have*

a) For all $T > 0$, equation (10) has at most one solution in $\mathbf{F}_{0,p,T}$.

b) There is a positive constant $\Gamma_0(p)$ such that equation (10) has a solution in $\mathbf{F}_{0,p,T}$, for all $T > 0$ and $w_0 \in L^p_3$ satisfying

$$T^{1-\frac{3}{2p}} \|w_0\|_p < \frac{1}{\Gamma_0(p)}.$$

To prove global uniqueness, we shall proceed in a similar way as in [13] using next lemma.

Lemma 3.3 *Let $g : [0,T] \to]0,\infty[$ be a bounded measurable function, and suppose there exist constants $C \geq 0$ and $\theta > 0$ such that $g(t) \leq C \int_0^t (t-s)^{\theta-1} g(s)\, ds$ for all $t \in [0,T]$. Then,*

$$g(t) \leq C^2 \beta(\theta,\theta) \int_0^t (t-s)^{2\theta-1} g(s)\, ds.$$

The proof of local existence will rely on a standard contraction argument for the abstract equation (27), based on Banach's fixed point theorem (see for instance Cannone [7]):

Lemma 3.4 *Let $(\mathbf{F}, \|\!|\cdot|\!\|)$ be a Banach space, $\mathbf{B} : \mathbf{F} \times \mathbf{F} :\to \mathbf{F}$ a bilinear application and $\mathbf{y} \in \mathbf{F}$. Suppose there exists a positive constant Λ such that*

$$\|\!|\mathbf{B}(\mathbf{x}_1, \mathbf{x}_2)|\!\| \leq \Lambda \|\!|\mathbf{x}_1|\!\|\ \|\!|\mathbf{x}_2|\!\|$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in F$. If $4\Lambda \|\!|\mathbf{y}|\!\| < 1$, then for all $\gamma \in [\|\!|\mathbf{y}|\!\|, \frac{1}{4\Lambda}[$ there exists a unique solution of

$$\mathbf{x} = \mathbf{y} + \mathbf{B}(\mathbf{x}, \mathbf{x})$$

in the ball $\mathcal{B}_{R_\gamma} = \{\mathbf{x} \in \mathbf{F} : \|\!|\mathbf{x}|\!\| \leq R_\gamma\}$, $R_\gamma = \frac{1-\sqrt{1-4\Lambda\gamma}}{2\Lambda}$. The solution $\mathbf{x}$ satisfies $\|\!|\mathbf{x}|\!\| \leq 2\gamma$.

Proof of Theorem 3.1: *a)* Let $\mathbf{w}$ and $\mathbf{v}$ be two solutions in $\mathbf{F}_{0,p,T}$. Proceeding as in Proposition 3.1*i)* (with $r = p$) we obtain

$$\begin{aligned}
\|\mathbf{w}(t) - \mathbf{v}(t)\|_p \leq & C \int_0^t (t-s)^{-\frac{3}{2p}} \|\mathbf{w}(s)\|_p\ \|\mathbf{w}(s) - \mathbf{v}(s)\|_p\, ds \\
& + C \int_0^t (t-s)^{-\frac{3}{2p}} \|\mathbf{v}(s)\|_p\ \|\mathbf{w}(s) - \mathbf{v}(s)\|_p\, ds \\
\leq & C\, (\|\!|\mathbf{w}|\!\|_{0,p,T} + \|\!|\mathbf{v}|\!\|_{0,p,T}) \int_0^t (t-s)^{-\frac{3}{2p}} \|\mathbf{w}(s) - \mathbf{v}(s)\|_p\, ds.
\end{aligned}$$

Let $\theta_N := 2^N(1 - \frac{3}{2p}) > 0$ and $N(p)$ be the first integer for which $\theta_N - 1 > 0$. Then, by applying $N(p)$ times Lemma 3.3 , it follows that

$$\|\mathbf{w}(t) - \mathbf{v}(t)\|_p \leq C(T)\, (\|\!|\mathbf{w}|\!\|_{0,p,T} + \|\!|\mathbf{v}|\!\|_{0,p,T}) \int_0^t \|\mathbf{w}(s) - \mathbf{v}(s)\|_p\, ds,$$

for some $C(T) > 0$. We conclude by Gronwall's lemma.

b) From Proposition 3.1 *i)*, one has for all $T > 0$ and $\mathbf{w}, \mathbf{v} \in \mathbf{F}_{0,p,T}$ that

$$\|\mathbf{B}(\mathbf{w}, \mathbf{v})\|_{0,p,T} \le C_0(p,p) T^{1-\frac{3}{2p}} \|\mathbf{w}\|_{0,p,T} \; \|\mathbf{v}\|_{0,p,T} \tag{28}$$

where $C_0(p,p) > 0$ does not depend on T. On the other hand, Lemma 3.2 *i)* provides a positive constant $\overline{C}_0(p)$ such that $\|\mathbf{w}_0\|_{0,p,T} \le \overline{C}_0(p)\|w_0\|_p$. Therefore, by Lemma 3.4, a solution $\mathbf{w} \in \mathbf{F}_{0,p,T}$ to the abstract equation (27) exists if

$$4C_0(p,p) T^{1-\frac{3}{2p}} \overline{C}_0(p) \|w_0\|_p < 1. \tag{29}$$

The conclusion follows taking $\Gamma_0(p) = 4\overline{C}_0(p) \cdot C_0(p,p)$.

□

3.2 Integrable solutions, Sobolev regularity and continuity in time

The probabilistic interpretation of the vortex equation we will develop requires the existence of integrable solutions. However, the non-continuity of $\mathbf{K}$ in L^1_3 prevents us from using a contraction argument in that space. In turn, as a consequence of Proposition 3.1, a solution in $\mathbf{F}_{0,p,T}$ (with p as before) will also belong to $\mathbf{F}_{0,1,T}$, as soon as the natural probabilistic condition $w_0 \in L^1_3$ is added.

Lemma 3.5 *Assume that* $w_0 \in L^p \cap L^{p'}$, *with* $\frac{3}{2} < p < 3$ *and* $\frac{3p}{6-p} \le p' < \frac{3p}{6-2p}$.

i) If $\mathbf{w} \in \mathbf{F}_{0,p,T}$ *is a solution of (27), then* $\mathbf{w} \in \mathbf{F}_{0,r,T}$ *for all* $r \in [\min\{p,p'\}, \max\{p,p'\}]$.

ii) We deduce that if $w_0 \in L^1_3 \cap L^p_3$, *then* $\mathbf{w} \in \mathbf{F}_{0,r,T}$ *for all* $r \in [1,p]$.

Proof: Recall that if $1 \le r_1 \le r \le r_2 < \infty$, then $L^{r_1} \cap L^{r_2} \subseteq L^r$ with

$$\|f\|^r_r \le \|f\|^{r_1}_{r_1} + \|f\|^{r_2}_{r_2}, \; \forall f \in L^{r_1} \cap L^{r_2}. \tag{30}$$

i) Let $\mathbf{w} \in \mathbf{F}_{0,p,T}$ be a solution of (27). Since $\frac{3p}{6-p} \le p < \frac{3p}{6-2p}$, we have from Proposition 3.1*i)* that $\mathbf{B}(\mathbf{w}, \mathbf{w}) \in \mathbf{F}_{0,r,T}$ for all $r \in [\min\{p,p'\}, \max\{p,p'\}]$. Thanks to (30), one has also $w_0 \in L^r_3$ and by the Lemma 3.2, $\mathbf{w}_0 \in \mathbf{F}_{0,r,T}$. We conclude that $\mathbf{w} \in \mathbf{F}_{0,r,T}$.
ii) Assume that $w_0 \in L^1_3 \cap L^p_3$ and that $r \in [1,p[$. The sequence defined by $r_0 = r$, $r_{n+1} = \frac{6r_n}{3+r_n}$ is increasing and converges to 3. Let $N \in \mathbb{N}$ be such that $r_N < p \le r_{N+1}$. The function $s \mapsto \frac{3s}{6-s}$ is strictly increasing on $[0,6]$, so we have $\frac{3p}{6-p} \le \frac{3r_{N+1}}{6-r_{N+1}} = r_N$. As w_0 belongs to $L^p_3 \cap L^{r_N}_3$ thanks to (30), we deduce from *i)* (with $p' = r_N$) that $\mathbf{w} \in \mathbf{F}_{0,r_N,T}$. Now, (30) also implies that $w_0 \in L^{r_N}_3 \cap L^{r_{N-1}}_3$. By applying *i)* with p replaced by r_N and $p' = \frac{3r_N}{6-r_N} = r_{N-1}$ we obtain that $\mathbf{w} \in \mathbf{F}_{0,r_{N-1},T}$. We conclude by repeating $N-1$ times this argument.

□

To prove Sobolev regularity of the solution, we need some simple technical facts:

Lemma 3.6 *Let* $\mathbf{w} \in \mathbf{F}_{0,p,T}$ *be a solution of (27) with* $\frac{3}{2} < p < 3$. *If for each* $\tau \in]0,T]$ *we write*

$$\mathbf{w}_{0,\tau}(t) := G^\nu_t * \mathbf{w}(\tau), \; \textit{and } \mathbf{w}_\tau(t) := \mathbf{w}(\tau + t),$$

then the function $\mathbf{w}_\tau$ *is a solution in* $\mathbf{F}_{0,p,T-\tau}$ *of the equation*

$$\mathbf{v}(t,x) = \mathbf{w}_{0,\tau}(t,x) + \mathbf{B}(\mathbf{v}, \mathbf{v})(t,x). \tag{31}$$

Proof: The proof follows from the semigroup property of G^ν and Fubini's theorem, using estimates (18) and (19) and similar arguments as in the proof of Proposition 3.1.

$\square$

Remark 3.2 *If $p \leq r_1 \leq r \leq r_2 < \infty$, then we have $\mathbf{F}_{i,(p;r_1),T} \cap \mathbf{F}_{i,(p;r_2),T} \subseteq \mathbf{F}_{i,(p;r),T}$ for $i = 0, 1$, and*

$$\|\mathbf{v}\|^r_{i,(p;r),T} \leq \|\mathbf{v}\|^{r_1}_{i,(p;r_1),T} + \|\mathbf{v}\|^{r_2}_{i,(p;r_2),T}, \quad \text{for all } \mathbf{v} \in \mathbf{F}_{i,(p;r_1),T} \cap \mathbf{F}_{i,(p;r_2),T}. \tag{32}$$

For $i = 0$ (resp. $i = 1$) this follows by taking in (30) the function $t^{\frac{3}{2p}}\mathbf{v}(t)$ (resp. $t^{\frac{1}{2}+\frac{3}{2p}}\frac{\partial \mathbf{v}(t)}{\partial x_k}$), and then multiplying by $t^{-\frac{3}{2}}$.

We are now ready to prove the regularity properties of $\mathbf{w}$ we need in the sequel:

Theorem 3.2 *Let $p \in]\frac{3}{2}, 3[$ and $\mathbf{w} \in \mathbf{F}_{0,p,T}$ be a solution of (10).*

i) One has $\mathbf{w} \in \mathbf{F}_{1,p,T}$, and $\|\mathbf{w}\|_{1,p,T} \leq C(T,p)\|\mathbf{w}\|_{0,p,T}$, with $C(T,p)$ a constant not depending on $\mathbf{w}$.

ii) For all $p \leq r < \infty$ one has $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$. Further, if $A > 0$ is an upper bound for $\|\mathbf{w}\|_{0,p,T}$, there exists a finite constant $C(T,p,r,A)$ depending on $\mathbf{w}$ only through w_0 and A, such that

$$\|\mathbf{w}\|_{1,r,(T;p)} \leq C(T,p,r,A).$$

iii) If $w_0 \in L^{p'}_3$ with $p' \in [1,p]$, then the function $\mathbf{w} : [0,T] \to L^{p'}_3$ is continuous.

Proof : *i)* The proof is exactly the same as in Lemma 4.4 in [13]. We will repeat it here since it is an important point for the sequel. Notice that all results for (27) obtained so far apply also to equations (31) with the *same* constants for all $\varepsilon \geq 0$. From Lemma 3.2 one has $\|\mathbf{w}_{0,\tau}\|_{1,p,T'\wedge(T-\tau)} \leq \overline{C}_1(p)\|\mathbf{w}\|_{0,p,T}$ for all $0 < T' < T$. If we choose T' small enough so that

$$(T')^{1-\frac{3}{2p}}\|\mathbf{w}\|_{0,p,T} < \frac{1}{\Gamma_1(p)},$$

where $\Gamma_1(p) = 4\overline{C}_1(p) \cdot C_1(p,p)$ and $C_1(p,p)$ given in the proof of Proposition 3.1 *iii)*, then

$$\|\mathbf{w}_{0,\tau}\|_{1,p,T'\wedge(T-\tau)} \leq \overline{C}_1(p)\|\mathbf{w}\|_{0,p,T} < \frac{1}{4(T')^{1-\frac{3}{2p}}C_1(p,p)}$$

for all $\tau \in [0,T]$. Thus, from Lemma 3.4 we deduce for each $\tau \in [0,T]$ that (31) has a solution in $\mathbf{F}_{1,p,T'}$ (in the ball of radius R_γ defined in Lemma 3.4, with $\gamma = \overline{C}_1(p)\|\mathbf{w}\|_{0,p,T}$). Define $\tau_k := k\frac{T'}{2}$ for $k = 0 \dots N := [\frac{2T}{T'}]$. Uniqueness for (31) in the space $\mathbf{F}_{0,p,T'\wedge(T-\tau_N)}$, for each $\tau = \tau_k$, implies that the functions $\mathbf{w}_{(\tau_k)} := \mathbf{w}(\tau_k + \cdot)$ belong to $\mathbf{F}_{1,p,T'\wedge(T-\tau_N)}$ for all $k = 0, \dots, N$. But one has $\mathbf{w}_{(\tau_k)}(t) = \mathbf{w}_{(\tau_{k-1})}(\frac{T'}{2} + t)$ for all $t \in [0, \frac{T'}{2} \wedge T]$ and $k = 1, \dots, N$, so we conclude that $\mathbf{w}_{(\tau_k)}, \frac{\partial \mathbf{w}_{(\tau_k)}}{\partial x_j} \in \mathbf{F}_{0,p,T'\wedge(T-\tau_N)}$ for $k = 1, \dots, N$, implying that $\mathbf{w} \in \mathbf{F}_{1,p,T}$.
The estimate for the norm follows from the fact that for all $t \in [0, \frac{T'}{2} \wedge T]$ and $k = 1...N$, one has $(\tau_k + t)^{\frac{1}{2}}\|\frac{\partial \mathbf{w}(\tau_k+t)}{\partial x_j}\|_p \leq C(T')(\tau_k + t)^{\frac{1}{2}}(t + \frac{T'}{2})^{\frac{1}{2}}\|\frac{\partial \mathbf{w}_{\tau_{k-1}}(t+\frac{T'}{2})}{\partial x_j}\|_p \leq C(T)\|\mathbf{w}\|_{0,p,T}$.

ii) First we notice that $\mathbf{w}_0 \in \mathbf{F}_{1,r,(T;p)}$ for all $r \geq p$, and that $\mathbf{w} \in \mathbf{F}_{1,p,T}$ by *i)*.

Consider the function $g(s) := \frac{3s}{6-2s}$ defined on the interval $]\frac{3}{2}, 3[$ and define a sequence l_n by $l_0 = p$, $l_{n+1} = g(l_n)$. Since $g'(s) > 2$, for all $\frac{3}{2} < s < t < 3$ one has $g(t) - g(s) > 2(t-s)$ and consequently there exists $N \in \mathbb{N}$ such that $l_N < 3$ and $l_{N+1} \geq 3$ (*e.g.* $N = 0$ if $p \in [2, 3[$).
Observe that for all $n = 0, \dots, N-1$, we have $g([l_n, l_{n+1}[) = [l_{n+1}, l_{n+2}[\subseteq [\frac{3l_n}{6-l_n}, \frac{3l_n}{6-2l_n}[$.
We can therefore apply Proposition 3.1 *iv)* with $l = l_0 = p$ and $l' = \frac{l_0+l_1}{2}$ and deduce that $\mathbf{w} \in \mathbf{F}_{1,\frac{l_0+l_1}{2},(T;p)}$. Taking then $l = \frac{l_0+l_1}{2}$ and $l' = l_1$ we deduce that $\mathbf{w} \in \mathbf{F}_{1,l_1,(T;p)}$. We apply the previous two-step argument starting now from l_1 and we deduce that $\mathbf{w} \in \mathbf{F}_{1,l_2,(T;p)}$.
Iterating this two-step procedure N times we conclude that $\mathbf{w} \in \mathbf{F}_{1,l_N,(T;p)}$, and taking then $l = l_N$ and $l' = 3$ we establish that $\mathbf{w} \in \mathbf{F}_{1,3,(T;p)}$.
Let us point out that by Remark 3.2 and the latter we have $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$ for all $r \in [p, 3]$. On the other hand, at each time we apply Proposition 3.1 *iv)* we can obtain an estimate for the norm of $|||\mathbf{w}|||_{1,l',(T;p)}$ in terms of $|||\mathbf{w}|||_{1,l,(T;p)}$, of $\mathbf{w}_0$, and of a fixed upper bound for the norm of the operator $\mathbf{B} : (\mathbf{F}_{1,l,(T;p)})^2 \to \mathbf{F}_{1,l',(T;p)}$.
Therefore, we can exhibit an upper bound for $|||\mathbf{w}|||_{1,(p;3),T}$ in terms of T,$\|w_0\|_p$, $|||\mathbf{w}_0|||_{0,p,T}$ and of the norm of the operators $\mathbf{B} : (\mathbf{F}_{1,l,(T;p)})^2 \to \mathbf{F}_{1,l',(T;p)}$, with indexes $l \in [p, 3[$ and $l' \in [l, \frac{3l}{6-l}[$ chosen among a fixed finite subset of $[p, 3]$. Thanks now to Remark 3.2, for every $r \in [p, 3]$ we can obtain an upper bound for the norm $|||\mathbf{w}|||_{1,r,(T;p)}$ in terms of the same (fixed) data about $\mathbf{B}$.
To obtain the result for $r \in]3, \infty[$, we take $l := g^{-1}(2r)$ (which belongs to $[2, 3[$), and $l' = r$ and conclude as before with Proposition 3.1 *iv)*, with an upper for the norm $|||\mathbf{w}|||_{1,r,(T;p)}$ obtained in a similar way as before.

iii) We will check that the operator $\mathbf{B} : C([0,T], L^p_3)^2 \to C([0,T], L^{p'}_3)$ is continuous when p and p' are chosen as in Proposition 3.1 *i)*. Indeed, if t_n is a sequence in $[0,T]$ converging to t_*, then for all $\mathbf{v}, \mathbf{v}' \in C([0,T], L^p_3)$ one has

$$\|\mathbf{B}(\mathbf{v},\mathbf{v}')(t_n) - \mathbf{B}(\mathbf{v},\mathbf{v}')(t_*)\|_{p'} \leq \int_0^T s^{\frac{3}{p}-\frac{3}{2p'}} \Big[\|\mathbf{v}(t_n - s) - \mathbf{v}(t_* - s)\|_p \|\mathbf{v}'(t_n - s)\|_p \mathbf{1}_{s \leq t_n} + \|\mathbf{v}'(t_n - s) - \mathbf{v}'(t_* - s)\|_p \|\mathbf{v}(t_* - s)\|_p \mathbf{1}_{s \leq t_*} \Big] ds$$

and therefore $\mathbf{B}(\mathbf{v},\mathbf{v}')(t_n) \to \mathbf{B}(\mathbf{v},\mathbf{v}')(t_*)$ in $L^{p'}_3$. Since clearly $\mathbf{w}_0 \in C([0,T], L^p_3)$, we have a local existence result in that space for equation (27). Together with uniqueness in $\mathbf{F}_{0,p,T}$, this shows that any mild solution $\mathbf{w}$ in that space is continuous in t. The rest of the proof is achieved by the same arguments of Lemma 3.5, using the continuity property of $\mathbf{B}$ we have just established.

□

Denote by $\mathcal{C}^\alpha$ the space of functions $\mathbb{R}^3 \to \mathbb{R}^3$ that are Hölder continuous of index $\alpha \in]0, 1[$.
We recall the following standard embbeding of Sobolev spaces (see e.g. [4]):

Lemma 3.7 *For all $m > 3$, the space $W^{1,m}_3$ is continuously embedded into $L^\infty_3 \cap \mathcal{C}^{1-\frac{3}{m}}$.*

From this and Theorem 3.2 we deduce

Corollary 3.1 *Let $p \in]\frac{3}{2}, 3[$ and $\mathbf{w} \in F_{0,p,T}$ be a solution of the mild equation (10). Write $\mathbf{u}(s,x) := \mathbf{K}(\mathbf{w})(s,x)$. Then, the following hold:*

i)

$$\sup_{t\in[0,T]} t^{\frac{1}{2}} \left\{ \|\mathbf{u}(t)\|_\infty + \|\mathbf{u}(t)\|_{\mathcal{C}^{\frac{2p-3}{p}}} \right\} < \hat{C}(T,p)\|\mathbf{w}\|_{0,p,T} \tag{33}$$

for a constant $\hat{C}(T,p) > 0$ not depending on $\mathbf{w}$.

ii) For all $r \in]3,\infty[$, $i = 1,2,3$, and any upper bound $A \in \mathbb{R}$ of $\|\mathbf{w}\|_{0,p,T}$

$$\sup_{t\in[0,T]} t^{\frac{1}{2}+\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \left\{ \left\| \frac{\partial \mathbf{u}(t)}{\partial x_i} \right\|_\infty + \left\| \frac{\partial \mathbf{u}(t)}{\partial x_i} \right\|_{\mathcal{C}^{1-\frac{3}{r}}} \right\} < \hat{C}(T,p,r,A) \tag{34}$$

with $\hat{C}(T,p,r,A) > 0$ a constant depending on $\mathbf{w}$ only through w_0 and A.

iii) By taking in ii) $r \in]3, \frac{3p}{3-p}[$, we deduce that $\int_0^T \left[\|\mathbf{u}(t)\|_\infty + \sum_{i=1}^3 \|\frac{\partial \mathbf{u}(t)}{\partial x_i}\|_\infty\right] dt < \infty$.

Proof: By Lemma 2.2, one has $\mathbf{u} \in \mathbf{F}_{1,q,T}$, with $q = \frac{3p}{3-p}$, and by Lemma 3.7, we deduce that for $t \in [0, \min\{T,1\}]$

$$t^{\frac{1}{2}} \left(\|\mathbf{u}(t)\|_\infty + \|\mathbf{u}(t)\|_{\mathcal{C}^{\frac{2p-3}{p}}} \right) \le C t^{\frac{1}{2}} \|\mathbf{u}(t)\|_{1,q} \le C\|\mathbf{u}(t)\|_q + t^{\frac{1}{2}} \|\nabla \mathbf{u}(t)\|_q.$$

On the other hand, if $t \in [\min\{T,1\}, T]$, one has

$$t^{\frac{1}{2}} \left(\|\mathbf{u}(t)\|_\infty + \|\mathbf{u}(t)\|_{\mathcal{C}^{\frac{2p-3}{p}}} \right) \le C t^{\frac{1}{2}} \|\mathbf{u}(t)\|_{1,q} \le C T^{\frac{1}{2}} \left(\|\mathbf{u}(t)\|_q + t^{\frac{1}{2}} \|\nabla \mathbf{u}(t)\|_q \right).$$

Since $\|\mathbf{u}(t)\|_q + t^{\frac{1}{2}} \|\nabla \mathbf{u}(t)\|_q \le C\|\mathbf{w}(t)\|_{1,p,T}$ for all $t \in [0,T]$, the statement *i)* follows from Theorem 3.2 *i)*. Statement *ii)* is proved in a similarly way, noting that $\frac{\partial \mathbf{u}}{\partial x_i} \in \mathbf{F}_{1,r,(T;p)}$ by Lemma 2.3 and using Theorem 3.2 *ii)*. Part *iii)* is immediate.

□

3.3 Proofs of the continuity properties of K and ∇K

We first recall some basic facts about Riesz potentials. Let $0 < \alpha < 3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be a measurable function. If well defined, the function $\mathcal{I}_\alpha(f)$ given by

$$\mathcal{I}_\alpha(f)(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^{3-\alpha}} dy, \tag{35}$$

is called the Riesz potential of f (we omit the multiplicative constant usually appearing in the definition of $\mathcal{I}_\alpha$).

Theorem 3.3 *Let $p \in [1, \frac{3}{\alpha}[$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{3}$. For every $f \in L^p(\mathbb{R}^3)$, the integral (35) converges absolutely for almost every x.*
If further $p \in]1, \frac{3}{\alpha}[$, then $\mathcal{I}_\alpha(f) \in L^q(\mathbb{R}^3)$, and there exists a positive constant $C_{p,q}$ such that

$$\|\mathcal{I}_\alpha(f)\|_q \le C_{p,q}\|f\|_p$$

for all $f \in L^p(\mathbb{R}^3)$.

The proof can be found in Stein [29], Ch. 5. The same reference contains the proof of

Lemma 3.8 *Let $\gamma(\alpha)$ be the constant $\gamma(\alpha) := \pi^{\frac{3}{2}} 2^{\alpha}\Gamma(\frac{\alpha}{2})/\Gamma(\frac{3-\alpha}{2})$. The following identities hold in the distribution sense:*

$$\mathcal{F}(|x|^{-3+\alpha})(\xi) = \gamma(\alpha)(2\pi|\xi|)^{-\alpha}, \tag{36}$$

and

$$\mathcal{F}(\mathcal{I}_\alpha(g))(\xi) = \gamma(\alpha)(2\pi|\xi|)^{-\alpha}\mathcal{F}(g)(\xi), \tag{37}$$

where $\mathcal{F}$ is the Fourier transform as defined in **1.1**.

Proof of Lemma 2.2: Denote by $K_j(x)$ the j-th component of the vector $K(x)$, and consider the operator

$$f \mapsto \mathcal{K}_j(f) = \int_{\mathbb{R}^3} K_j(\cdot - y)f(y)dy \tag{38}$$

acting on real valued functions f. Let *i')* (resp. *ii')*) be the analogous statement of *i)* (resp. *ii)*) in the space $L^p(\mathbb{R}^3)$ (resp. $W^{1,p}(\mathbb{R}^3)$) for the operator $\mathcal{K}_j$ instead of $\mathbf{K}$. Clearly, it is enough to prove *i')* and *ii')*.
The proof of the statement *i')* follows readily from Theorem 3.3 with $\alpha = 1$. We now prove *ii')*. By using *i')*, we just need to check that for all $f \in W^{1,p}(\mathbb{R})$ the identity

$$\frac{\partial}{\partial x_k}\mathcal{K}_j(f) = \mathcal{K}_j\left(\frac{\partial f}{\partial x_k}\right) \tag{39}$$

holds in the distribution sense. Let us take $f \in \mathcal{S}$. Since $K_j(x) = -\frac{1}{4\pi}\frac{\partial}{\partial x_j}\left(\frac{1}{|x|}\right)$, it follows from (37) with $\alpha = 2$ that

$$\mathcal{F}(\mathcal{K}_j(f))(\xi) = c(2)\mathcal{F}(f)(\xi)\frac{\xi_j}{|\xi|^2} \in L^2, \quad \mathcal{F}(\mathcal{K}_j(\frac{\partial f}{\partial x_k}))(\xi) = c(2)\mathcal{F}(f)(\xi)\frac{i\xi_j\xi_k}{|\xi|^2} \in L^2,$$

for all $k = 1, 2, 3$ and a constant $c(2)$ that can be explicited. On the other hand, one has $|\mathcal{F}(\mathcal{K}_j(f))(\xi)||\xi| \leq C|\mathcal{F}(f)(\xi)| \in L^2$, and then $\frac{\partial}{\partial x_k}\mathcal{K}_j(f)$ is also in L^2. Thus,

$$\mathcal{F}\left(\frac{\partial}{\partial x_k}\mathcal{K}_j(f)\right)(\xi) = i\xi_k c(2)\mathcal{F}(f)(\xi)\frac{\xi_j}{|\xi|^2},$$

and we obtain the identity (39) in L^2 for all $f \in \mathcal{S}$. Now, from this and an integration by parts in $W^{1,2}$, we conclude that

$$\int_{\mathbb{R}^3} \mathcal{K}\left(\frac{\partial f}{\partial x_k}\right)(x)g(x)dx = -\int_{\mathbb{R}^3} \mathcal{K}(f)(x)\frac{\partial g}{\partial x_k}(x)dx$$

for all $g \in \mathcal{S}$. By the continuity of $\mathcal{K}_j : L^p \to L^q$ and the density of $\mathcal{S}$ in $W^{1,p}$, the previous identity holds for arbitrary $f \in W^{1,p}$. This completes the proof. □

To prove Lemma 2.3 we will use a result on singular integrals (proved in [29], Ch. 2.):

Theorem 3.4 *Let $\kappa : \mathbb{R}^3 \to \mathbb{R}$ be an homogeneous function of degree 0 such that*

- $\int_{S^2} \kappa(x)d\gamma = 0$, *where γ is the Euclidean measure on the sphere S^2, and*
- $\int_0^1 \frac{1}{\delta} \sup_{\substack{|x-y|\leq\delta \\ |x|=|y|=1}} \{|\kappa(x) - \kappa(y)|\}d\delta < \infty.$

Let $p \in]1,\infty[$ and $f \in L^p$, and consider for each $x \in \mathbb{R}^3$ the singular integral

$$\mathcal{H}(f)(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\kappa(x-y)}{|x-y|^3} f(y) dy.$$

Then, the limit $\mathcal{H}(f)$ exists in L^p norm. Further, there exists a real positive constant C_p depending only on p, such that

$$\|\mathcal{H}(f)\|_p \leq C_p \|f\|_p \tag{40}$$

for all $f \in L^p$. Finally, if $p = 2$ one has the relation $\mathcal{F}(\mathcal{H}(f))(\xi) = m(\xi)\mathcal{F}(f)(\xi)$, where $m : \mathbb{R}^3 \to \mathbb{R}$ is a homogeneous function of degree 0.

Proof of Lemma 2.3: Notice that

$$\frac{\partial K_j(x)}{\partial x_j} = \frac{\kappa_j(x)}{|x|^3}, \quad \text{and} \quad \frac{\partial K_j(x)}{\partial x_k} = \frac{\kappa_{k,j}(x)}{|x|^3},$$

where the functions $\kappa_j(x) = -\frac{1}{4\pi}\left(1 - \frac{3x_j^2}{|x|^2}\right)$ and $\kappa_{k,j}(x) = \frac{3x_k x_j}{4\pi|x|^2}$ satisfy the conditions of Theorem 3.4 (cf.: $\sum_{j=1}^3 \kappa_j(x) = 0$ and $\kappa_{k,j}(x)$ is odd in each component of x). Thus, the singular integrals

$$\mathcal{H}_j(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\partial K_j(x-y)}{\partial x_j} f(y)dy \text{ and } \mathcal{H}_{k,j}(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\partial K_j(x-y)}{\partial x_k} f(y)dy$$

of real valued functions f define continuous linear operators $L^p \to L^p$ for $1 < p < \infty$. If $f \in \mathcal{S}$, it follows as in the proof of Lemma 2.2 that

$$\mathcal{F}(\mathcal{H}_{k,j}(f))(\xi) = c(2)\frac{i\xi_j\xi_k}{|\xi|^2}\mathcal{F}(f)(\xi)$$

in L^2, and then

$$\mathcal{H}_{k,j}(f) = \frac{\partial}{\partial x_k}\mathcal{K}_j(f).$$

Using integration by parts, density, and the continuity of $\mathcal{H}_{k,j}$ we conclude that this holds for all $f \in L^p$. The result for $\mathcal{H}_j$ is proved in exactly the same way.
The continuity estimate *i)* is now immediate thanks to Theorem 3.4. The estimate *ii)* is proved in a similar way as Lemma 2.2 *ii)* by considering Fourier transforms (see also Lemma 2.2 in [13]).

□

4 The nonlinear martingale problem

We consider now a fixed time $0 < T < \infty$ and we make the following assumption:

- w_0 is a function in $L_3^1 \cap L_3^p$ for some $\frac{3}{2} < p < 3$.

In this section, we will identify the solution $\mathbf{w}$ of the mild vortex equation in $\mathbf{F}_{0,p,T}$, with a flow of $\mathbb{R}^3$-valued vector measures associated with a generalized nonlinear diffusion of the McKean-Vlasov type. Let us establish some notation required in the sequel:

- We denote by $\mathcal{P}(\mathcal{C}_T)$ the space of probability measures on $\mathcal{C}_T = C([0,T],\mathbb{R}^3 \times \mathcal{M}_{3\times 3})$.

- For any element $P \in \mathcal{P}(\mathcal{C}_T)$, we will write P° for the first marginal $P^\circ = P|_{C([0,T],\mathbb{R}^3)}$, and P' for the second marginal $P' = P|_{C([0,T],\mathcal{M}_{3\times 3})}$.

- The canonical process in $C([0,T],\mathbb{R}^3 \times \mathcal{M}_{3\times 3})$ will be denoted by (X,Φ).

- We use the notation $\mathcal{P}_b(\mathcal{C}_T)$ for the subspace of $\mathcal{P}(\mathcal{C}_T)$ of probability measures Q such that the support of Q' is bounded. (Equivalently, under each law $Q \in \mathcal{P}_b(\mathcal{C}_T)$, the process Φ is bounded independently of t and of the randomness.)

- By $F_{0,p,T}$, $F_{1,p,T}$, $F_{0,r,(T;p)}$ and $F_{1,r,(T;p)}$ we denote the subspaces of $\mathcal{M}^T$ that are the real-valued analogous of the spaces $\mathbf{F}$ defined in Section **3**. We use the same notation as therein for the norms.

We define now a "vectorial weight function" in terms of the initial condition w_0, by setting

$$h_0(x) := w_0(x)\frac{\|w_0\|_1}{|w_0(x)|} \tag{41}$$

(with the convention "$\frac{0}{0} = 0$"). Observe that h_0 takes values in the sphere $\|w_0\|_1 \cdot S^2$ or 0. With each $Q \in \mathcal{P}_b(\mathcal{C}_T)$ we can associate a family of $\mathbb{R}^3$-valued vector measures $(\tilde{Q}_t)_{t\in[0,T]}$ on $\mathbb{R}^3$, defined by

$$\tilde{Q}_t(\mathbf{f}) = E^Q(\mathbf{f}(X_t)\Phi_t h_0(X_0)), \tag{42}$$

for all $\mathbf{f} \in \mathcal{D}_3$. Since Φ is bounded, the vector measure $\tilde{Q}_t$ is absolutely continuous with respect to Q_t°, with density given by

$$h_t^Q(x) := \frac{d\tilde{Q}_t}{dQ_t^\circ}(x) = E^Q(\Phi_t h_0(X_0)|X_t = x), \tag{43}$$

and its total mass is bounded by $\|w_0\|_1(\sup_{\phi\in supp(Q')}\sup_{t\in[0,T]}|\phi_t|)$.

Notice that $(t,x)\mapsto h_t^Q(x)$ is measurable. With the notation (43), we can rewrite (42) as

$$\tilde{Q}_t(\mathbf{f}) = E^Q(\mathbf{f}(X_t)h_t^Q(X_t)). \tag{44}$$

Thus, we can think of $h_t^Q(x)$ as a bounded vectorial weight found at position x at time t.

If now $P \in \mathcal{P}_b(\mathcal{C}_T)$ is such that for each t the probability measure P_t° is absolutely continuous with respect to Lebesgue's measure, then the same holds for the vector measure $\tilde{P}_t$. In that case, and if $\rho : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ is the family of densities of P_t°, we will denote by $\tilde{\rho} : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ the family of densities of $\tilde{P}_t$ (taking always bi-measurable versions of both of them if they exist). We stress the fact that $\tilde{\rho}_t$ is defined in terms of the joint law of (X_0, X_t, Φ_t).

We will study the following nonlinear martingale problem: to find $P \in \mathcal{P}_b(\mathcal{C}_T)$ such that

$$
\begin{aligned}
&\bullet\quad P^\circ|_{t=0}(dx) = \tfrac{|w_0(x)|}{\|w_0\|_1}dx \text{ and for all } 0\le t\le T, P_t^\circ(dx) = \rho_t(x)dx \text{ and } \tilde{P}_t(dx) = \tilde{\rho}_t(x)dx.\\
&\bullet\quad f(t,X_t) - f(0,X_0) - \int_0^t \left[\tfrac{\partial f}{\partial s}(s,X_s) + \nu\triangle f(s,X_s)ds + \mathbf{K}(\tilde{\rho})(s,X_s)\nabla f(s,X_s)\right]ds,\\
&\qquad 0\le t\le T, \text{ is a continuous } P^\circ\text{-martingale for all } f\in\mathcal{C}_b^{1,2};\\
&\bullet\quad \Phi_t = Id_{3\times 3} + \int_0^t \nabla\mathbf{K}(\tilde{\rho})(s,X_s)\Phi_s\, ds, \text{ for all } 0\le t\le T,\ P \text{ almost surely.}
\end{aligned}
\tag{45}
$$

To state our the main result on the probabilistic interpretation of the vortex equation, we need

Definition 4.1 *We define $\mathcal{P}^T_{\frac{3}{2},b,0}$ as the space of probability measures $P \in \mathcal{P}_b(\mathcal{C}_T)$ satisfying the following conditions:*

- *For each $t \in [0,T]$, the time marginal P°_t is absolutely continuous with respect to the Lebesgue measure, with a bi-measurable family of densities $(t,x) \mapsto \rho(t,x)$ that belongs to the space $F_{0,p,T}$ for some $\frac{3}{2} < p < 3$.*
- *For all $t \in [0,T]$, the condition div $\tilde{\rho}_t = 0$ holds.*

Theorem 4.1 *Assume that $w_0 \in L^1_3 \cap L^p_3$ for some $p \in]\frac{3}{2}, 3[$. For every $T > 0$, the nonlinear martingale problem (45) has at most one solution P in the class $\mathcal{P}^T_{\frac{3}{2},b,0}$.*
Further, there exists a solution P in $\mathcal{P}^T_{\frac{3}{2},b,0}$ such that P° has a density family $\rho \in F_{0,p,T}$, if and only if there exists in $\mathbf{F}_{0,p,T}$ a solution $\mathbf{w}$ of the mild equation (10) with initial condition w_0. In that case, for all $t \in [0,T]$ one has the relations

$$\mathbf{w}(t,x) = \tilde{\rho}(t,x), \qquad \rho(t,x)\left| E^P(\Phi_t h_0(X_0)|X_t = x)\right| = |\mathbf{w}(t,x)|,$$

and for all $1 \le r \le p$ and $p \le r' < \infty$, it holds that $\rho \in F_{0,r,T} \cap F_{1,r',(T;p)}$.

If $\Gamma_0(p)$ is the constant of Theorem 3.1, we immediately deduce

Corollary 4.1 *If $w_0 \in L^1_3 \cap L^p_3$ for some $p \in]\frac{3}{2}, 3[$ and $T^{1-\frac{3}{2p}} \|w_0\|_p < \frac{1}{\Gamma_0(p)}$, then the problem (45) has a unique solution in $\mathcal{P}^T_{\frac{3}{2},b,0}$.*

The proof of Theorem 4.1 will be done in several steps. First of all, we shall dwell upon the properties of the evolution equation satisfied by the densities ρ of the marginal P° of a given solution P. The study of this equation will provide *a priori* regularity estimates for the drift term $\mathbf{K}(\tilde{\rho})$ in (45).

4.1 A nonlinear Fokker-Planck equation associated with the vortex equation

Assume for a while that (45) has a solution $P \in \mathcal{P}_b(\mathcal{C}_T)$ with densities $\rho \in \mathcal{M}es^T$. Assume furthermore that

$$\int_0^T \int_{\mathbb{R}^3} |\mathbf{K}(\tilde{\rho})(t,x)|\rho(t,x)\, dxdt < \infty \tag{46}$$

(which is a minimal condition ensuring that $\int_0^t \mathbf{K}(\tilde{\rho})(s,X_s)ds$ has finite variation). Then, by applying Itô's formula to $f(t,X_t)$ for an arbitrary function $f \in C^{1,2}_b$ and taking expectations, we deduce that the couple $(\rho,\tilde{\rho})$ satisfies the weak evolution equation:

$$\int_{\mathbb{R}^3} f(t,y)\rho(t,y)dy = \int_{\mathbb{R}^3} f(0,y)\rho_0(y)dy + \int_0^t \int_{\mathbb{R}^3} \left[\frac{\partial f}{\partial s}(s,y) + \nu \triangle f(s,y) + \mathbf{K}(\tilde{\rho})(s,y)\nabla f(s,y)\right]\rho(s,y)\, dy\, ds, \tag{47}$$

where $\rho_0(x) = \frac{|w_0(x)|}{\|w_0\|_1} dx$. Observe that by (44), one has

$$\tilde{\rho}_t(x) = h_t^P(x)\rho_t(x).$$

If P is fixed, the function h^P is also fixed, and (47) is a nonlinear Fokker-Planck equation for the unknown ρ, which can be treated to a large extent as a scalar of analog vortex equation. To obtain its mild form, we fix $\psi \in \mathcal{D}$ and $t \in [0,T]$ and take in (47) the function $f_t : [0,t] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $f_t(s,y) = G_{t-s}^{\nu} * \psi(y)$, which is of class $C_b^{1,2}$, and solves the backward heat equation on $[0,t] \times \mathbb{R}^3$ with final condition $f(t,y) = \psi(y)$. By Lemma 3.1 and condition (46), it is easily checked that

$$\int_0^t \int_{(\mathbb{R}^3)^2} \sum_{j=1}^3 \left| \frac{\partial G_{t-s}^{\nu}}{\partial y_j}(x-y) \right| |\mathbf{K}(\tilde{\rho})_j(s,y)||\psi(x)|\rho(s,y) dx\ dy\ ds < \infty,$$

and by Fubini's theorem we deduce that

$$\rho(t,x) = G_t^{\nu} * \rho_0(x) + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^{\nu}}{\partial y_j}(x-y)\mathbf{K}(h\rho)_j(s,y)\rho(s,y)\ dy\ ds \tag{48}$$

for all $t \in [0,T]$, where $h = h^P$ and $h\rho$ stands for the function $h\rho(t,x) = h_t(x)\rho_t(x)$.

We will now study some of the analytical properties of equation (48) in a more general situation. Namely, we assume that $h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ is a fixed but arbitrary function of class $L^{\infty}([0,T], L_3^{\infty})$, and define for $\rho, \eta \in \mathcal{M}es^T$ a function $\mathbf{b}^h(\rho,\eta) : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{b}^h(\rho,\eta)(t,x) = \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^{\nu}}{\partial y_j}(x-y)\mathbf{K}(h\eta)_j(s,y)\rho(s,y) dy\ ds.$$

The next observation will be important:

Remark 4.1 *For each $p \in [1,\infty]$ (resp. each $p \in [1,\infty]$ and $r \geq p$), the mapping $\eta \mapsto h\eta$ is continuous from $F_{0,p,T}$ to $\mathbf{F}_{0,p,T}$ (resp. from $F_{0,r,(T;p)}$ to $\mathbf{F}_{0,r,(T;p)}$).*

Thus, the following continuity properties of $\mathbf{b}^h$ can be proved in exactly the same way as Proposition 3.1 *i)* and *ii)*:

Lemma 4.1 *Let $p, p' \in [1,\infty]$. Then, $\mathbf{b}^h : (F, \||\cdot\||)^2 \to (F', \||\cdot\||')$ is well defined and continuous whenever*

i) $\frac{3}{2} \leq p < 3$, $\frac{3p}{6-p} \leq p' < \frac{3p}{6-2p}$, $F = F_{0,p,T}$ *and* $F' = F_{0,p',T}$.

ii) $\frac{3}{2} \leq p < 3$, $p \leq r < 3$, $\frac{3r}{6-r} \leq r' < \frac{3r}{6-2r}$, $F = F_{0,r,(T;p)}$ *and* $F' = F_{0,r',(T;p)}$.

Write now

$$\gamma_0(t,x) := G_t^{\nu} * \rho_0(x) = G_t^{\nu} * \frac{|w_0|}{\|w_0\|_1}(x). \tag{49}$$

Since $w_0 \in L_3^p$, Lemma 3.1 and Young's inequality imply that $\gamma_0 \in F_{0,r,(T;p)}$ for all $r \geq p$. This and the previous lemma give sense to the abstract equation

$$\rho = \gamma_0 + \mathbf{b}^h(\rho,\rho) \tag{50}$$

in $F_{0,p,T}$ if $\frac{3}{2} < p < 3$, and (48) is equivalent to (50) in that space.

As we did before in the case of the vortex equation (10), we deduce now some additional properties for (48):

Lemma 4.2 *Assume that $w_0 \in L^p_3$, with $\frac{3}{2} < p < 3$.*

i) For all $T > 0$ and every fixed $h \in L^\infty([0,T], L^\infty_3)$ the nonlinear Fokker-Planck equation (48) has at most one solution ρ in $F_{0,p,T}$.

ii) If $\rho \in F_{0,p,T}$ is a solution of (48), then $\rho \in F_{0,r,(T;p)}$ and $\|\rho\|_{0,r,(T;p)} \leq C(T,p,r,\|\rho\|_{0,p,T})$ for all $p \leq r < \infty$.

iii) We deduce that $\tilde{\rho} = h\rho$ satisfies $\tilde{\rho} \in \mathbf{F}_{0,r,(T;p)}$ and $\|\tilde{\rho}\|_{0,r,(T;p)} \leq \tilde{C}(T,h,p,r,\|\tilde{\rho}\|_{0,p,T})$ for all $p \leq r < \infty$.

Proof: *i)* is proved in the same way as Theorem 3.1 *a)*. The proof of *ii)* can be adapted from the proof of Theorem 3.2 *ii)*, reasoning in the spaces $F_{0,r;(T,p)}$ instead of the spaces $\mathbf{F}_{1,r,(T;p)}$, and using Lemma 4.1 and Remark 4.1. Part *iii)* is clear from *ii)* and Remark 4.1. □

We can now prove *a priori* regularity estimates for ρ, $\tilde{\rho}$ and the drift term $\mathbf{K}(\tilde{\rho})$ of (45):

Proposition 4.1 *Assume that P is a solution of (45) in the class $\mathcal{P}^T_{\frac{3}{2},b,0}$ with densities $\rho \in F_{0,p,T}$ and $\frac{3}{2} < p < 3$, and write $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$. Then, the following hold:*

i) $\tilde{\rho} \in \mathbf{F}_{0,r,(T;p)}$ for all $r \in [p,\infty[$ and $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,l,(T;q)}$ for all $l \in [q,\infty[$.

ii) $\rho \in F_{1,r,(T;p)}$ for all $r \in [p,\infty[$.

Proof : *i)* First notice that ρ belongs to $F_{0,\frac{3}{2},T}$ by inequality (30) since $\rho \in F_{0,1,T} \cap F_{0,p,T}$. Thus, (46) holds by Remark 4.1 and Lemma 2.2 *i)*. We deduce that $\rho \in F_{0,p,T}$ solves the mild Fokker-Planck equation (48).
If we take $l \geq q$ and define $r := (\frac{1}{l} + \frac{1}{3})^{-1}$, then one has $r \geq p$, and so Lemma 4.2 *iii)* and Lemma 2.2 *i)* imply that

$$\sup_{t\in[0,T]} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|\mathbf{K}(\tilde{\rho}(t))\|_l < \infty.$$

As $\frac{1}{p} - \frac{1}{r} = \frac{1}{q} - \frac{1}{l}$, this means that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{0,l,(T;q)}$.
We next check that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,l,(T;q)}$. From the fact that $\tilde{\rho} \in \mathbf{F}_{0,l,(T;p)}$ holds in particular for all $l \geq q$, we get from Lemma 2.3 *i)* that $\frac{\partial \mathbf{K}(\tilde{\rho})}{\partial x_k} \in \mathbf{F}_{0,l,(T;p)}$ for all $k = 1,2,3$. Therefore

$$\sup_{t\in[0,T]} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{l})} \left\| \frac{\partial \mathbf{K}(\tilde{\rho})}{\partial x_k} \right\|_l < \infty.$$

Since $\frac{3}{2}(\frac{1}{p} - \frac{1}{l}) = \frac{1}{2} + \frac{3}{2}(\frac{1}{q} - \frac{1}{l})$, we conclude that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,l,(T;q)}$.
ii) We claim that for each $p \leq r < \infty$, the *linear* operator (with ρ fixed) defined by

$$\eta(t,x) \mapsto \mathbf{b}^h(\eta,\rho)(t,x) = \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}}{\partial y_j}(x-y)\mathbf{K}(\tilde{\rho})_j(s,y)\eta(s,y)dy\, ds.$$

is continuous from $F_{1,r,(T;p)}$ to $F_{1,r',(T;p)}$, where r and r' satisfy the same constraints as in Lemma 4.1. The proof of this fact is similar as in Lemma 4.1, taking therein $p = p'$

and leaving one of the arguments in the bilinear function $\mathbf{b}^h$ fixed and equal to ρ. (More precisely, the norm $\|\mathbf{b}^h(\rho,\eta)\|_{1,(p;r'),T}$ can be bounded by certain constant times the product $\|\eta\|_{1,r,(T;p)}\|\mathbf{K}(\tilde{\rho})\|_{1,l,(T;q)}$, with l satisfying $\frac{1}{l}-\frac{1}{r}=\frac{1}{q}-\frac{1}{p}=-\frac{1}{3}$.)
Notice that the norm of $\mathbf{b}^h(\rho,\cdot): F_{1,r,(T;p)} \to F_{1,r',(T;p)}$ has the same dependence on T as the functional $\mathbf{B}$ has in Proposition 3.1 *i)* and *ii)*. Thus, by Banach's fixed point theorem applied to $\eta \mapsto \gamma_0 + \mathbf{b}^h(\rho,\eta)$, we have a local existence result in $F_{1,r,(T';p)}$ for some positive T' (possibly smaller than T) for the linear equation

$$\eta = \gamma_0 + \mathbf{b}^h(\rho,\eta). \tag{51}$$

Using this and uniqueness for (51) in $F_{0,p,T}$, together with the fact that $u \in \mathbf{F}_{1,q,T}$, we can adapt the arguments of Theorem 3.2 *i)* to the linear equation (51) to show that any solution $\eta \in F_{0,p,T}$ belongs to $F_{1,p,T}$. By following then the proof of Theorem 3.2 *ii)*, we prove that $\eta \in F_{1,r,(T;p)}$ for all $r \in [p,\infty[$. Since $\eta = \rho$ is a solution of (51), the statement follows.

□

A straightforward consequence is the regularity of the process Φ in (45):

Corollary 4.2 *Assume that P is a solution of (45) in the class $\mathcal{P}^T_{\frac{3}{2},b,0}$. Then, under P, the process Φ is continuous and with finite variation. We deduce that the associated function $\tilde{\rho}$ is a weak solution of the vortex equation with initial condition w_0.*

Proof: Since condition (46) holds, the process $\mathbf{f}(t,X_t)$ is a semi-martingale under P for any $\mathbf{f} \in C^{1,2}_{b,3}$. On the other hand, from Lemma 4.2 *iii)* with $r = 3$ and Lemma 2.3 *i)* we get that

$$\int_0^T \int_{\mathbb{R}^3} |\nabla \mathbf{K}(\tilde{\rho})(t,x)| \rho(t,x)\, dxdt < \infty. \tag{52}$$

As the process Φ is bounded under P, the equation verified by Φ in (45), together with (52) imply that $t \mapsto \Phi_t$ has finite variation.
We can thus apply Itô's formula to the product $\mathbf{f}(t,X_t)\Phi_t$ and see that

$$\mathbf{f}(t,X_t)\Phi_t - \mathbf{f}(0,X_0) - \int_0^t \Big[\frac{\partial \mathbf{f}}{\partial s}(s,X_s) + \nu\triangle \mathbf{f}(s,X_s) + \mathbf{K}(\tilde{\rho})(s,X_s)\nabla \mathbf{f}(s,X_s) + \mathbf{f}(s,X_s)\nabla \mathbf{K}(\tilde{\rho})(s,X_s)\Big]\Phi_s\, ds$$

is a martingale for all $\mathbf{f} \in C^{1,2}_{b,3}$. By multiplying the previous equation by $h_0(X_0)$ and taking expectations, we conclude from the definition of $\tilde{P}_s$ and Fubini's theorem (thanks also to (46) and (52)), that $\tilde{\rho}$ is a solution of the weak vortex equation (12).

□

Remark 4.2 *We have not used the fact that $\nabla\tilde{\rho}_t = 0$ to establish any of the previous results. This condition will allow us to conclude that $\tilde{\rho}$ is a mild solution of the vortex equation, and this will provide the additional regularity for the function $\nabla \mathbf{K}(\tilde{\rho})$ required to prove that (45) is well posed.*

4.2 Existence

In this section we will assume that

- $p \in]\frac{3}{2}, 3[$ and $\mathbf{w} \in \mathbf{F}_{0,p,T}$ is a given solution of the mild equation (10) defined in some time interval $[0,T]$.
- $\mathbf{w}(0) = w_0 \in L^1_3$.

We will associate with $\mathbf{w}$ a solution P of the martingale problem (45) in the class $\mathcal{P}^T_{\frac{3}{2},b,0}$, and such that the corresponding flow $\tilde{\rho}$ of vector measures defined as in (42) satisfies $\tilde{\rho} = \mathbf{w}$.

By Corollary 3.1, the drift term $\mathbf{K}(\mathbf{w})(t)$ and its gradient $\nabla\mathbf{K}(\mathbf{w})(t)$ are continuous and bounded functions on x for each $t \in]0,T]$, and with singularities in L^∞ and Hölder norm at time $t = 0$. To construct the probability measure P, we will follow a similar strategy as in [13] by an approximation argument by suitable processes involving regularized kernels instead of K. The additional difficulty here is that we have to approximate simultaneously both processes X and Φ, and therefore to take care of both drift terms $\mathbf{K}(\mathbf{w})(s, X_s)$ and $\nabla\mathbf{K}(\mathbf{w})(s, X_s)\Phi_s$.

Consider $\varphi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ a regular approximation of the Dirac mass, that is, $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^3}\varphi(\frac{x}{\varepsilon})$ for all x and $\varepsilon > 0$, with $\varphi : \mathbb{R}^3 \to \mathbb{R}$ a positive function in $\mathcal{S}$ such that $\int_{\mathbb{R}^3} \varphi(x)dx = 1$. We define regularized kernels $K_\varepsilon = \varphi_\varepsilon * K$, and associated mollified operators $\mathbf{K}^\varepsilon$ by

$$\mathbf{K}^\varepsilon(w)(x) := \int_{\mathbb{R}^3} K_\varepsilon(x-y) \wedge w(y)dy$$

The following are key remarks:

Remark 4.3 *If $w \in L^r_3$ for some $r \in]1,3[$, then*

$$\mathbf{K}^\varepsilon(w) = \mathbf{K}(\varphi_\varepsilon * w).$$

This is easily seen first for $w \in \mathcal{D}_3$ by taking Fourier transforms, and then for general $w \in L^r_3$ by density and Lemma 2.2 i). By similar reasons (using Lemma 2.3 i)), one has for all $r \in]1,\infty[$ and $w \in L^r_3$ that

$$\nabla\mathbf{K}^\varepsilon(w) = \nabla\mathbf{K}(\varphi_\varepsilon * w).$$

In particular, the continuity estimates of Lemmas 2.2 and 2.3 hold true (with the same constants as therein) for each of the operators $\mathbf{K}^\varepsilon$.

Remark 4.4 *We also deduce that $\mathbf{K}^\varepsilon(w)$ converges in L^l_3 to $\mathbf{K}(w)$, for all $r \in]1,3[$ and $\frac{1}{l} = \frac{1}{r} - \frac{1}{3}$, and that $\frac{\partial\mathbf{K}^\varepsilon(w)}{\partial x_k}$ converges in L^r_3 to $\frac{\partial\mathbf{K}(w)}{\partial x_k}$ for all $r \in]1,\infty[$ and $k = 1,2,3$.*

Let (ε_n) be a sequence converging to 0, and take in a fixed probability space a standard three dimensional Brownian motion B, and a $\mathbb{R}^3$-valued r.v. X_0 independent of B with law

$$\rho_0(x)dx := \frac{|w_0(x)|}{\|w_0\|_1}dx.$$

Consider moreover the following family of linear stochastic differential equations:

$$\xi^{(n)}_t(x) = x + \sqrt{2\nu}B_t + \int_0^t \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s, \xi^{(n)}_t(x))ds, \quad t \in [0,T]. \tag{53}$$

We have

Lemma 4.3 *The function $(s,y) \mapsto \mathbf{K}^\varepsilon(\mathbf{w})(s,y)$ is bounded and continuous on $[0,T] \times \mathbb{R}^3$, and infinitely many times differentiable on $y \in \mathbb{R}^3$, with uniformly bounded and continuous derivatives on $[0,T] \times \mathbb{R}^3$.*

Proof: The kernel K_ε belongs to $\mathcal{S}$, and $t \in [0,T] \mapsto \mathbf{w}(t) \in L^1_3$ is continuous by Theorem 3.2 *iii)*. We have

$$|\mathbf{K}^\varepsilon(\mathbf{w})(t,x) - \mathbf{K}^\varepsilon(\mathbf{w})(s,y)| \leq |\mathbf{K}^\varepsilon(\mathbf{w})(t,y) - \mathbf{K}^\varepsilon(\mathbf{w})(t,z)| \leq \tilde{C}(\mathbf{w},\varepsilon)|y-z| \\ + \int_{\mathbb{R}^3} |\mathbf{w}(t,z) - \mathbf{w}(s,z)||K_\varepsilon(z-y)|dz,$$

for constant $\tilde{C}(\mathbf{w},\varepsilon)$ a depending on $\|\mathbf{w}\|_{0,1,T}$ and on K_ε. The continuity of $\mathbf{K}^\varepsilon(\mathbf{w})$ follows from these considerations, and for the derivatives the proof is similar using the derivatives of K_ε.

$\square$

Therefore, equation (53) has a unique trajectorial solution for each x, and we can further take a version of the process $(t,x) \mapsto \xi_t^{(n)}(x)$ that is continuously differentiable in x for all t, and with continuous derivative $\nabla\xi_t^{(n)}(x)$ (see Kunita [19] Ch.2). For each x, $t \mapsto \nabla\xi_t^{(n)}(x)$ is the solution of the ordinary differential equation in $\mathcal{M}_{3\times 3}$

$$\nabla\xi_t^{(n)}(x) = Id + \int_0^t \nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,\xi_s^{(n)}(x))\nabla\xi_s^{(n)}(x)ds, \quad t \in [0,T].$$

We will denote by $(X^{(n)}, \Phi^{(n)})$ the couple of processes defined on $[0,T]$ by

$$X_t^{(n)} := \xi_t^{(n)}(X_0), \text{ and } \Phi_t^{(n)} = \nabla\xi_t^{(n)}(X_0),$$

so that

$$\begin{aligned} X_t^{(n)} &= X_0 + \sqrt{2\nu}B_t + \int_0^t \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,X_s^{(n)})ds \\ \Phi_t^{(n)} &= Id + \int_0^t \nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,X_s^{(n)})\Phi_s^{(n)}ds \end{aligned} \tag{54}$$

The law of $(X^{(n)}, \Phi^{(n)})$ clearly belongs to $\mathcal{P}_b(\mathcal{C}_T)$ and will be denoted by $Q^{(n)}$. Moreover, since the drift term in the first equation in (54) is bounded, $(Q^{(n)})_t^\circ$ has a density with respect to Lebesgue's measure. For each $n \in \mathbb{N}$, there exists a bi-measurable version of $(t,x) \mapsto \rho^{(n)}(t,x)$ of the densities of $(Q^{(n)})_t^\circ$ (see [26], p. 194), and thus, a bi-measurable version $(t,x) \mapsto \tilde{\rho}^{(n)}(t,x)$ of the densities of $\tilde{Q}_t^{(n)}$.

In what follows, we will prove that the sequence $Q^{(n)}$ is uniformly tight and that its accumulation points are solutions of (45). A first step is to prove the convergence, in a strong enough sense, of the one dimensional time marginals. We need an auxiliary regularity result, that will allow us to identify weak and mild solutions of the linear equation satisfied by $\tilde{\rho}^{(n)}$. This result is proved in Section **4.5**.

Lemma 4.4 *Consider $\tau > 0$, and let $\mathbf{v} : [0,\tau] \times \mathbb{R}^3 \to \mathbb{R}^3$ be a bounded continuous function, with bounded continuous derivatives up to the third order in the space variable $x \in \mathbb{R}^3$, which are Hölder continuous in x, uniformly in $(t,x) \in [0,\tau] \times \mathbb{R}^3$.*

Let $\phi \in \mathcal{D}$. Then, the unique solution g of the backward Cauchy problem

$$
\begin{aligned}
\frac{\partial}{\partial s} g(s,y) + \nu \Delta g(s,y) + \mathbf{v}(s,y) \nabla g(s,y) &= 0, \quad (s,y) \in [0,\tau] \times \mathbb{R}^3, \\
g(\tau, y) &= \phi(y).
\end{aligned}
\tag{55}
$$

is of class $C_b^{1,3}$ on $[0,\tau] \times \mathbb{R}^3$.

Lemma 4.5 *For all $t \in [0,T]$ and $n \in \mathbb{N}$, we have $\tilde{\rho}^{(n)}(t) \in L_3^p$, $\rho^{(n)}(t) \in L^p$, and*

$$
\sup_{n\in\mathbb{N}} \|\tilde{\rho}^{(n)}\|_{0,p,T} < \infty, \sup_{n\in\mathbb{N}} \|\rho^{(n)}\|_{0,p,T} < \infty. \tag{56}
$$

Moreover, $\tilde{\rho}^{(n)}(t)$ converges to $\mathbf{w}(t)$ in L_3^p for each $t \in [0,T]$, and in $L^1([0,T], L_3^p)$. Similarly, $\rho^{(n)}(t)$ converges to $\rho(t)$ in L^p for each $t \in [0,T]$, and in $L^1([0,T], L^p)$, ρ being a solution of the linear equation

$$
\rho(t,x) = G_t^\nu * \rho_0(x) + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y) \mathbf{K}(\mathbf{w})_j(s,y) \rho(s,y) \; dy \; ds \tag{57}
$$

for all $t \in [0,T]$.

Proof : By writing Itô's formula for the product $\mathbf{f}(t, X_t^{(n)}) \Phi_t^{(n)}$ for an arbitrary function $\mathbf{f} \in (C_b^{1,2})_3([0,T], \mathbb{R}^3)$, and taking expectations after multiplying by $h_0(X_0)$, we see that $\tilde{\rho}^{(n)}(t)$ is a solution of the following weak equation

$$
\begin{aligned}
\int_{\mathbb{R}^3} \mathbf{f}(t,y) \tilde{\rho}^{(n)}(t,y) dy = \int_{\mathbb{R}^3} \mathbf{f}(0,y) w_0(y) dy + \int_0^t \int_{\mathbb{R}^3} \bigg[\frac{\partial \mathbf{f}}{\partial s}(s,y) + \nu \triangle \mathbf{f}(s,y) \\
+ \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y) \nabla \mathbf{f}(s,y) + \mathbf{f}(s,y) \nabla \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y) \bigg] \tilde{\rho}^{(n)}(s,y) \; dy \; ds.
\end{aligned}
\tag{58}
$$

Hence, by similar arguments as in Lemma 2.1, the function $\tilde{\rho}^{(n)}$ solves the linear equation

$$
\begin{aligned}
\tilde{\rho}^{(n)}(t,x) = G_t^\nu * w_0(x) + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \bigg[\frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y) [\mathbf{K}^{\varepsilon_n}(\mathbf{w})_j(s,y) \tilde{\rho}^{(n)}(s,y)] \\
+ G_{t-s}^\nu(x-y) [\tilde{\rho}_j^{(n)}(s,y) \frac{\partial \mathbf{K}^{\varepsilon_n}(\mathbf{w})}{\partial y_j}(s,y)] \bigg] dy \; ds.
\end{aligned}
\tag{59}
$$

We will prove that $\tilde{\rho}^{(n)}$ also solves the linear mild equation

$$
\begin{aligned}
\tilde{\rho}^{(n)}(t,x) =& \int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy \\
&+ \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y) \left[\mathbf{K}^{\varepsilon_n}(\mathbf{w})_j(s,y) \tilde{\rho}^{(n)}(s,y) - \tilde{\rho}_j^{(n)}(s,y) \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y) \right] dy \; ds.
\end{aligned}
\tag{60}
$$

To that end, we first check that $\tilde{\rho}^{(n)}(s)$ has null divergence in the distribution sense. By Lemma 4.3 and Lemma 4.4, for each $\phi \in \mathcal{D}$ and $t \in]0,T[$, the backward Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial s} g(s,y) + \nu\Delta g(s,y) + \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y)\nabla g(s,y) &= 0, \quad (s,y) \in [0,t[\times\mathbb{R}^3, \\ g(t,y) &= \phi(y), \end{aligned} \tag{61}$$

has a unique solution g which is of class $C_b^{1,3}$ on $[0,t] \times \mathbb{R}^3$.
We can therefore plug the function $\mathbf{f} = \nabla g$ in (58), and after simple computations obtain

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla\phi(y)\tilde{\rho}^{(n)}(t,y)dy &= \int_0^t \int_{\mathbb{R}^3} \nabla\left[\frac{\partial g}{\partial s}(s,y) + \nu\triangle g(s,y) + \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y)\nabla g(s,y)\right]\tilde{\rho}^{(n)}(s,y)\ dy\ ds \\ &= 0 \end{aligned} \tag{62}$$

for all $\phi \in \mathcal{D}$ and $t \in [0,T]$. Thus, $div\ \tilde{\rho}^{(n)} = 0$.
Next, we claim that $\tilde{\rho}^{(n)}(t)$ belongs to L_3^p for all $t \in [0,T]$. To verify this, notice that the integral in the l.h.s of (59) belongs to $F_{0,r',T}$ if $\tilde{\rho}^{(n)} \in \mathbf{F}_{0,r,T}$ and $r' \in [r, \frac{3r}{3-r}[$ for given $r \in [1, \frac{3}{2}[$. This is seen by using Young's inequality, and the facts that $\|G^\nu_{t-s}\|_m, \|\nabla G^\nu_{t-s}\|_m \in L^1([0,t], ds)$ for $m \in [1, \frac{3}{2}[$, and that $\mathbf{K}^{\varepsilon_n}(\mathbf{w})$ and its gradient are bounded functions.
Then, using moreover the fact that $\mathbf{w}_0 \in \mathbf{F}_{0,r,T}$ for all $r \in [1,p]$ we state inductively that $\tilde{\rho}^{(n)} \in \mathbf{F}_{0,r_k,T}$ for a finite sequence $r_k, k = 0, \cdots, N$, such that $r_0 = 1$, $r_{k+1} \in [r_k, \frac{3r_k}{3-r_k}[$ and $r_N = p$, and our claim is proved.
From the latter and (62) we deduce that $\int_{\mathbb{R}^3} \nabla\phi(y)\tilde{\rho}^{(n)}(t,y)dy = 0$ for all $\phi \in W^{1,p^*}$, and from (59) and the fact $G^\nu_{t-s}(x-\cdot)\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,\cdot) \in W_3^{1,p^*}$, we conclude that $\tilde{\rho}^{(n)}$ solves (60).
We will now derive an upper bound for $\|\tilde{\rho}^{(n)}\|_{0,p,T}$ independent of n.
Let us take L_3^p norm in (60). By standard arguments (as in Proposition 3.1 *i)*), and using Lemma 2.2 and Remark 4.3, it follows that

$$\|\tilde{\rho}^{(n)}(t)\|_p \leq \|\mathbf{w}_0\|_{0,p,T} + C\|\mathbf{w}\|_{0,p,T} \int_0^t (t-s)^{-\frac{3}{2p}}\ \|\tilde{\rho}^{(n)}(s)\|_p\ ds.$$

Iterating this inequality $N(p)$ times, with $N(p)$ the first integer for which $2^N(1-\frac{3}{2p}) > 0$, we deduce that

$$\|\tilde{\rho}^{(n)}(t)\|_p \leq C + C' \int_0^t \|\tilde{\rho}^{(n)}(s)\|_p\ ds,$$

with constants independent of n, and we conclude by Gronwall's lemma.
The same arguments establish the corresponding results for the functions $\rho^{(n)}$, starting this time from the fact that $\rho^{(n)}$ solves the linear equation

$$\rho^{(n)}(t,x) = G^\nu_t * \rho_0(x) + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\mathbf{K}^{\varepsilon_n}(\mathbf{w})_j(s,y)\rho^{(n)}(s,y)\ dy\ ds, \tag{63}$$

which is seen by similar arguments as in Section **4.1**.
Now we prove the asserted convergence for $\tilde{\rho}^{(n)}$. By taking the L_3^p norm to the difference $\mathbf{w}(t) - \tilde{\rho}^{(n)}(t)$ and proceeding as above, it is seen that

$$\begin{aligned} \|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p \leq & C \int_0^t (t-s)^{-\frac{3}{2p}}\ \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q\ ds \\ & + C \int_0^t (t-s)^{-\frac{3}{2p}}\ \|\tilde{\rho}^{(n)}(s) - \mathbf{w}(s)\|_p\ ds. \end{aligned}$$

We have also used here the estimates (56). Writing $\theta_0 = 1 - \frac{3}{2p}$ and using induction, we get

$$\|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p \leq C \int_0^t \sum_{k=1}^{N} (t-s)^{k\theta_0 - 1} \ \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \ ds$$
$$+ C \int_0^t (t-s)^{N\theta_0 - 1} \ \|\tilde{\rho}^{(n)}(s) - \mathbf{w}(s)\|_p \ ds.$$

The identity $\int_0^t (t-s)^{\theta-1} \int_0^s (s-r)^{\epsilon-1} dr \ ds = \beta(\theta,\epsilon) \int_0^t (t-s)^{\theta+\epsilon-1} ds$ for all $\theta, \epsilon > 0$ has also been used. Thus, taking a fixed $N = \tilde{N}(p)$ such that $= \tilde{N}(p) > \theta_0^{-1}$, yields

$$\|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p \leq C \int_0^t \alpha(t-s) \ \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \ ds$$
$$+ C(T) \int_0^t \ \|\tilde{\rho}^{(n)}(s) - \mathbf{w}(s)\|_p \ ds, \tag{64}$$

with $\alpha(s) = \sum_{k=1}^{\tilde{N}(p)} s^{k\theta_0 - 1}$. Integrating now between 0 and $\tau \in [0,T]$ gives

$$\int_0^\tau \|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p dt \leq C \int_0^T \int_0^t \alpha(t-s) \ \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \ ds \ dt$$
$$+ C \int_0^\tau \ \int_0^t \|\tilde{\rho}^{(n)}(s) - \mathbf{w}(s)\|_p \ ds \ dt,$$

and by Gronwall's lemma,

$$\int_0^\tau \|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p dt \leq C \int_0^T \int_0^t \alpha(t-s) \ \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \ ds \ dt.$$

Thanks to the Remark 4.4, the right hand side converges to 0 by a double application of Lebesgue's theorem. Taking $\tau = T$ gives us the convergence in $L^1([0,T], L^p_3)$, and point-wise convergence in L^p_3 on $[0,T]$ follows then from (64).

Repeating this reasoning with the difference $\rho^{(m)}(t) - \rho^{(n)}(t)$, $n, m \in \mathbb{N}$, shows that $\rho^{(n)}$ is Cauchy in $L^1([0,T], L^p)$, and that $\rho^{(n)}(t)$ is also Cauchy in L^p, $\forall t \in [0,T]$. Consequently, there is point-wise convergence of $\rho^{(n)}$ in L^p on the interval $[0,T]$ to a limit $\rho \in L^1([0,T], L^p)$. Estimate (56) implies that $\rho \in F_{0,p,T}$, and using the fact that

$$\left\| \int_0^t \sum_{j=1}^{3} \int \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y) \ \Big(\rho^{(n)}(s,y) \ \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y) - \rho(s,y) \ \mathbf{K}(\mathbf{w})(s,y) \Big) \ dy \ ds \right\|_p$$

is bounded above by $C \int_0^t \ (t-s)^{-\frac{3}{2p}} \left[\|\rho^{(n)}(s) - \rho(s)\|_p + \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \right] ds$ (which goes to 0 as $n \to \infty$), we pass to the limit on n in equation (63) to conclude that ρ solves (57). This completes the proof of the lemma.

∎

Next step is to prove tightness of the sequence $Q^{(n)}$. We will use the following version of Gronwall's lemma:

Lemma 4.6 *Let $g : [0,T] \to]0,\infty[$ be a bounded function satisfying*

$$g(t) \le C + \int_0^t g(s)k(s)ds$$

for all $t \in [0,T]$, where $k : [0,T] \to]0,\infty[$ is a positive function such that $\int_0^T k(s)ds < \infty$. Then, for all $t \in [0,T]$,

$$g(t) \le C \exp \int_0^T k(s)ds.$$

Lemma 4.7 *The sequence $(Q^{(n)}, n \in \mathbb{N})$ is tight.*

Proof: It is enough to prove that each of the two sequences of process $X^{(n)}$ and $\Phi^{(n)}$ have laws that are uniformly tight in n. We will use Aldous' criterion for both of them.
Let R_n, S_n be stopping times in the filtration of $(X^{(n)}, \Phi^{(n)})$ such that $0 \le R_n \le S_n \le T$ and $S_n - R_n \le \Delta$. Thanks to Remark 4.3, Lemma 2.2 *ii)*, Lemma 3.7 *i)*, and the arguments of Corollary 3.1, we have

$$\int_{R_n}^{S_n} |\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(t, X_t^{(n)})|dt \le C \int_{R_n}^{S_n} t^{-\frac{1}{2}} \| \mathbf{K}^{(\varepsilon_n)}(\mathbf{w}) \|_{1,q,T} dt \le C \left(S_n^{\frac{1}{2}} - R_n^{\frac{1}{2}} \right) \| \mathbf{w} \|_{1,p,T} \le C \Delta^{\frac{1}{2}},$$

and the criterion applies.
Consider now the processes $\Phi^{(n)}$. Since $\mathbf{K}^{\varepsilon_n}(\mathbf{w})(t)$ is bounded, each process $\Phi^{(n)}$ is bounded on $[0,T]$. On the other hand, by Remark 4.3, Lemmas 2.3 *ii)* and Lemma 3.7 *ii)*, we have

$$\left\| \frac{\partial \mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(t)}{\partial x_k} \right\|_\infty \le C t^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} \| \mathbf{w} \|_{1,r,(T;p)} \tag{65}$$

for all $r \in]3, \frac{3p}{3-p}[$ and $k = 1,2,3$. From this and Lemma 4.6 we deduce that

$$|\Phi_t^{(n)}| \le \exp \left(C T^{\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} \| \mathbf{w} \|_{1,r,(T;p)} \right) \tag{66}$$

for all $t \in [0,T]$ and a constant $C > 0$ which does not depend on n. Let now R_n, S_n be stopping times as before, and fix $r \in]3, \frac{3p}{3-p}[$. By using (65) and (66) we establish that

$$\int_{R_n}^{S_n} |\nabla \mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(t, X_t^{(n)})||\Phi_t^{(n)}|dt \le C(R_n^{\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} - S_n^{\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})}) \le C \Delta^{\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})}$$

for a constant $C > 0$ not depending on n, and the result follows since $\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r}) > 0$. □

We can now prove

Proposition 4.2 *Every accumulation point of the sequence $Q^{(n)}$ is a solution of the martingale problem (45) in the class $\mathcal{P}^T_{\frac{3}{2},b,0}$.*

Proof : Denote by P the limit of a convergent subsequence renamed $Q^{(n)}$. From the weak convergence $(Q^{(n)})_t^\circ \to P_t^\circ$ and Lemma 4.5, we deduce that $P_t^\circ(dx) = \rho(t,x)dx$ for all $t \in [0,T]$, with $\rho \in F_{0,p,T}$ the unique solution of (57).

Now we take $f \in C_b^{1,2}$, $0 \le s_1 \le \cdots \le s_m \le s < t \le T$ and $\lambda : \mathbb{R}^m \to \mathbb{R}$ a continuous bounded function. We will first show that

$$E^P\Big[\Big(\int_s^t \Big\{\frac{\partial f}{\partial \tau}(\tau, X_\tau) + \nu \Delta f(\tau, X_\tau) + \mathbf{K}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau)\Big\}\, d\tau + f(t, X_t) - f(s, X_s)\Big) \times \lambda(X_{s_1}, \ldots, X_{s_m})\Big] = 0, \quad (67)$$

and that

$$E^P\left[\left|\Phi_t - Id - \int_0^t \nabla \mathbf{K}(\mathbf{w})(\tau, X_\tau)\Phi_\tau d\tau\right|\right] = 0, \quad (68)$$

with (X, Φ) the canonical process and $\mathbf{w} \in \mathbf{F}_{0,p,T}$ the solution of (10) we are given. Notice that the result will follow from (67) and (68) by proving that the density family $\tilde{\rho}$ of $\tilde{P}$ is equal to $\mathbf{w}$.

Define a function $\kappa : C([0,T], \mathbb{R}) \to \mathbb{R}$ by

$$\kappa(\xi) = \Big(\int_s^t \Big\{\frac{\partial f}{\partial \tau}(\tau, \xi(\tau)) + \nu \Delta f(\tau, \xi(\tau)) + \mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\nabla f(\tau, \xi(\tau))\Big\}\, d\tau + f(t, \xi(t)) - f(s, \xi(s))\Big) \times \lambda(\xi(s_1), \ldots, \xi(s_m)) \quad (69)$$

We now check that it is continuous and bounded. From Corollary 3.1 *i)* we see that

$$\int_s^t |\mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\nabla f(\tau, \xi(\tau))|\, d\tau \le C(T)\|\nabla f\|_\infty \|\mathbf{w}\|_{0,p,T} \text{ and}$$

$$|\mathbf{K}(\mathbf{w})(\tau, x) - \mathbf{K}(\mathbf{w})(\tau, y)| \le C\tau^{-\frac{1}{2}}|x - y|^{\frac{2p-3}{p}} \|\mathbf{w}\|_{0,p,T}, \ \forall x, y \in \mathbb{R}^3.$$

Thus,

$$\int_s^t |\mathbf{K}(\mathbf{w})(\tau, \xi_1(\tau))\nabla f(\tau, \xi_1(\tau)) - \mathbf{K}(\mathbf{w})(\tau, \xi_2(\tau))\nabla f(\tau, \xi_2(\tau))|\, d\tau$$
$$\le \ C(T)\|\nabla f\|_\infty \|\xi_1 - \xi_2\|_\infty^{\frac{2p-3}{p}} \|\mathbf{w}\|_{0,p,T} + C'(T)\|\Delta f\|_\infty \|\xi_1 - \xi_2\|_\infty \|\mathbf{w}\|_{0,p,T},$$

for all $\xi_1, \xi_2 \in C([0,T], \mathbb{R})$. It follows that the mapping $\xi \mapsto \int_s^t \mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\nabla f(\tau, \xi(\tau))\, d\tau$ is continuous and bounded on $C([0,T], \mathbb{R})$, and then the same holds for κ.

Therefore, we have $E^{Q^{(n)}}(\kappa(X)) \to E^P(\kappa(X))$ as $n \to \infty$. Now, from (54) and the definition of $Q^{(n)}$, it follows that

$$E^{Q^{(n)}}\Big[\Big(\int_s^t \Big\{\frac{\partial f}{\partial \tau}(\tau, X_\tau) + \nu \Delta f(\tau, X_\tau) + \mathbf{K}^{\varepsilon_n}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau)\Big\}\, d\tau + f(t, X_t) - f(s, X_s)\Big) \times \lambda(X_{s_1}, \ldots, X_{s_m})\Big] = 0,$$

and then

$$E^{Q^{(n)}}(\kappa(X)) = E^{Q^{(n)}}\Big[\int_s^t \Big(\mathbf{K}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau) - \mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau)\Big)\, d\tau \times \lambda(X_{s_1}, \ldots, X_{s_m})\Big].$$

As $\rho^{(n)}(t)$ is a probability density and $q* = \frac{3p}{4p-3} < \frac{3}{2} < p$, we have

$$\sup_{k\in\mathbb{N}} \|\rho^{(k)}\|_{0,q*,T} < \infty$$

thanks to the estimate (56) for $\rho^{(n)}$. It follows that

$$\begin{aligned}\left|E^{Q^{(n)}}(\kappa(X))\right| &\leq CE^{Q^{(n)}}\left[\int_s^t \left|\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau, X_\tau) - \mathbf{K}(\mathbf{w})(\tau, X_\tau)\right| d\tau\right] \\ &\leq C\sup_{k\in\mathbb{N}} \|\rho^{(k)}\|_{0,q*,T} \int_0^T \left\|\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau) - \mathbf{K}(\mathbf{w})(\tau)\right\|_q d\tau,\end{aligned}$$

and by Remark 4.4, we conclude that $E^{Q^{(n)}}(\kappa(X)) \to 0$. This proves (67).

We next prove (68). Consider an arbitrary continuous truncation function on matrices $\chi_R : \mathcal{M}_{3\times 3} \to \mathcal{M}_{3\times 3}$, with $R > 0$, such that $|\chi_R(z)| \leq R$ for all $z \in \mathcal{M}_{3\times 3}$.
By (66) there exists a constant $R = R_{\mathbf{w}}$ independent of n such that $\sup_{t\in[0,T]} |\Phi^{(n)}| \leq R_{\mathbf{w}}$ for all $n \in \mathbb{N}$. We will check that the function $\zeta : C([0,T], \mathbb{R}^3) \times C([0,T], \mathcal{M}_{3\times 3}) \to \mathbb{R}$ given by

$$\zeta(\xi, z) := \left|\chi_{R_{\mathbf{w}}}(z_t) - Id - \int_0^t \nabla\mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\chi_{R_{\mathbf{w}}}(z_\tau)d\tau\right| \tag{70}$$

is bounded and continuous. To that end, it is enough to state that the mapping $(\xi, z) \mapsto \int_0^t \nabla\mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\chi_{R_{\mathbf{w}}}(z(\tau))d\tau$ is bounded and continuous. The first fact is consequence of (65). The continuity follows easily from the estimate

$$|\nabla\mathbf{K}(\mathbf{w})(\tau, x) - \nabla\mathbf{K}(\mathbf{w})(\tau, y)| \leq C\|\mathbf{w}\|_{1,r,(T;p)}\tau^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}|x-y|^{1-\frac{3}{r}},\ \forall x, y \in \mathbb{R}^3.$$

for any fixed $r \in]3, \frac{3p}{3-p}[$, given by Lemma 3.7. Thus, proving (68) amounts to check that

$$E^{Q^{(n)}}\left(\zeta(\xi, z)\right) \to 0 \tag{71}$$

when $n \to \infty$. Since

$$E^{Q^{(n)}}\left|\chi_{R_{\mathbf{w}}}(z_t) - Id - \int_0^t \nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w})(\tau, \xi(\tau))\chi_{R_{\mathbf{w}}}(z_\tau)d\tau\right| = 0$$

by (54), we have

$$E^{Q^{(n)}}(\zeta(X, \Phi)) \leq R_{\mathbf{w}}E^{Q^{(n)}}\left[\int_s^t \left|\nabla\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau, X_\tau) - \nabla\mathbf{K}(\mathbf{w})(\tau, X_\tau)\right| d\tau\right]. \tag{72}$$

If $p \geq 2$, the r.h.s. of (72) is bounded above by

$$C\sup_{k\in\mathbb{N}} \|\rho^{(k)}\|_{0,p*,T} \int_0^T \left\|\nabla\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau) - \nabla\mathbf{K}(\mathbf{w})(\tau)\right\|_p d\tau.$$

The fact that the supremum is finite is immediate from (56) since $p* \leq 2$, and (71) follows then from Remark 4.4.
If $p < 2$ we bound the r.h.s. of (72) above by

$$C\sup_{k\in\mathbb{N}} \|\rho^{(k)}\|_{0,p*,(T;p)} \int_0^T t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{p*})} \left\|\nabla\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau) - \nabla\mathbf{K}(\mathbf{w})(\tau)\right\|_p d\tau.$$

To conclude (71) we just have to stablish that

$$\sup_{k\in\mathbb{N}} \|\rho^{(k)}\|_{0,p*,(T;p)} < \infty. \tag{73}$$

This follows by our usual iterative argument applied to the equation (63), that is, using repeatedly the fact that solution of the equation (63) in $F_{0,r,(T;p)}$ belongs to $F_{0,r',(T;p)}$ for $p \le r < 3$ and $r \le r' \le \frac{3r}{6-2r}$. The key point here is that, thanks to Lemma 2.2 and Remark 4.3, the norm of the linear functional

$$\eta(t,x) \mapsto \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\mathbf{K}^{\varepsilon_n}(\mathbf{w})_j(s,y)\eta(s,y)\;dy\;ds,$$

defined from $F_{0,r,(T;p)}$ to $F_{0,r',(T;p)}$, can be estimated in terms of $\|\mathbf{K}^{\varepsilon_n}(\mathbf{w})\|_{0,l,(T;q)}$ (where $l = \frac{3r}{3-r}$) and therefore in terms of $\|\mathbf{w}\|_{0,r,(T;p)}$ only, thanks to Lemma 2.2 and Remark 4.3. Therefore, the norm $\|\tilde{\rho}^{(n)}\|_{0,r',(T;p)}$ is bounded independently of $n \in \mathbb{N}$, and the rest of the argument follows in a standard way. Thus, (71) being proved in all cases, we conclude (68).

To finish the proof, we just have to check that for each $t \in [0,T]$, the function $\tilde{\rho}(t)$ given by

$$\int_{\mathbb{R}^3} \mathbf{f}(x)\tilde{\rho}(t,x)dx := E^P(\mathbf{f}(X_t)\Phi_t h_0(X_0)),$$

for all $\mathbf{f} \in \mathcal{D}$, is equal to $\mathbf{w}(t)$. Thanks to the convergence $\tilde{\rho}^{(n)} \to \mathbf{w}$ stated in Lemma 4.5, this will hold as soon as the convergence

$$E^{Q^{(n)}}(\mathbf{f}(X_t)\Phi_t h_0(X_0)) \to E^P(\mathbf{f}(X_t)\Phi_t h_0(X_0))$$

is proved. Here we must be careful because the function h_0 is not necessarily continuous. We will use fact (proved in [16]) that for every $k \in \mathbb{N}$, one can find a continuous bounded function h_0^k such that $\frac{|w_0|}{\|w_0\|_1}(\{h_0^k \neq h_0\}) \le \frac{1}{k}$, and $|h_0^k| \le |h_0|$. We also notice that under P, the process Φ is bounded by the same constant $R_{\mathbf{w}}$ as it is under each law $Q^{(n)}$.
Hence, for any $k \in \mathbb{N}$ we have

$$\begin{aligned}
&|E^{Q^{(n)}}(\mathbf{f}(X_t)\Phi_t h_0(X_0)) - E^P(\mathbf{f}(X_t)\Phi_t h_0(X_0))| \\
&\qquad \le C(E^{Q^{(n)}}|h_0^k(X_0) - h_0(X_0)| + E^P|h_0^k(X_0) - h_0(X_0)|) \\
&\qquad\qquad + |E^{Q^{(n)}}(\mathbf{f}(X_t)\chi_{R_{\mathbf{w}}}(\Phi_t)h_0(X_0)) - E^P(\mathbf{f}(X_t)\chi_{R_{\mathbf{w}}}(\Phi_t)h_0(X_0))|.
\end{aligned}$$

We conclude by taking $\limsup$ as $n \to \infty$ and then limit as $k \to \infty$.

∎

4.3 Uniqueness

Proposition 4.3 *i) If $P \in \mathcal{P}^T_{\frac{3}{2},b,0}$ is a solution of (45) with $\rho \in F_{0,p,T}$ and $p \in]\frac{3}{2},3[$, then $\mathbf{w} := \tilde{\rho}$ is a solution of the mild vortex equation (10) in the space $\mathbf{F}_{0,p,T}$.*

ii) We deduce that uniqueness holds for (45) in the class $\mathcal{P}^T_{\frac{3}{2},b,0}$.

Proof: *i)* Let $P \in \mathcal{P}^T_{\frac{3}{2},b,0}$ be a solution of (45). By Proposition 4.1 and Corollary 4.2, $\tilde{\rho}$ is a weak solution in the spaces $\mathbf{F}_{0,p,T} \cap \mathbf{F}_{0,r,(T;p)}$ for all $r \in [p, \infty[$. As in Corollary 4.2, conditions (46) and (52) can be seen to hold, and then it is not hard to check that $\tilde{\rho}$ satisfies the assumptions of Lemma 2.1. Thus, it solves the intermediate mild equation (13). To conclude it is enough to verify that

$$\sum_{j=1}^{3} \int_{\mathbb{R}^3} G^{\nu}_{t-s}(x-y)[\tilde{\rho}_j(s,y)\frac{\partial \mathbf{K}(\tilde{\rho})}{\partial y_j}(s,y)] + \frac{\partial G^{\nu}_{t-s}}{\partial y_j}(x-y)[\tilde{\rho}_j(s,y)\mathbf{K}(\tilde{\rho})(s,y)]dy = 0 \quad (74)$$

for all $s \in [0,T]$. Since $1 < q* < \frac{3}{2}$ (where $q = \frac{3p}{3-p}$), the function $\tilde{\rho} = h^P \rho$ belongs to $\mathbf{F}_{0,q*,T}$. This and the fact that $div\ \tilde{\rho}(s) = 0$ in the distribution sense yield (74), because $G^{\nu}_{t-s}(x-\cdot)\mathbf{K}(\tilde{\rho})(s,\cdot) \in W^{1,q}_3$ thanks to Proposition 4.1 *i)*.

ii) Assume that P^1 and P^2 are two solutions of (45) in $\mathcal{P}^T_{\frac{3}{2},b,0}$, with density families $\rho^1 \in F_{0,p^1,T}$ and $\rho^2 \in F_{0,p^2,T}$ respectively. Then, we have $\tilde{\rho}^1, \tilde{\rho}^2 \in F_{0,p,T}$, and from *i)* and the uniqueness statement for the mild vortex equation (10) in $\mathbf{F}_{0,p,T}$ (Theorem 3.1 *a)*), we deduce that $\tilde{\rho}^1 = \tilde{\rho}^2$.

On the other hand, by the arguments in the proof of Proposition 4.1, ρ^1 and ρ^2 solve equation (48) with $h = h^{P^1}$ and $h = h^{P^2}$ respectively. Since by *i)* and Theorem 3.1 *i)*, the function $h^{P^1}\rho^1 = h^{P^2}\rho^2 = \mathbf{w} \in \mathbf{F}_{0,p,T}$ is uniquely determined, it follows that ρ^1 and ρ^2 are solutions of the linear equation

$$\rho(t,x) = G^{\nu}_t * \rho_0(x) + \int_0^t \sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{\partial G^{\nu}_{t-s}}{\partial y_j}(x-y)\mathbf{K}(\mathbf{w})_j(s,y)\rho(s,y)\ dy\ ds, \quad (75)$$

and thus they equal the unique solution of (75) in $F_{0,p,T}$, that we denote by ρ.

Thus, we have established that P^1 and P^2 solve the linear martingale problem in $\mathcal{P}^T_{\frac{3}{2},b,0}$:

- $Q^{\circ}|_{t=0}(dx) = \frac{|w_0(x)|}{\|w_0\|_1}dx$ and for all $0 \le t \le T$ and $Q^{\circ}_t(dx) = \rho_t(x)dx$ which is fixed.
- $f(t,X_t) - f(0,X_0) - \int_0^t \left[\frac{\partial f}{\partial s}(s,X_s) + \nu\triangle f(s,X_s)ds + \mathbf{K}(\mathbf{w})(s,X_s)\nabla f(s,X_s)\right] ds,$ (76)

  $0 \le t \le T,$ is a continuous Q°-martingale for all $f \in \mathcal{C}^{1,2}_b$;
- $\Phi_t = Id + \int_0^t \nabla\mathbf{K}(\mathbf{w})(s,X_s)\Phi_s\ ds,$ for all $0 \le t \le T,$ Q almost surely.

We can now follow the arguments of [25] or [13] to prove the fact that $(P^1)^{\circ} = (P^2)^{\circ}$. Indeed, by Corollary 3.1 *i)*, the coefficient $\mathbf{K}(\mathbf{w})$ in (76) satisfies $|\mathbf{K}(\mathbf{w})(t)| \le Ct^{-\frac{1}{2}}$. Consequently, if Q is a solution of (76), and if D_n with $n \in \mathbb{N}$ denotes the shift operator on $C([0,T],\mathbb{R}^3)$ defined by $D_n(\xi) = \xi(\frac{1}{n}+\cdot)$, then the probability measure $Q^{\circ} \circ D_n^{-1}$ solves a martingale problem with bounded coefficients, and with a fixed initial law given by $\rho(\frac{1}{n},x)dx$. By classic results of Stroock and Varadhan [30], $Q^{\circ} \circ D_n^{-1}$ is uniquely determined, and thus $(P^1)^{\circ} \circ D_n^{-1} = (P^2)^{\circ} \circ D_n^{-1}$ for all $n \in \mathbb{N}$. By letting $n \to \infty$ we conclude that $(P^1)^{\circ} = (P^2)^{\circ}$

It remains us to prove the identity $(P^1)' = (P^2)'$. In virtue of the estimate in L^{∞}-norm in Corollary 3.1 *iii)*, it is an elementary fact that for each $\xi \in C([0,T],\mathbb{R}^3)$ the O.D.E.

$$z(t) = Id + \int_0^t \nabla\mathbf{K}(\mathbf{w})(s,\xi(s))z(s)ds$$

has a unique continuous solution $t \in [0,T] \mapsto z(t) \in \mathcal{M}_{3\times 3}$. Further, using the estimate in Hölder norm of Corollary 3.1 *iii)*, and Gronwall's lemma, it is easily seen that the mapping $\xi \mapsto z$ is continuous. This clearly implies that $(P^1)' = (P^2)'$, and the proof is finished.

□

4.4 Proof of Lemma 4.4

Under the assumptions on the function $\mathbf{v}$ in Lemma 4.4, standard results (see e.g. [14]) provide existence of a unique solution $g : [0,\tau] \times \mathbb{R}^3 \to \mathbb{R}^3$ of (55), which is continuous, bounded and of class $C_b^{1,2}$ on each domain $[0,\theta] \times \mathbb{R}^3$ with $0 < \theta < \tau$. We could try to adapt available analytical results to prove that $g \in C_b^{1,3}([0,\tau] \times \mathbb{R}^3)$, but this is rather tedious. (Indeed, standard statements require Hölder continuity in time of the coefficients, and do not provide the complete regularity result we need here up to the final time τ.) We will thus give a direct proof of Lemma 4.4 by a probabilistic argument, inspired from Theorem 7.1, Ch. 3 in Kunita's course [19].

Proof of Lemma 4.4: Consider a three dimensional Brownian motion B on a filtered probability space. By the assumptions on $\mathbf{v}$ and the results in [19], there exists on that space a continuous three parameter process $\xi_{s,t}(x)$ defined for all $0 \leq s \leq t \leq \tau$ and $x \in \mathbb{R}^3$, such that for each (s,x),

$$\xi_{s,t}(x) = x + \sqrt{2\nu}(B_t - B_s) + \int_s^t \mathbf{v}(\theta, \xi_{s,\theta}(x))d\theta, \text{ for all } t \in [s,\tau]$$

almost surely. Further, the function $\xi_{s,t} : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism of class C^3. Since

$$\nabla \xi_{s,t}(x) = Id + \int_s^t \nabla \mathbf{v}(\theta, \xi_{s,\theta}(x)) \nabla \xi_{s,\theta}(x) d\theta,$$

it follows that $\nabla \xi_{s,t}(x)$ is bounded and then $\nabla \xi_{s,t}(x) \to Id$ when $s \to t^-$ for each $t \in [0,\tau]$. Considering the equations satisfied by the higher order derivatives, one can also conclude that $D^\alpha \xi_{s,t}$ is bounded, and that $D^\alpha \xi_{s,t}(x) \to D^\alpha x$ when $s \to t^-$, for any multi-index $|\alpha| \leq 3$. It follows that the function $f(s,x) := E(\phi(\xi_{s,\tau}(x))$ has derivatives in x up to the third order, and f and its derivatives are bounded and continuous on $[0,\tau] \times \mathbb{R}^3$.

We will show that $f = g$, which achieves the proof. Write $L_\theta \phi(x) := \nu \Delta \phi(x) + \mathbf{v}(\theta, x) \nabla \phi(x)$. By the backward Itô formula (Theorem 1.1 in [19], Ch. 3), one has

$$\phi(\xi_{s,t}(x)) = \phi(x) + \sqrt{2\nu} \int_s^t \nabla(\phi \circ \xi_{\theta,t})(x) \widehat{d}B_\theta + \int_s^t L_\theta(\phi \circ \xi_{\theta,t})(x) d\theta \tag{77}$$

where $\int_s^t \cdot \, \widehat{d}B_\theta$ is the backward stochastic integral with respect to B on $[s,t]$ (i.e. the stochastic integral with respect to the standard Brownian motion $(\hat{B}_s^t = B_{t-s} - B_t, s \in [0,t])$ and its natural filtration). Using (77), we check that

$$L_\theta \phi(y) = \lim_{\theta' \to \theta^-} \frac{1}{\theta - \theta'} \left[E(\phi \circ \xi_{\theta',\theta}(y)) - \phi(y) \right],$$

and then the commutation relation $E[L_\theta(\phi \circ \xi_{\theta,t})(x)] = L_\theta E[(\phi \circ \xi_{\theta,t})(x)]$ is obtained, thanks also to the independence of $\xi_{s',s}(x)$ and $\xi_{s,t}(y)$ for $s' < s < t$. It follows then from (77) that

$$f(s,x) - f(s',x) = -\int_{s'}^s L_\theta f(\theta, x) d\theta$$

for all $s, s' \in [0,\tau]$. Whence, f and g are equal.

□

4.5 Strong statements and stochastic flow

Corollary 4.3 *Let* $\mathbf{w} \in \mathbf{F}_{0,p,T}$ *be a solution of the mild equation (10) with* $w_0 \in L^1_3$*.*

a) There is strong existence and uniqueness for the **linear** *stochastic differential equation*

$$\begin{aligned} \overline{X}_t &= \overline{X}_0 + \sqrt{2\nu}B_t + \int_0^t \mathbf{K}(\mathbf{w})(s, \overline{X}_s)ds \\ \overline{\Phi}_t &= Id + \int_0^t \nabla\mathbf{K}(\mathbf{w})(s, \overline{X}_s)\overline{\Phi}_s ds \qquad t \in [0,T] \\ law(\overline{X}_0) &= \frac{|w_0(x)|}{\|w_0\|_1}dx, \end{aligned} \tag{78}$$

and one has

$$law((\overline{X}, \overline{\Phi})) = P,$$

the unique solution in $\mathcal{P}^T_{\frac{3}{2},b,0}$ *of the nonlinear martingale problem (45) such that* $\tilde{\rho} = \mathbf{w}$*.*

b) The family of SDE's

$$\xi_{s,t}(x) = x + \sqrt{2\nu}B_t + \int_s^t \mathbf{K}(\mathbf{w})(t, \xi_{s,r}(x))dr, \quad t \in [s,T] \tag{79}$$

with $x \in \mathbb{R}^3$ *and* $s \in [0,T]$*, defines a* C^1-*stochastic flow* ξ*, and one has*

$$(\overline{X}, \overline{\Phi}) = (\xi_{0,\cdot}(X_0), \nabla\xi_{0,\cdot}(X_0)).$$

Proof: *a)* By Theorem 4.1 and Proposition 5.4.11 in [18], there exists a weak solution (X, Φ) of the SDE (78) in some probability space. If now $(\overline{X}, \overline{\Phi})$ and $(\overline{Y}, \overline{\Psi})$ are two solutions of (78) in the same given probability space, then

$$|\overline{X}_t - \overline{Y}_t| \le C\int_0^t s^{\frac{-1}{2}}|\overline{X}_s - \overline{Y}_s|ds$$

by Corollary 3.1 *i)* for all $t \in [0,T]$, and we conclude that $\overline{X} = \overline{Y}$ by Lemma 4.6. The fact that $\overline{\Phi} = \overline{\Psi}$ follows as in the last part of Proposition 4.3. Thus, trajectorial uniqueness holds for (78) which yields the result.

b) By Corollary 3.1 and the results in [19], the stochastic flow (79) is well defined for $s \in [0,T]$ and of class C^1 in x for all $s,t \in]0,T], s < t$. We just have to check that $\xi_{0,t} : \mathbb{R}^3 \to \mathbb{R}^3$ is also C^1. By Lemma 4.6 and similar arguments as in *a)*, the function $\xi_{0,t}$ is globally Lipschitz continuous (independently of the randomness and of $t \in [0,T]$), and the quotients $\delta_t(x,y) := \frac{1}{|x-y|}|\xi_{0,t}(x) - \xi_{0,t}(y)|$ are bounded. With this and the relation

$$\begin{aligned} \xi_{0,t}(y) - \xi_{0,t}(x) =& y - x + \int_0^t \int_0^1 \Big[\nabla\mathbf{u}\big(s, \xi_{0,s}(x) + \theta(\xi_{0,s}(y) - \xi_{0,s}(x))\big) - \nabla\mathbf{u}\big(s, \xi_{0,s}(x)\big)\Big]d\theta \\ & \cdot \big(\xi_{0,s}(y) - \xi_{0,s}(x)\big)ds \\ & + \int_0^t \nabla\mathbf{u}\big(s, \xi_{0,s}(x)\big)\big(\xi_{0,s}(y) - \xi_{0,s}(x)\big)ds \end{aligned}$$

we deduce for a fixed $r \in]3, \frac{3p}{3-p}[$ that

$$|\delta_t(x,y) - \delta_t(x,y')| \le C\int_0^t s^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\Big[|y-y'|^{1-\frac{3}{r}} + |\delta_s(x,y) - \delta_s(x,y')|\Big]ds$$

for all $x, y, y' \in \mathbb{R}^3$ thanks also to Corollary 3.1 *ii)*. By Lemma 4.6,

$$|\delta_t(x,y) - \delta_t(x,y')| \le C(T)|y-y'|^{1-\frac{3}{r}}$$

for an absolute constant $C(T) > 0$, and therefore

$$|\delta_t(x,y) - \delta_t(x',y')| \le C\left[|x-x'|^{1-\frac{3}{r}} + |y-y'|^{1-\frac{3}{r}}\right]$$

for all x, x', y, y', which easily yields the conclusion.

□

5 A cutoffed and mollified mean field model for the vortex equation

This section provides the theoretical framework to construct stochastic approximations of the vortex equation (2).

5.1 A generalized McKean-Vlasov equation

Consider a filtered probability space endowed with an adapted standard 3-dimensional Brownian motion B and with a $\mathbb{R}^3$-valued random variable X_0 independent of B. Let $\chi_R : \mathcal{M}_{3\times 3} \to \mathcal{M}_{3\times 3}$ be a Lipschitz continuous truncation function such that $|\chi_R(\phi)| \le R$. We will study the following system of nonlinear stochastic differential equations of the McKean-Vlasov type:

$$\begin{aligned} X_t &= X_0 + \sqrt{2\nu}B_t + \int_0^t u_{\varepsilon,R}(s, X_s)ds \\ \Phi_t &= Id + \int_0^t \nabla u_{\varepsilon,R}(s, X_s)\chi_R(\Phi_s)ds \end{aligned} \tag{80}$$

with

$$u_{\varepsilon,R}(s,x) = E\left[K_\varepsilon(x - X_s) \wedge \chi_R(\Phi_s)h_0(X_0)\right] \tag{81}$$

Theorem 5.1 *There is existence and uniqueness (trajectorial and in law) for (80), (81).*

Proof: The proof is adapted from Theorem 1.1 in [31], so we will skip details. Consider the closed subspace $\mathcal{P}(\mathcal{C}_T^0)$ of $\mathcal{P}(\mathcal{C}_T)$ of probability measures Q such that $Q|_{t=0} = law(X_0) \otimes \delta_{Id}$. We define a mapping $\Xi : \mathcal{P}(\mathcal{C}_T^0) \to \mathcal{P}(\mathcal{C}_T^0)$ associating to Q the law $\Xi(Q)$ of the solution of

$$\begin{aligned} X_t^Q &= X_0 + \sqrt{2\nu}B_t + \textstyle\int_0^t u_Q(s, X_s^Q)ds \\ \Phi_t^Q &= Id + \textstyle\int_0^t \nabla u_Q(s, X_s^Q)\chi_R(\Phi_s^Q)ds, \end{aligned} \tag{82}$$

where

$$u_Q(s,x) = \int_{\mathcal{C}_T} \left[K_\varepsilon(x - y(s)) \wedge \chi_R(\Psi(s))h_0(y(0))\right] Q(dy, d\psi). \tag{83}$$

The coefficients in equation (82) are Lipschitz continuous and bounded functions, and so Ξ is well defined (path-wise). Also by Lipschitz continuity, we just have to prove existence and uniqueness in law for (80), (81), which is equivalent to existence of a unique fixed point for Ξ. The Kantorovitch-Rubinstein (or Vaserstein) distance

$$D_T(Q^1,Q^2) := \inf\left\{ \int_{\mathcal{C}_T^2} \sup_{0\le t\le T} \left[\min\{|x(t)-y(t)|,1\}+\min\{|\phi(t)-\psi(t)|,1\}\right] \Pi(dx,d\phi,dy,d\psi), \right.$$
$$\left. \Pi \text{ has marginals } Q^1 \text{ and } Q^2 \right\}, \quad (84)$$

induces on $\mathcal{P}(\mathcal{C}_T^0)$ the usual weak topology. The required fixed point result can be deduced in a standard way from the following inequality: for all $t \le T$ and $Q^1, Q^2 \in \mathcal{P}(\mathcal{C}_T^0)$,

$$D_t(\Xi(Q^1),\Xi(Q^2)) \le C_T \int_0^t D_s(Q^1,Q^2)ds, \qquad (85)$$

with C_T a positive constant, and $D_t(Q^1,Q^2)$ the distance between the projections of Q^1 and Q^2 on $C([0,t],\mathbb{R}^3 \times \mathcal{M}_{3\times 3})$. To prove (85), consider for each $i=1,2$ processes (X^i,Φ^i) defined in terms of Q^i as in (82),(83). Take on a different probability space (Ω',P') a coupling $(Y^i,\Psi^i)_{i=1,2}$ of two processes such that $law(Y^i,\Psi^i)=Q^i$. Then,

$$\begin{aligned}
|X_t^1 - X_t^2| \le& \int_0^t |u_{Q^1}(s,X_s^1) - u_{Q^1}(s,X_s^2)| + |u_{Q^1}(s,X_s^2) - u_{Q^2}(s,X_s^2)|ds \\
\le& \int_0^t \left| E'\left[\left(K_\varepsilon(X_s^1 - Y_s^1) - K_\varepsilon(X_s^2 - Y_s^1)\right) \wedge \chi_R(\Psi_s^1) h_0(X_0)\right]\right| ds \\
&+ \int_0^t \left| E'\left[K_\varepsilon(X_s^2 - Y_s^1) \wedge \left(\chi_R(\Psi_s^1) - \chi_R(\Psi_s^2)\right) h_0(X_0)\right]\right| ds \\
&+ \int_0^t \left| E'\left[\left(K_\varepsilon(X_s^2 - Y_s^1) - K_\varepsilon(X_s^2 - Y_s^2)\right) \wedge \chi_R(\Psi_s^2) h_0(X_0)\right]\right| ds \\
\le& C\int_0^t \min\{|X_s^1 - X_s^2|,1\}ds + C\int_0^t E'\left[\min\{|Y_s^1 - Y_s^2|,1\} + \min\{\left|\Psi_s^1 - \Psi_s^2\right|,1\}\right] ds.
\end{aligned}$$

On the other hand, the processes Φ^i, with $i=1,2$, are bounded on $[0,T]$:

$$\sup_{t\in[0,T]} \left|\Phi_t^i\right| \le 1 + L_\varepsilon R \|h_0\|_\infty T, \qquad (86)$$

with L_ε a Lipschitz constant of K_ε. Thus,

$$\begin{aligned}
\left|\Phi_t^1 - \Phi_t^2\right| \le& \int_0^t \left|\left(\nabla u_{Q^1}(s,X_s^1) - \nabla u_{Q^1}(s,X_s^2)\right)\Phi_s^1\right| + \left|\nabla u_{Q^2}(s,X_s^2)\left(\Phi_s^1 - \Phi_s^2\right)\right| ds \\
\le& C\left[\int_0^t \left|\nabla u_{Q_1}(s,X_s^1) - \nabla u_{Q_1}(s,X_s^2)\right| ds + \int_0^t \min\{\left|\Phi_s^1 - \Phi_s^2\right|,1\}ds\right] \\
\le& C\left[\int_0^t \min\{|X_s^1 - X_s^2|,1\} + \min\{\left|\Phi_s^1 - \Phi_s^2\right|,1\}ds \right. \\
&\left. + \int_0^t E'\left[\min\{|Y_s^1 - Y_s^2|,1\} + \min\{\left|\Psi_s^1 - \Psi_s^2\right|,1\}\right] ds\right].
\end{aligned}$$

The conclusion follows with help of Gronwall's lemma. □

5.2 Propagation of chaos

Consider now a probability space endowed with a sequence $(B^i)_{i\in\mathbb{N}}$ of independent 3-dimensional Brownian motions, and a sequence of independent random variables $(X_0^i)_{i\in\mathbb{N}}$ with same law as X_0 and independent of the Brownian motions. For each $n\in\mathbb{N}$ and $R,\varepsilon>0$, we define the following system of interacting particles:

$$\begin{aligned} X_t^{i,n,\varepsilon,R} &= X_0^i + \sqrt{2\nu}B_t^i + \int_0^t \frac{1}{n}\sum_{j\neq i} K_\varepsilon(X_s^{i,n,\varepsilon,R} - X_s^{j,n,\varepsilon,R}) \wedge \chi_R(\Phi_s^{j,n,\varepsilon,R}) h_0(X_0^j)ds \\ \Phi_t^{i,n,\varepsilon,R} &= Id + \int_0^t \frac{1}{n}\sum_{j\neq i} \left[\nabla K_\varepsilon(X_s^{i,n,\varepsilon,R} - X_s^{j,n,\varepsilon,R}) \wedge \chi_R(\Phi_s^{j,n,\varepsilon,R}) h_0(X_0^j)\right] \chi_R(\Phi_s^{i,n,\varepsilon,R})ds, \end{aligned} \tag{87}$$

for $i=1\dots n$, and with $\nabla K(y)\wedge z = \nabla_y(K(y)\wedge z)$ for $y,z\in\mathbb{R}^3, y\neq 0$. Notice that the coefficients in the system of SDE's (87) are globally Lipschitz continuous and bounded, so that there is a unique strong solution. We also consider in the same probability space the sequence

$$\begin{aligned} X_t^{i,\varepsilon,R} &= X_0^i + \sqrt{2\nu}B_t^i + \int_0^t u_{\varepsilon,R}(s, X_s^{i,\varepsilon,R})ds \\ \Phi_t^{i,\varepsilon,R} &= Id + \int_0^t \nabla u_{\varepsilon,R}(s, X_s^{i,\varepsilon,R})\chi_R(\Phi_s^{i,\varepsilon,R})ds \end{aligned}, \qquad i\in\mathbb{N} \tag{88}$$

of independent copies of (80). Their common law is denoted by $P^{\varepsilon,R}$, and $\bar h, M_\varepsilon, L_\varepsilon, J_\varepsilon, R$ and L_R are positive constants such that for all $x,y\in\mathbb{R}$,

- $|h_0(x)|\leq \bar h$, and
- $|K_\varepsilon(x)|\leq M_\varepsilon, |K_\varepsilon(x)-K_\varepsilon(y)|\leq L_\varepsilon|x-y|, |\nabla K_\varepsilon(x)-\nabla K_\varepsilon(y)|\leq J_\varepsilon|x-y|$.

Recall that $|\chi_R(\phi)|\leq R$ for all $\phi\in\mathcal{M}_{3\times 3}$, and that χ_R is Lipschitz continuous function. Moreover, we will assume for simplicity that its Lipschitz constant is equal to 1:

$$|\chi_R(\phi)-\chi_R(\psi)|\leq|\phi-\psi| \text{ for all } \phi,\psi\in\mathcal{M}_{3\times 3}.$$

Theorem 5.2 *For $\varepsilon>0$ sufficiently small and all $R>0$, we have*

$$E\left[\sup_{t\in[0,T]}\left\{|X_t^{i,\varepsilon,R,n}-X_t^{i,\varepsilon,R}|+|\Phi_t^{i,\varepsilon,R,n}-\Phi_t^{i,\varepsilon,R}|\right\}\right]\leq\frac{1}{\sqrt{n}}C(\varepsilon,R,\bar h,T) \tag{89}$$

for all $i\leq n$, where

$$C(\varepsilon,R,\bar h,T)=C_1\varepsilon(1+R\bar hT)(R\bar hT)\exp\{C_2\varepsilon^{-9}\bar hT(R+1)(\bar h+RT)\}$$

for some positive constants C_1, C_2 independent of R, ε, T and $\bar h$. We deduce that the system (87) is chaotic with limiting law $P^{\varepsilon,R}\in\mathcal{P}(\mathcal{C}_T)$. That is, for all $k\in\mathbb{N}$,

$$law\left((X^{1,\varepsilon,R,n},\Phi^{1,\varepsilon,R,n}),(X^{2,\varepsilon,R,n},\Phi^{2,\varepsilon,R,n}),\dots,(X^{k,\varepsilon,R,n},\Phi^{k,\varepsilon,R,n})\right)\Longrightarrow(P^{\varepsilon,R})^{\otimes k} \tag{90}$$

when $n\to\infty$ in the space $\mathcal{P}((\mathcal{C}_T)^k)$.

Proof: The convergence (90) is a simple consequence of (89), which we now prove. Since they are fixed, we will drop the superscripts ε and R of all processes. The proof is an extension of the arguments of Theorem 1.4 in [31], but we we shall make the computations explicit in order to keep track of the constants. We have

$$\begin{aligned}|X_t^{i,n} - X_t^i| \leq \int_0^t \Big|\frac{1}{n}\sum_{j=1}^n \big(K_\varepsilon(X_s^{i,n} - X_s^{j,n}) - K_\varepsilon(X_s^i - X_s^{j,n})\big) \wedge \chi_R(\Phi_s^{j,n}) h_0(X_0^j)\Big| ds \\ + \int_0^t \Big|\frac{1}{n}\sum_{j=1}^n \big(K_\varepsilon(X_s^i - X^{j,n}) - K_\varepsilon(X_s^i - X_s^j)\big) \wedge \chi_R(\Phi_s^{j,n}) h_0(X_0^j)\Big| ds \\ + \int_0^t \Big|\frac{1}{n}\sum_{j=1}^n K_\varepsilon(X_s^i - X_s^j) \wedge \big(\chi_R(\Phi_s^{j,n}) - \chi_R(\Phi_s^j)\big)\, h_0(X_0^j)\Big| ds \\ + \int_0^t \Big|\frac{1}{n}\sum_{j=1}^n K_\varepsilon(X_s^i - X_s^j) \wedge \chi_R(\Phi_s^j) h_0(X_0^j) \\ - \int K_\varepsilon(X_s^i - x(s)) \wedge \chi_R(\phi(s)) h_0(x(0)) P^{\varepsilon,R}(dx, d\phi)\Big| ds\end{aligned}$$

Hence,

$$\begin{aligned}|X_t^{i,n} - X_t^i| \leq & L_\varepsilon R\bar{h} \int_0^t \big\{|X_s^{i,n} - X_s^i| + \frac{1}{n}\sum_{j=1}^n |X_s^{j,n} - X_s^j|\big\} ds \\ & + L_R M_\varepsilon \bar{h} \int_0^t \frac{1}{n}\sum_{j=1}^n |\Phi_s^{j,n} - \Phi_s^j| ds + \int_0^t I(n, R, \varepsilon, s) ds,\end{aligned}$$

where

$$\begin{aligned}I(n, R, \varepsilon, s) = \Big|\frac{1}{n}\sum_{j=1}^n \big[K_\varepsilon(X_s^i - X_s^j) \wedge \chi_R(\Phi_s^j) h_0(X_0^j) \\ - \int K_\varepsilon(X_s^i - x(s)) \wedge \chi_R(\phi(s)) h_0(x(0)) P(dx, d\phi)\big]\Big|.\end{aligned}$$

Thanks to the exchangeability of the system (87), we obtain

$$\begin{aligned}E\{\sup_{r\leq t} |X_r^{i,n} - X_r^i|\} \leq & 2L_\varepsilon R\bar{h} \int_0^t E\{\sup_{r\leq s} |X_r^{i,n} - X_r^i|\} ds + L_R M_\varepsilon \bar{h} \int_0^t E\{\sup_{r\leq s} |\Phi_r^{i,n} - \Phi_r^i|\} ds \\ & + \int_0^t E(I(n, R, \varepsilon, s)) ds.\end{aligned}$$

Now, each of the n squared terms in the sum $I(n, R, \varepsilon, s)^2$ is bounded by $\frac{1}{n^2}(2M_\varepsilon R\bar{h})^2$, and by using the independence of the sequence $(X^i, \Phi^i)_{i\in\mathbb{N}}$, all the "crossed terms" are seen to have null expectation. We conclude that

$$E(I(n, R, \varepsilon, s)^2) \leq \frac{1}{n}(2M_\varepsilon R\bar{h})^2.$$

Then,

$$E\{\sup_{r\le t}|X^{i,n}_r - X^i_r|\} \le 2L_\varepsilon R\bar{h}\int_0^t E\{\sup_{r\le s}|X^{i,n}_r - X^i_r|\}ds + L_R M_\varepsilon \bar{h}\int_0^t E\{\sup_{r\le s}|\Phi^{i,n}_r - \Phi^i_r|\}ds + \frac{1}{\sqrt{n}}(2M_\varepsilon R\bar{h})t. \tag{91}$$

On the other hand, we have

$$\begin{aligned}
|\Phi^{i,n}_t - \Phi^i_t| \le &\int_0^t \big|\big[\frac{1}{n}\sum_{j=1}^n \big(\nabla K_\varepsilon(X^{i,n}_s - X^{j,n}_s) - \nabla K_\varepsilon(X^i_s - X^{j,n}_s)\big)\wedge \chi_R(\Phi^{j,n}_s)h_0(X^j_0)\big]\chi_R(\Phi^{i,n}_s)\big|ds \\
&+ \int_0^t \big|\big[\frac{1}{n}\sum_{j=1}^n \big(\nabla K_\varepsilon(X^i_s - X^{j,n}) - \nabla K_\varepsilon(X^i_s - X^j_s)\big)\wedge \chi_R(\Phi^{j,n}_s)h_0(X^j_0)\big]\chi_R(\Phi^{i,n}_s)\big|ds \\
&+ \int_0^t \big|\big[\frac{1}{n}\sum_{j=1}^n \nabla K_\varepsilon(X^i_s - X^j_s)\wedge \big(\chi_R(\Phi^{j,n}_s) - \chi_R(\Phi^j_s)\big)\, h_0(X^j_0)\big]\chi_R(\Phi^{i,n}_s)\big|ds \\
&+ \int_0^t \big|\big[\frac{1}{n}\sum_{j=1}^n \nabla K_\varepsilon(X^i_s - X^j_s)\wedge \chi_R(\Phi^j_s)h_0(X^j_0)\big]\big(\chi_R(\Phi^{i,n}_s) - \chi_R(\Phi^i_s)\big)\big|ds \\
&+ \int_0^t \big|\big[\frac{1}{n}\sum_{j=1}^n \nabla K_\varepsilon(X^i_s - X^j_s)\wedge \chi_R(\Phi^j_s)h_0(X^j_0)\big]\chi_R(\Phi^i_s) \\
&\qquad - \big[\int \nabla K_\varepsilon(X^i_s - x(s))\wedge \chi_R(\phi(s))h_0(x(0))P^{\varepsilon,R}(dx,d\phi)\big]\chi_R(\Phi^i_s)\big|ds.
\end{aligned}$$

Notice that

$$\sup_{t\in[0,T]}|\Phi^{i,n}_t|,\ \sup_{t\in[0,T]}|\Phi^i_t| \le C_{\varepsilon,R,T} := (1 + L_\varepsilon R\bar{h}T) \tag{92}$$

for all $n\in\mathbb{N}$. Thus,

$$\begin{aligned}
|\Phi^{i,n}_t - \Phi^i_t| \le &J_\varepsilon R\bar{h}C_{\varepsilon,R,T}\int_0^t \big\{|X^{i,n}_s - X^i_s| + \frac{1}{n}\sum_{j=1}^n |X^{j,n}_s - X^j_s|\big\}ds \\
&+ L_\varepsilon L_R\bar{h}C_{\varepsilon,R,T}\int_0^t \frac{1}{n}\sum_{j=1}^n |\Phi^{j,n}_s - \Phi^j_s|\ ds \\
&+ L_\varepsilon R\bar{h}\int_0^t |\Phi^{i,n}_s - \Phi^i_s|\ ds + \int_0^t I'(n,R,\varepsilon,s)\ ds,
\end{aligned}$$

with

$$\begin{aligned}
I'(n,R,\varepsilon,s) = \Bigg|\frac{1}{n}\sum_{j=1}^n \Bigg[&\big(\nabla K_\varepsilon(X^i_s - X^j_s)\wedge \chi_R(\Phi^j_s)h_0(X^j_0)\big)\chi_R(\Phi^{i,n}_s) \\
&- \int \nabla K_\varepsilon(X^i_s - x(s))\wedge \chi_R(\phi(s))h_0(x(0))P^{\varepsilon,R}(dx,d\phi)\chi_R(\Phi^i_s)\Bigg]\Bigg|.
\end{aligned}$$

We conclude in a similar way as before that

$$E\{\sup_{r\le t}|\Phi_r^{i,n}-\Phi_r^i|\} \le 2J_\varepsilon R\bar h C_{\varepsilon,R,T}\int_0^t E\{\sup_{r\le s}|X_r^{i,n}-X_r^i|ds + \left(L_\varepsilon L_R\bar h C_{\varepsilon,R,T}+L_\varepsilon R\bar h\right)\int_0^t E\{\sup_{r\le s}|\Phi_r^{i,n}-\Phi_r^i|\}\,ds + \frac{1}{\sqrt n}(2L_\varepsilon R\bar h C_{\varepsilon,R,T})t. \tag{93}$$

Putting together (91) and (93), we get

$$E\{\sup_{r\le t}|X_r^{i,n}-X_r^i|+|\Phi_t^{i,n}-\Phi_t^i|\} \le 2R\bar h\left(L_\varepsilon+J_\varepsilon C_{\varepsilon,R,T}\right)\int_0^t E\{\sup_{r\le s}|X_s^{i,n}-X_s^i|\}ds + \bar h\left(L_R M_\varepsilon+L_\varepsilon L_R C_{\varepsilon,R,T}+L_\varepsilon R\right)\int_0^t E\{\sup_{r\le s}|\Phi_r^{i,n}-\Phi_r^i|\}ds + \frac{2R\bar h}{\sqrt n}(M_\varepsilon C_{\varepsilon,R,T}+L_\varepsilon)t. \tag{94}$$

Finally, we notice that

$$|K*\varphi_\varepsilon(x)| \le C\sup_{z\in\mathbb{R}^3}\{\varphi_\varepsilon(z)\}\int_{|x-y|\le 1}|x-y|^{-2}dy + C\int_{|x-y|\ge 1}\varphi_\varepsilon(y)dy \le \frac{C}{\varepsilon^3}+C$$

and then, $M_\varepsilon \le C\varepsilon^{-3}$ for all ε small enough. We deduce in a similar way that $L_\varepsilon \le C\varepsilon^{-4}$ and $J_\varepsilon \le C\varepsilon^{-5}$ (since for functions $\varphi\in\mathcal{S}$, convoluting with K commutes with derivation). As observed in Jourdain and Méléard [17], if $g:\mathbb{R}^+\to\mathbb{R}^+$ is a bounded function such that $g(t)\le c_1\int_0^t g(s)ds + c_2 t$ for all $t\in[0,T]$, then $g(t)\le \frac{c_2}{c_1}\exp(c_1 T)$. This and (94) provide an upper bound for the r.h.s. of (89) by the constant $\frac{c_2}{c_1}\exp(c_1T)$, where

$$c_1 = 2R\bar h\left(L_\varepsilon + J_\varepsilon(1+RL_\varepsilon\bar hT)\right) + \bar h(M_\varepsilon + L_\varepsilon(1+RL_\varepsilon\bar hT)+L_\varepsilon R)$$

and $c_2 = \frac{2R\bar h}{\sqrt n}(M_\varepsilon(1+RL_\varepsilon\bar hT)+L_\varepsilon)$. The statement follows by noting the existence of universal positive constants $\mathbf{C},\mathbf{C}',\mathbf{C}''$ and ε_0 (in particular independent of R, $\bar h$ and T) such that

$$\mathbf{C}J_\varepsilon L_\varepsilon(R\bar h)^2T \le c_1 \le \mathbf{C}'J_\varepsilon L_\varepsilon\bar h(R+1)(\bar h+RT)$$

and for all $\varepsilon\in]0,\varepsilon_0[$

$$c_2 \le \mathbf{C}''\frac{L_\varepsilon^2}{\sqrt n}R\bar h(1+R\bar hT).$$

□

We can take for instance χ_R defined by

$$\chi_R(\phi) = \begin{cases} \phi & if|\phi|\le R,\\ \frac{R}{|\phi|}\,\phi & \text{if } |\phi|\ge R.\end{cases}$$

(which is the truncation function proposed in [12]).

Remark 5.1 *In [12], Esposito and Pulvirenti claimed (without proving) the existence of a nonlinear process satisfying analogous conditions as (80),(81), but without the truncation* χ_R *on the process* Φ *inside the expectation that we imposed in (81). Indeed, truncating "outside the expectation" in (80) is not strictly necessary: the two previous theorems can also be proved in that case, by bounding* $|\Phi|$ *and* $|\Phi^i|$ *above by* $\exp\{L_\varepsilon R\bar{h}T\}$ *(thanks to Gronwall's lemma), instead of the bounds (86) and (92). In turn, it seems not possible to obtain these results in the way conjectured in [12] (truncating* Φ *only outside the expectation). In fact, one cannot provide in that case a bound like (86) for the process* Φ *by absolute constants, which is crucial for estimate (85) to hold (or for an analogous to it with a different metric), and therefore to ensure that an iteration (fixed point) procedure will converge.*

6 The 3 dimensional stochastic vortex method

We will now state and prove our main result. We assume the following:

- $w_0 \in L_3^p \cap L_3^1$ with $p \in]\frac{3}{2}, 3[$ and $0 < T < \infty$.
- $T^{1-\frac{3}{2p}} \|w_0\|_p < \frac{1}{\Gamma_0(p)}$, where $\Gamma_0(p)$ is the constant given by Theorem 3.1.
- $\mathbf{w} \in \mathbf{F}_{0,p,T}$ is the solution of the mild vortex equation (10) given by Theorem 3.1

Let us write
$$\mathbf{u}(t,x) = \mathbf{K}(\mathbf{w})(t,x)$$
for all $(t,x) \in [0,T] \times \mathbb{R}^3$, and fix a real number
$$r_\circ \in]3, \frac{3p}{3-p}[.$$

By the proof of Theorem 3.1, one has $\|\mathbf{w}\|_{0,p,T} \leq 2\overline{C}_0(p)\|w_0\|_p$, where $\overline{C}_0(p)$ is the constant given in Lemma 3.2 *i)*. Thus, we can make the following

Remark 6.1 *By taking in Corollary 3.1 ii)* $A = 2\overline{C}_0(p)\|w_0\|_p$ *and* $r = r_\circ$ *, we deduce the existence of a constant* $\hat{C}(\|w_0\|_p, T, p)$*, depending on* $\mathbf{w}$ *only through the norm* $\|w_0\|_p$*, such that*
$$\|\nabla \mathbf{u}(t)\|_\infty \leq t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r_\circ})} \hat{C}(\|w_0\|_p, T, p) \text{ for all } t \in [0,T].$$

Define now a positive constant $R(w_0, T)$ by
$$R(w_0,T) := \exp\left\{\hat{C}(\|w_0\|_p, T, p) \int_0^T t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r_\circ})} dt\right\}, \tag{95}$$
and recall that C_2 is a universal constant provided by Theorem 5.2.

Theorem 6.1 *Assume that* $w_0 \in L_3^p \cap L_3^1$ *with* $p \in]\frac{3}{2}, 3[$*, and that* $T^{1-\frac{3}{2p}}\|w_0\|_p < \frac{1}{\Gamma_0(p)}$ *holds, with* $\Gamma_0(p)$ *the constant of Theorem 3.1. Let* (ε_n) *be a sequence converging to* 0 *in such way that*
$$\frac{1}{\sqrt{n}} \exp\left\{C_2 \varepsilon_n^{-9} \|w_0\|_1 T \left(R(w_0,T)+1\right)\left(\|w_0\|_1 + TR(w_0,T)\right)\right\} \to 0.$$

Furthermore, define for each $n \in \mathbb{N}$ a system of interacting particles on $\mathbb{R}^3 \times \mathcal{M}_{3\times 3}$ by

$$Z^{i,n} := (X^{i,\varepsilon_n,R,n}, \Phi^{i,\varepsilon_n,R,n}),$$

and let P be the unique solution in $\mathcal{P}^T_{\frac{3}{2},b,0}$ of the nonlinear martingale problem (45). Then, for all $k \in \mathbb{N}$, when $n \to \infty$,

$$law(Z^{1,n}, Z^{2,n}, ..., Z^{k,n}) \Longrightarrow P^{\otimes k}$$

in the space $\mathcal{P}(\mathcal{C}^k_T)$.

Remark 6.2 *Theorem 6.1 will hold if for instance $\varepsilon_n = (c \ln n)^{-9}$, with*

$$0 < c < C_2^{-1} \left((R(w_0,T)+1)(\|w_0\|_1 + 1)(T+1)\right)^{-2}.$$

The proof of Theorem 6.1 will mainly use similar techniques as those in [25] or [13] for the equations considered therein. First we will prove that under the conditions ensuring existence of the solution $\mathbf{w}$, P can be approximated by a family of solutions P^{ε_n} of some nonlinear martingale problems with regular interactions. Each P^{ε_n} is associated with the solution $\mathbf{w}^{\varepsilon_n} \in \mathbf{F}_{0,p,T}$ of a mollified vortex equation involving a smooth kernel K_{ε_n}.

6.1 The mollified equations

Consider the operator $\mathbf{K}^\varepsilon$ defined as in Section **4.3**, and for each $\varepsilon > 0$ define

$$\mathbf{B}^\varepsilon(\mathbf{v}', \mathbf{v})(t,x) = \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}}{\partial y_j}(x-y) \left[\mathbf{K}^\varepsilon(\mathbf{v}')_j(s,y)\mathbf{v}(s,y) - \mathbf{v}_j(s,y)\mathbf{K}^\varepsilon(\mathbf{v}')(s,y)\right] dy\, ds. \tag{96}$$

Remark 6.3 *In virtue of Remark 4.3, the functional $\mathbf{B}^\varepsilon : \mathbf{F}^2 \to \mathbf{F}'$ satisfies the same continuity properties as the functional $\mathbf{B}$ in the spaces $\mathbf{F}, \mathbf{F}'$ considered in Proposition 3.1. Moreover, in such spaces the norm of $\mathbf{B}^\varepsilon$ is smaller or equal than the norm of $\mathbf{B}$.*

Therefore, the same existence and regularity results of Theorem 3.1 and Theorem 3.2 hold true *with the same constants* for the family of mollified equations

$$\begin{aligned} \mathbf{v}(t,x) = &\int_{\mathbb{R}^3} G^\nu_t(x-y) w_0(y) dy \\ &+ \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y) \left[\mathbf{K}^\varepsilon(\mathbf{v})_j(s,y)\mathbf{v}(s,y) - \mathbf{v}_j(s,y)\mathbf{K}^\varepsilon(\mathbf{v})(s,y)\right] dy\, ds. \end{aligned} \tag{97}$$

Theorem 6.2 *Assume that $w_0 \in L^p_3 \cap L^1_3$ with $p \in]\frac{3}{2}, 3[$, and that $T^{1-\frac{3}{2p}} \|w_0\|_p < \frac{1}{\Gamma_0(p)}$ holds, with $\Gamma_0(p)$ the constant of Theorem 3.1. There exists a unique solution $\mathbf{w}^\varepsilon$ of equation (97) in $\mathbf{F}_{0,p,T}$. The solution satisfies $\|\mathbf{w}^\varepsilon\|_{0,p,T} \leq 2\overline{C}_0(p)\|w_0\|_p$, with $\overline{C}_0(p)$ the constant in Lemma 3.2 i), and $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$ for all $r \in [p, \infty[$. Moreover, if A is an upper bound for $\|\mathbf{w}^\varepsilon\|_{0,p,T}$, and if $C(T,p)$ and $C(T,p,r,A)$ are the constants in Theorem 3.2 ii) and iii) respectively, then one has*

$$\|\mathbf{w}^\varepsilon\|_{1,p,T} \leq C(T,p)\|\mathbf{w}^\varepsilon\|_{0,p,T},$$

and

$$\|\mathbf{w}^\varepsilon\|_{1,r,(T,p)} \leq C(T,p,r,A).$$

Finally, $\mathbf{w}^\varepsilon$ *belongs to* $\mathbf{F}_{0,1,T}$ *and the function* $t \in [0,T] \mapsto \mathbf{w}^\varepsilon(t,\cdot) \in L^1_3$ *is continuous.*

Hence, as in Section **4.3**, for each $\varepsilon > 0$ the stochastic differential equations

$$\xi^\varepsilon_t(x) = x + \sqrt{2\nu} B_t + \int_0^t \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(s, \xi^\varepsilon_s(x)) ds, \quad t \in [0,T], x \in \mathbb{R}^3,$$

define a process $(t,x) \mapsto \xi^\varepsilon_t(x)$ which is continuously differentiable on x for all t.
Let (ε_n) be a sequence converging to 0 and denote by (X^n, Φ^n) the couple of processes

$$X^n_t := \xi^{\varepsilon_n}_t(X_0), \text{ and } \Phi^n_t = \nabla \xi^{\varepsilon_n}_t(X_0) \text{ with } t \in [0,T].$$

The law of (X^n, Φ^n) is denoted by P^n, and we write

$$\rho^n(t,x) \text{ and } \tilde{\rho}^n(t,x)$$

for bi-measurable versions of the densities of $(P^n)^\circ_t$ and $\tilde{P}^n_t$ respectively. By similar arguments as in the proof of Lemma 4.5, it is seen that $\tilde{\rho}^n \in \mathbf{F}_{0,p,T}$ and it satisfies the linear mild equation

$$\begin{aligned}\tilde{\rho}^n(t,x) &= \int_{\mathbb{R}^3} G^\nu_t(x-y) w_0(y) dy \\ &+ \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y) \left[\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})_j(s,y) \tilde{\rho}^n(s,y) - \tilde{\rho}^n_j(s,y) \mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(s,y)\right] dy\, ds.\end{aligned} \tag{98}$$

Since uniqueness holds in $\mathbf{F}_{0,p,T}$ for (98), we deduce that

$$\tilde{\rho}^n = \mathbf{w}^{\varepsilon_n} \tag{99}$$

for all $n \in \mathbb{N}$. Thus, (X^n, Φ^n) solves the nonlinear stochastic differential equation

$$\begin{aligned} X^n_t &= X_0 + \sqrt{2\nu} B_t + \int_0^t u_{\varepsilon_n}(s, X^n_s) ds \\ \Phi^n_t &= Id + \int_0^t \nabla u_{\varepsilon_n}(s, X^n_s) \Phi^n_s ds \end{aligned} \tag{100}$$

with

$$u_{\varepsilon_n}(s,x) = E\left[K_{\varepsilon_n}(x - X^n_s) \wedge \Phi^n_s h_0(X_0)\right]. \tag{101}$$

(The reader should compare this process *without truncation* the with the process (80),(81).)

Proposition 6.1

i) For all $t \in [0,T]$ *and* $n \in \mathbb{N}$, *one has* $\tilde{\rho}^n(t) \in L^p_3$, $\rho^n(t) \in L^p$ *, and*

$$\sup_{n \in \mathbb{N}} \|\tilde{\rho}^n\|_{0,p,T} < \infty, \sup_{n \in \mathbb{N}} \|\rho^n\|_{0,p,T} < \infty. \tag{102}$$

Moreover, $\tilde{\rho}^n(t)$ *converges to* $\mathbf{w}(t)$ *in* L^p_3 *for each* $t \in [0,T]$, *and in* $L^1([0,T], L^p_3)$. *Similarly,* $\rho^n(t)$ *converges to* $\rho(t)$ *in* L^p *for each* $t \in [0,T]$, *and in* $L^1([0,T], L^p)$, *with* ρ *the solution of the linear equation (57).*

ii) The sequence $(P^n, n \in \mathbb{N})$ is uniformly tight.

iii) When $n \to \infty$, one has $P^n \Longrightarrow P$.

Proof: *i)* The uniform bound for $\|\tilde{\rho}^n\|_{0,p,T}$ is clear from (99) and Theorem 6.2, and the bound for $\|\rho^n\|_{0,p,T}$ follows as in Lemma 4.5. The proof of the convergence $\tilde{\rho}^n \to \mathbf{w}$ is also similar as therein. Indeed, one has

$$
\begin{aligned}
\|\tilde{\rho}^n(t) - \mathbf{w}(t)\|_p \leq & C \int_0^t (t-s)^{-\frac{3}{2p}} \; [\|\mathbf{K}^{\varepsilon_n}(\tilde{\rho}^n)(s) - \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s)\|_q + \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q] \; ds \\
& + C \int_0^t (t-s)^{-\frac{3}{2p}} \; \|\tilde{\rho}^n(s) - \mathbf{w}(s)\|_p \; ds \\
\leq & C \int_0^t (t-s)^{-\frac{3}{2p}} \; \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \; ds \\
& + C \int_0^t (t-s)^{-\frac{3}{2p}} \; \|\tilde{\rho}^n(s) - \mathbf{w}(s)\|_p \; ds
\end{aligned}
$$

thanks to Remark 4.3, and then one can follow the same arguments of Lemma 4.5. The convergence of ρ^n is obtained in a similar way.

ii) In virtue of the uniform estimates for $\|\mathbf{w}^{\varepsilon_n}\|_{1,p,T}$ and $\|\mathbf{w}^{\varepsilon_n}\|_{1,r,(T;p)}$ in Theorem 6.2, the proof is done exactly in the same way as Lemma 4.7.

iii) We just have to identify the limiting points in a similar way as in Proposition 4.2. If Q is the limit of a convergent subsequence renamed P^n, we only need to check that $E^Q(\kappa(X)) = 0$ and $E^Q(\zeta(X, \Phi)) = 0$, where $\kappa : C([0,T], \mathbb{R}^3) \to \mathbb{R}$ and $\zeta : C([0,T], \mathbb{R}^3) \times C([0,T], \mathcal{M}_{3\times 3}) \to \mathbb{R}$ are the functions defined in (69) and (70). We know that

$$
\begin{aligned}
E^{P^n}\Bigg[\Bigg(\int_s^t \left\{ \frac{\partial f}{\partial \tau}(\tau, X_\tau) + \nu \Delta f(\tau, X_\tau) + \mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(\tau, X_\tau) \nabla f(\tau, X_\tau) \right\} d\tau \\
+ f(t, X_t) - f(s, X_s) \Bigg) \times \lambda(X_{s_1}, \dots, X_{s_m}) \Bigg] = 0,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
E^{P^n}(\kappa(X)) = E^{P^n}\Bigg[\int_s^t \big(\mathbf{K}(\mathbf{w})(\tau, X_\tau) \nabla f(\tau, X_\tau) - \mathbf{K}^{(\varepsilon_n)}(\mathbf{w}^{(\varepsilon_n)})(\tau, X_\tau) \nabla f(\tau, X_\tau) \big) d\tau \\
\times \lambda(X_{s_1}, \dots, X_{s_m}) \Bigg].
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\left| E^{P^n}(\kappa(X)) \right| &\leq C \sup_{k \in \mathbb{N}} \|\rho^k\|_{0,q*,T} \int_0^T \|\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(\tau) - \mathbf{K}(\mathbf{w})(\tau)\|_q \; d\tau \\
&\leq C \int_0^T \left(\|\mathbf{w}^{\varepsilon_n}(\tau) - \mathbf{w}(\tau)\|_p + \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(\tau) - \mathbf{K}(\mathbf{w})(\tau)\|_q \right) \; d\tau
\end{aligned}
$$

thanks to Remark 4.3 (and with C a finite constant), and we conclude with Remark 4.4 that $E^Q(\kappa(X)) = 0$.

In a similar way, one can adapt the arguments of Proposition 4.2 to prove that $E^{Q}(\zeta(X,\Phi)) = 0$. The only point that needs special attention is to establish the uniform bound

$$\sup_{k\in\mathbb{N}} \|\rho^k\|_{0,p*,(T;p)} < \infty,$$

when $p < 2$. This can be justified by similar arguments as in Proposition 4.2 using the fact that the norm of the linear functional

$$\eta(t,x) \mapsto \int_0^t \sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{\partial G^{\nu}_{t-s}}{\partial y_j}(x-y)\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})_j(s,y)\eta(s,y)\ dy\ ds,$$

defined from $F_{0,r,(T;p)}$ to $F_{0,r',(T;p)}$, can be estimated in terms of $\|\mathbf{w}^{\varepsilon_n}\|_{0,r,(T;p)}$ by Remark 4.3, and the last is bounded independently of n as asserted in Theorem 6.2.

□

6.2 Convergence of the particle approximations

We consider now a sequence (ε_n) as in Theorem 6.1. To prove the theorem, we will combine the convergence result we have just obtained with the propagation of chaos result obtained *for fixed* R *and* ε in Section **5**.
Clearly, by (101), (99) and the definition of $\tilde{\rho}^n$, the drift term u_{ε_n} of the nonlinear process (X^n, Φ^n) satisfies

$$u_{\varepsilon_n}(t,x) = \mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(t,x).$$

We deduce the following crucial remark (which is analogous to the one pointed out in Esposito and Pulvirenti [12] in a more restrictive functional setting):

Remark 6.4 *Since* $\|\mathbf{w}^{\varepsilon}\|_{0,p,T} \le 2\overline{C}_0(p)\|w_0\|_p$ *holds, we have*

$$\|\nabla u_{\varepsilon_n}(t)\|_\infty \le t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\hat{C}(\|w_0\|_p, T, p)$$

for all $t \in [0,T], n \in \mathbb{N}$*, and* $\hat{C}(\|w_0\|_p, T, p)$ *the same constant of Remark 6.1. Thus, it follows from (100) and Lemma 4.6 that for all* $n \in \mathbb{N}$*, almost surely*

$$\sup_{t\in[0,T]} |\Phi^n_t| \le R(w_0, T). \tag{103}$$

Consequently, if $R \ge R(w_0,T)$, the nonlinear process (X^n, Φ^n) defined by (100) and (101) is a weak solution of the nonlinear McKean-Vlasov equation (80),(81). Since uniqueness in law holds for the latter, this proves

Proposition 6.2 *Let the pairs* $(X^{\varepsilon_n,R}, \Phi^{\varepsilon_n,R})$ *and* (X^n, Φ^n) *be respectively defined on* $[0,T]$ *by (80),(81), and by (100),(101). Then, for all* $R \ge R(w_0,T)$ *and all* $n \in \mathbb{N}$,

$$law(X^n, \Phi^n) = law(X^{\varepsilon_n,R}, \Phi^{\varepsilon_n,R}),$$

We proceed now to the

Proof of Theorem 6.1: Let $k \in \mathbb{N}$ be fixed. Consider the set $\mathcal{P}(\mathcal{C}^k_T)$ of probabilities Q on the space

$$\mathcal{C}^k_T := C([0,T], (\mathbb{R}^3)^k \times (\mathcal{M}_{3\times3})^k),$$

and the Kantorovich-Rubinstein distance

$$\widehat{D}(Q,Q') :=$$
$$\inf\left\{ \int_{(\mathcal{C}_T^k)^2} \sup_{0\le t\le T} \min\{|x(t)-y(t)|,1\} + \min\{|\phi(t)-\psi(t)|,1\}\widehat{\Pi}((dx,d\phi),(dy,d\psi)) : \widehat{\Pi} \text{ has marginals } Q \text{ and } Q' \right\}$$

(which is a distance on $\mathcal{P}(\mathcal{C}_T^k)$, compatible with the topology of the weak convergence). Let $\overline{Z}^{i,n} := (X^{i,\varepsilon_n,R}, \Phi^{i,\varepsilon_n,R})$ be the process defined in (88), with $R = R(w_0, T)$ and $\varepsilon = \varepsilon_n$. If P is the solution in $\mathcal{P}^T_{\frac{3}{2},b,0}$ of the nonlinear martingale problem (45), we have

$$\begin{aligned}
&\widehat{D}\left((law(Z^{1,n},\dots,Z^{k,n}), P^{\otimes k}\right)\\
&\le \widehat{D}\left((law(Z^{1,n},\dots,Z^{k,n}), law(\overline{Z}^{1,n}\dots,\overline{Z}^{k,n})\right) + \widehat{D}\big(law(\overline{Z}^{1,n}\dots,\overline{Z}^{k,n}), law(\overline{Z}^{1},\dots,\overline{Z}^{k})\big)\\
&\le C\sum_{i=1}^k E\left[\sup_{t\in[0,T]}\left\{|X_t^{i,\varepsilon_n,R,n} - X_t^{i,\varepsilon_n,R}| + |\Phi_t^{i,\varepsilon_n,R,n} - \Phi_t^{i,\varepsilon_n,R}|\right\}\right] + \widehat{D}\big(law(\overline{Z}^{1,n}\dots,\overline{Z}^{k,n}), P^{\otimes k}\big).
\end{aligned}$$

The term involving the sum is bounded by $k\frac{C\varepsilon_n}{\sqrt{n}}\exp\left\{C_2\varepsilon_n^{-9}\|w_0\|_1 T\,(R+1)\,(\|w_0\|_1 + RT)\right\}$ thanks to Theorem 5.2, and goes to 0 by the choice of ε_n. The last term goes to 0 thanks to Proposition 6.2 and Proposition 6.1 *iii)*, and this finishes the proof.

□

Remark 6.5 *In order to obtain an explicit rate in the previous convergence, it is necessary to estimate in terms of ε_n the distance between P^{ε_n} and P. This distance could be deduced from L^∞ estimates of $\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n}) - \mathbf{K}(\mathbf{w})$ and $\nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n}) - \nabla\mathbf{K}(\mathbf{w})$. The explosion at $t = 0$ of the Sobolev norms prevents us from obtaining such estimates in the functional setting we have chosen, but this should possible under additional regularity assumptions on w_0. On the other hand, it seems hard to improve the propagation of chaos estimates in Section* **5**, *at least by the approach we have followed there (which does not depend on the specific form of the interaction kernel).*

A first consequence is convergence at the level of empirical processes. Consider the space $\mathcal{M}_3(\mathbb{R}^3)$ of finite $\mathbb{R}^3$-valued measures on $\mathbb{R}^3$, endowed with the weak topology, and the space $C([0,T],\mathcal{M}_3(\mathbb{R}^3))$ with the topology of uniform convergence.

Corollary 6.1 *The family $(\tilde{\mu}_t^{n,\varepsilon_n,R})_{0\le t\le T}$ of $\mathbb{R}^3$-weighted empirical measures on $\mathbb{R}^3$*

$$\tilde{\mu}_t^{n,\varepsilon_n,R} = \frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,\varepsilon_n,R,n}} \cdot \left(\chi_R(\Phi_t^{i,\varepsilon_n,R,n})h_0(X_0^i)\right)$$

converges in law and in probability to $(\mathbf{w}(t,x)dx)_{0\le t\le T}$ in the space $C([0,T],\mathcal{M}_3(\mathbb{R}^3))$.

Proof: Since $law(Z^{1,n},...,Z^{n,n})$ is exchangeable, the propagation of chaos in Theorem 6.1 is equivalent to the convergence in law (and in probability) of the empirical measure of the system to P, as a probability measure in the path space (see [31]). This implies that

$$E\left(\frac{1}{n}\sum_{i=1}^n \mathbf{f}(X_t^{i,\varepsilon_n,R,n})\chi(\Phi_t^{i,\varepsilon_n,R,n})\mathbf{f}_0(X_0^i)\right) \to E^P\left(\mathbf{f}(X_t)\chi(\Phi_t)\mathbf{f}_0(X_0)\right),$$

for all continuous bounded functions $\mathbf{f}_0, \mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\chi : \mathcal{M}_{3\times 3} \to \mathcal{M}_{3\times 3}$. Let $k \in \mathbb{N}$ and h_0^k be a continuous bounded function approximating h_0 as in Proposition 4.2. Since under P we have $\chi_R(\Phi_t) = \Phi_t$, it follows that

$$\begin{aligned}
&\left| E\langle \tilde{\mu}_t^{n,\varepsilon_n,R}, \mathbf{f}\rangle - \int_{\mathbb{R}^3} \mathbf{f}(x)\mathbf{w}(t,x)dx \right| \\
&\quad \le E\left| \frac{1}{n}\sum_{i=1}^n \mathbf{f}(X_t^{i,\varepsilon_n,R,n})\chi_R(\Phi_t^{i,\varepsilon_n,R,n})h_0(X_0^i) - E^P(\mathbf{f}(X_t)\chi_R(\Phi_t)h_0(X_0)))\right| \\
&\quad \le E\left| \frac{1}{n}\sum_{i=1}^n \mathbf{f}(X_t^{i,\varepsilon_n,R,n})\chi_R(\Phi_t^{i,\varepsilon_n,R,n})(h_0(X_0^i) - h_0^k(X_0^i))\right| \\
&\qquad + E\left| \frac{1}{n}\sum_{i=1}^n \mathbf{f}(X_t^{i,\varepsilon_n,R,n})\chi_R(\Phi_t^{i,\varepsilon_n,R,n})h_0^k(X_0^i) - E^P(\mathbf{f}(X_t)\chi_R(\Phi_t)h_0^k(X_0)\right| \\
&\qquad + E^P|\mathbf{f}(X_t)\chi_R(\Phi_t)(h_0^k(X_0) - h_0(X_0))|.
\end{aligned}$$

By similar arguments as in the proof of Proposition 4.2 we conclude that

$$\limsup_{n\to\infty} \left| E\langle \tilde{\mu}_t^{n,\varepsilon_n,R}, \mathbf{f}\rangle - \int_{\mathbb{R}^3} \mathbf{f}(x)\mathbf{w}(t,x)dx \right| = 0.$$

□

6.3 Stochastic approximations of the velocity field

Finally, we prove the convergence of the "approximated velocity field", defined by

$$\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) := \int_{\mathbb{R}^3} K_{\varepsilon_n}(x-y) \wedge \tilde{\mu}_t^{n,\varepsilon_n,R}(dy),$$

to the local solution $\mathbf{u}(t) = \mathbf{K}(\mathbf{w})(t)$ of the Navier-Stokes equation in $\mathbf{F}_{0,q,T}$. We need a technical lemma:

Lemma 6.1 *Under the assumptions of Theorem 6.2, we have*

$$\|\nabla \mathbf{w}^\varepsilon(t) - \nabla \mathbf{w}(t)\|_p \to 0 \text{ for all } t \in]0,T], \text{ and } \int_0^T \|\nabla \mathbf{w}^\varepsilon(t) - \nabla \mathbf{w}(t)\|_p dt \to 0$$

when $\varepsilon \to 0$.

Proof: We write $\mathbf{w}^{\varepsilon;\tau} := \mathbf{w}^\varepsilon(\tau + \cdot)$ (so that $\mathbf{w}^{0;\tau} = \mathbf{w}(\tau + \cdot)$). For each $\varepsilon > 0$ and $\tau \in [0,T]$, $\mathbf{w}^{\tau,\varepsilon} \in \mathbf{F}_{1,p,T-\tau}$ solves the "shifted" equation $\mathbf{w}^{\varepsilon;\tau} = \mathbf{w}_0^{\varepsilon;\tau} + \mathbf{B}^\varepsilon(\mathbf{w}^{\varepsilon;\tau}, \mathbf{w}^{\varepsilon;\tau})$, with $\mathbf{w}_0^{\varepsilon;\tau} := G_t^\nu * \mathbf{w}^\varepsilon(\tau)$. On the other hand, it is clear that *div* $\mathbf{w}^\varepsilon(t) = 0$. Taking derivatives in the previous equation yields, for $k = 1,2,3$,

$$\begin{aligned}
\frac{\partial \mathbf{w}^{\varepsilon;\tau}}{\partial x_i}(t,x) &= \int_{\mathbb{R}^3} \frac{\partial G_t^\nu}{\partial x_i}(x-y)(\mathbf{w}^\varepsilon)_k(\tau,y)dy \\
&\quad - \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial x_i}(x-y)\left[\mathbf{K}^{\varepsilon;\tau}(\mathbf{w}^{\varepsilon;\tau})_j(s,y)\frac{\partial \mathbf{w}_k^{\varepsilon;\tau}(s,y)}{\partial y_j}\right. \\
&\qquad \left. - \mathbf{w}_j^{\varepsilon;\tau}(s,y)\frac{\partial \mathbf{K}^\varepsilon(\mathbf{w}^{\varepsilon;\tau})_k(s,y)}{\partial y_j}\right] dy\ ds.
\end{aligned}$$

By taking now L_3^p norm of the differences $\frac{\partial \mathbf{w}^{\varepsilon;\tau}}{\partial x_i}(t,x) - \frac{\partial \mathbf{w}^{0;\tau}}{\partial x_i}(t,x)$, $i=1,2,3$, we deduce by similar arguments as in Theorem 3.1 *a)* that

$$\begin{aligned}
\|\nabla \mathbf{w}^{\varepsilon;\tau}(t) - \nabla \mathbf{w}^{0;\tau}(t)\|_p \le & Ct^{-\frac{1}{2}}\|\mathbf{w}^{\varepsilon}(\tau) - \mathbf{w}(\tau)\|_p \\
& + C\int_0^t (t-s)^{-\frac{3}{2p}}(\tau+s)^{-\frac{1}{2}}\big[\|\mathbf{w}^{\varepsilon;\tau}(s) - \mathbf{w}^{0;\tau}(s)\|_p \\
& \qquad + \|\mathbf{K}^{\varepsilon}(\mathbf{w}^{0,\tau})(s) - \mathbf{K}(\mathbf{w}^{0,\tau})(s)\|_q\big]ds \\
& + C\int_0^t (t-s)^{-\frac{3}{2p}}\big[\|\nabla \mathbf{w}^{\varepsilon;\tau}(s) + \nabla \mathbf{w}^{0;\tau}(s)\|_p \\
& \qquad + \|\nabla \mathbf{K}^{\varepsilon}(\mathbf{w}^{0;\tau})(s) - \nabla \mathbf{K}(\mathbf{w}^{0;\tau})(s)\|_q\big]ds
\end{aligned}$$

Now define $\delta^{\varepsilon,\tau}(t) := \|\nabla \mathbf{w}^{\varepsilon;\tau}(t) - \nabla \mathbf{w}^{0;\tau}(t)\|_p$, and

$$\begin{aligned}
\Delta^{\varepsilon,\tau}(t) := & \tau^{-\frac{1}{2}}\left(\|\mathbf{w}^{\varepsilon;\tau}(t) - \mathbf{w}^{0;\tau}(t)\|_p + \|\mathbf{K}^{\varepsilon}(\mathbf{w}^{0,\tau})(t) - \mathbf{K}(\mathbf{w}^{0,\tau})(t)\|_q\right) \\
& + \|\nabla \mathbf{K}^{\varepsilon}(\mathbf{w}^{0,\tau})(t) - \nabla \mathbf{K}(\mathbf{w}^{0,\tau})(t)\|_q.
\end{aligned}$$

Observe that since $\mathbf{w}^{0,\tau} \in \mathbf{F}_{0,p,T-\tau} \cap \mathbf{F}_{0,q,T-\tau}$, the convergence $\Delta^{\varepsilon,\tau}(t) \to 0$ holds for each $t \in]0, T-\tau[$ when $\varepsilon \to 0$ (cf. Remark 4.4). Now, for all $t \in]0, T-\tau]$ we have

$$\delta^{\varepsilon,\tau}(t) \le Ct^{-\frac{1}{2}}\|\mathbf{w}^{\varepsilon}(\tau) - \mathbf{w}(\tau)\|_p + \int_0^t (t-s)^{-\frac{3}{2p}}(\Delta^{\varepsilon,\tau}(s) + \delta^{\varepsilon,\tau}(s))ds.$$

As in the proof of Lemma 4.5 (and with the same notation), it follows by induction that

$$\delta^{\varepsilon,\tau}(t) \le C\|\mathbf{w}^{\varepsilon}(\tau) - \mathbf{w}(\tau)\|_p \sum_{k=1}^{\tilde{N}(p)} t^{(k-1)\theta_0 - \frac{1}{2}} + C\int_0^t \alpha(t-s)\Delta^{\varepsilon,\tau}(s)ds + C(T)\int_0^t \delta^{\varepsilon,\tau}(s)ds. \tag{104}$$

Thus, integrating and using Gronwall's lemma yield, for all $\lambda \in [0, T-\tau]$,

$$\int_0^{\lambda} \delta^{\varepsilon,\tau}(t)dt \le C(T)\|\mathbf{w}^{\varepsilon}(\tau) - \mathbf{w}(\tau)\|_p + C'(T)\int_0^{T-\tau}\int_0^t \alpha(t-s)\Delta^{\varepsilon,\tau}(s)ds\ dt.$$

Therefore, $\int_0^{\lambda} \delta^{\varepsilon,\tau}(t)dt \to 0$ and from this and (104) we deduce that $\delta^{\varepsilon,\tau}(t) \to 0$ for all $t \in]0, T-\tau[$. Consequently, $\nabla \mathbf{w}^{\varepsilon}(t) \to \nabla \mathbf{w}(t)$ in $(L_3^p)^3$ for all $t \in]0,T]$. Since $\mathbf{w}^{\varepsilon}$ is bounded in $\mathbf{F}_{1,p,T}$ uniformly in ε, the convergence takes also place in $L^1([0,T], (L_3^p)^3)$.

□

Corollary 6.2 *Let* $T^{1-\frac{3}{2p}}\|w_0\|p < \frac{1}{\Gamma_0(p)}$ *and denote* $\mathbf{u} = \mathbf{K}(\mathbf{w})$. *Let* $\varepsilon_n = (c \ln n)^{-9}$ *be a sequence satisfying the condition of Theorem 6.1. Then, when* $n \to \infty$, *we have*

$$\sup_{x \in \mathbb{R}^3} E\left(\left|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)\right|\right) \to 0$$

for each $t \in]0,T]$, *and*

$$\sup_{x \in \mathbb{R}^3} E\left(\int_0^T \left|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)\right| dt\right) \to 0.$$

Proof: For all $(t,x) \in [0,T] \times \mathbb{R}^3$, it holds that

$$\begin{aligned}
\left|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)\right| \leq & \Bigg|\mathbf{K}_{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) \\
& \qquad - \frac{1}{n}\sum_{i=1}^{n} K_{\varepsilon_n}(x - X_t^{i,\varepsilon_n,R}) \wedge (\chi_R(\Phi_t^{i,\varepsilon_n,R})h_0(X_0^i))\Bigg| \\
& + \Bigg|\frac{1}{n}\sum_{i=1}^{n} K_{\varepsilon_n}(x - X_t^{i,\varepsilon_n,R}) \wedge (\chi_R(\Phi_t^{i,\varepsilon_n,R})h_0(X_0^i)) \\
& \qquad - \int_{\mathcal{C}_T} K_{\varepsilon_n}(x - y(s)) \wedge \chi_R(\phi(s))h_0(x(0))P^{\varepsilon_n,R}(dy,d\phi)\Bigg| \\
& + |\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(t,x) - \mathbf{u}(t,x)|
\end{aligned} \tag{105}$$

with $P^{\varepsilon_n,R} = P^{\varepsilon_n} = law(X^{i,\varepsilon_n,R}, \Phi^{i,\varepsilon_n,R})$. The independence of the processes $(X^{i,\varepsilon_n,R}, \Phi^{i,\varepsilon_n,R})$, $i \in \mathbb{N}$, imply that the expectation of the second term is bounded by $\frac{1}{\sqrt{n}}(2M_{\varepsilon_n}R\|w_0\|_1)$. Thus,

$$\begin{aligned}
E\left|\mathbf{K}_{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)\right| \leq & (L_{\varepsilon_n}R\|w_0\|_1 + M_{\varepsilon_n}\|w_0\|_1)\frac{C\varepsilon(1 + R\|w_0\|_1T)}{\sqrt{n}(R\|w_0\|_1T)} \\
& \times \exp\{C_2\varepsilon_n^{-9}\|w_0\|_1T(R+1)(\|w_0\|_1 + RT)\} \\
& + \frac{1}{\sqrt{n}}(2M_{\varepsilon_n}R\|w_0\|_1) \\
& + \|\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(t) - \mathbf{K}^{\varepsilon_n}(\mathbf{w})(t)\|_\infty \\
& + \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(t) + \mathbf{u}(t)\|_\infty.
\end{aligned} \tag{106}$$

The first term is bounded by $C\frac{(\ln n)^{\alpha_1}}{n^{\alpha_2}}$ for some constants $C, \alpha_1, \alpha_2 > 0$ and goes to 0 when $n \to \infty$. The same holds for the second term for some other constants. The third is bounded by $C\|\mathbf{w}^{\varepsilon_n}(t) - \mathbf{w}(t)\|_{W^{1,p}}$ by Remark 4.3, and goes to 0 for each $t \in]0,T]$ and in $L^1([0,T],\mathbb{R})$, thanks to Proposition 6.1 *i)* and Lemma 6.1. The convergence of the last term for each $t \in]0,T]$ and in $L^1([0,T],\mathbb{R})$ is obtained by standard arguments.

□

Remark 6.6 *In order to improve the estimate of the first term in the l.h.s. of (105) (i.e. by avoiding the dependence on the divergent constants L_{ε_n} and M_{ε_n} we used in the l.h.s. of (106)), one could envisage to adapt the argument of Méléard [25] for the two dimensional vortex equation. That argument used uniform estimates in Lebesgue spaces for the densities of the approximating processes, following from results on generators in generalized divergence form in Osada [28], together with a representation formula for the Biot-Savart kernel in 2dimensions also given therein. We have yet not been able to generalize that formula (and the consequent argument) to the three dimensional case.*

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