Exponential estimates for spatially homogeneous Landau equations via the Malliavin calculus

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Abstract

The aim of this paper is to show how a probabilistic approach and the use of Malliavin calculus provide exponential estimates for the solution of a spatially homogeneous Landau equation, for a generalization of Maxwellian molecules. We recall how this solution can be obtained as the density of a nonlinear process. This process is a diffusion driven by a space-time white noise, with linear growth, but unbounded coefficients, and a degenerate diffusion matrix. However, the nonlinearity gives some non-degeneracy which implies the existence and regularity of the density. We use some ideas introduced by A. Kohatsu-Higa and developed by V. Bally, adapted to our situation to show that this density can be upper and lower bounded by some exponential-type estimates.

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1 Introduction and main results

The Landau equation, also called the Fokker-Planck-Landau equation, is a nonlinear partial differential equation that describes the collisions of particles in a plasma. This equation can be obtained as limit of Boltzmann equations when collisions become grazing (for this convergence, see for example [5], [13], [8]). The Landau equation has a physical sense in dimension 3 but can be generalized in any d-dimension.

In this paper, we consider Landau equations in $\mathbb{R}^d$, $d \geq 2$, in the spatially homogeneous case and for a generalization of Maxwellian molecules, given by

$$\frac{\partial f}{\partial t}(t,v) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d} dv_* a_{ij}(v-v_*)\left[f(t,v_*)\frac{\partial f}{\partial v_j}(t,v) - f(t,v)\frac{\partial f}{\partial v_{*j}}(t,v_*)\right]\right\}, \tag{1.1}$$

where $f(t,v) \geq 0$ is the density of particles with velocity $v \in \mathbb{R}^d$ at time $t \geq 0$ and $(a_{ij}(z))_{1\leq i,j\leq d}$ is a non-negative symmetric matrix depending on the interaction between particles given by

$$a_{ij}(z) = h(|z|^2)(|z|^2\delta_{ij} - z_i z_j),$$

where δ_{ij} denotes Kronecker symbol, $|\cdot|$ denotes Euclidean norm in $\mathbb{R}^d$, and h is a positive continuous function on $\mathbb{R}_+$ such that for all $z \in \mathbb{R}^d$ there exist $m, M > 0$ such that

$$m \leq h(|z|^2) \leq M.$$

When h is a constant, we recognize the coefficients of the spatially homogeneous Landau equation for Maxwellian molecules, c.f. [14].

By integrating by parts, one obtains (c.f. [13]) a weak formulation of the Landau equation (1.1): for $\varphi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ with bounded derivatives,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\varphi(v)f(t,v)\,dv &= \frac{1}{2}\sum_{i,j=1}^{d}\iint_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(t,v) f(t,v_*) a_{ij}(v-v_*)\frac{\partial^2\varphi}{\partial v_i \partial v_j}(v) \\ &\quad + \sum_{i=1}^{d}\iint_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(t,v) f(t,v_*) b_i(v-v_*)\frac{\partial\varphi}{\partial v_i}(v), \end{aligned}\tag{1.2}$$

where

$$b_i(z) = \sum_{j=1}^{d}\frac{\partial a_{ij}}{\partial z_j}(z) = -(d-1)h(|z|^2)z_i.$$

As a is a non-negative symmetric real matrix, there exists a $d\times d$ matrix $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ such that

$$a = \sigma.\sigma^*$$

where σ^* denotes the transpose matrix of σ. Note that the choice of σ is not unique.

In what follows, we will assume that σ and b are of class $\mathcal{C}^\infty$ with bounded derivatives of order greater or equal to one (in particular, Lipschitz continuous with constants K_σ and K_b).

For example, the choices of

$$\sigma(z) = \sqrt{h(|z|^2)}\begin{bmatrix} z_2 & 0 \\ -z_1 & 0 \end{bmatrix}$$

in dimension two, and of

$$\sigma(z) = \sqrt{h(|z|^2)} \begin{bmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{bmatrix}$$

in dimension three are convenient, as soon as h is a bounded function of class $\mathcal{C}^\infty$ with $h^{(l)}(x) = O\left(\frac{1}{|x|^l}\right)$ for each $l \in I\!N^*$, when $x \to +\infty$.

In [7], the Landau equation (1.1) is solved in a probabilistic way as follows. Due to the conservation of mass, one is looking for a solution defined as a family of probability measures $(P_t)_{t\geq 0}$, when the initial condition is a probability measure $P_0(dv)$ which can be - but not necessarily - equal to $f_0(v)dv$, with f_0 a probability density function. Remark that if $P_t(dv) = f(t,v)dv$ for any $t > 0$, f will be a solution of the Landau equation in the standard weak sense.

We deduce from (1.2) the following definition of weak solutions of the Landau equation.

Definition 1.1 *Let P_0 be a probability measure on $I\!R^d$. A measure solution of the Landau equation* (1.1) *with initial data P_0 is a family of probability measures $(P_t)_{t\geq 0}$ on $I\!R^d$ satisfying for any $\varphi \in \mathcal{C}^2\left(I\!R^d, I\!R\right)$ with bounded derivatives,*

$$\begin{aligned} \frac{d}{dt}\int_{I\!R^d} \varphi(v)\, P_t(dv) &= \frac{1}{2}\sum_{i,j=1}^{d} \int_{I\!R^d} P_t(dv) \left(\int_{I\!R^d} P_t(dv_*)\, a_{ij}(v - v_*)\right) \partial_{ij}\varphi(v) \\ &\quad + \sum_{i=1}^{d} \int_{I\!R^d} P_t(dv) \left(\int_{I\!R^d} P_t(dv_*)\, b_i(v - v_*)\right) \partial_i\varphi(v). \end{aligned} \tag{1.3}$$

The probabilistic approach developed in [6] and [7] consists in associating with this equation a nonlinear process with law P such that the time-marginals (P_t) are solutions of (1.3). This process is the solution of a nonlinear diffusion driven by a space-time white noise, with linear growth, but unbounded coefficients, and a degenerate diffusion matrix, as will be described next.

Let $W = (W^1, ..., W^d)$ be a d-dimensional space-time white noise on $[0,\infty) \times [0,1]$, defined on its canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, Ω is the space of continuous functions from $[0,\infty) \times [0,1]$ to $I\!R^d$ vanishing on the axes, $\mathcal{F}$ is the Borel σ-field of Ω and $\mathbb{P}$ is the Wiener measure associated with W. Here the W^i are independent space-time white noises with covariance measure $dtd\alpha$ on $[0,\infty) \times [0,1]$ (according to the definition of Walsh, [15]), where $d\alpha$ denotes Lebesgue measure on $[0,1]$. We consider its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t = \sigma\{W([0,s] \times A) : 0 \leq s \leq t, A \in \mathcal{B}([0,1])\}$.

In order to describe the nonlinearity, we also consider the probability space

$$([0,1], \mathcal{B}([0,1]), d\alpha),$$

and we denote by $\mathbb{E}$, $\mathbb{E}_\alpha$ the expectations and $\mathcal{L}$, $\mathcal{L}_\alpha$ the distributions of a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $([0,1], \mathcal{B}([0,1]), d\alpha)$, respectively.

For $k \geq 2$, we denote by $\mathcal{P}_k$ the space of continuous adapted processes $X = (X_t)_{t\geq 0}$ from $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ to $I\!R^d$, such that for any $T > 0$, $\mathbb{E}[\sup_{0\leq t\leq T} |X_t|^k] < \infty$, and by $\mathcal{P}_{k,\alpha}$

the space of continuous processes $Y = (Y_t)_{t\geq 0}$ from $([0,1], \mathcal{B}([0,1]), d\alpha)$ to $I\!R^d$, such that $\mathbb{E}_\alpha[\sup_{0\leq t\leq T} |Y_t|^k] < \infty$, for any $T > 0$.

Let X_0 be an integrable vector on $I\!R^d$, independent of W.

Let us consider the following nonlinear stochastic differential equation.

Definition 1.2 *A couple of processes (X, Y) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}) \times ([0,1], \mathcal{B}([0,1]), d\alpha)$ is said to be a solution of the Landau stochastic differential equation if, for any $t \geq 0$,*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s\left(\alpha\right)\right).W\left(d\alpha, ds\right) + \int_0^t \int_0^1 b\left(X_s - Y_s\left(\alpha\right)\right) d\alpha ds, \tag{1.4}$$

and $\mathcal{L}\left(X\right) = \mathcal{L}_\alpha\left(Y\right)$, where σ and b are the coefficients of the Landau equation (1.1).

Using Itô's Formula, one easily proves that if (X, Y) is a solution of the Landau SDE (1.4), then the family of distribution $(P_t)_{t\geq 0}$ of X (or of Y) is a measure solution of the Landau equation with initial data $P_0 = \mathcal{L}\left(X_0\right)$ according to Definition 1.1.

Under standard assumptions, one can obtain existence and uniqueness of the solution of the Landau SDE.

Theorem 1.3 [6, Theorem 5] *Assume that X_0 has finite moments of order k, $k \geq 2$ and the coefficients σ and b of the Landau equation are Lipschitz continuous. Then there exists a couple (X, Y), unique in law, solution of the Landau SDE with $(X, Y) \in \mathcal{P}_k \times \mathcal{P}_{k,\alpha}$. The family of distributions $(P_t)_{t\geq 0}$ of X is the unique measure solution of the Landau equation with initial data P_0, and with finite moments of order k.*

Using this interpretation, Guérin proves in [6], by tools of Malliavin calculus, the existence and uniqueness of a smooth solution of the Landau equation. She obtains in fact the existence and regularity of a density for each $P_t, t > 0$, the degeneracy of the matrix σ being compensated by the effect of nonlinearity.

Theorem 1.4 [6, Theorems 13 and 18] (1) *Assume that X_0 is square integrable and that its distribution is not a Dirac mass. Assume that the coefficients σ and b are Lipschitz continuous of class $\mathcal{C}^1$ and let (X, Y) denote the solution of the Landau SDE. Then, for any $t > 0$, the regular version of the conditional distribution of X_t given X_0 is absolutely continuous with respect to Lebesgue measure. Denote by $f_{X_0}(t, v)$ its density function.*

(2) *Assume moreover that σ and b are infinitely differentiable with bounded derivatives of order greater or equal to one, and X_0 has finite moments of order k, $k \geq 2$. Then, for each $t > 0$, the density $f_{X_0}(t, .)$ is (P_0-a.s.) bounded and of class $\mathcal{C}^\infty$ with bounded derivatives.*

A consequence of Theorem 1.4 is the existence of a weak function solution of the Landau equation (1.1), which is given by

$$f(t, v) = \int_{I\!R^d} f_{x_0}(t, v) P_0(dx_0).$$

The aim of this paper is to obtain some exponential-type upper and lower bounds for the solution of the Landau equation. The research of a lower-bound was partially developed in Villani [14]. In that paper, the author obtains (in Section 7-Theorem 3) a lower bound for the solution of the spatially homogeneous Landau equation in the case of Maxwellian molecules, assuming that the initial condition is a lower bounded function. The general case is more complicated and a conjecture is stated in [14] Proposition 6, but never proved.

Here, we prove exponential-type upper and lower bounds for the conditional density of the solution of the Landau SDE X_t given X_0, from which one can deduce bounds for the solution of the Landau equation.

Theorem 1.5 *Assume that the hypotheses of Theorem 1.4* (2) *hold and let $f_{X_0}(t,v)$ denote the conditional density given X_0 of the solution of the Landau SDE.*

(a) *Assume the following hypothesis.*

(H) *For all $\xi \in I\!\!R^d$, $\mathbb{E}[|X_0|^2|\xi|^2 - < X_0, \xi >^2] > 0$.*

Then, for all $t \in (0,T]$ and for fixed $v \in I\!\!R^d$, there exist positive constants c and C_T, which can be explicitly given, such that P_0-a.s.,

$$f_{X_0}(t,v) \geq \frac{1}{C_T t^{d/2}(1+|v|^2)^{d/2}} e^{-M_T(t,v,X_0)\ln(C_T(1+|v|^2+|X_0|^2)^{d/2})},$$

where

$$M_T(t,v,X_0) = 2\left(c_T(v,X_0)\, t \vee \frac{c|v-X_0|^2}{t} \vee 1\right) + 1,$$

with $c_T(v,X_0) = C_T(1+|v|^{2(d^2+d+1)} + |X_0|^{2(d^2+d+1)})^2$.

(b) *There exist positive finite constants c_T, c, C such that P_0-a.s.,*

$$f_{X_0}(t,v) \leq c_T t^{-d/2} e^{-\frac{(\ln(1+|v|^2)-\ln(1+|X_0|^2)-Ct)^2}{ct}},$$

for all $t \in (0,T]$ and $v \in I\!\!R^d$.

Remark 1.6 *Hypothesis* (H) *means that the support of the law of X_0 is not embedded in a line. In particular, hypothesis* (H) *holds for the two extreme cases, if either the law P_0 of X_0 has a density f_0 with respect to Lebesgue measure, or if $P_0 = \frac{\delta_{x_1}+\delta_{x_2}}{2}$, with x_1 and x_2 non collinear vectors, where δ_z denotes Dirac function at $z \in I\!\!R^d$.*

We obtain (a) by adapting ideas introduced in Kohatsu-Higa [9] and developed by Bally [2] in the case of non elliptic diffusion processes. The main tool is the conditioned Malliavin calculus. The trick consists in discretizing the time-interval and use a recursive argument in order to obtain the lower bound. In [9], general random processes are studied, but the techniques necessitate some ellipticity and boundedness of the coefficients, that we do not have in our situation. So we use an additional idea introduced in Bally. On every discretization interval, we only consider the diffusion in a tube around a deterministic trajectory, in which we have some estimates on the coefficients. We have then to estimate the probability for the diffusion to stay in the tube. We will see that choosing the discretization meshes sufficiently small, and a simple deterministic path, we are able to evaluate all quantities we need. The obtention of the upper bound (b) is simpler and uses Malliavin calculus in a more standard way. As the coefficients are not bounded, the upper bound in not of Gaussian type. The trick here consists in introducing the SDE satisfied by $\ln(1+|X_t|^2)$, which has bounded coefficients.

A consequence of the lower bound of Theorem 1.5 is the strict positivity of the solution. Such a result was obtained by Fournier in [4] for the solution of the Boltzmann equation, adapting a probabilistic approach due to Bally and Pardoux [1]. Let us remark that such an approach could also be adapted without difficulty to the Landau framework, proving that the solution of the equation is strictly positive for each positive time. Our aim here is to obtain the a refine result giving precise estimates.

2 The Malliavin calculus

In this section we present some elements of Malliavin calculus and conditional Malliavin calculus that will be used for the proof of Theorem 1.5.

2.1 Elements of Malliavin calculus

In this subsection, we recall, following [10], some elements of Malliavin calculus related to W.

Let H be the Hilbert space $H = \mathcal{L}^2([0,T]\times[0,1];\mathbb{R}^d)$. For any $h \in H$, we set $W(h) = \sum_{j=1}^d \int_0^T \int_0^1 h^j(r,z)W^j(dr,dz)$. The Gaussian subspace $\mathcal{H} = \{W(h), h \in H\}$ of $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ is isomorphic to H.

Let $\mathcal{S}$ denote the class of smooth random variables $F = f(W(h_1),...,W(h_n))$, where $h_1,...,h_n$ are in H, $n \geq 1$, and f belongs to $\mathcal{C}_p^\infty(\mathbb{R}^n)$, the space of functions of class $\mathcal{C}^\infty$ such that all its partial derivatives have at most polynomial growth order.

Given F in S, its derivative is the d-dimensional stochastic process $DF = (D_{(r,z)}F = (D^1_{(r,z)}F,...,D^d_{(r,z)}F), (r,z) \in [0,T]\times[0,1])$, where the $D_{(r,z)}F$ are H-valued random vectors given by

$$D^l_{(r,z)}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1),...,W(h_n))h_i^l(r,z), \;\; l = 1,...,d.$$

More generally, the k-th order derivative of F is obtained by iterating k times the derivative operator: if F is a smooth random variable, k is an integer and $(l_1,...,l_k)$ is a k-uplet of $\{1,...,d\}^k$, we denote the iterated derivative as $D^{l_1}_{\alpha_1}\cdots D^{l_k}_{\alpha_k}F$, where $\alpha_i = (r_i,z_i) \in [0,T]\times[0,1]$. Then for every $p \geq 1$ and any natural number m, we denote by $\mathbb{D}^{m,p}$ the closure of $\mathcal{S}$ with respect to the semi-norm $\|\cdot\|_{m,p}$ defined by

$$\|F\|_{m,p} = (\mathbb{E}[|F|^p] + \sum_{k=1}^m \mathbb{E}[\|D^{(k)}F\|^p_{H^{\otimes k}}])^{1/p},$$

where

$$\|D^{(k)}F\|^2_{H^{\otimes k}} = \sum_{l_1,...,l_k=1}^d \int\cdots\int_{([0,T]\times[0,1])^k} |D^{l_1}_{\alpha_1}\cdots D^{l_k}_{\alpha_k}F|^2 d\alpha_1\cdots d\alpha_k.$$

We set $\mathbb{D}^\infty = \cap_{p\geq 1}\cap_{m\geq 1}\mathbb{D}^{m,p}$.

Similarly, for any separable Hilbert space V, one can define the analogous spaces $\mathbb{D}^{m,p}(V)$ and $\mathbb{D}^\infty(V)$ of V-valued random variables, and the related $\|\cdot\|_{m,p,V}$ semi-norms (the related smooth functionals being of the form $F = \sum_{j=1}^n F_j v_j$, where $F_j \in \mathcal{S}$ and $v_j \in V$).

We denote by δ the adjoint of the operator D, which is an unbounded operator on $\mathcal{L}^2(\Omega; H)$ taking values in $\mathcal{L}^2(\Omega)$ (see [10, Def.1.3.1]). In particular, if u belongs to Dom δ, then $\delta(u)$ is the element of $\mathcal{L}^2(\Omega)$ characterized by the following duality relation:

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\sum_{j=1}^{d} \int_0^T \int_0^1 D^j_{(r,z)} F \, u^j(r,z) dz dr], \quad \text{for any } F \in \mathbb{D}^{1,2}. \tag{2.1}$$

Recall that if $u \in \mathcal{L}^2([0,T] \times [0,1] \times \Omega; \mathbb{R}^d)$ is an adapted process, then (cf. [10, Prop.1.3.4]) u belongs to Dom δ and $\delta(u)$ coincides with the Itô integral:

$$\delta(u) = \sum_{j=1}^{d} \int_0^T \int_0^1 u^j(r,z) W^j(dz, dr).$$

One of the multiple applications of the Malliavin calculus is the study of existence and smoothness of densities for laws of random vectors on the Wiener space. The basic assumptions are introduced in the following definition of a *non-degenerate* random vector.

Definition 2.1 *A random vector $F = (F^1, ..., F^d) \in (\mathbb{D}^\infty)^d$ is said to be non-degenerate if the Malliavin matrix of F defined by $\gamma_F = (\langle DF^i, DF^j \rangle_H)_{1 \le i,j \le d}$ is invertible a.s. and*

$$(\det \gamma_F)^{-1} \in \cap_{p \ge 1} \mathcal{L}^p(\Omega).$$

For a nondegenerate random vector, the following *integration by parts formula* plays a key role.

Proposition 2.2 [11, Prop.3.2.1] *Let $F = (F^1, ..., F^d) \in (\mathbb{D}^\infty)^d$ be a non-degenerate random vector, let $G \in \mathbb{D}^\infty$ and let $g \in \mathcal{C}_p^\infty(\mathbb{R}^d)$. Fix $k \ge 1$. Then for any multi-index $\alpha = (\alpha_1, ..., \alpha_k) \in \{1, \dots, d\}^k$, there exists an element $H_\alpha(F, G) \in \mathbb{D}^\infty$ such that*

$$\mathbb{E}[\partial_\alpha g(F) G] = \mathbb{E}[g(F) H_\alpha(F, G)],$$

where the random variables $H_\alpha(F, G)$ are recursively given by

$$H_{(i)}(F, G) = \sum_{j=1}^{d} \delta(G \, (\gamma_F^{-1})_{ij} \, DF^j),$$
$$H_\alpha(F, G) = H_{(\alpha_k)}(F, H_{(\alpha_1, \dots, \alpha_{k-1})}(F, G)).$$

¿From Proposition 2.2 it follows that the density of a non-degenerate random vector is infinitely differentiable. Moreover, taking $G = 1$ and $\alpha = (1, ..., d)$, one obtains the following expression for the density of a non-degenerate random vector, that gives a powerful tool to get upper bounds for this density.

Corollary 2.3 [11, Corollary 3.2.1] *Let $F = (F^1, ..., F^d) \in (\mathbb{D}^\infty)^d$ be a non-degenerate random vector and let $p_F(x)$ denote the density of F. Then*

$$p_F(x) = \mathbb{E}[1_{\{F^i > x^i, 1 \le i \le d\}} H_{(1, \dots, d)}(F, 1)],$$

where

$$H_{(1, \dots, d)}(F, 1) = \delta((\gamma_F^{-1} DF)^d \delta((\gamma_F^{-1} DF)^{d-1} \cdots \delta((\gamma_F^{-1} DF)^1) \cdots)).$$

2.2 Conditional Malliavin calculus

In this section, we give the conditional version of some of the results established in Section 3.1.

We need a preliminary result, which is a consequence of [10, Prop.1.2.4].

Lemma 2.4 *Let $s \in [0,T]$ and let F_s be an $\mathcal{F}_s$-measurable random variable in $\mathbb{D}^{1,2}$. Then, $D_t F_s$ is zero almost everywhere in $[s,T] \times \Omega$.*

A consequence of this result is the conditional version of the duality relation (2.1), which follows similarly to the one-parameter case [12, (2.12)].

Proposition 2.5 [3, Proposition 4.4] *Let $s \in [0,T]$. Let F be a random variable in $\mathbb{D}^{1,2}$ and let u be an adapted process such that $\mathbb{E}[\int_0^T \int_0^1 |u(r,z)|^2 dzdr] < \infty$. Then the following duality relation holds:*

$$\mathbb{E}[F \int_s^T \int_0^1 u(r,z) \cdot W(dz,dr) \,|\mathcal{F}_s] = \mathbb{E}[\int_s^T \int_0^1 D_{(r,z)}F \cdot u(r,z)\, dzdr \,|\mathcal{F}_s].$$

The following norms are the white noise versions of those in [12, Def.1]. Let $s \in [0,T]$. For any function $f \in \mathcal{L}^2(([0,T] \times [0,1])^n; \mathbb{R}^d)$, any random variable $F \in \mathbb{D}^{m,p}$, and any process u such that $u(r,z) \in \mathbb{D}^{m,p}$, for all $r \in [0,T]$, we define

$$H_s = \mathcal{L}^2([s,T] \times [0,1]; \mathbb{R}^d),$$

$$\|f\|_{H_s^{\otimes n}} = (\int_{([s,T]\times[0,1])^n} |f(r,z)|^2 dz_1 \cdots dz_n dr_1 \cdots dr_n)^{1/2},$$

$$\|F\|_{m,p,s} = \{\mathbb{E}[|F|^p \,|\mathcal{F}_s] + \sum_{k=1}^m \mathbb{E}[\|D^{(k)}F\|^p_{H_s^{\otimes k}} \,|\mathcal{F}_s]\}^{1/p},$$

and

$$\|u\|_{m,p,s} = \{\mathbb{E}[\|u\|^p_{H_s} |\mathcal{F}_s] + \sum_{k=1}^m \mathbb{E}[\|D^{(k)}u\|^p_{H_s^{\otimes k+1}} |\mathcal{F}_s]\}^{1/p}.$$

Moreover, we write $\gamma_F(s)$ for the Malliavin covariance matrix with respect to H_s, that is,

$$\gamma_F(s) = (\langle DF^i, DF^j \rangle_{H_s})_{1 \le i,j \le d}.$$

We next give a conditional version of the integration by parts formula (Proposition 2.2). The proof follows similarly as the non-conditional version using conditional expectations and Lemma 2.4, and is therefore omitted.

Proposition 2.6 *Let $F, Z_s \in (\mathbb{D}^\infty)^d$ be two non-degenerate random vectors where Z_s is $\mathcal{F}_s$-measurable. Let $G \in \mathbb{D}^\infty$ and let $g \in \mathcal{C}_p^\infty(\mathbb{R}^d)$. Fix $k \ge 1$. Then for any multi-index $\alpha = (\alpha_1, ..., \alpha_k) \in \{1, \dots, d\}^k$, there exists a random variable $H_\alpha^s(F,G) \in \mathbb{D}^\infty$ such that*

$$\mathbb{E}[\partial_\alpha g(F + Z_s) G \,|\mathcal{F}_s] = \mathbb{E}[g(F+Z_s) H_\alpha^s(F,G) \,|\mathcal{F}_s],$$

where the $H_\alpha^s(F,G)$ are recursively given by

$$H_{(i)}^s(F,G) = \sum_{j=1}^d \delta(G\, (\gamma_F(s)^{-1})_{ij}\, DF^j),$$

$$H_\alpha^s(F,G) = H_{(\alpha_k)}^s(F, H_{(\alpha_1,\dots,\alpha_{k-1})}^s(F,G)).$$

Finally, the next result gives an estimate of the $\mathcal{L}^2(H_s,\Omega)$-norm of the random variables $H^s_\alpha(F,G)$ for $\alpha=(1,...,d)$, which gives the explicit exponents that appear in the non-conditional version [11, Proposition 3.2.2].

Proposition 2.7 *Let $F\in(I\!D^\infty)^d$ be a non-degenerate random vector. Assume that there exist positive $\mathcal{F}_s$-measurable random variables X_s and Y_s (eventually deterministic) such that, for any $p>1$ and $k\geq 1$, there exists constants $c_1(p)>0$ and $c_2(k,p)\geq 0$ such that*

(a) $\mathbb{E}[(\det\gamma_F(s))^{-p}\,|\mathcal{F}_s]^{1/p}\leq c_1 X_s$;

(b) $\mathbb{E}[\|D^{(k)}(F^i)\|^p_{H_s^{\otimes k}}\,|\mathcal{F}_s]^{1/p}\leq c_2 Y_s,\ i=1,...,d.$

Then there exists a constant $c>0$ and indices m,q depending only on d such that

$$\|H^s_{(1,\dots,d)}(F,G)\|_{0,2,s}\leq c\|G\|_{m,q,s}X_s^dY_s^{d(2d-1)}.$$

Proof. The proof of this result uses the same arguments in the proof of [3, Lemma 4.11], but in a general setting. We will only give the main steps.

Using the continuity of δ (c.f. [3, Proposition 4.5]) and Hölder inequality for the conditional Malliavin norms (c.f. [16, Proposition 1.10, p.50], we obtain

$$\|H^s_{(1,\dots,d)}(F,G)\|_{0,2,s}\leq c\|H^s_{(1,\dots,d-1)}(F,G)\|_{1,4,s}\sum_{j=1}^d\|(\gamma_F(s)^{-1})_{dj}\|_{1,8,s}\,\|D(F^j)\|_{1,8,s}. \quad (2.2)$$

For the third factor we use hypothesis (b). For the second factor, note that

$$\|(\gamma_F(s)^{-1})_{ij}\|_{m,p,s}=\{\mathbb{E}[|(\gamma_F(s)^{-1})_{ij}|^p\,|\mathcal{F}_s]+\sum_{k=1}^m\mathbb{E}[\|D^{(k)}(\gamma_F(s)^{-1})_{ij}\|^p_{H_s^{\otimes k}}\,|\mathcal{F}_s]\}^{1/p}.$$

For the first term, we use Cramer's formula to get that

$$|(\gamma_F(s)^{-1})_{ij}|=|A_{ij}(\det\gamma_F(s))^{-1}|,$$

where A_{ij} denotes the adjoint of $(\gamma_F(s))_{ij}$. From the Cauchy-Schwarz inequality for conditional expectations and hypotheses (a) and (b) we find that

$$\begin{aligned}\mathbb{E}[((\gamma_F(s)^{-1})_{ij})^p\,|\mathcal{F}_s]&\leq c_{d,p}\mathbb{E}[(\det\gamma_F(s))^{-2p}\,|\mathcal{F}_s]^{1/2}\times\mathbb{E}[\|D(F)\|^{4p(d-1)}_{H_s}\,|\mathcal{F}_s]^{1/2}\\&\leq c_{d,p}X_s^pY_s^{2p(d-1)}.\end{aligned}$$

For the second term, we iterate the equality (cf. [10, Lemma 2.1.6]),

$$D(\gamma_F(s)^{-1})_{ij}=-\sum_{k,l=1}^d(\gamma_F(s)^{-1})_{ik}D(\gamma_F(s))_{kl}(\gamma_F(s)^{-1})_{jl},$$

in the same way as in the proof of [3, Lemma 4.11]. Then, again using hypotheses (a) and (b) and iterating the process (2.2) we conclude the desired bound. △

3 The lower bound

In this section we adapt the techniques introduced by Kohatsu-Higa [9] and developed by Bally [2], in the situation of the Landau equation in order to prove the lower bound of Theorem 1.5.

Consider the Landau equation introduced in Section 2, that is,

$$X_t^i = X_0^i + \sum_{j=1}^d \int_0^t \int_0^1 \sigma_{ij}(X_s - Y_s(\alpha)) W^j(d\alpha, ds) + \int_0^t \int_0^1 b_i(X_s - Y_s(\alpha)) d\alpha ds,$$

where $i = 1, ..., d$, $0 \leq t \leq T$, and X_0, σ and b satisfy the hypotheses of Theorem 1.4.

Consider a time grid $0 = t_0 < t_1 < \cdots < t_N = t$ and let

$$\Delta_k = t_k - t_{k-1}.$$

We define the following evolution sequence,

$$X_{t_k}^i = X_{t_{k-1}}^i + J_k^i + \Gamma_k^i, \quad i = 1, ..., d, \tag{3.1}$$

where

$$J_k^i = \sum_{j=1}^d \int_{t_{k-1}}^{t_k} \int_0^1 \sigma_{ij}(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)) W^j(d\alpha, ds),$$

and

$$\begin{aligned}\Gamma_k^i = &\sum_{j=1}^d \int_{t_{k-1}}^{t_k} \int_0^1 (\sigma_{ij}(X_s - Y_s(\alpha)) - \sigma_{ij}(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha))) W^j(d\alpha, ds) \\ &+ \int_{t_{k-1}}^{t_k} \int_0^1 b_i(X_s - Y_s(\alpha)) d\alpha ds.\end{aligned}$$

3.1 Preliminary estimates

We start by establishing some preliminary estimates that will be needed for the proof of the lower bound of Theorem 1.5.

Consider the conditional covariance matrix of the (Gaussian) random variable J_k with respect to $\mathcal{F}_{t_{k-1}}$, which is given by

$$\begin{aligned}C_{lr}(J_k) &= \sum_{j=1}^d \int_{t_{k-1}}^{t_k} \int_0^1 \sigma_{lj}(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)) \sigma_{rj}(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)) d\alpha ds \\ &= (t_k - t_{k-1}) \sum_{j=1}^d \int_0^1 \sigma_{lj}(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)) \sigma_{rj}(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)) d\alpha, \ 1 \leq l, r \leq d.\end{aligned}$$

Note that, as J_k is Gaussian, we have that $C(J_k) = \gamma_{J_k}(t_{k-1})$.

The first two results concern a minoration of the lower eigenvalue and a majoration of the upper eigenvalue of the matrix $C(J_k)$.

Proposition 3.1 *Assume hypothesis* (H) *of Theorem 1.5. Then,*

$$\inf_{\xi \in \mathbb{R}^d, |\xi|=1} \sum_{l,r=1}^{d} C_{lr}(J_k)\xi_l\xi_r \geq c\Delta_k > 0,$$

where $c > 0$ is a constant not depending on k.

Proof. In one of the steps of the proof of Theorem 1.4, Guérin showed that for each $\xi \in \mathbb{R}^d$, one has

$$\xi^* C(J_k)\xi \geq \Delta_k m F(\xi, t_{k-1}),$$

where

$$F(\xi, t) = \mathbb{E}[|X_t|^2|\xi|^2 - \langle X_t, \xi\rangle^2], \tag{3.2}$$

and m is a lower bound of the function h appearing in the Landau equation.

Since the law of X_t has a density, $F(\xi, t) > 0$ for each $t > 0$. Moreover, assuming hypothesis (H), we obtain that $F(\xi, t) > 0$ for each $t \geq 0$. Then, as the function $F(\xi, t)$ is positive and continuous on the compact set $[0, T] \times \{\xi \in \mathbb{R}^d : |\xi| = 1\}$, a strictly positive minimum is reached on this set.

Hence, for all $\xi \in \mathbb{R}^d$, $|\xi| = 1$, we have that

$$\xi^T C(J_k)\xi \geq c\Delta_k,$$

where $c > 0$ is independent of k. In particular,

$$\inf_{\xi \in \mathbb{R}^d, |\xi|=1} \sum_{l,r=1}^{d} C_{lr}(J_k)\xi_l\xi_r \geq c\Delta_k > 0.$$

△

Proposition 3.2 *There exists a finite constant $C_T > 0$ not depending on k such that*

$$\sup_{\xi \in \mathbb{R}^d, |\xi|=1} \sum_{l,r=1}^{d} C_{lr}(J_k)\xi_l\xi_r \leq C_T\Delta_k(|X_{t_{k-1}}|^2 + 1).$$

Proof. Let $\xi \in \mathbb{R}^d$, $|\xi| = 1$. Using the Lipschitz property of σ and the fact that $X \in \mathcal{P}_2$,

$$\begin{aligned}
\sum_{l,r=1}^{d} C_{lr}(J_k)\xi_l\xi_r &\leq \Delta_k K_\sigma^2 \int_0^1 (|X_{t_{k-1}}|^2 + |Y_{t_{k-1}}(\alpha)|^2)d\alpha \\
&= \Delta_k K_\sigma^2(|X_{t_{k-1}}|^2 + \mathbb{E}[|X_{t_{k-1}}|^2]) \\
&\leq \Delta_k K_\sigma^2(|X_{t_{k-1}}|^2 + \mathbb{E}[\sup_{0\leq s\leq T} |X_s|^2]) \\
&\leq C_T\Delta_k(|X_{t_{k-1}}|^2 + 1).
\end{aligned}$$

Therefore,

$$\sup_{\xi \in \mathbb{R}^d, |\xi|=1} \sum_{l,r=1} C_{lr}(J_k)\xi_l\xi_r \leq C_T\Delta_k(|X_{t_{k-1}}|^2 + 1).$$

△

The next two results concern estimates for the conditional Sobolev norms given $\mathcal{F}_{t_{k-1}}$ of the terms J_k and Γ_k of the evolution sequence (3.1). Note that as the coefficients of the Landau equation are unbounded these conditional bounds will depend on the random variable $X_{t_{k-1}}$.

Lemma 3.3 *For any $p > 1$ and $m \geq 1$, there exists a finite constant $C_T > 0$ such that*

$$\|D(J_k^i)\|_{m,p,t_{k-1}} \leq C_T \Delta_k^{1/2}(|X_{t_{k-1}}| + 1), \text{ for all } i = 1, ..., d.$$

Proof. Let $(r, z) \in [0, t] \times [0, 1]$. Then

$$D_{r,z}^l(J_k^i) = \sigma_{il}(X_{t_{k-1}} - Y_{t_{k-1}}(z))1_{\{r \in [t_{k-1}, t_k]\}}, \quad 1 \leq i, l \leq d.$$

Hence, from Lipschitz property of σ, Hölder's inequality, the fact that $X \in \mathcal{P}_p$ and the inequality $(a + b)^\alpha \leq a^\alpha + b^\alpha$ for $a, b \geq 0$ and $0 < \alpha \leq 1$, we obtain that

$$\begin{aligned}
\|D(J_k^i)\|_{m,p,t_{k-1}} &= \mathbb{E}[(\int_{t_{k-1}}^{t_k} \int_0^1 \sum_{l=1}^d (\sigma_{il}(X_{t_{k-1}} - Y_{t_{k-1}}(z)))^2 drdz)^{p/2} \,|\mathcal{F}_{t_{k-1}}]^{1/p} \\
&\leq (\Delta_k^{p/2} K_\sigma^p (|X_{t_{k-1}}|^p + \mathbb{E}[\sup_{0 \leq s \leq T} |X_s|^p]))^{1/p} \\
&\leq C_T \Delta_k^{1/2}(|X_{t_{k-1}}| + 1).
\end{aligned}$$

$\triangle$

Lemma 3.4 *For any $p > 1$ and $m \geq 1$, there exists a finite constant $C_T > 0$ such that*

$$\|\Gamma_k^i\|_{m,p,t_{k-1}} \leq C_T \Delta_k(|X_{t_{k-1}}| + 1), \text{ for all } i = 1, ..., d.$$

Proof. By definition,

$$\|\Gamma_k^i\|_{m,p,t_{k-1}} = \{\mathbb{E}[|\Gamma_k^i|^p \,|\mathcal{F}_{t_{k-1}}] + \sum_{j=1}^m \mathbb{E}[\|D^{(j)}(\Gamma_k^i)\|_{H_{t_{k-1}}^{\otimes j}}^p \,|\mathcal{F}_{t_{k-1}}]\}^{1/p}. \tag{3.3}$$

For the first term in (3.3), note that

$$\mathbb{E}[|\Gamma_k^i|^p \,|\mathcal{F}_{t_{k-1}}] \leq 2^{p-1}(A + B), \tag{3.4}$$

where

$$\begin{aligned}
A &:= \mathbb{E}[(\int_{t_{k-1}}^{t_k} \int_0^1 \sum_{j=1}^d (\sigma_{ij}(X_s - Y_s(\alpha)) - \sigma_{ij}(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)))^2 d\alpha ds)^{p/2} \,|\mathcal{F}_{t_{k-1}}], \\
B &:= \mathbb{E}[(\int_{t_{k-1}}^{t_k} \int_0^1 b_i(X_s - Y_s(\alpha)) d\alpha ds)^p \,|\mathcal{F}_{t_{k-1}}].
\end{aligned}$$

Now, using Hölder's inequality and Lipschitz property of σ, we have

$$A \leq C\Delta_k^{p/2-1} \int_{t_{k-1}}^{t_k} \int_0^1 \mathbb{E}[|X_s - X_{t_{k-1}}|^p + |Y_s(\alpha) - Y_{t_{k-1}}(\alpha))|^p \,|\mathcal{F}_{t_{k-1}}] \, d\alpha ds. \tag{3.5}$$

In order to evaluate the first term in (3.5) we use the stochastic differential equation satisfied by the increment $X_s - X_{t_{k-1}}$. Then, from Burkholder's inequality and again Lipschitz property,

$$\begin{aligned}
\mathbb{E}[|X_s - X_{t_{k-1}}|^p \,|\mathcal{F}_{t_{k-1}}] \quad &\leq \quad C_T \Delta_k^{p/2-1} \Big\{ \int_{t_{k-1}}^s \int_0^1 \mathbb{E}[|X_u|^p + |Y_u(\alpha)|^p \,|\mathcal{F}_{t_{k-1}}] \, d\alpha du \\
&\quad + \Delta_k^{p/2} \int_{t_{k-1}}^s \int_0^1 \mathbb{E}[|X_u|^p + |Y_u(\alpha)|^p \,|\mathcal{F}_{t_{k-1}}] \, d\alpha du \Big\} \\
&\leq \quad C_T \Delta_k^{p/2-1} \Big(\int_{t_{k-1}}^s \mathbb{E}[|X_u|^p \,|\mathcal{F}_{t_{k-1}}] + \mathbb{E}[|X_u|^p] \, du \Big).
\end{aligned}$$

Moreover, using the fact that $X \in \mathcal{P}_p$,

$$\begin{aligned}\mathbb{E}[|X_u|^p \,|\mathcal{F}_{t_{k-1}}] &\leq \mathbb{E}[|X_u - X_{t_{k-1}}|^p \,|\mathcal{F}_{t_{k-1}}] + |X_{t_{k-1}}|^p \\ &\leq C_T\left(\int_{t_{k-1}}^{u} \mathbb{E}[|X_r|^p \,|\mathcal{F}_{t_{k-1}}]\, dr + 1\right) + |X_{t_{k-1}}|^p.\end{aligned}$$

By Gronwall's Lemma,

$$\mathbb{E}[|X_u|^p \,|\mathcal{F}_{t_{k-1}}] \leq (C_T + |X_{t_{k-1}}|^p)\exp(C_T). \tag{3.6}$$

This implies that

$$\mathbb{E}[|X_s - X_{t_{k-1}}|^p \,|\mathcal{F}_{t_{k-1}}] \leq C_T \Delta_k^{p/2}(|X_{t_{k-1}}|^p + 1). \tag{3.7}$$

In particular,

$$\mathbb{E}[|X_s - X_{t_{k-1}}|^p] \leq C_T \Delta_k^{p/2}. \tag{3.8}$$

Therefore, substituting (3.7) and (3.8) into (3.5), we obtain that

$$A \leq C_T \Delta_k^{p}(|X_{t_{k-1}}|^p + 1). \tag{3.9}$$

In the same way, from Hölder's inequality, Lipschitz property of b and (3.6) we obtain that

$$B \leq C_T \Delta_k^{p}(|X_{t_{k-1}}|^p + 1). \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.4) we finally obtain

$$\mathbb{E}[|\Gamma_k^i|^p \,|\mathcal{F}_{t_{k-1}}] \leq C_T \Delta_k^{p}(|X_{t_{k-1}}|^p + 1). \tag{3.11}$$

We will now treat the second term in (3.3). Assume $m = 1$. Let $(r,z) \in [0,T] \times [0,1]$. Then, for $r \in [t_{k-1}, t_k]$, the Malliavin derivative of Γ_k^i satisfies the following SDE

$$\begin{aligned}D_{r,z}^l(\Gamma_k^i) &= \sigma_{il}(X_r - Y_r(z)) - \sigma_{il}(X_{t_{k-1}} - Y_{t_{k-1}}(z)) \\ &\quad + \sum_{j=1}^{d} \int_r^{t_k} \int_0^1 D_{r,z}^l(\sigma_{ij}(X_s - Y_s(\alpha)))W^j(d\alpha, ds) \\ &\quad + \int_r^{t_k} \int_0^1 D_{r,z}^l(b_i(X_s - Y_s(\alpha)))d\alpha ds,\ 1 \leq i, l \leq d.\end{aligned}$$

Note that

$$\mathbb{E}[\|D(\Gamma_k^i)\|_{H_{t_{k-1}}}^p \,|\mathcal{F}_{t_{k-1}}] \leq 3^{p-1}(A' + B' + C'), \tag{3.12}$$

where

$$\begin{aligned}A' &:= \mathbb{E}[(\int_{t_{k-1}}^{t_k} \int_0^1 \sum_{l=1}^{d} (\sigma_{il}(X_r - Y_r(z)) - \sigma_{il}(X_{t_{k-1}} - Y_{t_{k-1}}(z)))^2 drdz)^{p/2} |\, \mathcal{F}_{t_{k-1}}], \\ B' &:= \mathbb{E}[(\int_{t_{k-1}}^{t_k} \int_0^1 \sum_{l=1}^{d} (\sum_{j=1}^{d} \int_r^{t_k} \int_0^1 D_{r,z}^l(\sigma_{ij}(X_s - Y_s(\alpha)))W^j(d\alpha, ds))^2 drdz)^{p/2} \,|\mathcal{F}_{t_{k-1}}] \\ C' &:= \mathbb{E}[(\int_{t_{k-1}}^{t_k} \int_0^1 \sum_{l=1}^{d} (\int_r^{t_k} \int_0^1 D_{r,z}^l(b_i(X_s - Y_s(\alpha)))d\alpha ds)^2 drdz)^{p/2} \,|\mathcal{F}_{t_{k-1}}].\end{aligned}$$

For the first term in (3.12), we use (3.9) to get

$$A' \leq C_T \Delta_k^p (|X_{t_{k-1}}|^p + 1). \tag{3.13}$$

For the second term in (3.12), we use Burkholder's and Hölder's inequality for conditional expectations, and the bounds for the derivatives of the coefficients of σ, to get

$$B' \leq C_T \Delta_k^p \sup_{s \in [t_{k-1}, t_k]} \sum_{l,n=1}^d \mathbb{E}[\int_{t_{k-1}}^{t_k} \int_0^1 |D_{r,z}^l (X_s^n)|^p dr dz \,|\mathcal{F}_{t_{k-1}}].$$

We now consider the stochastic differential equation satisfied by the Malliavin derivative of X_{t_k}, that is, for $(r,z) \in [t_{k-1}, t_k] \times [0,1]$,

$$\begin{aligned} D_{(r,z)}^l (X_{t_k}^i) = \sigma_{il}(X_r - Y_r(z)) &+ \int_r^{t_k} \int_0^1 \sum_{j,n=1}^d \partial_n \sigma_{ij}(X_s - Y_s(\alpha)) D_{(r,z)}^l (X_s^n) W^j(d\alpha, ds) \\ &+ \int_r^t \int_0^1 \sum_{n=1}^d \partial_n b_i (X_s - Y_s(\alpha)) D_{(r,z)}^l (X_s^n) d\alpha ds. \end{aligned}$$

Again from Burkholder's and Hölder's inequalities for conditional expectations, the bounds of the derivatives of σ and b, and Gronwall's lemma, we obtain that

$$\sup_{s \in [t_{k-1}, t_k]} \sum_{j,n=1}^d \mathbb{E}[\int_{t_{k-1}}^{t_k} \int_0^1 |D_{r,z}^l (X_s^n)|^p dr dz \,|\mathcal{F}_{t_{k-1}}] \leq C_T (|X_{t_{k-1}}|^p + 1).$$

Therefore, we have that

$$B' \leq C_T \Delta_k^p (|X_{t_{k-1}}|^p + 1). \tag{3.14}$$

Finally, in the same way, using the bounds for the derivative of b, we obtain

$$\begin{aligned} C' &\leq C_T \Delta_k^p \sup_{s \in [t_{k-1}, t_k]} \sum_{n,l=1}^d \mathbb{E}[\int_{t_{k-1}}^{t_k} \int_0^1 |D_{r,z}^l (X_s^n)|^p dr dz \,|\mathcal{F}_{t_{k-1}}] \\ &\leq C_T \Delta_k^p (|X_{t_{k-1}}|^p + 1). \end{aligned} \tag{3.15}$$

Substituting (3.13), (3.14) and (3.15) into (3.12) we conclude that

$$\mathbb{E}[\|D(\Gamma_k^i)\|_{H_{t_{k-1}}}^p \,|\mathcal{F}_{t_{k-1}}] \leq C_T \Delta_k^p (|X_{t_{k-1}}|^p + 1).$$

In order to treat the case $m > 1$ in (3.3) we use the stochastic differential equation satisfied by the iterated derivatives of Γ_k^i, that is, for $(l_1, ..., l_m) \in \{1, ..., d\}^m$, $(\beta_1, ..., \beta_m)$ with $\beta_i = (r_i, z_i) \in [0,T] \times [0,1]$, and $r_1, ..., r_m \in [t_{k-1}, t_k]$,

$$\begin{aligned} D_{\beta_1}^{l_1} \cdots D_{\beta_m}^{l_m} (\Gamma_k^i) = &\sum_{n=1}^m D_{\beta_1}^{l_1} \cdots D_{\beta_{n-1}}^{l_{n-1}} D_{\beta_{n+1}}^{l_{n+1}} \cdots D_{\beta_m}^{l_m} (\sigma_{i\, l_n} (X_{r_n} - Y_{r_n}(z_n)) \\ &+ \sum_{j=1}^d \int_{r_1 \vee \cdots \vee r_m}^{t_k} \int_0^1 D_{\beta_1}^{l_1} \cdots D_{\beta_m}^{l_m} (\sigma_{ij}(X_s - Y_s(\alpha)))\, W^j(d\alpha, ds) \\ &+ \int_{r_1 \vee \cdots \vee r_m}^{t_k} \int_0^1 D_{\beta_1}^{l_1} \cdots D_{\beta_m}^{l_m} (b_i(X_s - Y_s(\alpha)))\, d\alpha ds. \end{aligned}$$

Then, similar arguments as above conclude that, for $j = 1, ..., m$,

$$\mathbb{E}[\|D^{(j)}(\Gamma_k^i)\|^p_{H^{\otimes j}_{t_{k-1}}} |\mathcal{F}_{t_{k-1}}] \leq C_T \Delta_k^p (|X_{t_{k-1}}|^p + 1). \tag{3.16}$$

Finally, substituting (3.11) and (3.16) into (3.3) we conclude the proof of the lemma. $\triangle$

Finally, we will need a lower bound for the determinant of the conditional Malliavin matrix of the random variable $J_k + \rho\Gamma_k$, for $\rho \in (0,1)$. For this we will use a truncating argument using the idea of [2].

Using Proposition 3.1, we have that

$$\begin{aligned}
\det \gamma_{J_k+\rho\Gamma_k}(t_{k-1}) &\geq \left(\frac{1}{2} \inf_{\xi \in \mathbb{R}^d, |\xi|=1} \sum_{l,r=1}^d C_{lr}(J_k)\xi_l\xi_r - \sup_{\xi \in \mathbb{R}^d, |\xi|=1} \|D(\Gamma_k)\cdot\xi\|^2_{H_{t_{k-1}}}\right)^d \\
&\geq \left(\frac{c}{2}\Delta_k - \sup_{\xi \in \mathbb{R}^d, |\xi|=1} \|D(\Gamma_k)\cdot\xi\|^2_{H_{t_{k-1}}}\right)^d,
\end{aligned}$$

where, from Cauchy-Schwarz inequality, for all $\xi \in \mathbb{R}^d$, $|\xi| = 1$,

$$\|D(\Gamma_k)\cdot\xi\|^2_{H_{t_{k-1}}} \leq \int_{t_{k-1}}^{t_k} \int_0^1 \sum_{i,l=1}^d (D^l_{r,z}(\Gamma^i_k))^2 dr dz := V_k.$$

We will now localize on the set where $\Lambda_k := \frac{V_k}{c\Delta_k} \leq 1/4$. Let $\theta \in \mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R}_+)$ with bounded derivatives such that $0 \leq \theta \leq 1$, $\theta(x) = 1$ if $x < 1/8$ and $\theta(x) = 0$ if $x > 1/4$ and we define the random variable

$$Q = \theta(\Lambda_k). \tag{3.17}$$

Note that, on the set $\{\Lambda_k \leq 1/4\}$, we have

$$\det \gamma_{J_k+\rho\Gamma_k} \geq C\Delta_k^d. \tag{3.18}$$

Finally, the next lemma evaluates some Sobolev norms of Q.

Lemma 3.5 *There exists a finite positive constant C_T such that:*

(i) $\|Q\|_{m,p,t_{k-1}} \leq C_T \Delta_k (|X_{t_{k-1}}|^2 + 1)$;

(ii) $\|1 - Q\|_{m,p,t_{k-1}} \leq C_T \Delta_k^{1/2} (|X_{t_{k-1}}| + 1 + \Delta_k^{1/2}(|X_{t_{k-1}}|^2 + 1))$.

Proof. Note that

$$\begin{aligned}
\|Q\|_{m,p,t_{k-1}} &= \{\mathbb{E}[|Q|^p|\mathcal{F}_{t_{k-1}}] + \sum_{j=1}^m \mathbb{E}[\|D^{(j)}(Q)\|^p_{H^{\otimes j}_{t_{k-1}}} |\mathcal{F}_{t_{k-1}}]\}^{1/p} \\
&\leq \{1 + \sum_{j=1}^m \mathbb{E}[\|D^{(j)}(Q)\|^p_{H^{\otimes j}_{t_{k-1}}} |\mathcal{F}_{t_{k-1}}]\}^{1/p}.
\end{aligned}$$

Using the bounds on the derivatives of θ, we have

$$\begin{aligned}
&\mathbb{E}[\|D^{(j)}(Q)\|^p_{H^{\otimes j}_{t_{k-1}}}|\mathcal{F}_{t_{k-1}}] \\
&\leq C\mathbb{E}[\|D^{(j)}(\Lambda_k)\|^p_{H^{\otimes j}_{t_{k-1}}}|\mathcal{F}_{t_{k-1}}] \\
&= C\Delta_k^{-p}\mathbb{E}[\|D^{(j)}(\|D(\Gamma^i_k)\|^2_{H_{t_{k-1}}})\|^p_{H^{\otimes j}_{t_{k-1}}}|\mathcal{F}_{t_{k-1}}] \\
&\leq C\Delta_k^{-p}(j+1)^{p-1}\sum_{l=0}^{j}\binom{j}{l}^p\{(\mathbb{E}[\|D^{(l+1)}(\Gamma^i_k)\|^{2p}_{H^{\otimes(l+1)}_{k-1}}])^{1/2} \\
&\qquad\qquad\qquad\qquad\qquad \times(\mathbb{E}[\|D^{(j-l+1)}(\Gamma^i_k)\|^{2p}_{H^{\otimes(j-l+1)}_{k-1}}])^{1/2}\} \\
&\leq C_T\Delta_k^p(|X_{t_{k-1}}|+1)^{2p}.
\end{aligned}$$

We now evaluate the Sobolev norm of $1-Q$, that is,

$$\|1-Q\|_{m,p,t_{k-1}} = \{\mathbb{E}[|1-Q|^p|\mathcal{F}_{t_{k-1}}] + \sum_{j=1}^m \mathbb{E}[\|D^{(j)}(1-Q)\|^p_{H^{\otimes j}_{t_{k-1}}}|\mathcal{F}_{t_{k-1}}]\}^{1/p}.$$

Since $0\leq Q\leq 1$ and $Q=1$ on the set $\{\Lambda_k\leq 1/8\}$, by Chebychev's inequality,

$$\begin{aligned}
\mathbb{E}[|1-Q|^p|\mathcal{F}_{t_{k-1}}] &\leq \mathbb{P}\{\Lambda_k\geq 1/8|\mathcal{F}_{t_{k-1}}\} \\
&\leq 8^{p/2}\mathbb{E}[\Lambda_k^{p/2}|\mathcal{F}_{t_{k-1}}] \\
&= \frac{8^{p/2}}{(c\Delta_k)^{p/2}}\mathbb{E}[\|D(\Gamma^i_k)\|^p_{H_{t_{k-1}}}|\mathcal{F}_{t_{k-1}}] \\
&\leq C_T\Delta_k^{p/2}(|X_{t_{k-1}}|+1)^p.
\end{aligned}$$

As

$$\begin{aligned}
\mathbb{E}[\|D^{(j)}(1-Q)\|^p_{H^{\otimes j}_{t_{k-1}}}|\mathcal{F}_{t_{k-1}}] &= \mathbb{E}[\|D^{(j)}(Q)\|^p_{H^{\otimes j}_{t_{k-1}}}|\mathcal{F}_{t_{k-1}}] \\
&\leq C_T\Delta_k^p(|X_{t_{k-1}}|+1)^{2p},
\end{aligned}$$

we conclude the desired bound. △

3.2 Proof of the lower bound of Theorem 1.5

Let $t>0$ and $v\in\mathbb{R}^d$ be fixed, and assume that hypotheses of Theorem 1.4 (2) and hypothesis (H) of Theorem 1.5 are satisfied for a random variable X_0. The aim of this section is to prove the lower bound of Theorem 1.5 for the density of the solution of the Landau SDE $f_{X_0}(t,v)$.

As explained in the introduction, we start by describing the path we will choose to define some evaluation tubes. We consider the straight line linking X_0 at time 0 and v at time t, that is,

$$x(s,\omega)=\frac{s}{t}v+\frac{t-s}{t}X_0(\omega).$$

Then $x:[0,t]\times\Omega\mapsto\mathbb{R}^d$ is a continuous differentiable function, measurable with respect to $\mathcal{F}_0$, such that $x(0)=X_0$ and $x(t)=v$.

We consider the time grid defined at the start of this Section. The tubes A_k are defined, for $k = 1, ..., N$, as

$$A_k := \{\omega : |X_{t_{i-1}}(\omega) - x(t_i)| < \frac{\sqrt{c\Delta_k}}{2}, i = 1, ..., k\} \in \mathcal{F}_{t_{k-1}},$$

where c is the constant obtained in Proposition 3.1.

We will start by proving a lower bound for the conditional density of the random variable X_{t_k} (from the evolution sequence (3.1)) given $\mathcal{F}_{t_{k-1}}$ on A_k. Note that this conditional density exists and from Watanabe's notation can be written as $\mathbb{E}[\delta_z(X_{t_k})|\mathcal{F}_{t_{k-1}}]$, where δ_z denotes the Dirac function at the point $z \in I\!R^d$.

We will work with the following approximation of δ. Let $\phi \in \mathcal{C}_b^\infty(I\!R^d)$, $0 \le \phi \le 1$, $\int \phi = 1$ and $\phi(x) = 0$ for $|x| > 1$. Let

$$\phi_\eta(x) = \eta^{-d}\phi(\eta^{-1}x).$$

Remark that $\phi_\eta(x) = 0$ for $|x| > \eta$.

Then, our goal is to find a lower bound for the quantity $\mathbb{E}[\phi_\eta(X_{t_k} - z)\,|\mathcal{F}_{t_{k-1}}]$ on the tube A_k, independent of η.

Let Q be the random variable defined in (3.17). Let us apply the mean value theorem. We have

$$\begin{aligned}
&\mathbb{E}[\phi_\eta(X_{t_k} - z)\,|\mathcal{F}_{t_{k-1}}] = \mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k + \Gamma_k - z)\,|\mathcal{F}_{t_{k-1}}]\\
&\ge\ \mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k + \Gamma_k - z)Q\,|\mathcal{F}_{t_{k-1}}]\\
&\ge\ \mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)Q\,|\mathcal{F}_{t_{k-1}}] + \sum_{i=1}^d \int_0^1 \mathbb{E}\left[\frac{\partial \phi_\eta}{\partial x^i}(X_{t_{k-1}} + J_k - z + \rho\Gamma_k)\Gamma_k^i Q \,\middle|\, \mathcal{F}_{t_{k-1}}\right] d\rho\\
&\ge\ \mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)Q\,|\mathcal{F}_{t_{k-1}}] - \left|\sum_{i=1}^d \int_0^1 \mathbb{E}\left[\frac{\partial \phi_\eta}{\partial x^i}(X_{t_{k-1}} + J_k - z + \rho\Gamma_k)\Gamma_k^i Q \,\middle|\, \mathcal{F}_{t_{k-1}}\right] d\rho\right|
\end{aligned} \tag{3.19}$$

In order to obtain a lower bound for $\mathbb{E}[\phi_\eta(X_{t_k} - z)\,|\mathcal{F}_{t_{k-1}}]$ on A_k we use the method in [9] which consists in finding a lower bound for the first (Gaussian) term in (3.19), and an upper bound for the second term by the use of integration by parts formula. Then, it suffices to choose the meshes Δ_k sufficiently small in order to have a positive lower bound.

In the same spirit, in order to obtain the lower bound for $\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)Q\,|\mathcal{F}_{t_{k-1}}]$ on A_k, we write

$$\begin{aligned}
&\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)Q\,|\mathcal{F}_{t_{k-1}}]\\
&= \mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)\,|\mathcal{F}_{t_{k-1}}] - \mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)(1 - Q)\,|\mathcal{F}_{t_{k-1}}].
\end{aligned} \tag{3.20}$$

Again we will need to find a lower bound for the first term in and an upper bound for the second term in (3.20) and choose Δ_k sufficiently small in order to obtain a positive lower bound.

We start by establishing a lower bound for $\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)\,|\mathcal{F}_{t_{k-1}}]$ on A_k. For this, we will use the estimates of the eigenvalues of the conditional covariance matrix of J_k obtained in Section 4.1.

Proposition 3.6 *Assume* $0 < \eta \le \sqrt{c\Delta_k}$, *and choose* $z \in I\!R^d$ *such that*

$$|x(t_k) - z| \le \frac{\sqrt{c\Delta_k}}{2}.$$

Then there exists a constant $C_T > 0$ *not depending on* k, *such that, on* A_k,

$$\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)|\mathcal{F}_{t_{k-1}}] \ge \frac{1}{\Delta_k^{d/2} c_1(z)}. \tag{3.21}$$

where $c_1(z) = C_T(|z|^2 + 1)^{d/2}$.

Proof. Note firstly that using Proposition 3.2 and the assumption on z, we has

$$\begin{aligned}\sup_{\xi \in I\!R^d, |\xi|=1} \sum_{l,r=1}^d C_{lr}(J_k)\xi_l \xi_r &\le C\Delta_k(|X_{t_{k-1}}|^2 + 1) \le C\Delta_k(|z|^2 + c\Delta_k + 1) \\ &\le C_T \Delta_k(|z|^2 + 1), \end{aligned}\tag{3.22}$$

since $\Delta_k \le T$.

As J_k is Gaussian, we have, on A_k,

$$\begin{aligned}&\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)|\mathcal{F}_{t_{k-1}}] \\ &= \int_{I\!R^d} \phi_\eta(X_{t_{k-1}} + x - z)\frac{1}{(2\pi)^{d/2}\det(C(J_k))^{1/2}} \exp\left(-\frac{x^* C(J_k)^{-1} x}{2}\right) dx \\ &= \int_{I\!R^d} \phi_\eta(\tilde{z})\frac{1}{(2\pi)^{d/2}\det(C(J_k))^{1/2}} \exp\left(-\frac{(\tilde{z} + z - X_{t_{k-1}})^* C(J_k)^{-1}(\tilde{z} + z - X_{t_{k-1}})}{2}\right) d\tilde{z}.\end{aligned}$$

Since $|\tilde{z}| \le \eta \le \sqrt{c\Delta_k}$, and using the assumption on z, we have that, on A_k,

$$|\tilde{z} + z - X_{t_{k-1}}|^2 \le 3(|\tilde{z}|^2 + |z - x(t_k)|^2 + |x(t_k) - X_{t_{k-1}}|^2) \le \frac{9}{2}c\Delta_k.$$

Then, using Propositions 3.1 and 3.2 and (3.22), we obtain

$$\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)|\mathcal{F}_{t_{k-1}}] \ge \frac{1}{C_T \Delta_k^{d/2}(|z|^2 + 1)^{d/2}}.$$

△

The next result gives an upper bound for the second term in (3.20) which uses the integration by parts formula from the Malliavin calculus and some of the Sobolev norm estimates obtained in Section 4.1.

Proposition 3.7 *Choose* $z \in I\!R^d$ *such that* $|x(t_k) - z| \le \frac{\sqrt{c\Delta_k}}{2}$. *Then, there exists a constant* $C_T > 0$ *not depending on* k *such that, on* A_k,

$$\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)(1 - Q)\ |\mathcal{F}_{t_{k-1}}] \le c_2(z)\Delta_k^{(1-d)/2},$$

where $c_2(z) := C_T(1 + |z| + |z|^2)(1 + |z|)^{2d^2 - d}$.

Proof. Note that from the assumption on z, and from the estimates of Lemmas 3.3 and 3.5 (ii), we have that

$$\begin{aligned} \|D(J_k^i)\|_{m,p,t_{k-1}} &\leq C\Delta_k^{1/2}(|X_{t_{k-1}}|+1) \leq C\Delta_k^{1/2}(|z|+c\Delta_k+1) \\ &\leq C_T\Delta_k^{1/2}(|z|+1), \end{aligned} \tag{3.23}$$

as $\Delta_k \leq T$, and

$$\|1-Q\|_{m,p,t_{k-1}} \leq C_T\Delta_k^{1/2}(1+|z|+|z|^2). \tag{3.24}$$

Now, in order to apply the integration by parts formula, define

$$\Phi_\eta(x) = \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_d}\phi_\eta(u)du, \quad x \in I\!R^d.$$

By the conditional version of the integration by parts formula (Proposition 2.6), we have

$$\mathbb{E}[\phi_\eta(X_{t_{k-1}}+J_k-z)(1-Q)\,|\mathcal{F}_{t_{k-1}}] = \mathbb{E}[\Phi_\eta(X_{t_{k-1}}+J_k-z)H^{t_{k-1}}_{(1,\dots,d)}(J_k,1-Q)\,|\mathcal{F}_{t_{k-1}}].$$

As $\int\phi_\eta = 1$, by the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}[\phi_\eta(X_{t_{k-1}}+J_k-z)(1-Q)\ |\mathcal{F}_{t_{k-1}}] \leq \|H^{t_{k-1}}_{(1,\dots,d)}(J_k,1-Q)\|_{0,2,t_{k-1}}.$$

Using the estimates of Proposition 3.1, (3.23) and (3.24), in addition with Proposition 2.7, we obtain

$$\|H^{t_{k-1}}_{(1,\dots,d)}(J_k,1-Q)\|_{0,2,t_{k-1}} \leq C_T\Delta_k^{(1-d)/2}(1+|z|+|z|^2)(1+|z|)^{2d^2-d}.$$

$\triangle$

The next result gives an upper bound for the second term in (3.19) again with the use of integration by parts formulas and some of the Sobolev norms estimates of Section 4.1.

Proposition 3.8 *Choose $z \in I\!R^d$ such that $|x(t_k)-z| \leq \frac{\sqrt{c\Delta_k}}{2}$. Then there exists a constant $C_T > 0$ independent of k such that, on A_k,*

$$\left|\mathbb{E}\left[\frac{\partial\phi_\eta}{\partial x^i}(X_{t_{k-1}}+J_k-z+\rho\Gamma_k)\Gamma_k^iQ\ \middle|\mathcal{F}_{t_{k-1}}\right]\right| \leq c_3(z)\Delta_k^{(1-d)/2},$$

where $c_3(z) := C_T(|z|+1)^{2d^2+d+2}$.

Proof. Let Φ_η as in Proposition 3.7 so that

$$\frac{\partial\phi_\eta}{\partial x^i}(X_{t_{k-1}}+J_k-z+\rho\Gamma_k) = \frac{\partial^{d+1}\Phi_\eta}{\partial x^i\partial x^1\cdots\partial x^d}(X_{t_{k-1}}+J_k-z+\rho\Gamma_k).$$

By Proposition 2.6,

$$\begin{aligned} &\mathbb{E}\left[\frac{\partial\phi_\eta}{\partial x^i}(X_{t_{k-1}}+J_k-z+\rho\Gamma_k)\Gamma_k^iQ\ \middle|\mathcal{F}_{t_{k-1}}\right] \\ &\qquad = \mathbb{E}[\Phi_\eta(X_{t_{k-1}}+J_k-z+\rho\Gamma_k)H^{t_{k-1}}_{(1,\dots,d,i)}(J_k+\rho\Gamma_k,\Gamma_k^iQ)\ |\mathcal{F}_{t_{k-1}}]. \end{aligned}$$

As $\int \phi_\eta = 1$, by the Cauchy-Schwarz inequality, we obtain

$$\left|\mathbb{E}\left[\frac{\partial \phi_\eta}{\partial x^i}(X_{t_{k-1}} + J_k - z + \rho\Gamma_k)\Gamma_k^i Q \;\middle|\mathcal{F}_{t_{k-1}}\right]\right| \leq \|H^{t_{k-1}}_{(1,\ldots,d,i)}(J_k + \rho\Gamma_k, \Gamma_k^i Q)\|_{0,2,t_{k-1}}.$$

Finally, from Proposition 2.7 and the estimates of Lemmas 3.3, 3.4 and 3.5 and (3.18), we obtain

$$\begin{aligned}
&\|H^{t_{k-1}}_{(1,\ldots,d,i)}(J_k + \rho\Gamma_k, \Gamma_k^i Q)\|_{0,2,t_{k-1}} \\
\leq\ & C_T\Delta_k(|z|+1)^3((\Delta_k^{1/2}(|z|+1))^{2(d+1)^2(d-1)} + (\Delta_k(|z|+1))^{2(d+1)^2(d-1)}) \times \Delta_k^{-d(d+1)} \\
\leq\ & C_T\Delta_k^{(1-d)/2}(|z|+1)^{2d^2+d+2}.
\end{aligned}$$

$\triangle$

We will now chose Δ_k sufficiently small to arrive at the proof of the lower bound for the approximation of the conditional density of the random variable X_{t_k} given $\mathcal{F}_{t_{k-1}}$ on A_k.

Proposition 3.9 *Let $v \in I\!R^d$ be fixed at the start of the Section. There exists a finite constant $C_T > 0$ such that if we choose Δ_k satisfying*

$$\sqrt{\Delta_k} < \frac{1}{C_T c(v, X_0)}, \tag{3.25}$$

with

$$c(v, X_0) = 1 + |v|^{2(d^2+d+1)} + |X_0|^{2(d^2+d+1)},$$

then, for fixed $0 < \eta \leq \sqrt{c\Delta_k}$, and all $z \in I\!R^d$ such that $|x(t_k) - z| \leq \sqrt{c\Delta_k}/2$, we have, on A_k,

$$\mathbb{E}[\phi_\eta(X_{t_k} - z)\,|\mathcal{F}_{t_{k-1}}] \geq \frac{1}{4c_1(z)\Delta_k^{d/2}},$$

where $c_1(z)$ is the constant obtained in Proposition 3.6.

Proof. We first note that, on the set $\{|z - x(t_k)| \leq \sqrt{c\Delta_k}/2\}$, and using the fact that $|x(t_k) - v| \leq |v - X_0|$ and $\Delta_k \leq T$, we have, for all $q \geq 1$,

$$\begin{aligned}
|z|^q &\leq 3^{q-1}(|z - x(t_k)|^q + |x(t_k) - v|^q + |v|^q) \\
&\leq C_T(1 + |v|^q + |X_0|^q).
\end{aligned}$$

Then, using the last inequality, one can easily see that there exists a finite positive constant C_T independent of z such that, if we assume condition (3.25) with this constant, we have

$$\sqrt{\Delta_k} < \frac{1}{4c_1(z)c_3(z)} \wedge \frac{1}{2c_1(z)c_2(z)}, \tag{3.26}$$

where $c_1(z), c_2(z), c_3(z)$ are the constants obtained in Propositions 3.6, 3.7 and 3.8.

Now, substituting the lower bound of Proposition 3.6 and the upper bound of Proposition 3.7 into (3.20) and using condition (3.26) on Δ_k, it yields

$$\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)Q|\mathcal{F}_{t_{k-1}}] \geq \left(\frac{1}{c_1(z)\Delta_k^{d/2}} - \frac{c_2(z)\Delta_k^{1/2}}{\Delta_k^{d/2}}\right) \geq \frac{1}{2c_1(z)\Delta_k^{d/2}}.$$

Finally, applying this lower bound and the upper bound of Proposition 3.8 into (3.19), and again using condition (3.26), we obtain

$$\mathbb{E}[\phi_\eta(X_{t_k}-z)\,|\mathcal{F}_{t_{k-1}}] \geq \left(\frac{1}{2c_1(z)\Delta_k^{d/2}} - \frac{c_3(z)\Delta_k^{1/2}}{\Delta_k^{d/2}}\right) \geq \frac{1}{4c_1(z)\Delta_k^{d/2}}.$$

△

The next step is now to find a large deviation evaluation for the probability of the event A_N conditioning on X_0. For this, we will use a recursive argument in the same way as in [2]. We choose

$$\Delta := \Delta_k = \Delta_{k-1}, \quad \text{for } k = 1, ..., N-1,$$

and $\Delta_N = t - t_{N-1}$ such that $\Delta_N \leq \Delta$. In particular, note that $\Delta \leq t$ and

$$N \leq \left[\frac{t}{\Delta}\right] + 1,$$

where $[x]$ denotes the integer part of x. We will then choose Δ sufficiently small and use the lower bounds from Proposition 3.9 in a recursive way in order to obtain a lower bound for the probabilities $\mathbb{P}_{X_0}(A_N)$.

Proposition 3.10 *Let $v \in \mathbb{R}^d$ be fixed at the start of this Section. Choose Δ defined above such that*

$$\sqrt{\Delta} < \frac{1}{C_T c(v, X_0)} \wedge \frac{t\sqrt{c}}{4|v - X_0|}, \tag{3.27}$$

where C_T and $c(v, X_0)$ are the constant and function obtained in Proposition 3.9. Then, there exists a finite constant $C_T > 0$ such that

$$\mathbb{P}_{X_0}(A_N) \geq e^{-N \ln(C_T(1+|v|^2+|X_0|^2)^{d/2})}, \quad P_0\text{-}a.s.$$

Proof. Note that condition (3.27) implies that, for $k = 1, ..., N-1$,

$$|x(t_k) - x(t_{k-1})| = \frac{\Delta}{t}|v - X_0| \leq \frac{\sqrt{c\Delta}}{4}. \tag{3.28}$$

Let $0 < \eta < \frac{\sqrt{c\Delta}}{4}$. As $A_k = A_{k-1} \cap \{|X_{t_{k-1}} - x(t_k)| \leq \frac{\sqrt{c\Delta}}{2}\}$ and using the fact that $\int \phi_\eta = 1$ and that Lebesgue measure is invariant to translations, we have, for $k = 1, ..., N-1$,

$$\begin{aligned}
& \mathbb{P}_{X_0}(A_k) \\
= \; & \mathbb{E}_{X_0}[1_{A_{k-1}}\mathbb{E}[1_{\{|X_{t_{k-1}}-x(t_k)|\leq \frac{\sqrt{c\Delta}}{2}\}}|\mathcal{F}_{t_{k-2}}]] \\
= \; & \mathbb{E}_{X_0}[1_{A_{k-1}}\int_{I\!R^d}\mathbb{E}[\phi_\eta(X_{t_{k-1}}-z)1_{\{|X_{t_{k-1}}-x(t_k)|\leq \frac{\sqrt{c\Delta}}{2}\}}|\mathcal{F}_{t_{k-2}}]dz] \\
\geq \; & \mathbb{E}_{X_0}[1_{A_{k-1}}\int_{|z-x(t_{k-1})|\leq\sqrt{c\Delta}/4-\eta}\mathbb{E}[\phi_\eta(X_{t_{k-1}}-z)1_{\{|X_{t_{k-1}}-x(t_k)|\leq \frac{\sqrt{c\Delta}}{2}\}}|\mathcal{F}_{t_{k-2}}]dz] \\
= \; & \mathbb{E}_{X_0}[1_{A_{k-1}}\int_{|z-x(t_{k-1})|\leq\sqrt{c\Delta}/4-\eta}\mathbb{E}[\phi_\eta(X_{t_{k-1}}-z)|\mathcal{F}_{t_{k-2}}]dz].
\end{aligned}$$

The last equality follows from the fact that

$$\begin{aligned}
|X_{t_{k-1}} - x(t_k)| &\leq |X_{t_{k-1}} - z| + |z - x(t_{k-1})| + |x(t_{k-1}) - x(t_k)| \\
&\leq \eta + \frac{\sqrt{c\Delta}}{4} - \eta + \frac{\sqrt{c\Delta}}{4} = \frac{\sqrt{c\Delta}}{2}.
\end{aligned}$$

Take $\eta = \frac{\sqrt{c\Delta}}{8}$. Using condition on Δ and Proposition 3.9 we obtain, for $k = 1, ..., N-1$,

$$\begin{aligned}
\mathbb{P}_{X_0}(A_k) &\geq \mathbb{E}_{X_0}[1_{A_{k-1}} \int_{|z-x(t_{k-1})|\leq\sqrt{c\Delta}/8} \mathbb{E}[\phi_\eta(X_{t_{k-1}} - z)|\mathcal{F}_{t_{k-2}}]dz] \\
&\geq \mathbb{E}_{X_0}[1_{A_{k-1}} \int_{|z-x(t_{k-1})|\leq\sqrt{c\Delta}/8} \frac{1}{C_T \Delta^{d/2}(1+|z|^2)^{d/2}} dz].
\end{aligned}$$

Note that, on the set $\{|z - x(t_{k-1})| \leq \sqrt{c\Delta}/8\}$, and using the fact that $|x(t_{k-1}) - v| \leq |v - X_0|$ and $\Delta \leq T$, we have

$$\begin{aligned}
|z|^2 &\leq 3(|z - x(t_{k-1})|^2 + |x(t_{k-1}) - v|^2 + |v|^2) \\
&\leq C_T(1 + |v|^2 + |X_0|^2).
\end{aligned}$$

Hence, we finally get that, for $k = 1, ..., N-1$,

$$\begin{aligned}
\mathbb{P}_{X_0}(A_k) &\geq \frac{1}{C_T \Delta^{d/2}(1+|v|^2+|X_0|)^{d/2}} \lambda(\{|z - x(t_{k-1})| \leq \sqrt{c\Delta}/8\}) \mathbb{P}_{X_0}(A_{k-1}) \\
&= \frac{1}{C_T(1+|v|^2+|X_0|)^{d/2}} \mathbb{P}_{X_0}(A_{k-1}),
\end{aligned}$$

where $\lambda(E)$ denotes Lebesgue measure of $E \subset I\!R^d$.

Iterating this process and using the fact that $\mathbb{P}_{X_0}(A_1) = 1$ P_0-a.s. we obtain that P_0-a.s.

$$\mathbb{P}_{X_0}(A_N) \geq \left(\frac{1}{C_T(1+|v|^2+|X_0|)^{d/2}}\right)^{N-1} \mathbb{P}_{X_0}(A_1) \geq e^{-N\ln(C_T(1+|v|^2+|X_0|^2)^{d/2})},$$

which concludes the desired bound. △

We are now able to prove the lower bound of Theorem 1.5. For this, we write $X_t = X_{t_N}$ and we consider the approximation of the (conditional) density $f_{X_0}(t, v)$ given by

$$\mathbb{E}[\phi_\eta(X_{t_N} - v)|X_0] \geq \mathbb{E}[\mathbb{E}[\phi_\eta(X_{t_N} - v)|\mathcal{F}_{t_{N-1}}]1_{A_N}|X_0].$$

We now choose Δ such that

$$\Delta = \frac{1}{2}\left(\frac{1}{C_T^2 c^2(v, X_0)} \wedge \frac{t^2 c}{16|v - X_0|^2} \wedge t\right),$$

where C_T and $c(v, X_0)$ are the constant and function from Proposition 3.9.

We then apply Proposition 3.9 with $k = N$ and $z = v$ and Proposition 3.10, we use the fact that $\Delta_N \leq \Delta$, and we let η tend to zero to finally get that P_0-a.s.

$$f_{X_0}(t, v) \geq \frac{1}{C_T \Delta^{d/2}(1+|v|^2)^{d/2}} e^{-N\ln(C_T(1+|v|^2+|X_0|^2)^{d/2})}.$$

We finally use the fact that $N \leq \frac{t}{\Delta} + 1$ to ultimately conclude that P_0-a.s.

$$f_{X_0}(t, v) \geq \frac{1}{C_T \Delta^{d/2}(1+|v|^2)^{d/2}} e^{(-t/\Delta - 1)\ln(C_T(1+|v|^2+|X_0|^2)^{d/2})},$$

which proves the lower bound of Theorem 1.5. △

4 The upper bound

In this section we prove the upper bound of Theorem 1.5. Let $t > 0$ and $v \in I\!R^d$ be fixed and assume that the hypotheses of Theorem 1.4 (2) hold. Let $f_{X_0}(t,v)$ denote the conditional density of the solution of the Landau SDE given X_0.

The proof of the upper bound for $f_{X_0}(t,v)$ is classical: apply the Cauchy-Schwarz inequality to the expression of the density obtained in Corollary 2.3 to get

$$f_{X_0}(t,v) \leq \mathbb{P}_{X_0}\{|X_t| \geq |v|\}^{1/2} \mathbb{E}[(H^0_{(1,\dots,d)}(X_t,1))^2 \ |X_0]^{1/2}, \quad P_0\text{-a.s.} \tag{4.1}$$

Then, one evaluates the factor $\mathbb{P}_{X_0}\{|X_t| \geq |y|\}$ using an exponential martingale inequality in order to get a large deviation type bound. In order to obtain an upper bound for the second factor $\mathbb{E}[|H_{(1,\dots,d)}(X_t,1)|^2 \ |X_0]^{1/2}$ and to get the factor $t^{-d/2}$ of the upper bound of Theorem 1.5, we will use precise estimates on the Sobolev norms of X_t similar of those obtained in [6], in addition with Proposition 2.7.

This is given in the following two lemmas.

Lemma 4.1 *Under the hypotheses of Theorem 1.3, there exist finite constants $c, C > 0$ such that P_0-a.s.*

$$\mathbb{P}_{X_0}\{|X_t| \geq |v|\} \leq \exp\left(-\frac{(\ln(1+|v|^2) - \ln(1+|X_0|^2) - Ct)^2}{ct}\right),$$

for all $t \in (0,T]$ and $v \in I\!R^d$.

Proof. Consider $Z_t = \ln(1+|X_t|^2)$. From the d-dimensional Itô's formula,

$$\begin{aligned} Z_t = \ln(1+|X_0|^2) &+ \int_0^t \int_0^1 \sum_{i,j=1}^d \frac{2X_s^i}{1+|X_s|^2} \sigma_{ij}(X_s - Y_s(\alpha)) W^j(d\alpha, ds) \\ &+ \int_0^t \int_0^1 \sum_{i=1}^d \frac{2X_s^i}{1+|X_s|^2} b_i(X_s - Y_s(\alpha)) d\alpha ds \\ &+ \int_0^t \int_0^1 \sum_{i,j=1}^d \frac{1}{1+|X_s|^2} (\sigma_{ij}(X_s - Y_s(\alpha)))^2 d\alpha ds \\ &- \int_0^t \int_0^1 \sum_{i,j,k=1}^d \frac{2X_s^i X_s^k}{(1+|X_s|^2)^2} \sigma_{ij}(X_s - Y_s(\alpha)) \sigma_{kj}(X_s - Y_s(\alpha)) d\alpha ds. \end{aligned}$$

Using the Lipschitz property of b and the fact that $X \in \mathcal{P}_2$, we have

$$\begin{aligned} \left| \int_0^t \int_0^1 \sum_{i=1}^d \frac{2X_s^i b_i(X_s - Y_s(\alpha))}{1+|X_s|^2} d\alpha ds \right| &\leq 2K_b t + K_b \int_0^t \int_0^1 |Y_s(\alpha)| d\alpha ds \\ &\leq 2K_b t + K_b t \mathbb{E}[\sup_{0\leq s\leq T} |X_s|] \\ &\leq c_1 t. \end{aligned}$$

Equally, from the Lipschitz property of σ and the fact that $X \in \mathcal{P}_2$,

$$\left| \int_0^t \int_0^1 \sum_{i,j=1}^d \frac{1}{1+|X_s|^2} (\sigma_{ij}(X_s - Y_s(\alpha)))^2 d\alpha ds \right| \leq c_2 t,$$

and

$$\left|\int_0^t \int_0^1 \sum_{i,j,k=1}^d \frac{2X_s^i X_s^k}{(1+|X_s|^2)^2}\sigma_{ij}(X_s - Y_s(\alpha))\sigma_{kj}(X_s - Y_s(\alpha))d\alpha ds\right| \leq c_3 t.$$

Hence, we obtain

$$\begin{aligned}\mathbb{P}_{X_0}\{|X_t| \geq |v|\} &\leq \mathbb{P}_{X_0}\{Z_t \geq \ln(1+|v|^2)\}\\ &\leq \mathbb{P}_{X_0}\{M_t \geq \ln(1+|v|^2) - \ln(1+|X_0|^2) - Ct\}, \end{aligned} \tag{4.2}$$

where $C := c_1 + c_2 + c_3$ and

$$M_t = \int_0^t \int_0^1 \sum_{i,j=1}^d \frac{2X_s^i}{1+|X_s|^2}\sigma_{ij}(X_s - Y_s(\alpha))W^j(d\alpha, ds)$$

is a continuous martingale with respect to $\mathcal{F}_t$ and with increasing process given by

$$\langle M\rangle_t = \int_0^t \int_0^1 \sum_{j=1}^d \Big(\sum_{i=1}^d \frac{2X_s^i}{1+|X_s|^2}\sigma_{ij}(X_s - Y_s(\alpha))\Big)^2 d\alpha ds.$$

Again, using the Lipschitz property of σ and the fact that $X \in \mathcal{P}_2$, we get that

$$\langle M\rangle_t \leq ct.$$

Finally, applying the exponential martingale inequality to (4.2), we obtain that P_0-a.s.

$$\mathbb{P}_{X_0}\{|X_t| \geq |v|\} \leq \exp\Big(-\frac{(\ln(1+|v|^2) - \ln(1+|X_0|^2) - Ct)^2}{2ct}\Big).$$

△

Lemma 4.2 *Under the hypotheses of Theorem 1.4* (2), *there exists a finite constant $C_T > 0$ such that P_0-a.s.*

$$\mathbb{E}[(H^0_{(1,\dots,d)}(X_t,1))^2 \; |X_0]^{1/2} \leq C_T t^{-d/2},$$

for all $t \in (0,T]$.

Proof. In order to prove this result, it suffices to prove that for any $p > 1$ and $k \geq 1$ there exist finite constants $c_1(p,T) > 0$ and $c_2(k,p,T) \geq 0$ such that

(i) $\mathbb{E}[(\det \gamma_{X_t}(0))^{-p} \; |X_0]^{1/p} \leq c_1 t^{-d}$;

(ii) $\mathbb{E}[\|D^{(k)}(X_t^i)\|^p_{H_0^{\otimes k}} \; |X_0]^{1/p} \leq c_2 t^{1/2}, \; i = 1, ..., d.$

Then, Proposition 2.7 with $s = 0$ and $G = 1$ concludes the desired estimate.

In order to prove (i) we follow the proof of [6, Theorem 19]. Fix $\epsilon \in (0, 1/2]$ so that $t/2 \leq t(1-\epsilon) < t$. From the proof of [6, Theorem 19] it follows that

$$\begin{aligned}(\det \gamma_{X_t}(0))^{1/d} &\geq \inf_{\xi \in I\!R^d, |\xi|=1} \langle \gamma_{X_t}(0)\xi, \xi\rangle\\ &\geq \frac{2}{3}m\tilde{c}t\epsilon - 2\sup_{\xi \in I\!R^d, |\xi|=1} I,\end{aligned}$$

where m is the lower bound of the function h appearing in the Landau equation, $\tilde{c}$ denotes the infimum of the function $F(\xi, r)$ defined in (3.2) on the compact set $\{r \in [\frac{t}{2}, t]\} \times \{\xi \in I\!R^d : |\xi| = 1\}$, and

$$I := \sum_{k=1}^{d} \int_{t(1-\epsilon)}^{t} \int_0^1 \Big\{ \sum_{i=1}^{d} \xi_i \int_r^t \int_0^1 \sum_{j,l=1}^{d} \partial_l \sigma_{ij}(X_s - Y_s(\alpha)) D^k_{(r,z)}(X^l_s) W^j(d\alpha, ds)$$
$$+ \sum_{i=1}^{d} \xi_i \int_r^t \int_0^1 \sum_{l=1}^{d} \partial_l b_i(X_s - Y_s(\alpha)) D^k_{(r,z)}(X^l_s) d\alpha ds \Big\}^2 dz dr.$$

We now choose $y > 0$ such that $\frac{2}{3} m\tilde{c}t\epsilon = 3y^{-1/d}$, and notice that since $\epsilon \leq 1/2$ we have that $y \geq 9^d (m\tilde{c}t)^{-d} := k$. In addition, as ϵ varies in $(0, 1/2]$, y varies in $[k, \infty)$.

By Chebyshev's inequality, for $q \geq 2$.

$$\begin{aligned} \mathbb{P}_{X_0}\{\det\gamma_{X_t}(0) < \frac{1}{y}\} &\leq \mathbb{P}_{X_0}\{(\frac{2}{3}m\tilde{c}t\epsilon - 2 \sup_{\xi \in I\!R^d, |\xi|=1} I) < y^{-1/d}\} \\ &\leq \mathbb{P}_{X_0}\{ \sup_{\xi \in I\!R^d, |\xi|=1} I > y^{-1/d}\} \\ &\leq y^{q/d} \mathbb{E}[\sup_{\xi \in I\!R^d, |\xi|=1} |I|^q \, |X_0]. \end{aligned}$$

Again following the proof of [6, Theorem 19], using Burkholder's and Hölder's inequalities, we have that

$$\begin{aligned} \mathbb{E}[\sup_{\xi \in I\!R^d, |\xi|=1} |I|^q \, |X_0] &\leq c_{d,q,T} (t\epsilon)^{2q-1} \sup_{0 \leq s \leq T} \sum_{k,l=1}^{d} \mathbb{E}[\int_0^s \int_0^1 |D^k_{(r,z)}(X^l_s)|^{2q} dz dr \, |X_0] \\ &\leq c_{d,q,T} (t\epsilon)^{2q-1}. \end{aligned}$$

In the last inequality we have used the recurrence hypothesis (ii) in the proof of [6, Theorem 11]. Thus, by the definition of y, we obtain

$$\mathbb{E}[\sup_{\xi \in I\!R^d, |\xi|=1} |I|^q \, |X_0] \leq c_{d,q,T} 9^{2q-1} (2m\tilde{c})^{1-2q} y^{\frac{1-2q}{d}}.$$

Consequently, taking $q > pd - 1$, we get

$$\begin{aligned} \mathbb{E}[(\det \gamma_{X_t}(0))^{-p} \, |X_0] &\leq k^p + p \int_k^\infty y^{p-1} \mathbb{P}_{X_0}\{\det \gamma_{X_t}(0) < \frac{1}{y}\} \, dy \\ &\leq 9^{dp} (m\tilde{c})^{-dp} t^{-dp} + p \int_k^\infty y^{p-1+q/d} \mathbb{E}[\sup_{\xi \in I\!R^d, |\xi|=1} |I|^q \, |X_0] \, dy \\ &\leq 9^{dp} (m\tilde{c})^{-dp} t^{-dp} + c_{d,p,T} 9^{2q-1} (2m\tilde{c})^{1-2q} \int_k^\infty y^{p-1-q/d+1/d} dy \\ &\leq c_{d,p,T} t^{-dp}, \end{aligned}$$

which concludes the proof of (i).

We now prove (ii). For $k=1$, we consider the stochastic differential equation satisfied by the derivative (c.f. [6, Theorem 11]), that is, for $r \le t$,

$$D^l_{(r,z)}(X^i_t) = \sigma_{il}(X_r - Y_r(z)) + \int_r^t \int_0^1 \sum_{j,k=1}^d \partial_k \sigma_{ij}(X_s - Y_s(\alpha)) D^l_{(r,z)}(X^k_s) W^j(d\alpha, ds)$$
$$+ \int_r^t \int_0^1 \sum_{k=1}^d \partial_l b_i(X_s - Y_s(\alpha)) D^l_{(r,z)}(X^k_s) d\alpha ds.$$

Then, using Burkholder's and Hölder's inequality for conditional expectations, the Lipschitz property of σ and the bounds from the derivatives of σ and b we obtain,

$$\mathbb{E}[\|D(X^i_t)\|^p_{H_0} \;|X_0] = \mathbb{E}[(\int_0^t \int_0^1 \sum_{l=1}^d (D^l_{(r,z)}(X^i_t))^2 dr dz)^{p/2} \;|X_0]$$
$$\le c_T t^{p/2}(1 + \sup_{0\le t\le T} \sum_{k,l=1}^d \mathbb{E}[\int_0^t \int_0^1 |D^l_{(r,z)}(X^k_t)|^p dr dz \;|X_0]).$$

Then, the recurrence hypothesis (ii) in the proof of [6, Theorem 11] concludes the proof of (ii) for $k=1$. The case $k>1$ follows along the same lines using the stochastic differential equation satisfies by the iterated derivative, that is, for $(l_1,...,l_k) \in \{1,...,d\}^k$ and $(\beta_1,...,\beta_k)$ with $\beta_i = (r_i, z_i) \in [0,T] \times [0,1]$, if $t \ge r_1 \vee \cdots \vee r_k$,

$$D^{l_1}_{\beta_1} \cdots D^{l_k}_{\beta_k}(X^i_t) = \sum_{n=1}^k D^{l_1}_{\beta_1} \cdots D^{l_{n-1}}_{\beta_{n-1}} D^{l_{n+1}}_{\beta_{n+1}} D^{l_1}_{\beta_1} \cdots D^{l_k}_{\beta_k}(\sigma_{il_n}(X_{r_n} - Y_{r_n}(z_n))$$
$$+ \sum_{j=1}^d \int_{r_1\vee\cdots\vee r_k}^t \int_0^1 D^{l_1}_{\beta_1} \cdots D^{l_k}_{\beta_k}(\sigma_{ij}(X_s - Y_s(\alpha)))\, W^j(d\alpha, ds)$$
$$+ \int_{r_1\vee\cdots\vee r_k}^t \int_0^1 D^{l_1}_{\beta_1} \cdots D^{l_k}_{\beta_k}(b_i(X_s - Y_s(\alpha)))\, d\alpha ds.$$

Then, together with Burkholder's and Hölder's inequality for conditional expectations, the Lipschitz property of σ and the bounds from the derivatives of σ and b and the recurrence hypothesis (ii) of [6, Theorem 11] we conclude the proof of (ii). △

The proof of the upper bound of Theorem 1.5 follows then substituting the results of Lemmas 4.1 and 4.2 into (4.1).

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