# HYPERCONTRACTIVITY FOR PERTURBED DIFFUSION SEMI GROUPS. 

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#### Abstract

Log-Sobolev inequality, we give conditions on $F$ for the Boltzmann measure $\nu=e^{-2 F} \mu$ to also satisfy some Log-Sobolev inequality. This paper improves and completes the final section in [6]. A general sufficient condition is given and examples are explicitly studied.


Key words : Hypercontractivity, Boltzmann measure, Girsanov Transform.
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## 1. Introduction and Framework.

In [6] we have introduced a pathwise point of view in the study of classical inequalities. The last two sections of this paper were devoted to the transmission of Log-Sobolev and Spectral Gap inequalities to perturbed measures, without any explicit example. In the present paper we shall improve the results of section 8 in [6] and study explicit examples. Except for one point, the present paper is nevertheless self-contained.
In order to describe the contents of the paper we have first to describe the framework.

## Framework.

For a nonnegative measure $\mu$ on some measurable space E , let us first consider a $\mu$ symmetric diffusion process $\left(\mathbb{P}_{x}\right)_{x \in E}$ and its associated semi-group $\left(P_{t}\right)_{t \geq 0}$ with generator $A$. Here by a diffusion process we mean a strong Markov family of probability measures $\left(\mathbb{P}_{x}\right)_{x \in E}$ defined on the space of continuous paths $\mathcal{C}^{0}\left(\mathbb{R}^{+}, E\right)$ for some, say Polish, state space $E$, such that there exists some algebra $\mathbb{D}$ of uniformly continuous and bounded functions (containing constant functions) which is a core for the extended domain $D_{e}(A)$ of the generator (see [7]).
One can then show that there exists a countable orthogonal family $\left(C^{n}\right)$ of local martingales and a countable family $\left(\nabla^{n}\right)$ of operators s.t. for all $f \in D_{e}(A)$

$$
\begin{equation*}
M_{t}^{f}=f\left(X_{t}\right)-f\left(X_{0}\right)-\int_{0}^{t} A f\left(X_{s}\right) d s=\sum_{n} \int_{0}^{t} \nabla^{n} f\left(X_{s}\right) d C_{s}^{n}, \tag{1.1}
\end{equation*}
$$

in $\mathbb{M}_{\text {loc }}^{2}\left(\mathbb{P}_{\eta}\right)$ (local martingales) for all probability measure $\eta$ on $E$.
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One can thus define the "carré du champ" $\Gamma$ by

$$
\Gamma(f, g)=\sum_{n} \nabla^{n} f \nabla^{n} g \stackrel{\text { def }}{=}(\nabla f)^{2}
$$

so that the martingale bracket is given by

$$
<M^{f}>_{t}=\int_{0}^{t} \Gamma(f, f)\left(X_{s}\right) d s
$$

In terms of Dirichlet forms, all this, in the symmetric case, is roughly equivalent to the fact that the local pre-Dirichlet form

$$
\mathcal{E}(f, g)=\int \Gamma(f, g) d \mu \quad f, g \in \mathbb{D}
$$

is closable, and has a regular (or quasi-regular) closure $(\mathcal{E}, D(\mathcal{E}))$, to which the semi group $P_{t}$ is associated. Notice that with our definitions, for $f \in \mathbb{D}$

$$
\begin{equation*}
\mathcal{E}(f, f)=\int \Gamma(f, f) d \mu=-2 \int f A f d \mu=-\left.\frac{d}{d t}\left\|P_{t} f\right\|_{\mathbb{L}^{2}(\mu)}^{2}\right|_{t=0} \tag{1.2}
\end{equation*}
$$

It is then easy to check that

$$
\Gamma(f, g)=A(f g)-f A g-g A f
$$

and, that for $f_{i}$ in $\mathbb{D}$, the following composition formula holds

$$
A \Phi\left(f_{1}, \ldots, f_{n}\right)=\sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_{i}}\left(f_{1}, \ldots, f_{n}\right) A f_{i}+\frac{1}{2} \sum_{i, j=1}^{n} \frac{\partial^{2} \Phi}{\partial x_{i} \partial x_{j}}\left(f_{1}, \ldots, f_{n}\right) \Gamma\left(f_{i}, f_{j}\right)
$$

for $\Phi$ smooth enough, i.e. for instance $\mathcal{C}^{\infty}$ with compact support.

## Contents.

The aim of this paper is to give conditions on $F$ for the perturbed measure $\nu=e^{-2 F} \mu$ to share some properties with $\mu$, namely Log-Sobolev inequality or Spectral Gap property. As in the final section of [6] these conditions are first described in terms of some martingale properties in the spirit of the work by Kavian, Kerkyacharian and Roynette (see [13]).

After completing the main part of the paper, we (re)discovered the work by S. Kusuoka and D. Stroock ([15]). Published in 1985 this paper contains (in the framework of subelliptic operators) results on ultracontractivity (see Theorem (2.26) therein) neighboring those in [13] (themselves extending those in Davies [8] recalled in Theorem 2.5 below). At the same time, Theorem (2.21) in [15] is closed to our results on hypercontractivity (in particular the main sufficient condition (B.F) we shall introduce in section 4 below, appears in [15] (2.5)). A more precise comparison is done in section 6.2.
We were very surprised not to see [15] in the bibliography of almost all courses on the topic (at least all the ones we have looked at). Apparently it is due to the fact that these authors preferred "bounded" to "contractive".
It turns out that our approach (inspired by [13]) is completely different than the one in [15] and a little bit more general (we think that it also shows why (B.F) is a natural assumption). It also allows to partly recover Wang's recent results on the inverse Herbst argument (see
section 5) and can be used to obtain others contraction properties. So we believe that this approach still has its own interest.
Section 2 contains a short review on Log-Sobolev literature, and introduce the main definitions. Section 3 is devoted to the general perturbation results. The results extend those in the final sections of [6]. In section 4, following [13] we introduce the martingale method that yields the sufficient condition (B.F). The results of these sections are applied to the $\mathbb{R}^{N}$ case in section 5 . Examples and connection with concentration of measure property are detailed. Section 6 describes rapidly how to extend the previous results to the manifold value case (including uniformly elliptic operators), or to degenerate (strongly hypoelliptic) operators.

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## 2. Notation and general results.

In this section we shall recall some definitions and results in the literature. Because definitions and notations are varying from a paper to another we will be very accurate with the vocabulary. The material below can be found in many very goods textbooks or courses see e.g. [3], [4], [9], [10], [11], [17]. The reader has to be careful when comparing with these references where notations are not always the same as the ones here (some factors 2 for example).

In the framework of section 1 , we shall say that $\mu$ satisfies a Log-Sobolev inequality LSI if for some universal constants $a$ and $b$ and all $f \in \mathbb{D} \cap L^{1}(\mu)$,

$$
\begin{equation*}
\int f^{2} \log \left(\frac{f^{2}}{\|f\|_{\mathbb{L}^{2}(\mu)}^{2}}\right) d \mu \leq a \int \Gamma(f, f) d \mu+b\|f\|_{\mathbb{L}^{2}(\mu)}^{2} \tag{2.1}
\end{equation*}
$$

When $b=0$ we will say that the inequality is tight (TLSI), when $b>0$ we will say that the inequality is defective (DLSI). So we never will use (LSI) without specifying (TLSI) or (DLSI).
Note that when $\mu$ is bounded (2.1) easily extends to any $f \in D(\mathcal{E})$. It is not the case when $\mu$ is not bounded, in which case it only extends to $f \in D(\mathcal{E}) \cap \mathbb{L}^{1}(\mu)$ or to $f \in D(\mathcal{E})$ but replacing $\log$ by $\log ^{+}$in the left hand side of $(2.1)$. An example of such phenomenon is $f=(1+|x|)^{-\frac{1}{2}} \log ^{\alpha}(e+|x|)$ for $1<2 \alpha<2, E=\mathbb{R}$ and $d \mu=d x$.
These inequalities are known to be related to continuity (or contractivity) of the semi group $P_{t}$. We shall say that the semi-group is Hypercontractive if for some $t>0$ and $p>2, P_{t}$ maps continuously $\mathbb{L}^{2}(\mu)$ into $\mathbb{L}^{p}(\mu)$. In this case we shall denote the corresponding norm

$$
\left\|P_{t}\right\|_{\mathbb{L}^{2}(\mu) \rightarrow \mathbb{L}^{p}(\mu)}
$$

or simply $\left\|P_{t}\right\|_{2, p}$ when no confusion is possible.
A famous result of L. Gross tells that hypercontractivity is equivalent to a Log-Sobolev inequality. More precisely
Theorem 2.2. Gross Theorem.
(1)

$$
\text { Define } p(t)=1+e^{\frac{4 t}{C}} . \text { If for all } t>0 \text {, }
$$

$$
\left\|P_{t}\right\|_{2, p(t)} \leq \exp \left(4 b\left(\frac{1}{2}-\frac{1}{p(t)}\right)\right)
$$

then (DLSI) holds with $a=\frac{C}{2}$ and $b$.
(2) If (DLSI) holds then for all $q \geq p>1,\left\|P_{t}\right\|_{p, q} \leq \exp \left(4 b\left(\frac{1}{p}-\frac{1}{q}\right)\right)$ provided $t \geq \frac{a}{2} \log \left(\frac{q-1}{p-1}\right)$.
The point in the previous theorem is that continuity is required for all $t$ and some $p$. Using interpolation theorems one can show that actually (1) in Gross theorem holds (with appropriate constants) as soon as the semi-group is Hypercontractive (with our definition). This result is due to Hoegh-Krohn and Simon (see [4] Theorem 3.6 for such a proof). In [6] an alternate direct proof is given (Corollary 2.8.). More precisely
Theorem 2.3. If for some $t>0$ and $p>2$ one has $\left\|P_{t}\right\|_{2, p} \leq C$, then (DLSI) holds with $a=\frac{t p}{p-2}$ and $b=\frac{2 p}{p-2} \log (C)$.
In particular if the semi group is hypercontractive, $P_{t}$ is continuous from $\mathbb{L}^{p}$ into $\mathbb{L}^{q}$ for $q>p>1$ and $t$ large enough. Furthermore it is a contraction from $\mathbb{L}^{p}$ into $\mathbb{L}^{q}$ if and only if (TLSI) holds. We shall say in this case that the semi-group is strongly hypercontractive.

If $\left\|P_{t}\right\|_{2, p}$ is finite for some $p>2$ and all $t>0$ we shall say that the semi-group is Supercontractive (some authors are using immediately hypercontractive). Note that supercontractivity is equivalent to a family of (DLSI) namely for all $\varepsilon>0$ it holds

$$
\begin{equation*}
\int f^{2} \log \left(\frac{f^{2}}{\|f\|_{\mathbb{L}^{2}(\mu)}^{2}}\right) d \mu \leq \varepsilon \int \Gamma(f, f) d \mu+b(\varepsilon)\|f\|_{\mathbb{L}^{2}(\mu)}^{2} \tag{2.4}
\end{equation*}
$$

with

$$
b(\varepsilon)=\frac{2 p}{p-2} \log C\left(\frac{p-2}{p} \varepsilon\right) \quad \text { and } \quad\left\|\mathrm{P}_{\mathrm{t}}\right\|_{2, \mathrm{p}}=\mathrm{C}(\mathrm{t})
$$

A stronger notion is obtained when we replace $p$ by $+\infty$ in the definition of supercontractivity. This notion is called Ultracontractivity and is extensively studied in the book by E.B. Davies [8]. Links with Log-Sobolev inequalities are especially studied in chapter 2 of [8]. In particular the following is shown in [8] Theorem 2.2.3 and Corollary 2.2.8.
Theorem 2.5. Assume that $E$ is locally compact.
(1) If $\left\|P_{t}\right\|_{2,+\infty}=e^{M(t)}$ for all $t>0$ and $M($.$) being non increasing, then the family of$ (DLSI) (2.4) holds with $b(\varepsilon)=2 M(\varepsilon)$.
(2) If the family of (DLSI) (2.4) holds for some non increasing function b( $\varepsilon$ ) satisfying for all $t>0$,

$$
M(t)=\frac{1}{2 t} \int_{0}^{t} b(\varepsilon) d \varepsilon<+\infty
$$

then the semi-group is ultracontractive with $\left\|P_{t}\right\|_{2,+\infty} \leq e^{M(t)}$.
Note that this result can be obtained by taking limits when $p \rightarrow+\infty$ in the constants we obtained in Theorem 2.3. It is immediate if $\mu$ is bounded (without any assumption of local compactness). It is almost immediate if $\mu$ is not bounded, provided one can use some nice partition of unity (hence some topological assumptions are necessary).

As for Hypercontractivity, Ultracontractivity extends to any $1 \leq p$, i.e. one can easily show that

$$
\left\|P_{t}\right\|_{p,+\infty} \leq\left\|P_{\frac{t}{2}}\right\|_{2,+\infty}^{\frac{2}{p}}
$$

Finally we shall recall the relationship between (DLSI) and (TLSI) via spectral gaps properties.
As soon as we will use spectral gap properties we shall assume that $\mu$ is a Probability measure (or is bounded and normalized).
Let us introduce some definitions.
Definition 2.6. We say that the Spectral Gap Property (SGP) holds, if one of the following equivalent properties is satisfied
(1) $\lim _{t \rightarrow+\infty} \sup _{\|f\|_{2} \leq 1}\left\|P_{t} f-\int f d \mu\right\|_{2}=0$,
(2) there exists $\eta>0$ such that for all $f \in \mathbb{L}^{2}(\mu)$,

$$
\int\left(f-\int f d \mu\right)^{2} d \mu \leq \eta \int \Gamma(f, f) d \mu
$$

(3) there exists $\lambda>0$ such that for all $f \in \mathbb{L}^{2}(\mu)$,

$$
\left\|P_{t} f-\int f d \mu\right\|_{2} \leq e^{-\lambda t}\left\|f-\int f d \mu\right\|_{2}
$$

The best $\eta$ in (2) is called the inverse Spectral Gap, and the best $\lambda$ in (3) is then $\frac{1}{2 \eta}$. (2) is called the Poincaré Inequality.
Definition 2.7. We say that the Weak Spectral Gap Property (WSGP) holds, if one of the following equivalent properties is satisfied
(1) $\lim _{t \rightarrow+\infty} \sup _{\|f\|_{2} \leq 1}\left\|P_{t}^{*} f-\int f d \mu\right\|_{1}=0$,
(2) $\lim _{t \rightarrow+\infty} \sup _{\|f\|_{\infty} \leq 1}\left\|P_{t} f-\int f d \mu\right\|_{2}=0$,
(3) the weak Poincaré inequality

$$
\text { for all } r>0,\|g\|^{2} \leq \beta_{p}(r) \mathcal{E}(g, g)+r\|g\|_{p}^{2}
$$

holds for for all bounded $g \in D(\mathcal{E})$ such that $\int g d \mu=0$, some $+\infty \geq p>2$ and some non increasing function $\beta_{p}$,
(4) the previous weak Poincaré inequality holds for all $+\infty \geq p>2$,
(5) for any sequence $\left\{g_{n}\right\} \in D(\mathcal{E})$ such that $\int g_{n} d \mu=0,\left\|g_{n}\right\|_{\infty} \leq 1$ and $\mathcal{E}\left(g_{n}, g_{n}\right) \rightarrow 0$ as $n$ goes to $+\infty$, we have $g_{n} \rightarrow 0$ in $\mu$ probability,
(6) one can replace $\|.\|_{\infty}$ by $\|.\|_{2}$ in the previous statement.

If (WSGP) is satisfied, denoting by

$$
\xi_{p}(t)=\sup _{\|f\|_{p} \leq 1}\left\|P_{t} f-\int f d \mu\right\|_{2}^{2}
$$

one may choose in (3)

$$
\beta_{p}(r)=2 r \inf _{s>0} \frac{1}{s} \xi_{p}^{-1}\left(s \exp \left(1-\frac{s}{r}\right)\right)
$$

Conversely if (3) holds for some non increasing $\beta_{p}$,

$$
\xi_{p}(t) \leq 2 \inf \left\{r>0,-\beta_{p}(r) \log (r) \leq 2 t\right\}
$$

The results concerning (WSGP) are due to Röckner and Wang (see [20]) and are discussed in section 5 of [6] (see in particular Remark 5.11 therein). Others equivalent formulations in terms of Uniform Positivity Improving are due to several authors (see [1] Definition 2.1. and references therein, also see [26] for an almost complete study of this notion).
The next result explains the relationship between spectral gap properties and (TLSI).

## Proposition 2.8.

(1) If (TLSI) holds for some $a$, (SGP) holds with $\eta \leq \frac{a}{2}$.
(2) If (DLSI) holds with constants $(a, b)$ and (SGP) holds with inverse spectral gap $\eta$, then (TLSI) holds with a constant $a^{\prime} \leq a+2 \eta(2 b+1)$.
(3) If on one hand, for some $t>0$ and $p>2$ one has $\left\|P_{t}\right\|_{2, p}=c$, and on the other hand (WSGP) holds, then (SGP) holds with inverse spectral gap

$$
\eta \leq \inf _{s>t} \frac{s}{\left(1-c^{2} \xi_{p}(s-t)\right) \vee 0}
$$

Accordingly (TLSI) holds with a constant $a^{\prime} \leq \frac{p t}{p-2}+2 \eta\left(1+\frac{4 p \log (c)}{p-2}\right)$.
The final statement (3) in the previous Proposition is originally due to Mathieu ([19]). The form given here is the one shown in [6] Proposition 5.13. The final argument is of course a consequence of (2) and Theorem 2.3.

We conclude this section by recalling the now well known Herbst argument (see Ledoux [17]) connecting (TLSI) and the concentration of measure phenomenon. Here we assume that $E$ is some metric space, $\mu$ is still a Probability measure.
Proposition 2.9. If (TLSI) is satisfied with some constant $a$, then for all $f \in \operatorname{Lip}(E)$ with $\|f\|_{\text {Lip }} \leq 1$ it holds for all $R>0$,

$$
\mu\left(f \geq \int f d \mu+R\right) \leq \exp -\frac{R^{2}}{a}
$$

## 3. Hypercontractivity for general Boltzmann measures.

We introduce in this section a general perturbation theory. In the framework of section 1 let $F$ be some real valued function defined on $E$.
Definition 3.1. The Boltzmann measure associated with $F$ is defined as $\nu_{F}=e^{-2 F} \mu$.
When no confusion is possible we may not write the subscript $F$ and simply write $\nu$.
The transmission of Log-Sobolev or Spectral Gap inequalities to Boltzmann measures has been extensively studied in various contexts. The first classical result goes back to Holley and Stroock.
Proposition 3.2. Assume that $\mu$ is a Probability measure and $F$ is bounded. Then if $\mu$ satisfies (DLSI) with constants $(a, b), \nu_{F}$ satisfies (DLSI) with constants $\left(a e^{O s c(F)}, b e^{O s c(F)}\right)$ where $O s c(F)=\sup (F)-\inf (F)$.

This result is often stated with $2 O s c(F)$ i.e. with an useless factor 2 (see [21] Proposition 3.1.18).

When $F$ is no more bounded, general (though too restrictive) results have been shown by Aida and Shigekawa [2]. Other results can be obtained through the celebrated Bakry-Emery criterion.
In [6] section 7, we have given a new proof of Aida-Shigekawa results. In section 8 of [6] we followed a beautiful idea of Kavian, Kerkyacharian and Roynette (see [13]) in order to get better results (with a little bit more regularity). The main idea in [13] is that Ultracontractivity for a Boltzmann measure builded on $\mathbb{R}$ with $\mu$ the Lebesgue measure and $F$ regular enough, reduces to check the boundedness of one and only one function. In section 8 of [6] we proved similar results (in our general framework) for Hypercontractivity and Strong Hypercontractivity.
The aim of this section is to improve these results. In particular we shall be accurate with constants, i.e. we shall give some explicit controls. First let us state the hypotheses we need for $F$.

### 3.3 Assumptions H(F)

(1) $\nu_{F}$ is a Probability measure, $F \in D(\mathcal{E})$,
(2) for all $f \in \mathbb{D}, \mathcal{E}_{F}(f, f)=\int \Gamma(f, f) d \nu_{F}<+\infty$,
(3) for all $f \in \mathbb{D}, A f \in \mathbb{L}^{1}\left(\nu_{F}\right)$,
(4) $\int \Gamma(F, F) d \nu_{F}<+\infty$.

The Girsanov martingale $Z_{t}^{F}$ is then defined as

$$
\begin{equation*}
Z_{t}^{F}=\exp \left\{-\int_{0}^{t} \nabla F\left(X_{s}\right) \cdot d C_{s}-\frac{1}{2} \int_{0}^{t} \Gamma(F, F)\left(X_{s}\right) d s\right\} \tag{3.4}
\end{equation*}
$$

When $\mathrm{H}(\mathrm{F})$ holds, we know that $Z_{.}^{F}$ is a $\mathbb{P}_{x}$ martingale for $\nu_{F}$, hence $\mu$ almost all $x$. Furthermore $\nu_{F}$ is then a symmetric measure for the perturbed process $\left\{Z^{F} \mathbb{P}_{x}\right\}_{x \in E}$, which is associated with $\mathcal{E}_{F}$ (see (3.3.2)). For all this see [6] (especially Lemma 7.1 and section 2 ).
If in addition $F \in D(A)$, it is enough to apply Ito's formula in order to get another expression for $Z_{t}^{F}$, namely

$$
\begin{equation*}
Z_{t}^{F}=\exp \left\{F\left(X_{0}\right)-F\left(X_{t}\right)+\int_{0}^{t}\left(A F\left(X_{s}\right)-\frac{1}{2} \Gamma(F, F)\left(X_{s}\right)\right) d s\right\} \tag{3.5}
\end{equation*}
$$

If $P_{t}^{F}$ denotes the associated ( $\nu_{F}$ symmetric) semi-group, it holds $\nu_{F}$ a.s.

$$
\begin{equation*}
\left(P_{t}^{F} h\right)(x)=e^{F}(x) \mathbb{E}^{\mathbb{P}_{x}}\left[h\left(X_{t}\right) e^{-F\left(X_{t}\right)} M_{t}\right] \tag{3.6}
\end{equation*}
$$

with

$$
M_{t}=\exp \left(\int_{0}^{t}\left(A F\left(X_{s}\right)-\frac{1}{2} \Gamma(F, F)\left(X_{s}\right)\right) d s\right) .
$$

When $\mu$ is a probability measure, $e^{F} \in \mathbb{L}^{2}\left(\nu_{F}\right)$, and a necessary condition for $\nu_{F}$ to satisfy (DLSI) is thus

$$
\begin{equation*}
P_{t}^{F}\left(e^{F}\right)=e^{F} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right] \in \mathbb{L}^{p}\left(\nu_{F}\right) \tag{3.7}
\end{equation*}
$$

for all (some) $p>2$ and $t$ large enough. When $\mu$ is no more bounded one can formulate similar statements. For instance, if $e^{F} \in \mathbb{L}^{r}\left(\nu_{F}\right)$ for some $r>1$, then (3.7) has to hold for some (all) $p>r$ and $t$ large enough. One can also take $r=1$ in some cases. Since the exact formulation depends on the situation we shall not discuss it here.

A remarkable fact is that the (almost always) necessary condition (3.7) is also a sufficient one. The next two theorems explain why. Though the proof of the first one is partly contained in [6] (Proposition 8.8) we shall give here the full proof for completeness.
Theorem 3.8. Assume that $P_{t}$ is Ultracontractive with $\left\|P_{t}\right\|_{p, \infty}=K(t, p)$ for all $p \geq 1$. Assume that $H(F)$ is in force, $F \in D(A)$ and $M_{t}$ is bounded by some constant $C(t)$. Then a sufficient condition for $\nu_{F}$ to satisfy ( $D L S I$ ) is that

$$
P_{t}^{F}\left(e^{F}\right)=e^{F} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right] \in \mathbb{L}^{q}\left(\nu_{F}\right)
$$

for some $t>0$ and some $q>2$.
If in addition
(1) either $\mu$ is a probability measure,
(2) or $e^{F} \in \mathbb{L}^{p}\left(\nu_{F}\right)$ for some $p>1$,
this condition is also necessary.
Proof. Pick some $f \in \mathbb{D}$. Since $|f| e^{-F} \in \mathbb{L}^{2}(\mu)$ and using the Markov property, for $t>0$, $q>2$ it holds

$$
\begin{aligned}
\int\left(P_{t+s}^{F}(|f|)\right)^{q} d \nu_{F} & =\int e^{q F}\left(\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t} \mathbb{E}^{\mathbb{P}_{X_{t}}}\left[M_{s}\left(e^{-F}|f|\right)\left(X_{s}^{\prime}\right)\right]\right]\right)^{q} d \nu_{F} \\
& \leq \int e^{q F}(C(s))^{q}\left(E^{\mathbb{P}_{x}}\left[M_{t}\left(P_{s}\left(|f| e^{-F}\right)\right)\left(X_{t}\right)\right]\right)^{q} d \nu_{F} \\
& \leq(C(s))^{q}\left(\left\|P_{s}\right\|_{2, \infty}\right)^{q}\|f\|_{\mathbb{L}^{2}\left(\nu_{F}\right)}^{q} \int\left(e^{F} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]\right)^{q} d \nu_{F}
\end{aligned}
$$

Hence

$$
\begin{equation*}
\left\|P_{t+s}^{F}\right\|_{2, q} \leq C(s) K(s, 2)\left\|e^{F} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]\right\|_{\mathbb{L}^{q}\left(\nu_{F}\right)} \tag{3.9}
\end{equation*}
$$

and Theorem 2.3 furnishes the sufficient condition in the Theorem. The necessary part has already been discussed.

When $P_{t}$ is only Hypercontractive, the previous arguments are no more available and one has to work harder to get the following analogue of Theorem 3.8
Theorem 3.10. Assume that $P_{t}$ is Hypercontractive. Assume that $H(F)$ is in force, $F \in$ $D(A)$ and that $M_{t}$ is bounded by some constant $C(t)$. Assume in addition that $e^{F} \in \mathbb{L}^{r}\left(\nu_{F}\right)$ for some $r>1$ (we may choose $r=2$ when $\mu$ is a Probability measure).
Then a necessary and sufficient condition for $\nu_{F}$ to satisfy (DLSI) is that

$$
P_{t}^{F}\left(e^{F}\right)=e^{F} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right] \in \mathbb{L}^{p}\left(\nu_{F}\right)
$$

for some $p>2$ and some $t>0$ large enough.
Proof. The proof is based on the following elementary consequence of Girsanov theory and the variational characterization of relative entropy (see [6] section 2) : if $\int f^{2} d \nu_{F}=1$ and $f$ is nonnegative, then

$$
\begin{equation*}
\int\left(\sum_{j} \log h_{j}\right) f^{2} d \nu_{F} \leq \frac{t}{2} \mathcal{E}_{F}(f, f)+\log \int f^{2} h_{1} P_{t}^{F}\left(h_{2}\right) d \nu_{F} \tag{3.11}
\end{equation*}
$$

Choose $j=1,2, h_{1}=f^{\alpha-1}$ and $h_{2}=f^{\beta}$. (3.11) becomes

$$
\begin{equation*}
\frac{(\alpha+\beta-1)}{2} \int f^{2} \log \left(f^{2}\right) d \nu_{F} \leq \frac{t}{2} \mathcal{E}_{F}(f, f)+\log \int f^{1+\alpha} P_{t}^{F}\left(f^{\beta}\right) d \nu_{F} . \tag{3.12}
\end{equation*}
$$

Let $(q, s)$ a pair of conjugate real numbers. Then

$$
P_{t}^{F}\left(f^{\beta}\right) \leq\left(P_{t}^{F}\left(f^{q \beta} e^{-\frac{q}{s} F}\right)\right)^{\frac{1}{q}}\left(P_{t}^{F}\left(e^{F}\right)\right)^{\frac{1}{s}},
$$

and accordingly

$$
\begin{align*}
& \int f^{1+\alpha} P_{t}^{F}\left(f^{\beta}\right) d \nu_{F} \leq \int f^{1+\alpha}\left(P_{t}^{F}\left(f^{q \beta} e^{-\frac{q}{s} F}\right)\right)^{\frac{1}{q}}\left(P_{t}^{F}\left(e^{F}\right)\right)^{\frac{1}{s}} d \nu_{F}  \tag{3.13}\\
\leq & \left(\int f^{1+\alpha} e^{-q \delta F} P_{t}^{F}\left(f^{q \beta} e^{-\frac{q}{s} F}\right) d \nu_{F}\right)^{\frac{1}{q}}\left(\int f^{1+\alpha} e^{s \delta F} P_{t}^{F}\left(e^{F}\right) d \nu_{F}\right)^{\frac{1}{s}} \\
\leq & \left(\int e^{-\frac{2 q \delta}{1-\alpha} F}\left(P_{t}^{F}\left(f^{q \beta} e^{-\frac{q}{s} F}\right)\right)^{\frac{2}{1-\alpha}} d \nu_{F}\right)^{\frac{1-\alpha}{2 q}}\left(\int e^{\frac{2 s \delta}{1-\alpha} F}\left(P_{t}^{F}\left(e^{F}\right)\right)^{\frac{2}{1-\alpha}} d \nu_{F}\right)^{\frac{1-\alpha}{2 s}}
\end{align*}
$$

where we have used Hölder's inequality successively with $f^{1+\alpha} d \nu_{F}$ and $d \nu_{F}$, and we also used $\int f^{2} d \nu_{F}=1$ to get the last expression. We have of course to choose $\alpha<1$. We shall also choose $\beta=1$. The first factor in the latter expression can be rewritten

$$
\int e^{-\frac{2 q \delta}{1-\alpha} F}\left(P_{t}^{F}\left(f^{q} e^{-\frac{q}{s} F}\right)\right)^{\frac{2}{1-\alpha}} d \nu_{F}=\int e^{\theta F}\left(\mathbb{E}^{\mathbb{P}_{x}}\left(f^{q}\left(X_{t}\right) e^{-\left(1+\frac{q}{s}\right) F\left(X_{t}\right)} M_{t}\right)\right)^{\frac{2}{1-\alpha}} d \mu
$$

with

$$
\theta=-\frac{2 q \delta}{1-\alpha}+\frac{2}{1-\alpha}-2 .
$$

Hence if we choose $\alpha=q \delta<1, \theta=0$. Furthermore $q=1+\frac{q}{s}$ and $f^{q} e^{-q F} \in \mathbb{L}^{\frac{2}{q}}(\mu)$ with norm 1, provided $q<2$. Using our hypotheses we thus obtain

$$
\begin{equation*}
\int e^{-\frac{2 q \delta}{1-\alpha} F}\left(P_{t}^{F}\left(f^{q} e^{-\frac{q}{s} F}\right)\right)^{\frac{2}{1-\alpha}} d \nu_{F} \leq\left(C(t)\left\|P_{t}\right\|_{\frac{2}{q}, \frac{2}{1-\alpha}}\right)^{\frac{2}{1-\alpha}} . \tag{3.14}
\end{equation*}
$$

For the second factor we choose

$$
\frac{2 s \delta}{1-\alpha}<r,
$$

and since $\alpha=q \delta$, this choice imposes

$$
\delta<\frac{r}{2 s+r q} \quad \text { hence } \quad \alpha<\frac{r q}{2 s+r q} .
$$

Note that the condition $\alpha<1$ is then automatically satisfied. Applying Hölder again we get

$$
\begin{equation*}
\int e^{\frac{2 s \delta}{1-\alpha} F}\left(P_{t}^{F}\left(e^{F}\right)\right)^{\frac{2}{1-\alpha}} d \nu_{F} \leq\left(\int e^{r F} d \nu_{F}\right)^{\frac{2 s \delta}{r(1-\alpha)}}\left(\int\left(P_{t}^{F}\left(e^{F}\right)\right)^{p} d \nu_{F}\right)^{\frac{r(1-\alpha)-2 s \delta}{r(1-\alpha)}}, \tag{3.15}
\end{equation*}
$$

if

$$
p=\frac{2 r}{r(1-\alpha)-2 s \delta} \quad \text { hence } \quad \alpha=\frac{r(p-2)}{p(2(s-1)+r))} .
$$

It remains to check that all these choices are compatible, i.e

$$
\frac{r(p-2)}{p(2(s-1)+r))}<\frac{r q}{2 s+r q}
$$

which is easy.

Plugging (3.14) and (3.15) into (3.12) we obtain

$$
\begin{equation*}
\alpha \int f^{2} \log \left(f^{2}\right) d \nu_{F} \leq t \mathcal{E}_{F}(f, f)+2 A \tag{3.16}
\end{equation*}
$$

where

$$
A=\frac{1}{q} \log \left(C(t)\left\|P_{t}\right\|_{\frac{2}{q}, \frac{2}{1-\alpha}}\right)+\frac{\alpha}{q} \log \left(\left\|e^{F}\right\|_{\mathbb{L}^{r}\left(\nu_{F}\right)}\right)+\frac{1}{s} \log \left(\left\|P_{t}^{F}\left(e^{F}\right)\right\|_{\mathbb{L}^{p}\left(\nu_{F}\right)}\right)
$$

For a fixed $p$ we may choose any pair $(q, s)$ with $q<2$, and the corresponding $\alpha$ yields the result for

$$
t \geq \frac{a}{2} \log \left(\frac{q(1+\alpha)}{(2-q)(1-\alpha)}\right)
$$

according to Gross theorem, if $\mu$ satisfies (DLSI) with constants $(a, b)$.
Remark 3.17. (1) The previous proof seems to be more general as we claimed. Actually the proof of (3.11) in [6] requires $\mathrm{H}(\mathrm{F})$. The only point is that with an ad-hoc definition of $M_{t}$ we do not really need that $F \in D(A)$.
(2) We have some degrees of freedom in our choices in (3.16), for $(q, s)$ and possibly $p$ or $r$. Due to the expression for $t$ at the end of the previous proof, it seems not useful to try to get the optimal constant $a^{\prime}=\frac{t}{\alpha}$ in full generality. Moreover the previous methods cannot furnish the best constants. Indeed assume for example that $P_{t}^{F}$ is strongly hypercontractive and that $\mu$ is a Probability measure. Looking at (3.9) we see that we may choose independently $t$ and $s$. But it is clear that the best estimate for $\left\|e^{F} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]\right\|_{\mathbb{L}^{q}\left(\nu_{F}\right)}$ is obtained for $t=+\infty$ and is equal to

$$
\int e^{-F} d \mu \geq 1
$$

At the same time it is not difficult to see that $C(s) K(s, 2)$ is also greater than 1 , so that the right hand side in (3.9) will always be strictly greater than 1 while the left hand side is expected to be 1 for some pair $(t, s)$.
In view of the previous remark it is thus natural to look at the spectral gap properties too. The final result we shall recall is Lemma 2.2 in [1] (Proposition 7.11 in [6]) with a slightly different proof.
Theorem 3.18. Assume that $\mu$ is a probability measure satisfying (SGP) with some inverse spectral gap $\eta(\mu)$. Assume that $H(F)$ is in force and $\Gamma(F, F) \in \mathbb{L}^{1}(\mu)$. Then $\nu_{F}$ satisfies (WSGP).

Proof. Let $f \in \mathbb{D}$ be bounded. For any nonnegative $k$, introduce a non increasing smooth $\varphi_{k}$ defined on $\mathbb{R}$ such that for some $\delta>0$

$$
\varphi_{k}(x)=0, \text { if } x>k+1, \varphi_{k}(x)=1, \text { if } x<k,\left|\varphi_{k}^{\prime}(x)\right| \leq(1+\delta) \text { for all } x
$$

Then $\varphi_{k}(F) f \in D(\mathcal{E})$ and

$$
\Gamma\left(\varphi_{k}(F) f, \varphi_{k}(F) f\right) \leq 2\left(\left(\varphi_{k}(F)\right)^{2} \Gamma(f, f)+\left(\varphi_{k}^{\prime}(F)\right)^{2} f^{2} \Gamma(F, F)\right)
$$

Accordingly we get

$$
\begin{equation*}
\mathcal{E}\left(\varphi_{k}(F) f, \varphi_{k}(F) f\right) \leq 2 e^{2 k+2} \mathcal{E}_{F}(f, f)+2(1+\delta)^{2}\|f\|_{\infty}^{2}\left(\int \mathbb{1}_{k<F<k+1} \Gamma(F, F) d \mu\right) \tag{3.19}
\end{equation*}
$$

that extends to any bounded $f \in D\left(\mathcal{E}_{F}\right)$. We may then make $\delta$ go to 0 , hence $\varphi_{k}(x)=$ $1-(x-k)$ for $k<x<k+1$.
We may assume that $f$ is bounded by 1 and $\int f d \nu_{F}=0$. Denote by

$$
m(k)=\int f \varphi_{k}(F) d \mu
$$

Then, since $|m(k)| \leq 1$, for any $K>0$ one has

$$
\begin{align*}
& \int\left(f \varphi_{k}(F)-m(k)\right)^{2} d \nu_{F} \leq e^{2 K} \int\left(f \varphi_{k}(F)-m(k)\right)^{2} d \mu+4 \nu_{F}(F<-K)  \tag{3.20}\\
& \leq e^{2 K} \eta(\mu) \mathcal{E}\left(\varphi_{k}(F) f, \varphi_{k}(F) f\right)+4 \nu_{F}(F<-K) \\
& \leq 2 e^{2 K+2 k+2} \eta(\mu) \mathcal{E}_{F}(f, f)+2 e^{2 K} \eta(\mu)\left(\int \mathbb{1}_{k<F<k+1} \Gamma(F, F) d \mu\right)+4 \nu_{F}(F<-K) .
\end{align*}
$$

But since $\int f d \nu_{F}=0$ and $f$ is bounded by 1 , it holds

$$
\begin{align*}
|m(k)| & \leq\left|\int f \varphi_{k}(F) d \nu_{F}\right|+\int\left|f \varphi_{k}(F)-m(k)\right| d \nu_{F}  \tag{3.21}\\
& \leq \nu_{F}(k<F)+\left(\int\left(f \varphi_{k}(F)-m(k)\right)^{2} d \nu_{F}\right)^{\frac{1}{2}}
\end{align*}
$$

Finally

$$
\begin{gather*}
\int f^{2} d \nu_{F} \leq \int\left(f \varphi_{k}(F)\right)^{2} d \nu_{F}+\nu_{F}(k<F)  \tag{3.22}\\
\leq \int\left(f \varphi_{k}(F)-m(k)\right)^{2} d \nu_{F}+2|m(k)| \nu_{F}(k<F)-(m(k))^{2}+\nu_{F}(k<F) .
\end{gather*}
$$

For a given $0<s<1$ choose first $K(s)$ such that $4 \nu_{F}(F<-K(s))<s$. Since $\Gamma(F, F) \in$ $\mathbb{L}^{1}(\mu)$ one can find some $k(s)$ such that

$$
2 e^{2 K(s)} \eta(\mu)\left(\int \mathbb{I}_{k(s)<F} \Gamma(F, F) d \mu\right) \leq s
$$

thanks to Lebesgue bounded theorem, and $\nu_{F}(k(s)<F) \leq s$. Accordingly, using (3.22) and (3.20) we obtain

$$
\begin{equation*}
\int f^{2} d \nu_{F} \leq 2 e^{2 K(s)+2 k(s)+2} \eta(\mu) \mathcal{E}_{F}(f, f)+s(2|m(k(s))|+3) \tag{3.23}
\end{equation*}
$$

Furthermore, thanks to (3.21)

$$
\begin{align*}
|m(k(s))| & \leq \max \left(1, m(k(s))^{2}\right)  \tag{3.24}\\
& \leq \max \left(1,2 e^{2 K(s)+2 k(s)+2} \eta(\mu) \mathcal{E}_{F}(f, f)+3 s\right)
\end{align*}
$$

so that, plugging (3.24) into (3.23) we get for $r=9 s$ and $r<1$

$$
\begin{equation*}
\int f^{2} d \nu_{F} \leq \beta(r) \mathcal{E}_{F}(f, f)+r \tag{3.25}
\end{equation*}
$$

with

$$
\beta(r) \leq 2 e^{2 K(s)+2 k(s)+2} \eta(\mu)(1+2 s)
$$

(3.25) is exactly the weak Poincaré inequality of Definition 2.7.(3).

One can use Theorems 3.8 (or 3.10) and 3.18 together in order to show that the general Boltzmann measure satisfies (TLSI) provided $\mu$ is a Probability measure, thanks to Proposition 2.8. Otherwise one has to consider various reference measures $\mu$, as it will be clear in the next sections.

## 4. The "Well Method".

Our aim in this short section is to get sufficient general conditions for (3.7) to hold. To this end we shall slightly modify the "Well Method" of [13], i.e. use the martingale property of the Girsanov density. In the sequel we assume that $F \in D(A)$ satisfies $\mathrm{H}(\mathrm{F})$.

The main assumption we shall make is the following

### 4.1 Assumption B(F)

(1) $F$ is bounded from below by some constant $d$,
(2) there exist some $\lambda>0$ and some $c$ such that for all $x \in E$,

$$
\frac{1}{2} \Gamma(F, F)(x)-A F(x) \geq \lambda F(x)+c
$$

We may assume that $(\lambda d+c)<0$.
For $0<\varepsilon$ define the stopping time $\tau_{x}$ as

$$
\begin{equation*}
\tau_{x}=\inf \left\{s>0,\left(\frac{1}{2} \Gamma(F, F)-A F\right)\left(X_{s}\right) \leq \lambda F(x)+c-\varepsilon\right\} \tag{4.2}
\end{equation*}
$$

Note that for all $x \in E, \tau_{x}>0 \mathbb{P}_{x}$ a.s. and that on $\tau_{x}<+\infty$,

$$
\begin{equation*}
F\left(X_{\tau_{x}}\right) \leq \frac{1}{\lambda}\left(\left(\frac{1}{2} \Gamma(F, F)-A F\right)\left(X_{\tau_{x}}\right)-c\right) \leq F(x)-\frac{\varepsilon}{\lambda} \tag{4.3}
\end{equation*}
$$

provided $\left(\frac{1}{2} \Gamma(F, F)-A F\right)$ is (quasi)-left continuous on the paths. Not to introduce useless intricacies we shall assume in the sequel that $F, \nabla F$ and $A F$ are all finely continuous.

Recall that we want to estimate

$$
\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]=\mathbb{E}^{\mathbb{P}_{x}}\left[\exp \left(\int_{0}^{t}\left(A F-\frac{1}{2} \Gamma(F, F)\right)\left(X_{s}\right) d s\right)\right]
$$

Remark that 4.1 implies that

$$
M_{t} \leq e^{-(\lambda d+c) t}
$$

Introducing the previous stopping time we get

$$
\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]=\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t} \mathbb{I}_{t<\tau_{x}}\right]+\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t} \mathbb{I}_{\tau_{x} \leq t}\right]=A+B,
$$

with

$$
\begin{equation*}
A=\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t} \mathbb{I}_{t<\tau_{x}}\right] \leq \exp -((\lambda F(x)+c-\varepsilon) t) \tag{4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
B=\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t} \mathbb{I}_{\tau_{x} \leq t}\right] \tag{4.5}
\end{equation*}
$$

$$
\begin{array}{cc}
\leq & e^{-(\lambda d+c) t} \mathbb{E}^{\mathbb{P}_{x}}\left[\exp \left(\int_{0}^{t}\left(A F-\frac{1}{2} \Gamma(F, F)+(\lambda d+c)\right)\left(X_{s}\right) d s\right) \mathbb{I}_{\tau_{x} \leq t}\right] \\
\leq & e^{-(\lambda d+c) t} \mathbb{E}^{\mathbb{P}_{x}}\left[\exp \left(\int_{0}^{\tau_{x}}\left(A F-\frac{1}{2} \Gamma(F, F)+(\lambda d+c)\right)\left(X_{s}\right) d s\right) \mathbb{I}_{\tau_{x} \leq t}\right] \\
\leq & e^{-(\lambda d+c) t} \mathbb{E}^{\mathbb{P}_{x}}\left[\exp \left(\int_{0}^{\tau_{x}}\left(A F-\frac{1}{2} \Gamma(F, F)\right)\left(X_{s}\right) d s\right) \mathbb{I}_{\tau_{x} \leq t}\right] \\
= & e^{-(\lambda d+c) t} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{\tau_{x}} \mathbb{I}_{\tau_{x} \leq t}\right] .
\end{array}
$$

But $e^{-F\left(X_{s}\right)} M_{s}$ is a bounded (thanks to 4.1) $\mathbb{P}_{x}$ martingale. Hence, according to Doob stopping time Theorem

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}_{x}}\left[e^{-F\left(X_{\tau_{x}}\right)} M_{\tau_{x}} \mathbb{I}_{\tau_{x} \leq t}\right] \leq \mathbb{E}^{\mathbb{P}_{x}}\left[e^{-F\left(X_{t \wedge \tau_{x}}\right)} M_{t \wedge \tau_{x}}\right]=e^{-F(x)} \tag{4.6}
\end{equation*}
$$

According to (4.3),

$$
e^{-F\left(X_{\tau_{x}}\right)} \geq e^{\frac{\varepsilon}{\lambda}} e^{-F(x)}
$$

so that thanks to (4.6),

$$
\mathbb{E}^{\mathbb{P}_{x}}\left[M_{\tau_{x}} \mathbb{I}_{\tau_{x} \leq t}\right] \leq e^{-\left(\frac{\varepsilon}{\lambda}\right)} .
$$

Using this estimate in (4.5) and using (4.4) we finally obtain

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right] \leq e^{-((\lambda F(x)+c-\varepsilon) t)}+e^{-\left(\frac{\varepsilon}{\lambda}\right)} e^{-(\lambda d+c) t} \tag{4.7}
\end{equation*}
$$

We may thus state the following
Theorem 4.8. Assume that $F$ is such that, $H(F)$ and $B(F)$ are fulfilled. Assume in addition that $F, A F$ and $\nabla F$ are finely continuous and that there exists some $1<r$ such that $e^{F} \in$ $\mathbb{L}^{r}\left(\nu_{F}\right)$. Then $e^{F} \mathbb{E}^{\mathbb{P} x}\left[M_{t}\right] \in \mathbb{L}^{p}\left(\nu_{F}\right)$ as soon as

$$
t \geq \frac{p-r}{\lambda r}
$$

Proof. We will choose $\varepsilon=\beta(F(x)-d)+\xi$ for some $\xi>0$. Thus, according to (4.7)

$$
\left(e^{F(x)} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]\right)^{p} \leq 2^{p-1}\left(e^{p(1-(\lambda-\beta) t) F(x)} e^{p(\xi-\beta d-c) t}+e^{p\left(1-\frac{\beta}{\lambda}\right) F(x)} e^{-p(\lambda d+c) t} e^{p\left(\frac{\xi-\beta d}{\lambda}\right)}\right)
$$

It is thus enough to choose $\beta=\frac{p-r}{p} \lambda$ and then $p\left(1-\lambda \frac{r}{p} t\right) \leq r$.

It is quite natural to guess that the sufficient condition $\mathrm{B}(\mathrm{F})$ is in a sense almost necessary too. We shall not discuss this in full generality here (but we shall discuss this in some particular cases later). In particular one can see that supercontractivity (thanks to Theorem 3.10) is ensured as soon as $\mathrm{B}(\mathrm{F})(2)$ is satisfied for all $\lambda$ and some $c(\lambda)$, and again we may think that this condition is almost necessary. Note that this simple condition for supercontractivity is better than the corresponding one stated in Proposition 3.7. of [13].
For some particular $c(\lambda)$ one can expect obtain a sufficient condition for ultracontractivity (i.e. a $\mathbb{L}^{\infty}$ bound for $e^{F} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]$ according to the $p=\infty$ version of Theorem 3.8 first established in Lemme 2.1. of [13]) . Such a statement is not new. Indeed for the usual Laplace operator in $\mathbb{R}^{d}$, Theorem 4.7.1. in [8] tells that a sufficient condition for ultracontractivity is that

$$
c(\lambda) \geq-c \lambda^{\theta}
$$

for some $c>0$ and some $\theta>1$. Note that if $\mathrm{B}(\mathrm{F})$ (2) holds for all $\lambda$ then for all the $x^{\prime} s$ such that $F(x)>0$ (remark that the $L^{\infty}$ bound is automatic for the $x^{\prime} s$ such that $F(x) \leq 0$ )

$$
\begin{equation*}
\frac{1}{2} \Gamma(F, F)(x)-A F(x) \geq \max _{\lambda}\{\lambda F(x)+c(\lambda)\}=G(F(x)) \tag{4.9}
\end{equation*}
$$

Davies result corresponds to $G(y)=(\theta c)^{1-\theta}\left(1-\frac{1}{\theta}\right) y^{\frac{\theta}{\theta-1}}$. The natural suspicion is that a sufficient condition for ultracontractivity is that $G$ goes to $+\infty$ quickly enough (at least faster than $y$ ) when $y$ goes to $+\infty$. An example given in [13] (we shall recall later) shows that $G$ going to $\infty$ is not sufficient.
We shall thus conclude this section by recalling a (slightly modified and slightly weaker) version of Theorème 3.3. in [13]
Theorem 4.10. In addition to the hypotheses of Theorem 4.8, assume that (4.9) holds for some $G$ satisfying the following condition $U(F)$ : there exists some increasing sequence $\alpha_{k}$ such that $\sum_{k} e^{-\alpha_{k}}<+\infty$ and

$$
\sum_{k} \frac{\alpha_{k+1}}{G\left(\alpha_{k}\right)}<+\infty .
$$

Proof. We may mimic the proof of Theorème 3.3. in [13] choosing $A_{k}=\left\{F \leq \alpha_{k}\right\}, b_{k}=\alpha_{k}$ and $a_{k}=G\left(\alpha_{k}\right)$ therein. Denoting by $\beta_{k}=\frac{\alpha_{k+1}}{G\left(\alpha_{k}\right)}$, our hypothesis implies that $\sum_{k} \beta_{k}<$ $+\infty$. Hence by de la Vallée Poussin theorem, there exist two sequences $\varepsilon_{k}$ and $\gamma_{k}$ such that $\beta_{k}=\varepsilon_{k} \gamma_{k}, \sum_{k} \gamma_{k}<+\infty$ and $\varepsilon_{k}$ goes to 0 as $k$ goes to $\infty$. Thus

$$
-\varepsilon \gamma_{k} a_{k}+b_{k+1}=-\alpha_{k+1}\left(\frac{\varepsilon-\varepsilon_{k}}{\varepsilon_{k}}\right)
$$

and hypotheses 3 . and 4. of Theorème 3.3. in [13] are satisfied (there is no $\frac{1}{2}$ here because we replace their $u$ by $2 F$ ).

## 5. $\mathbb{R}^{N}$ valued Boltzmann measures.

In this section we shall deal with the $\mathbb{R}^{N}$ valued case, i.e. $E=\mathbb{R}^{N}, d x$ is Lebesgue measure, $A=\frac{1}{2} \Delta$ is one half of the Laplace operator and $\nabla$ is the usual gradient operator. $\mathbb{P}_{x}$ is thus the law of the Brownian motion starting at $x$, whose associated semi-group $P_{t}$ is $d x$ symmetric and ultracontractive with $\left\|P_{t}\right\|_{2,+\infty}=(4 \pi t)^{-\frac{N}{4}} . \mathbb{D}$ is the algebra generated by the usual set of test functions and the constants.
(TLSI) can thus be written

$$
\int f^{2} \log \left(\frac{f^{2}}{\|f\|_{\mathbb{L}^{2}\left(\nu_{F}\right)}^{2}}\right) e^{-2 F} d x \leq a \int|\nabla f|^{2} e^{-2 F} d x
$$

We may also use the Ornstein-Uhlenbeck semi group $P_{t}^{O U}$ associated with $A^{O U}=\frac{1}{2} \Delta-$ $\frac{1}{2} x . \nabla$ and $\gamma$ symmetric for $\gamma(d x)$ the standard gaussian measure on $\mathbb{R}^{N}$. It is well known that $\gamma$ satisfies (TLSI) with $a=2$ hence the Ornstein-Uhlenbeck semi group is strongly hypercontractive (Nelson's theorem) (see e.g. [3] for references of various proofs). Actually as we shall see below, our previous results allow to show hypercontractivity but not (immediately) strong hypercontractivity.
If we replace $\gamma$ by $\gamma_{\lambda}=e^{-\lambda|x|^{2}} d x$, then $\gamma_{\lambda}$ satisfies (TLSI) with $a_{\lambda}=\frac{1}{\lambda}$ for all $\lambda>0$.

As we shall see below another basic semi-group is of key interest. It is the one associated with

$$
A^{-O U(\lambda)}=\frac{1}{2} \Delta+\lambda x \cdot \nabla
$$

that is exchanging the sign of the drift term in $A^{O U}$ (that is why we are using -OU) and considering some $\lambda>0$. The existence of $\mathbb{P}_{x}^{-O U(\lambda)}$ is well known (using stochastic differential equations for instance). Furthermore, thanks to a result of Benes (see e.g. [12] Corollary 5.16 p .200 ), $\mathbb{P}_{x}^{-O U(\lambda)}$ is absolutely continuous (on $\mathcal{F}_{t}$ ) with respect to $\mathbb{P}_{x}$, with density

$$
\begin{align*}
Z_{t}^{-O U} & =\exp \left\{\int_{0}^{t} \lambda X_{s} \cdot d X_{s}-\frac{1}{2} \int_{0}^{t} \lambda^{2}\left|X_{s}\right|^{2} d s\right\}  \tag{5.1}\\
& =\exp \left\{\frac{\lambda}{2}\left(\left|X_{t}\right|^{2}-\left|X_{0}\right|^{2}-N t\right)-\frac{1}{2} \int_{0}^{t} \lambda^{2}\left|X_{s}\right|^{2} d s\right\}
\end{align*}
$$

Moreover the non bounded measure $\gamma_{\lambda}^{-}=e^{\lambda|x|^{2}}$ is a symmetric measure for the associated semi-group $P_{t}^{-O U(\lambda)}$ (easily seen on (5.1)).
It remains to study the contractivity properties of $P_{t}^{-O U(\lambda)}$. Actually using standard results in p.d.e. theory or Malliavin calculus one can show that the associated heat kernel is bounded for all $t>0$, hence $P_{t}^{-O U(\lambda)}$ is ultracontractive. We shall give below a proof in the spirit of Theorem 3.8 or [13].
Proposition 5.2. The semi group $P_{t}^{-O U(\lambda)}$ is ultracontractive, more precisely

$$
\left\|P_{t}^{-O U(\lambda)}\right\|_{\mathbb{L}^{2}\left(\gamma_{\lambda}^{-}\right), \mathbb{L}^{\infty}} \leq(4 \pi t)^{\frac{-N}{4}} e^{-\frac{\lambda}{2} N t}
$$

Proof. Pick some test function $f$ and define

$$
M_{t}=e^{-\frac{1}{2} \int_{0}^{t} \lambda^{2}\left|X_{s}\right|^{2} d s}
$$

Since $g=|f| e^{\frac{\lambda}{2}|x|^{2}} \in \mathbb{L}^{2}(d x)$ and since $M_{t} \leq 1$, for $t>0$, it holds

$$
\begin{aligned}
P_{t}^{-O U(\lambda)}(|f|)(x) & =e^{-\frac{\lambda}{2}\left(|x|^{2}+N t\right)} \mathbb{E}^{\mathbb{P}_{x}}\left[M_{t} g\left(X_{t}\right)\right] \\
& \leq e^{-\frac{\lambda}{2}\left(|x|^{2}+N t\right)} P_{t}(g)(x)
\end{aligned}
$$

The result follows by density.
Remark 5.3. The previous result (as well as its generalization to $e^{G}$ for some bounded below convex $G$ ) is already stated in [13] Remarque 2.2. However the proof lies on the Girsanov transform theory which is no more available when $G$ has a super-quadratic growth at infinity. Hence in order to use the perturbation theory we have built, the limitation to quadratic growth is essential. Actually Herbst argument shows that it is also an intrinsic limitation.
But of course the previous construction (5.1) extends to general sub-quadratic functions, i.e. to any $G$ such that $|\nabla G| \leq C(1+|x|)$. If $d \mu_{G}=e^{2 G} d x$ and assuming

$$
(\nabla G)^{2}-\Delta G \geq-2 c \text { and } G \geq-d
$$

one easily get that Proposition 5.2 is still available, more precisely

$$
\left\|P_{t}^{-G}\right\|_{\mathbb{L}^{2}\left(\mu_{G}\right), \mathbb{L}^{\infty}} \leq(4 \pi t)^{\frac{-N}{4}} e^{d+c t}
$$

In the sequel we will consider functions $F$ that are of class $C^{2}$ and according to Proposition 3.2 we shall then (if necessary) add to $F$ some bounded perturbation.

The most famous example of admissible perturbations is the strictly log-concave situation, that is when $\Delta F \geq \rho I d$ for some $\rho>0$. In this case $\nu_{F}$ satisfies (TLSI) with $a=\frac{1}{\rho}$. This is a consequence of the celebrated Bakry-Emery criterion (see e.g. [21] Théorème 3.1.29 or [3] chapter 5). The same is true (with a modified constant) for $F+U$ if $F$ is as above and $U$ either bounded or Lipschitz (see [18] for Miclo's argument in the latter case).

Another well known situation is the one dimensional case where a necessary and sufficient condition was obtained by Bobkov and Götze in 1999 (see [3] chapter 6). This beautiful criterion is written in terms of Muckenhoupt weights and does not require any regularity for $F$. In chapter 6.4 of [3] the authors Malrieu and Roberto study some related sufficient conditions for regular functions. Their result (théorème 6.4.3) reads as follows
Proposition 5.4. Malrieu and Roberto.
If $F$ is of $C^{2}$ class on $\mathbb{R}$ and satisfies both $\left|F^{\prime}(x)\right|>0$ for $|x|$ large enough and $\frac{F^{\prime \prime}(x)}{\left|F^{\prime}(x)\right|^{2}}$ goes to 0 as $|x|$ goes to $\infty$, then $\nu_{F}$ satisfies (TLSI) (resp. (SGP)) if and only if there exists some A such that

$$
\frac{F}{\left|F^{\prime}\right|^{2}}+\frac{\log \left|F^{\prime}\right|}{\left|F^{\prime}\right|^{2}} \quad\left(\text { resp. } \frac{1}{F^{\prime}}\right)
$$

is bounded on $\{|x| \geq A\}$.
Since (TLSI) implies (SGP) the condition for (TLSI) actually splits into $\frac{1}{F^{\prime}}$ and $\frac{F}{\left|F^{\prime}\right|^{2}}$ are bounded on $\{|x| \geq A\}$. Note that these conditions (together with $\frac{F^{\prime \prime}(x)}{\left|F^{\prime}(x)\right|^{2}}$ goes to 0 as $|x|$ goes to $\infty$ ) imply that $\mathrm{B}(\mathrm{F})(2)$ is satisfied. Since $F^{\prime}$ is strictly positive (resp. negative) near $+\infty$ (resp. $-\infty$ ) for $\nu_{F}$ to be a Probability measure, it also follows that $F$ is super-linear at infinity, thus $\mathrm{B}(\mathrm{F})(1)$ is satisfied and $e^{F}$ belongs to any $\mathbb{L}^{r}\left(\nu_{F}\right)$ for $0<r<2$. Furthermore, for any $u>0, e^{-u F}$ is convex at infinity and goes to 0 . Hence its derivative is non decreasing and negative near $+\infty$, thus has a limit which is necessarily 0 (the same holds at $-\infty$ ). It follows that $\left|F^{\prime}\right|^{2} e^{-2 u F}$ goes to 0 at infinity, for any $u>0$. Hence $\left|F^{\prime}\right|^{2} e^{-2 F}$ is integrable (since $e^{-v F}$ is integrable for $v>0$ ), so that $\mathrm{H}(\mathrm{F})$ holds too.
We achieve to see that conditions in Proposition 5.4 are more restrictive that the ones in Theorem 4.8. Together with Theorem 3.8 and the Ultracontractivity of the Brownian semigroup, Theorem 4.8 is showing that (DLSI) holds. Thus, provided we are able to directly prove (SGP) or (WSGP), the result by Malrieu and Roberto is contained in our results.
Actually what we obtained is some generalization of their result to the N -dimensional case. Before to see this point let us make an additional remark concerning (H.F). Instead of using the general perturbation theory for Dirichlet forms (yielding the statement of (H.F)) one can use the local compactness of $\mathbb{R}^{N}$ in order to build the perturbed semi group $P_{t}^{F}$. Indeed one may use Novikov criterion up to the exit times of compact subsets, integrate by parts and then look at the behaviour of the integrated form of the stopped Girsanov density. This study is done in [21] p.26-28. It is shown in particular that provided

$$
|\nabla F|^{2}-\Delta F
$$

is bounded from below (by a possibly negative constant) (3.6) is still hold. Of course (B.F) implies the above lower bound, and (H.F) is not necessary in this case.

We now quote a result on (SGP) we found in a recent paper by Kunz ([14] Proposition 3.7). For completeness we also give the main elements in Kunz's proof.
Proposition 5.5. Let $F$ be of $C^{2}$ class. If

$$
\liminf _{|x| \rightarrow+\infty}\left(|\nabla F|^{2}-\Delta F\right)=C>0
$$

then (SGP) holds for $\nu_{F}$.
Proof. First remark that the condition in the Theorem implies that $|\nabla F|^{2}-\Delta F$ is bounded below.
The unitary transform $U: \mathbb{L}^{2}\left(\mathbb{R}^{N}, d x\right) \rightarrow \mathbb{L}^{2}\left(\mathbb{R}^{N}, d \nu_{F}\right)$ defined by $U(f)=e^{F} f$ satisfies

$$
\int \nabla(U(f)) \cdot \nabla(U(g)) d \nu_{F}=\int\left(\nabla f . \nabla g+V_{F} f g\right) d x
$$

where $V_{F}=|\nabla F|^{2}-\Delta F$. The latter Dirichlet form is the one associated with the Schrödinger operator $H_{F}=\frac{1}{2}\left(-\Delta+V_{F}\right)$. Since $U$ is unitary the spectrum of $H_{F}$ on $\mathbb{L}^{2}(d x)$ and the one of $-A_{F}$ on $\mathbb{L}^{2}\left(\nu_{F}\right)$ coincide. It is known that the former is discrete on ] $-\infty, C$ (see [5] Theorem 3.1 p.165), thus so is the latter for which 0 is thus an isolated eigenvalue, i.e. (SGP) holds.

This result allows to extend Proposition 5.4 to the multidimensional setting as follows.
Theorem 5.6. Let $F$ be of $C^{2}$ class be such that
(1) $F(x) \rightarrow+\infty$ as $x$ goes to $\infty$ and $e^{-u F}$ is integrable for some $u<1$ (so that $e^{-2 F}$ is also integrable and as before we assume that it is normalized for $\nu_{F}$ to be a Probability measure),
(2) there exist some $\lambda>0$ and some $c$ such that for all $x \in \mathbb{R}^{N}$,

$$
\frac{1}{2}\left(|\nabla F|^{2}(x)-(\Delta F)(x)\right) \geq \lambda F(x)+c
$$

then $\nu_{F}$ satisfies (TLSI).
Proof. We shall apply Theorem 3.8 with the Brownian semi-group $P_{t}$. According to Theorem 4.8 (where (H.F) was only required for (3.6) to hold), (DLSI) will hold as soon as $\mathrm{B}(\mathrm{F})$ is fulfilled (since $e^{F} \in \mathbb{L}^{r}\left(\nu_{F}\right)$ for $r=2-u>1$ ). $\mathrm{B}(\mathrm{F})$ (2) is assumed while $\mathrm{B}(\mathrm{F})$ (1) is immediate since $F$ goes to $+\infty$ at infinity.
Since $|\nabla F|^{2}-\Delta F$ goes to $+\infty$ at infinity we may apply Proposition 5.5 in order to show that (SGP) holds. Since (SGP) together with (DLSI) implies (TLSI), the proof is completed.
Remark 5.7. If Malrieu-Roberto condition is some convexity assumption on $e^{-F}$, condition (3) seems difficult to express in a similar way (thanks to F . Barthe for pointing out to me this mistake). In particular we do not know whether (2) is a consequence of (a possibly strengthened version of) (3) or not.
Remark 5.8. The result in Proposition 5.5 is not quantitative, i.e. does not furnish estimates for the spectral gap. One can in many cases use instead Theorem 3.18 with some gaussian measure for $\mu$. In particular if $|\nabla F|^{2}(x) \leq e^{\alpha x^{2}}$ at infinity for some nonnegative $\alpha$, we obtain (TLSI) with explicit (but not sharp) bounds. Replacing the gaussian measure par $\nu_{G}$ with such a $G$ (for instance $e^{x}$ ) allows to cover the cases $|\nabla F|^{2}(x) \leq e^{G(x)}$ and so on. We
do not know whether it is possible to recover the full situation of Proposition 5.5 with this kind of argument or not.

Theorem 5.6 is available for many functions $F$ that are uniformly convex at infinity (i.e. such that $\operatorname{Hess}(F)(x) \geq \rho I d$ for some positive $\rho$ and all large enough $|x|)$. In particular it shows strong hypercontractivity for the Ornstein-Uhlenbeck process (but we cannot use Theorem 3.18 to get bounds, we really need Proposition 5.5). Here are others examples.

### 5.9 Examples.

(1) $F(x)=|x|^{\beta}$ for $\beta>2$. Then one can check the hypotheses of Theorem 4.10 with $G(y)=\frac{1}{2} y^{\frac{2(\beta-1)}{2}}$ and a sequence $\alpha_{k}=k^{\frac{1}{2}-\frac{1}{\beta}}$. Hence ultracontractivity holds and (TLSI) holds too.
(2) $F(x)=\left(1+|x|^{2}\right)\left(\log \left(1+|x|^{2}\right)\right)^{\beta}$. Then for $\beta>1$ one can again check the hypotheses of Theorem 4.10 with $\alpha_{k}=e^{k}$ and $G(y)=y(\log (y))^{\beta^{\prime}}$ for any $\beta^{\prime}$ such that

$$
1<\beta^{\prime}<\beta
$$

For $\beta=1$ it is easily seen that all the hypotheses in Theorem 5.6 are satisfied. In particular hypothesis 5.6 (3) is satisfied for all $\lambda>0$ so that the semi-group is supercontractive. As shown in [13] it is not ultracontractive.
(3) $F(x)=\left(1+|x|^{2}\right) \log \left(1+|x|^{2}\right)\left(\log \left(\log \left(e+|x|^{2}\right)\right)\right)^{\beta}$. Again one can show ultracontractivity for $\beta>1$.
But even in one dimension, one can find uniformly convex functions that do not satisfy the hypotheses in Theorem 5.6. Here is such an example. For simplicity the construction below does not furnish a $C^{2}$ function (the second derivative is not continuous on a discrete set), but can easily be modified in order to satisfy this condition too. Hence even in one dimension Theorem 5.6 (and consequently Malrieu-Roberto theorem) does not contain the uniformly convex case.

### 5.10 Example.

Let $\alpha_{n}, \beta_{n}$ and $g_{n}$ be three sequences of nonnegative real numbers such that $\alpha_{n}$ and $\beta_{n}$ go towards 0 and $g_{n}$ goes to $+\infty$. We define a function $g$ on $[1,+\infty[$ by

$$
g(x)=g_{n} \quad \text { if } n \leq x \leq n+\alpha_{n} \quad g(x)=\beta_{n} \quad \text { if } n+\alpha_{n}<x<n+1
$$

We will build $F$ such that $\frac{F^{\prime \prime}}{\left(F^{\prime}\right)^{2}}=g$ (at least on the continuity set of $g$ ). It thus holds

$$
F^{\prime \prime}(x)=g(x)\left(\int_{x}^{+\infty} g(t) d t\right)^{-2} \quad \text { and } \quad F^{\prime}(x)=\left(\int_{x}^{+\infty} g(t) d t\right)^{-1}
$$

It follows that

$$
\limsup _{x \rightarrow+\infty}\left(\frac{F^{\prime \prime}}{\left(F^{\prime}\right)^{2}}\right)=+\infty
$$

so that the hypotheses of Theorem 5.6 are not satisfied, while one can choose the sequences in such a way that $F$ is uniformly convex. To this end it is enough that there exists some $c>0$ such that

$$
\beta_{n} \geq c\left(\sum_{k \geq n} \alpha_{k} g_{k}+\beta_{k}\left(1-\alpha_{k}\right)\right)^{2}
$$

A possible choice is $\beta_{k}=k^{-2}$ and then any $g_{k} \gg k^{2}$ and $\alpha_{k}$ such that $\alpha_{k} g_{k} \ll k^{-2}$. In the previous example we have seen that one can build uniformly convex functions that do not satisfy the hypotheses of Theorem 5.6 because the second derivative can be locally much bigger that the first one. It clearly indicates that the third hypothesis in this Theorem has to be improved into a non pointwise one, certainly an integral one.

Remark 5.11. Note that if $\operatorname{Hess}(F)(x) \geq \rho I d$ for some positive $\rho$ and all $x$, then for all $x$ the function $g_{x}: t \rightarrow F(t x)$ satisfies $g_{x}^{\prime \prime} \geq \rho|x|^{2}$. Furthermore

$$
g_{x}^{\prime}(t)=x . \nabla F(t x) \geq t \rho|x|^{2}-b
$$

for some constant $b$. Thus, if $|x|=1$ for $t>t_{0}=\frac{b}{\rho}$ one has $g_{x}^{\prime}(t)>0$. Hence $2 g_{x}^{\prime} g_{x}^{\prime \prime} \geq 2 \rho g_{x}^{\prime}$ and integrating the previous we get $\left(g_{x}^{\prime}\right)^{2}(t) \geq 2 \rho\left(g_{x}(t)-g_{x}\left(t_{0}\right)\right)-\left(g_{x}^{\prime}\right)^{2}\left(t_{0}\right)$. Then using Cauchy-Schwartz and compactness of the unit sphere, one has

$$
|\nabla F|^{2}(t x) \geq\left(g_{x}^{\prime}\right)^{2}(t) \geq 2 \rho F(t x)-C
$$

where $C=\sup _{|x|=1}\left\{2 \rho g_{x}\left(t_{0}\right)+\left(g_{x}^{\prime}\right)^{2}\left(t_{0}\right)\right\}$.
Hence the only obstacle for an uniformly convex function to fulfill the conditions in 5.6 is the asymptotic behaviour of the second derivatives, as we have seen above.

Nevertheless remark the following : in Example (5.10) define $G^{\prime \prime}(x)=\frac{\beta_{k}}{g(x)} F^{\prime \prime}(x)$ for $k<$ $x \leq k+1$. With our choice $\beta_{k}=k^{-2}$, $G^{\prime \prime}$ goes to 1 at infinity, hence the corresponding $G$ is not only uniformly convex at infinity but also satisfies the hypotheses in Theorem 5.6. Furthermore $\left(F^{\prime}-G^{\prime}\right)(x)$ is bounded by $\sum_{k} g_{k} \alpha_{k} k^{2}$. Hence if the latter sum is finite $F-G$ is Lipschitz. Following the already quoted argument of Miclo, one can write

$$
F=(G+(F-G) * N(\sigma))+((F-G)-((F-G) * N(\sigma)))
$$

for the gaussian kernel $N(\sigma)$ with zero mean and variance $\sigma^{2}$, and show that for $\sigma$ large enough $(G+(F-G) * N(\sigma))$ is still satisfying the hypotheses in 5.6 , while $(F-G)-((F-G) * N(\sigma))$ is bounded (see the calculations in [18]). Hence we can combine Theorem 5.6 and Proposition 3.2 in order to prove (TLSI) for $\nu_{F}$.

The previous remark is suggesting that any uniformly convex function can be written as a sum of a nice function (satisfying 5.6) and a bounded function. We do not know whether this is true or not in full generality.

The set of uniformly convex functions (at infinity) is a natural benchmark for any study on (TLSI) since according to Herbst argument (see Proposition 2.9)

$$
\begin{equation*}
\text { if } \nu_{F} \text { satisfies (TLSI) with constant } a \text { then } \int e^{\varepsilon|x|^{2}} d \nu_{F}<+\infty \text { for all } \varepsilon<\frac{1}{a} \tag{5.12}
\end{equation*}
$$

It is thus natural to ask about what can happen when $F>\theta|x|^{2}$ at infinity for some $\theta>0$. This question is essentially solved in a series of papers by Wang (see [23], [22], [24] and [25]) whose main result is called "inverse Herbst argument" in [3] (chapter 7). The most achieved form of Wang's result reads as follows
Theorem 5.13. Wang (see [25] Theorem 1.1)
Assume that $\operatorname{Hess}(F) \geq-K$ for some $K \geq 0$.
Then $\nu_{F}$ satisfies (TLSI) provided $\int e^{\varepsilon|x|^{2}} d \nu_{F}<+\infty$ for some $\varepsilon>K$.

Conversely one can find an example of $F$ such that $\operatorname{Hess}(F) \geq-K, \int e^{\varepsilon|x|^{2}} d \nu_{F}<+\infty$ for $\varepsilon<\frac{K}{2}$ but $\nu_{F}$ does not satisfy (TLSI).
Wang's proof is a subtle application of semi-group methods, in particular Harnack's inequalities, using the curvature property of $P_{t}^{F}$ in the sense of Bakry.
Of course since $\operatorname{Hess}(F) \geq-K$ then $F_{\rho}=F+\rho|x|^{2}$ is uniformly convex for all $\rho>K$. So if we replace the Wiener measure by $\mathbb{P}_{x}^{-O U(2 \rho)}$, and $d x$ by $\gamma_{2 \rho}^{-}=e^{2 \rho|x|^{2}}$, we can see $\nu_{F}$ as a perturbation of $\gamma_{2 \rho}^{-}$replacing $F$ by $F_{\rho}$, and use the ultracontractivity of $\mathbb{P}_{t}^{-O U(2 \rho)}$ we have shown in Proposition 5.2 instead of the one of the Brownian semi-group. This yields the following modified version of Wang's result
Theorem 5.14. Let $F$ be of $C^{2}$ class and $\rho>0$. Define $F_{\rho}=F+\rho|x|^{2}$ and assume that
(1) $F_{\rho}(x) \rightarrow+\infty$ as $x$ goes to $\infty$ and $e^{-u F}$ is integrable for some $u<1$ (so that $e^{-2 F}$ is also integrable and as before we assume that it is normalized for $\nu_{F}$ to be a Probability measure),
(2) there exist some $\lambda>0$ and some $c$ such that for all $x \in \mathbb{R}^{N}$,

$$
\frac{1}{2}\left(\left|\nabla F_{\rho}\right|^{2}(x)-\left(\Delta F_{\rho}\right)(x)\right) \geq \lambda F_{\rho}(x)+c
$$

then provided

$$
\int e^{k \rho|x|^{2}} d \nu_{F}<+\infty
$$

for some $k>\frac{2-u}{1-u}, \nu_{F}$ satisfies (TLSI).
Proof. First notice that (3.6) holds for both $\nu_{F}$ and $\nu_{F_{\rho}}$. Since $F_{\rho}$ satisfies the hypotheses of Proposition 5.5 we know that (SGP) holds for $\nu_{F_{\rho}}$. But $\nu_{F}=e^{2 \rho|x|^{2}} \nu_{F_{\rho}}$ and $\int|x|^{2} d \nu_{F_{\rho}}<$ $+\infty$. Thus we may apply Theorem 3.18 in order to get that $\nu_{F}$ satisfies (WSGP). (We should have normalized all measures into probability measures, but this only introduces multiplicative constants. However constants play a role if one wants to determine precise bounds).
Next we shall apply Theorem 3.8 with $\mathbb{P}_{t}^{-O U(2 \rho)}$. According to Theorem 4.8, (DLSI) will hold since $\mathrm{B}(\mathrm{F})$ is fulfilled for $F_{\rho}$ provided there exists some $r>1$ with $e^{F_{\rho}} \in \mathbb{L}^{r}\left(\nu_{F}\right)$ i.e. if

$$
\int e^{r \rho|x|^{2}} e^{r F} d \nu_{F}<+\infty
$$

Applying Hölder and optimizing in $r>1$ one sees that a sufficient condition is the one in the statement of the Theorem.

Remark that if we can choose $u$ as small as we want, we recover the condition in [22] or [3] chapter 7, and not the improved bound in Theorem 5.13 (at least if $F_{\rho}$ is a nice uniformly convex function).

We shall finish this section by studying a final class of examples. If the situation for a bounded from below Hessian is almost completely understood, one may ask about the non bounded case. The following class of examples will also explain why we said that ( BF ) is almost necessary for (TLSI) to hold.

Example 5.15. Let us consider on $\mathbb{R}^{+}$the potential $F_{\beta}(x)=x^{2}+\beta x \sin (x)$ extended by symmetry to the full real line. We shall only look at its behaviour near $+\infty$.
The derivatives are given by $F_{\beta}^{\prime}(x)=(2+\beta \cos (x)) x+\beta \sin (x)$ and $F_{\beta}^{\prime \prime}(x)=(2-$ $\beta \sin (x)) x+2(1+\beta \cos (x))$. Hence $-\infty=\liminf _{x \rightarrow+\infty} F^{\prime \prime}(x)$.
For $|\beta|<2$ we may apply Malrieu-Roberto result (or Theorem 5.6) and show that (TLSI) holds.
For $|\beta| \geq 2$ the hypotheses of Theorem 5.6 are no more satisfied. It is easily seen however that (WSGP) holds (comparing again with some Ornstein-Uhlenbeck process). Of course one can try to see what happens with the Bobkov-Götze criterion both for (SGP) and (TLSI). We prefer to directly look at $P_{t}^{F_{\beta}}\left(e^{F_{\beta}}\right)$ in order to show that (DLSI) does not hold for $\beta=-2$. In what follows we simply use $F$ instead of $F_{-2}$.
$e^{F} \in \mathbb{L}^{p}\left(\nu_{F}\right)$ for all $p<2$. Hence for (DLSI) to hold it is necessary that for $t$ large enough $P_{t}^{F}\left(e^{F}\right) \in \mathbb{L}^{q}\left(\nu_{F}\right)$, that is $e^{(q-2) F}\left(\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]\right)^{q} \in \mathbb{L}^{1}(d x)$ for $q>2$.
Introduce $x_{k}=2 k \pi$. Then for $k$ large enough one can find $\varepsilon$ small enough and some constant $c$ such that
(5.16) for all $y$ such that $\left|y-x_{k}\right| \leq k^{-\frac{1}{2}}$ it holds $F^{\prime \prime}(y) \geq(2-\varepsilon) k \quad$ and $\quad\left|F^{\prime}(y)\right| \leq c$,

Introduce the stopping times $\tau_{k}=\inf \left\{s \geq 0,\left|X_{s}-y\right| \geq \frac{1}{2} k^{-\frac{1}{2}}\right\}$. Then according to (5.16), for $\left|y-x_{k}\right| \leq \frac{1}{2} k^{-\frac{1}{2}}$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}_{y}}\left(M_{t}\right) & \geq \mathbb{E}^{\mathbb{P}_{y}}\left(M_{t} \mathbb{I}_{t<\tau_{k}}\right) \\
& \geq e^{\frac{t}{2}\left((2-\varepsilon) k-c^{2}\right)} \mathbb{P}_{y}\left(t<\tau_{k}\right) \\
& \geq e^{\frac{t}{2}\left((2-\varepsilon) k-c^{2}\right)} e^{-\theta t k}
\end{aligned}
$$

for some constant $\theta$ such that for a standard Brownian motion

$$
\mathbb{P}\left(\sup _{0 \leq s \leq t}\left|B_{s}\right|<A\right) \geq e^{-\theta \frac{t}{A^{2}}}
$$

It follows

$$
\int^{+\infty} e^{(q-2) F}\left(\mathbb{E}^{\mathbb{P}_{x}}\left[M_{t}\right]\right)^{q} d x \geq \sum_{k} k^{-\frac{1}{2}} e^{4 \pi^{2}(q-2-\varepsilon) k^{2}} e^{-\theta^{\prime} t k}=+\infty
$$

Hence (DLSI) does not hold.
Note that for $|\beta|>2$ the previous scheme does not work so easily because we have to choose intervals of size $k^{-1}$ instead of $k^{-\frac{1}{2}}$. Actually we do not know what happens for $|\beta|>2$.

## 6. Others Examples.

In this section we shall briefly indicate some of the possible extensions of the results we have obtained in the previous section.
6.1. Riemannian Manifolds. We may replace the (flat) manifold $\mathbb{R}^{N}$ by some $N$-dimensional (noncompact) connected and complete Riemannian manifold $M .|x|$ will be replaced by the riemannian distance $\rho(x)$ between $x$ and some fixed $x_{0}, \Delta$ is then the Laplace-Beltrami operator.
If the Ricci curvature $\operatorname{Ric}(M)$ is nonnegative, the Brownian semi group on $M$ is known to be ultracontractive, provided the geometry is bounded (i.e. the volume of unit balls is uniformly bounded on $M$ ). If the riemannian distance is smooth enough, we may replace $|x|^{2}$ by $\rho^{2}(x)$ in order to define the O-U and the -O-U semigroups. According to [25] Corollary 1.6. the O-U semi-group is strongly hypercontractive, provided $\operatorname{Ric}(M)$ is bounded from below (by some possibly negative constant).
Hence we can use either Theorem 3.8 (when $\operatorname{Ric}(M)>0$ and a bounded geometry) or Theorem 3.10 (when $\operatorname{Ric}(M)$ is bounded from below), combined with (B.F) in order to get (DLSI). Theorem 3.18 can then be used in order to get (WSGP), hence (TLSI).
Of course we cannot recover Wang's result ([25] Theorem 1.1) when $\operatorname{Ric}(M)+\operatorname{Hess}(F)$ is bounded from below, but $\operatorname{Ric}(M)$ is not.
6.2. Subelliptic operators. On $\mathbb{R}^{N}$ (or on $M$ as above) consider $n$ vector fields $\left(V_{i}\right)_{i=1, \ldots, n}$ that are smooth $\left(C^{\infty}\right)$ bounded with bounded derivatives of any order. Then if some smooth $G$ satisfies

$$
\sum_{i=1}^{n}\left(V_{i} G\right) V_{i}+\sum_{i=1}^{n}\left(\operatorname{div} V_{i}\right) V_{i}=2 V_{0}^{G}
$$

the semi group generated by the second order operator $A^{G}=\frac{1}{2} \sum_{i=1}^{n}\left(V_{i}\right)^{2}+V_{0}^{G}$ written in Hörmander's form is symmetric with respect to $e^{G} d x$. We will use either $G=0$ or $G(x)= \pm \lambda|x|^{2}$.
If $F$ is $C^{2}$, the weighted logarithmic Sobolev inequality (TLSI) will be

$$
\begin{equation*}
\int f^{2} \log \left(\frac{f^{2}}{\|f\|_{\mathbb{L}^{2}\left(\nu_{F}\right)}^{2}}\right) e^{-2 F} d x \leq a \int \sum_{i=1}^{n}\left|V_{i} f\right|^{2} e^{-2 F} d x \tag{6.1}
\end{equation*}
$$

Conditions on $F$ for (TLSI) to hold, or the associated perturbed semi group to be hypercontractive, supercontractive or ultracontractive have been obtained by S. Kusuoka and D. Stroock in [15]. As we already said, this paper contains for the first time (up to our knowledge) condition (B.F) and actually a slightly weaker condition, namely

$$
\begin{equation*}
\text { (B.F. } \left.\alpha) \quad \frac{1}{2} \alpha \Gamma(F, F)(x)-A F(x)\right) \geq \lambda F(x)+c \tag{6.2}
\end{equation*}
$$

for some $\alpha$ satisfying $0 \leq \alpha<2$ (see [15] (2.5)). Hence our (B.F) is their (B.F.1). The main additional assumption in [15] is that $F$ is of $C^{\infty}$ class. This is due to the strategy of proof that is using in an essential way Malliavin Calculus for the perturbed semi-group $P_{t}^{F}$.
This strategy furnishes (in the present frame) an analogue to Theorem 3.8, i.e. that (DLSI) holds as soon as one can check the hypercontractive property for $e^{\alpha F}$ (see (2.1) and (2.2) in [15]). We do not know how to apply this strategy in the general framework of section 3.

However the second part of their strategy (showing that (B.F. $\alpha$ ) implies (DLSI)) works in a very general framework and furnishes an alternative to the "Well Method". We shall describe it quickly below in the case $\alpha=1$ (in order to use Theorem 3.8 or Theorem 3.10).

## Kusuoka-Stroock semi-group method.

Define

$$
\psi(t, x)=\exp \left(e^{-\lambda t} F(x)-\frac{c}{\lambda}\left(1-e^{-\lambda t}\right)\right)
$$

Assume that $e^{u F} \in \mathbb{L}^{1}\left(\nu_{F}\right)$ for all $0 \leq u<2$. Then applying Ito's formula to $\psi(T-$.,.) up to the exit time of the level sets of $F$, and using bounded convergence theorem it is not difficult to check that

$$
\begin{aligned}
P_{T}^{F} & \left(e^{F}\right)(x)-\psi(T, x)=\mathbb{E}_{x}^{F}\left[\psi\left(0, X_{T}\right)\right]-\mathbb{E}_{x}^{F}\left[\psi\left(T, X_{0}\right)\right]= \\
& =\int_{0}^{T} \mathbb{E}_{x}^{F}\left[-\frac{\partial \psi}{\partial t}\left(T-t, X_{t}\right)+A^{F} \psi\left(T-t, X_{t}\right)\right] d t
\end{aligned}
$$

But

$$
\begin{aligned}
-\frac{\partial \psi}{\partial t}+A^{F} \psi & =e^{-\lambda t} \psi\left(\lambda F+c+A F-\left(1-\frac{1}{2} e^{-\lambda t}\right) \Gamma(F, F)\right) \\
& \leq e^{-\lambda t} \psi\left(-\frac{1}{2}\left(1-e^{-\lambda t}\right) \Gamma(F, F)\right) \\
& \leq 0
\end{aligned}
$$

It follows that

$$
P_{T}^{F}\left(e^{F}\right)(x) \leq \exp \left(e^{-\lambda T} F(x)-\frac{c}{\lambda}\left(1-e^{-\lambda T}\right)\right)
$$

hence $P_{T}^{F}\left(e^{F}\right) \in \mathbb{L}^{p}\left(\nu_{F}\right)$ as soon as $T>\frac{1}{\lambda} \log \left(\frac{p}{2}\right)$.
Note that the lower bound for $T$ is better than the one obtained in Theorem $4.8\left(T>\frac{p-2}{2 \lambda}\right)$ with the same asymptotics when $p$ goes to 2 .

In order to combine this and Theorem 3.8 for subelliptic operators, we have first to know that $A$ is ultracontractive. This is an immediate consequence of the gaussian behaviour of the heat kernel associated with $A$ when the vector fields satisfy the (restricted) Hörmander condition

$$
\begin{equation*}
\inf _{x \in \mathbb{R}^{N}} \inf _{\eta \in \mathbb{S}^{N-1}} \sum_{|I| \leq L}\left|V_{I}(x) \cdot \eta\right|^{2} \geq \varepsilon>0 \tag{6.3}
\end{equation*}
$$

for some $L \geq 1$ where $I$ is some multi-index $\left(i_{1}, \ldots, i_{k}\right)$ with length $|I|=k$, and $V_{I}$ denotes the Lie bracket of order $k, V_{I}=\left[\left[\ldots\left[V_{i_{1}}, V_{i_{2}}\right], \ldots\right], V_{i_{k}}\right]$ (see e.g. [16]).

Finally, in order to show (TLSI) one can either use the spectral gap result shown in [15] Theorem (2.30) when $F$ is very smooth, or use Theorem 3.18 with $\mu$ the gaussian law, since $A^{G}$ satisfies (TLSI) for $G=x^{2}$ according to the smooth case, provided $F$ satisfies the integrability condition in Theorem 3.18 (one can then improve step by step this condition as we said in Remark 5.8).

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