A microscopic probabilistic description of a locally regulated population and macroscopic approximations

Nicolas FOURNIER[1] and Sylvie MÉLÉARD[2]

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Abstract

We consider a discrete model of locally regulated spatial population with mortality selection, introduced by Bolker and Pacala, [2]. We first generalize this model by adding spatial dependence, and give a pathwise description in terms of Poisson point measures. We then show that different renormalizations may lead to different macroscopic approximations of this model. We consider two specific cases. The first approximation is deterministic and gives a rigorous sense to the number density; the second one is a measure-valued process already studied by Etheridge [5]. Finally, we study in particular cases the long time behaviour of the system and reasonnable equilibria for the deterministic approximation.

Key words: Interacting measure-valued processes, Regulated population, Deterministic macroscopic approximation, Nonlinear superprocess, Equilibrium.

MSC 2000: 60J80, 60K35.

1 Introduction

We consider a spatial ecological system with mortality selection, where individuals, similarly to perennial plants, can reproduce, dispersing their offspring locally, and die, with a rate depending on their local density.
An approach to study this system was introduced in Bolker-Pacala [2] and consisted in modeling spatial interactions by deriving approximations for the time evolution of the moments (mean and spatial covariance) of distributions of individuals.
In this paper, we give a stochastic microscopic description of systems generalizing the one introduced in [2] by adding a spatial dependence in all the rates and in the interaction potential. We prove the existence and uniqueness of such systems thanks to a pathwise representation through Poisson point measures. The main difficulty is to take into account the mortality selection which appears as an interaction between the individuals, at the microscopic level.
Then, we refind in the Bolker-Pacala case the mean equation they intuitively obtained and give a rigorous sense to the covariance terms formally defined in [2].
Next, we prove how the empirical measure of such systems, conveniently renormalized, converges to the solution of a nonlinear partial integro-differential equation and we propose this as a rigorous interpretation of the density number. We also show that with another renormalization, the empirical measure of our system converges to the superprocess version of the Bolker-Pacala model,

[1]Institut Elie Cartan, Faculté des sciences, BP 239, 54506 Vandoeuvre-lès-Nancy Cedex, e-mail: fournier@iecn.u-nancy.fr
[2]Université Paris 10, MODALX, 200 av. de la République, 92000 Nanterre et Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 6, 4 place Jussieu, case 188, 75252 Paris cedex 05, e-mail: sylm@ccr.jussieu.fr

introduced and studied by Etheridge [5].
Finally, we study the long time behaviour of the system in specific cases. In Section 6, we prove the extinction of the process in a compact case, and a partial result on survival conditions. In Section 7, we are interested in the existence of non trivial equilibria in the Bolker-Pacala case. We firstly consider the deterministic limiting equation and show that under restrictive assumptions on the coefficients, there is a unique reasonnable equilibrium equal to the carrying capacity. In the specific detailed balance case, or if the initial condition is closed to the equilibrium, we prove the convergence to this equilibrium. Finally and again under the detailed balance condition, we exhibit a nontrivial equilibrium for the Bolker-Pacala process. We end the paper by some simulations.

2 The model

We consider a spatial ecological process where individuals can reproduce, dispersing locally their offspring and die, with rates depending on the position of each individual and on their local density. These events occur randomly, in continuous time. All individuals are identical and motionless once they have dispersed from their parents. As soon as they are born, as soon they disperse. Let us now describe the parameters of the model.

2.1 Definition of the parameters and heuristics

We will consider the following model. The plants have there locations in the closure $\bar{\mathcal{X}}$ of an open connected subset $\mathcal{X}$ of $\mathbb{R}^d$, for some $d \geq 1$. We will denote by $M_F(\bar{\mathcal{X}})$ the set of finite nonnegative measures on $\bar{\mathcal{X}}$, by $\mathcal{P}(\bar{\mathcal{X}})$ the set of probability measures on $\bar{\mathcal{X}}$ and by $\mathcal{M}$ the subset of $M_F(\bar{\mathcal{X}})$ consisting in all finite point measures, that is

$$\mathcal{M} = \left\{ \sum_{i=1}^{n} \delta_{x_i},\ n \geq 0, x_1, ...x_n \in \bar{\mathcal{X}} \right\} \tag{2.1}$$

where δ_x denotes the Dirac mass at x. For any $m = \sum_{i=1}^{n} \delta_{x_i} \in \mathcal{M}$, any measurable function f on $\bar{\mathcal{X}}$, we will denote $\langle m, f \rangle = \int_{\bar{\mathcal{X}}} f dm = \sum_{i=1}^{n} f(x_i)$.

Notation 2.1 *For all x in $\bar{\mathcal{X}}$, we introduce the following quantities:*
(i) $\mu(x) \in [0, \infty[$ is the rate of "natural" death of plants located at x,
(ii) $\gamma(x) \in [0, \infty[$ is the rate of seed production of plants located at x,
(iii) $D(x, dz)$ is the dispertion measure of the seeds of plants located at x, it is assumed to satisfy, for each $x \in \bar{\mathcal{X}}$, $\int_{z\in\mathbb{R}^d,\ x+z\in\bar{\mathcal{X}}} D(x, dz) = 1$ and $\int_{z\in\mathbb{R}^d,\ x+z\notin\bar{\mathcal{X}}} D(x, dz) = 0$.
(iv) $\alpha(x) \in [0, \infty[$ is the rate of interaction of plants located at x,
and, for x, y in $\bar{\mathcal{X}}$,
(v) $U(x, y) = U(y, x) \in [0, \infty[$ is the competition kernel.

The competition kernel $U(x, y)$ describes the power of competition between plants located at x and y, and thus can be thought of the form $U(x, y) = h(|x - y|)$, for some nonincreasing function h from $\mathbb{R}_+$ into $\mathbb{R}_+$.

We will be interested in the evolution of the stochastic process ν_t, taking its values in $\mathcal{M}$, and describing the "distribution" of plants at time t. We will write:

$$\nu_t = \sum_{i=1}^{I(t)} \delta_{X_t^i} \tag{2.2}$$

$I(t) \in \mathbb{N}$ standing for the number of alive plants at time t, and $X_t^1, ..., X_t^{I(t)}$ describing their locations (in $\bar{\mathcal{X}}$). The supposed dynamics for this population can be roughly resumed by:
(i) at the initial instant $t = 0$, we have a (possibly random) distribution $\nu_0 \in \mathcal{M}$,
(ii) each plant (located at some $x \in \bar{\mathcal{X}}$) has three independent exponential clocks: a "seed prodution" clock with parameter $\gamma(x)$, a "natural death" clock with parameter $\mu(x)$, and a "mortality selection" clock with parameter $\alpha(x) \sum_{i=1}^{I(t)} U(x, X_t^i)$,
(iii) if one of the two "death" clocks of a plant rings, then this plant disappears,
(iv) if the "seed production" clock rings, then it produces a seed. This seed immediately becomes a mature plant, at a location $y = x + z$, where z is chosen randomly according to the dispertion law $D(x, dz)$.

In [2], γ, μ, α, and D were assumed to be space-independent. Making them space-dependent might allow to take into account external effects, such relief, etc...

Note also that assuming that all these clocks are exponentially distributed, allows to set all the clocks to 0 at each time that one clock rings.

We describe the process by the evolution in time of the empirical measure ν. More precisely, we are looking for a $\mathcal{M}$-valued Markov process $(\nu_t)_{t\geq 0}$ with infinitesimal generator L, defined for a large class of functions ϕ from $\mathcal{M}$ into $\mathbb{R}$, for all $\nu \in \mathcal{M}$, by

$$
\begin{aligned}
L\phi(\nu) \quad = \quad & \int_{\bar{\mathcal{X}}} \nu(dx) \int_{\mathbb{R}^d} [\phi(\nu + \delta_{x+z}) - \phi(\nu)] \, \gamma(x) D(x, dz) \\
& + \int_{\bar{\mathcal{X}}} \nu(dx) \, [\phi(\nu - \delta_x) - \phi(\nu)] \left\{ \mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} U(x, y) \nu(dy) \right\}.
\end{aligned}
\tag{2.3}
$$

The first term is linear (in ν) and describes the seed production and dispertion phenomenon, while the second is nonlinear, and describes the death by oldness or competition. This infinitesimal generator can be compared with formula (3) in Bolker-Pacala [2] p. 182.

2.2 Description in terms of Poisson measures

We will now give a pathwise description of the $\mathcal{M}$-valued stochastic process $(\nu_t)_{t\geq 0}$. To this aim, we will use Poisson point measures. For the sake of simplicity, we assume that the spatial dependence of all the parameters is "bounded" in some sense.

Assumption (A): There exist some constants $\bar{\alpha}$, $\bar{\gamma}$ and $\bar{\mu}$ such that for all $x \in \bar{\mathcal{X}}$,

$$
\alpha(x) \leq \bar{\alpha}, \quad \gamma(x) \leq \bar{\gamma}, \quad \mu(x) \leq \bar{\mu} \tag{2.4}
$$

There exist a constant $C > 0$ and a probability density $\tilde{D}$ on $\mathbb{R}^d$ such that for all $x \in \bar{\mathcal{X}}$,

$$
D(x, dz) = D(x, z) dz \quad \text{with} \quad D(x, z) \leq C\tilde{D}(z) \tag{2.5}
$$

The competition kernel U is bounded by some constant $\bar{U}$.

We will also introduce the following notation.

Notation 2.2 *Let $H = (H^1, ..., H^k, ...)$ be the map from $\mathcal{M}$ into $(\mathbb{R}^d)^{\mathbb{N}^*}$ defined by*

$$
H\left(\sum_{i=1}^n \delta_{x_i}\right) = (x_{\sigma(1)}, ..., x_{\sigma(n)}, 0, ...0, ...) \tag{2.6}
$$

where $x_{\sigma(1)} \preccurlyeq ... \preccurlyeq x_{\sigma(n)}$, for some arbitrary order $\preccurlyeq$ on $\mathbb{R}^d$ (one may for example choose the lexicographic order).

This function H will allow us to solve the following purely notational problem: assume that a population of plants is described by a point measure $\nu \in \mathcal{M}$. Choosing a plant uniformly among all plants consists in choosing i uniformly in $\{1, ..., \langle \nu, 1\rangle\}$, and then in choosing the plant "number" i (from the arbitrary order point of view). The location of such a plant is thus $H^i(\nu)$.

Notation 2.3 *We consider the path space $\mathcal{T} \subset \mathbb{D}([0,\infty), M_F(\bar{\mathcal{X}}))$ defined by*

$$\mathcal{T} = \left\{ (\nu_t)_{t\geq 0} \;\Big/\; \begin{array}{l} \forall t \geq 0,\ \nu_t \in \mathcal{M}, \text{ and } \exists 0 = t_0 < t_1 < t_2 < ..., \\ \lim_n t_n = \infty \text{ and } \nu_t = \nu_{t_i}\ \forall t \in [t_i, t_{i+1}) \end{array} \right\} \tag{2.7}$$

Heuristically, $\mathcal{T}$ is the set of "step-measures". Note that for $(\nu_t)_{t\geq 0} \in \mathcal{T}$, and $t > 0$, we can define ν_{t-} in the following way: if $t \notin \cup\{t_i\}$, $\nu_{t-} = \nu_t$, while if $t = t_i$ for some $i \geq 1$, $\nu_{t-} = \nu_{t_{i-1}}$.

We finally introduce the probabilistic objects we will need.

Definition 2.4 *Let $(\Omega, \mathcal{F}, P)$ be a (sufficiently large) probability space. On this space, we consider the following four independent random elements:*
(i) a $\mathcal{M}$-valued random variable ν_0 (the initial distribution),
(ii) a Poisson point measure $N(ds, di, dz, d\theta)$ on $[0,\infty) \times \mathbb{N}^* \times \mathbb{R}^d \times [0,1]$, with intensity measure $\bar{\gamma} ds \left(\sum_{k\geq 1} \delta_k(di)\right) \left(C\tilde{D}(z)dz\right) d\theta$ (the seed production Poisson measure),
(iii) a Poisson point measure $M(ds, di, d\theta)$ on $[0,\infty) \times \mathbb{N}^* \times [0,1]$, with intensity measure $\bar{\mu} ds \left(\sum_{k\geq 1} \delta_k(di)\right) d\theta$ (the "natural" death Poisson measure),
(iv) a Poisson point measure $Q(ds, di, dj, d\theta, d\theta')$ on $[0,\infty) \times \mathbb{N}^* \times \mathbb{N}^* \times [0,1] \times [0,1]$, with intensity measure $\bar{U}\bar{\alpha} ds \left(\sum_{k\geq 1} \delta_k(di)\right) \left(\sum_{k\geq 1} \delta_k(dj)\right) d\theta d\theta'$ (the "competition" death Poisson measure).
We also consider the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by these processes.

We finally write the Bolker-Pacala model in terms of these stochastic objects.

Definition 2.5 *Assume (A). A $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process $\nu = (\nu_t)_{t\geq 0}$ belonging a.s. to $\mathcal{T}$ will be called a Bolker-Pacala process if a.s., for all $t \geq 0$,*

$$\begin{aligned} \nu_t = \nu_0 + \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{R}^d} \int_0^1 \mathbf{1}_{\{i \leq \langle \nu_{s-}, 1\rangle\}} \delta_{(H^i(\nu_{s-})+z)} \mathbf{1}_{\left\{\theta \leq \frac{\gamma(H^i(\nu_{s-}))D(H^i(\nu_{s-}),z)}{\bar{\gamma} C\tilde{D}(z)}\right\}} N(ds, di, dz, d\theta) \\ - \int_0^t \int_{\mathbb{N}^*} \int_0^1 \mathbf{1}_{\{i \leq \langle \nu_{s-}, 1\rangle\}} \delta_{H^i(\nu_{s-})} \mathbf{1}_{\left\{\theta \leq \frac{\mu(H^i(\nu_{s-}))}{\bar{\mu}}\right\}} M(ds, di, d\theta) \\ - \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{N}^*} \int_0^1 \mathbf{1}_{\{i \leq \langle \nu_{s-}, 1\rangle\}} \mathbf{1}_{\{j \leq \langle \nu_{s-}, 1\rangle\}} \delta_{H^i(\nu_{s-})} \mathbf{1}_{\left\{\theta' \leq \frac{U(H^i(\nu_{s-}), H^j(\nu_{s-}))}{\bar{U}}\right\}} \\ \mathbf{1}_{\left\{\theta \leq \frac{\alpha(H^i(\nu_{s-}))}{\bar{\alpha}}\right\}} Q(ds, di, dj, d\theta, d\theta') \end{aligned} \tag{2.8}$$

Although the formula is quite complicated, the principle is very simple, and describes exactely the Bolker-Pacala model. The indicator functions involving θ and θ' are related to the "rates" and appear when the parameters depend on the position variable. In the case where the rates are constant, which is the case studied by Bolker-Pacala, the probabilistic model is simpler, since one can cancel all the integrals and indicator functions involving θ.

Let us now show that if ν is solution of (2.8), then it follows the dynamics we are interested in.

Proposition 2.6 *Assume (A). Consider a solution $(\nu_t)_{t\geq 0}$ to equation (2.8). Then $(\nu_t)_{t\geq 0}$ is a Markov process with infinitesimal generator L, defined for all ϕ bounded and measurable from $\mathcal{M}$ into $\mathbb{R}$, all $\nu \in \mathcal{M}$, by (2.3).*

Proof The fact that $(\nu_t)_{t\geq 0}$ solution of (2.8) is Markovian is obvious. Let us now consider a function ϕ as in the statement. Recall that with our notations, $\nu_0 = \sum_{i=1}^{\langle \nu_0,1\rangle} \delta_{H^i(\nu_0)}$. Recall also that $L\phi(\nu_0) = \partial_t E\left[\phi(\nu_t)\right]_{t=0}$. A simple computation, using the fact that a.s., $\phi(\nu_t) = \phi(\nu_0) + \sum_{s\leq t}[\phi(\nu_{s-} + \Delta\nu_s) - \phi(\nu_{s-})]$, shows that

$$
\begin{aligned}
\phi(\nu_t) \quad = \quad & \phi(\nu_0) + \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{R}^d} \int_0^1 \left[\phi(\nu_{s-} + \delta_{(H^i(\nu_{s-})+z)}) - \phi(\nu_{s-})\right] \mathbf{1}_{\{i\leq\langle\nu_{s-},1\rangle\}} \\
& \qquad \mathbf{1}_{\left\{\theta\leq \frac{\gamma(H^i(\nu_{s-}))D(H^i(\nu_{s-}),z)}{\bar{\gamma}C\tilde{D}(z)}\right\}} N(ds,di,dz,d\theta) \\
& + \int_0^t \int_{\mathbb{N}^*} \int_0^1 \left[\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-})\right] \mathbf{1}_{\{i\leq\langle\nu_{s-},1\rangle\}} \mathbf{1}_{\left\{\theta\leq \frac{\mu(H^i(\nu_{s-}))}{\bar{\mu}}\right\}} M(ds,di,d\theta) \\
& + \int_0^t \int_{\mathbb{N}^*} \int_{\mathbb{N}^*} \int_0^1 \left[\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-})\right] \mathbf{1}_{\{i\leq\langle\nu_{s-},1\rangle\}} \mathbf{1}_{\{j\leq\langle\nu_{s-},1\rangle\}} \\
& \qquad \mathbf{1}_{\left\{\theta'\leq \frac{U(H^i(\nu_{s-}),H^j(\nu_{s-}))}{\bar{U}}\right\}} \mathbf{1}_{\left\{\theta\leq \frac{\alpha(H^i(\nu_{s-}))}{\bar{\alpha}}\right\}} Q(ds,di,dj,d\theta,d\theta')
\end{aligned}
$$

Taking expectations, we obtain

$$
\begin{aligned}
& E[\phi(\nu_t)] = E[\phi(\nu_0)] \\
& \quad + \int_0^t dsE\Big[\int_{\mathbb{R}^d} \bar{\gamma}C\tilde{D}(z)dz \sum_{i=1}^{\langle\nu_s,1\rangle} \frac{\gamma(H^i(\nu_{s-}))D(H^i(\nu_{s-}),z)}{\bar{\gamma}C\tilde{D}(z)} \left[\phi(\nu_{s-} + \delta_{(H^i(\nu_{s-})+z)}) - \phi(\nu_{s-})\right]\Big] \\
& \quad + \int_0^t dsE\Big[\bar{\mu} \sum_{i=1}^{\langle\nu_s,1\rangle} \frac{\mu(H^i(\nu_{s-}))}{\bar{\mu}} \left[\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-})\right]\Big] \\
& \quad + \int_0^t dsE\Big[\bar{U}\bar{\alpha} \sum_{i=1}^{\langle\nu_s,1\rangle} \sum_{j=1}^{\langle\nu_s,1\rangle} \frac{U(H^i(\nu_{s-}),H^j(\nu_{s-}))}{\bar{U}} \frac{\alpha(H^i(\nu_{s-}))}{\bar{\alpha}} \left[\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-})\right]\Big] \\
& \quad = E[\phi(\nu_0)] + \int_0^t dsE\Big[\int_{\bar{\mathcal{X}}} \nu_s(dx) \int_{\mathbb{R}^d} dz\gamma(x)D(x,z) \left[\phi(\nu_s + \delta_{(x+z)}) - \phi(\nu_s)\right]\Big] \\
& \quad + \int_0^t dsE\Big[\int_{\bar{\mathcal{X}}} \nu_s(dx) \left[\phi(\nu_s - \delta_x) - \phi(\nu_s)\right] \Big\{\mu(x) + \alpha(x)\int_{\bar{\mathcal{X}}} \nu_s(dy)U(x,y)\Big\}\Big]
\end{aligned}
$$

Differentiating this expression at $t = 0$ immediately drives to (2.3). $\square$

2.3 About simulation

This trajectorial definition of the Bolker-Pacala process leads to the following simulation algorithm:
Step 0: Simulate the initial state ν_0, and set $T_0 = 0$.
Step 1: Compute the total "event" rate, given by $m(0) = m_1(0) + m_2(0) + m_3(0)$, with

$$
m_1(0) = C\bar{\gamma}\langle\nu_0,1\rangle \ , \ m_2(0) = \bar{\mu}\langle\nu_0,1\rangle \ , \ m_3(0) = \bar{\alpha}\bar{U}\langle\nu_0,1\rangle^2 \tag{2.9}
$$

Simulate S_1 exponentially distributed, with parameter $m(0)$, and set $T_1 = T_0 + S_1$. Set $\nu_t = \nu_0$ for all $t < T_1$. Choose wether to go to Step 1.1, 1.2, or 1.3 with probability $m_1(0)/m(0)$, $m_2(0)/m(0)$ and $m_3(0)/m(0)$.
Step 1.1: choose i uniformly in $\{1,...,\langle\nu_0,1\rangle\}$. Choose $z \in \mathbb{R}^d$ according to the law $\tilde{D}(z)dz$. With probability $1 - \frac{\gamma(H^i(\nu_0))D(H^i(\nu_0),z)}{\bar{\gamma}C\tilde{D}(z)}$, do nothing (i.e. set $\nu_{T_1} = \nu_0$). Else, add a new plant at the location $H^i(\nu_0) + z$ (i.e. set $\nu_{T_1} = \nu_0 + \delta_{(H^i(\nu_0)+z)}$).

Step 1.2: choose i uniformly in $\{1, ..., \langle \nu_0, 1\rangle\}$. With probability $1 - \frac{\mu(H^i(\nu_0))}{\bar{\mu}}$, do nothing (i.e. set $\nu_{T_1} = \nu_0$). Else, remove the i-th plant (i.e. set $\nu_{T_1} = \nu_0 - \delta_{H^i(\nu_0)}$).
Step 1.3: choose i and j uniformly in $\{1, ..., \langle \nu_0, 1\rangle\}^2$. With probability $1-\frac{U(H^i(\nu_0)),H^j(\nu_0)))}{\bar{U}}\frac{\alpha(H^i(\nu_0))}{\bar{\alpha}}$, do nothing (i.e. set $\nu_{T_1} = \nu_0$). Else, remove the i-th plant (i.e. set $\nu_{T_1} = \nu_0 - \delta_{H^i(\nu_0)}$).
Step 2: Compute the total "event" rate, given by $m(T_1) = m_1(T_1) + m_2(T_1) + m_3(T_1)$, with

$$m_1(T_1) = C\bar{\gamma}\langle \nu_{T_1}, 1\rangle \ , \ m_2(T_1) = \bar{\mu}\langle \nu_{T_1}, 1\rangle \ , \ m_3(T_1) = \bar{\alpha}\bar{U}\langle \nu_{T_1}, 1\rangle^2 \tag{2.10}$$

Simulate S_2 exponentially distributed, with parameter $m(T_1)$, and set $T_2 = T_1 + S_1$. Set $\nu_t = \nu_{T_1}$ for all $t \in [T_1, T_2[$, etc...

3 Existence and first properties

We now show existence, uniqueness, and some moment estimates for the Bolker-Pacala process.

Theorem 3.1 *(i) Assume (A) and that $E(\langle \nu_0, 1\rangle) < \infty$. Then there exists a unique Bolker-Pacala process $(\nu_t)_{t\geq 0}$ in the sense of Definition 2.5. The law of this solution does not depend on the chosen order (see Notation 2.2).*
(ii) If furthermore for some $p \geq 1$, $E(\langle \nu_0, 1\rangle^p) < \infty$, then for any $T < \infty$,

$$E\left(\sup_{[0,T]}\langle \nu_t, 1\rangle^p\right) < \infty \tag{3.1}$$

Proof We first prove (ii). Consider thus a Bolker-Pacala process $(\nu_t)_{t\geq 0}$. We introduce for each n the stopping time $\tau_n = \inf\{t \geq 0,\ \langle \nu_t, 1\rangle \geq n\}$. Then a simple computation using (A) shows that, neglecting the nonpositive death terms,

$$\begin{aligned}\sup_{[0,t\wedge\tau_n]}\langle \nu_s, 1\rangle^p \leq \langle \nu_0, 1\rangle^p + \int_0^{t\wedge\tau_n}\int_{\mathbb{N}^*}\int_{\mathbb{R}^d}\int_0^1 [(\langle \nu_{s-}, 1\rangle + 1)^p - \langle \nu_{s-}, 1\rangle^p] \\ \mathbf{1}_{\{i\leq\langle\nu_{s-},1\rangle\}}\mathbf{1}_{\left\{\theta\leq\frac{\gamma(H^i(\nu_{s-}))D(H^i(\nu_{s-}),z)}{\bar{\gamma}C\tilde{D}(z)}\right\}}N(ds, di, dz, d\theta) \\ \leq \langle \nu_0, 1\rangle^p + C_p\int_0^{t\wedge\tau_n}\int_{\mathbb{N}^*}\int_{\mathbb{R}^d}\int_0^1\left[1 + \langle \nu_{s-}, 1\rangle^{p-1}\right]\mathbf{1}_{\{i\leq\langle\nu_{s-},1\rangle\}}N(ds, di, dz, d\theta)\end{aligned} \tag{3.2}$$

for some constant C_p. Taking expectations, we thus obtain:

$$\begin{aligned}E\left(\sup_{[0,t\wedge\tau_n]}\langle \nu_s, 1\rangle^p\right) &\leq C_p + C_pE\left(\int_0^{t\wedge\tau_n}ds\bar{\gamma}C\int_{\mathbb{R}^d}\tilde{D}(z)dz\,[\langle \nu_{s-}, 1\rangle + \langle \nu_{s-}, 1\rangle^p]\right) \\ &\leq C_p + C_pE\left(\int_0^t ds\,[1 + \langle \nu_{s\wedge\tau_n}, 1\rangle^p]\right)\end{aligned} \tag{3.3}$$

The Gronwall Lemma allows us to conclude that for any $T < \infty$, there exists a constant $C_p(T)$, not depending on n, such that $E\left(\sup_{[0,T\wedge\tau_n]}\langle \nu_t, 1\rangle^p\right) \leq C_p(T)$. Firstly, one easily concludes that τ_n tends a.s. to infinity. Then by Fatou's theorem, we obtain (3.1).
Point (i) is a consequence of point (ii). Indeed, one can for example build the solution $(\nu_t)_{t\geq 0}$ using the simulation algorithm previously described, choosing the rates and acceptance-rejection according to the Poisson measures N, M, and Q. One only has to check that the sequence of (effective or fictitious) jump instants T_n goes a.s. to infinity as n tends to infinity, which is a

consequence of (3.1) with $p = 1$. Uniqueness also holds, since one has no choice in the construction. $\square$

Let us observe that the existence of moments comes from the fact that the nonlinear terms, which might lead to explosion, are all nonpositive (since they represent the death terms) and are then neglected in the estimation.
We now prove a natural property: if there is initially at most one plant at each location, then this property propagates.

Proposition 3.2 *Assume (A), that $E(\langle \nu_0, 1\rangle) < \infty$. Assume also that a.s., $\sup_{x\in\bar{\mathcal{X}}} \nu_0(\{x\}) \leq 1$. Consider the Bolker-Pacala process $(\nu_t)_{t\geq 0}$. Then for all $t \geq 0$, a.s.,*

$$\int_{\bar{\mathcal{X}}} \nu_t(\{x\})\nu_t(dx) = \langle \nu_t, 1\rangle, \quad i.e. \quad \sup_{x\in\bar{\mathcal{X}}} \nu_t(\{x\}) \leq 1. \tag{3.4}$$

Proof Consider the nonnegative function ϕ defined on $\mathcal{M}$ by $\phi(\nu) = \int_{\bar{\mathcal{X}}} \nu(\{x\})\nu(dx) - \langle \nu, 1\rangle$. Then note that a.s., $\phi(\nu_0) = 0$, and that for any $\nu \in \mathcal{M}$, any $x \in \mathrm{supp}\, \nu$, $\phi(\nu - \delta_x) - \phi(\nu) \leq 0$. Consider, for each $n \geq 1$, the stopping time $\tau_n = \inf\{t \geq 0,\ \langle \nu_t, 1\rangle \geq n\}$. A simple computation allows to obtain, for all $t \geq 0$, all $n \geq 1$,

$$E\left[\phi(\nu_{t\wedge\tau_n})\right] \leq 0 + E\left[\int_0^{t\wedge\tau_n} ds \int_{\bar{\mathcal{X}}} \nu_s(dx) \int_{\mathbb{R}^d} D(x,dz)\gamma(x)\left\{\phi(\nu_s + \delta_{(x+z)}) - \phi(\nu_s)\right\}\right] \tag{3.5}$$

One easily checks that the RHS term identically vanishes, since $D(x, dz)$ has a density. Hence, a.s., $\phi(\nu_{t\wedge\tau_n}) = 0$. Thanks to (3.1) with $p = 1$, τ_n a.s. grows to infinity with n, which concludes the proof. $\square$

We carry on with another property, which deals with the absolute continuity of the expectation of ν_t. For ν a random measure, we define the deterministic measure $E(\nu)$ by $\langle E(\nu), f\rangle = E(\langle \nu, f\rangle)$.

Proposition 3.3 *Assume (A), that $E[\langle \nu_0, 1\rangle] < \infty$, and that $E(\nu_0)$ admits a density n_0 with respect to the Lebesgue measure. Consider the Bolker-Pacala process $(\nu_t)_{t\geq 0}$. Then for all $t \geq 0$, $E(\nu_t)$ has a density n_t: for all measurable nonnegative function f on $\bar{\mathcal{X}}$, $E[\langle \nu_t, f\rangle] = \int_{\bar{\mathcal{X}}} f(x)n_t(x)dx$.*

Proof Consider a Lebesgue-null subset A of $\mathbb{R}^d$. Consider also, for each $n \geq 1$, the stopping time $\tau_n = \inf\{t \geq 0,\ \langle \nu_t, 1\rangle \geq n\}$. A simple computation allows to obtain, for all $t \geq 0$, all $n \geq 1$,

$$\begin{aligned} E\left[\langle \nu_{t\wedge\tau_n}, \mathbf{1}_A\rangle\right] = {} & E(\langle \nu_0, \mathbf{1}_A\rangle) + E\bigg(\int_0^{t\wedge\tau_n} ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\gamma(x) \int_{\mathbb{R}^d} dz \mathbf{1}_A(x+z)D(x,z)\bigg) \\ & - E\bigg(\int_0^{t\wedge\tau_n} ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\mathbf{1}_A(x)\left(\mu(x) + \alpha(x)\int U(x,y)\nu_s(dy)\right)\bigg) \end{aligned} \tag{3.6}$$

By assumption, the first term in the RHS is null. The second term is also null, since for any $x \in \bar{\mathcal{X}}$, $\int_{\mathbb{R}^d} dz\mathbf{1}_A(x+z)D(x,z) = 0$. The third term is of course nonpositive. Hence for each n, $E(\langle \nu_{t\wedge\tau_n}, \mathbf{1}_A\rangle)$ is nonpositive, and thus null. Thanks to (3.1) with $p = 1$, τ_n a.s. grows to infinity with n, which concludes the proof. $\square$

We finally give some martingale properties of the process $(\nu_t)_{t\geq 0}$.

Proposition 3.4 *Assume (A), and that for some $p \geq 2$, $E[\langle \nu_0, 1\rangle^p] < \infty$. Consider the Bolker-Pacala process $(\nu_t)_{t\geq 0}$, and recall that L is defined by (2.3).*
(i) For all measurable function ϕ from $\mathcal{M}$ into $\mathbb{R}$ such that for some constant C, for all $\nu \in \mathcal{M}$, $|\phi(\nu)| + |L\phi(\nu)| \leq C(1 + \langle \nu, 1\rangle^p)$, the process

$$\phi(\nu_t) - \phi(\nu_0) - \int_0^t L\phi(\nu_s)ds \tag{3.7}$$

is a càdlàg L^1-$(\mathcal{F}_t)_{t\geq 0}$-martingale starting from 0.
(ii) Point (i) applies to any measurable ϕ satisfying $|\phi(\nu)| \leq C(1+\langle \nu, 1\rangle^{p-2})$.
(iii) Point (i) applies to any function $\phi(\nu) = \langle \nu, f\rangle^q$, with $0 \leq q \leq p-1$ and with f bounded and measurable on $\mathcal{M}$.
(iv) For any f bounded and measurable on $\bar{\mathcal{X}}$, the process

$$\begin{aligned} M_t^f &= \langle \nu_t, f\rangle - \langle \nu_0, f\rangle - \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\gamma(x) \int_{\mathbb{R}^d} dz f(x+z) D(x,z) \\ &\quad + \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx) f(x) \Big[\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} U(x,y)\nu_s(dy)\Big] \end{aligned} \tag{3.8}$$

is a càdlàg L^2-martingale starting from 0 *with (predictable) quadratic variation*

$$\langle M^f\rangle_t = \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx) \Big\{\gamma(x) \int_{\mathbb{R}^d} dz f^2(x+z) D(x,z) + f^2(x)\Big[\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} U(x,y)\nu_s(dy)\Big]\Big\} \tag{3.9}$$

Proof First of all note that point (i) is immediate thanks to Proposition 2.6 and (3.1). Points (ii) and (iii) follow from fair computations using (2.3). To prove (iv), we first assume that $E\left[\langle \nu_0, 1\rangle^3\right] < \infty$. We apply (i) with $\phi(\nu) = \langle \nu, f\rangle$. This yields that M^f is a martingale. To compute its bracket, we first apply (i) with $\phi(\nu) = \langle \nu, f\rangle^2$ and obtain that

$$\begin{aligned} &\langle \nu_t, f\rangle^2 - \langle \nu_0, f\rangle^2 - \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\gamma(x) \int_{\mathbb{R}^d} dz D(x,z) \Big[f^2(x+z) + 2f(x+z)\langle \nu_s, f\rangle\Big] \\ &\qquad - \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx) \Big\{f^2(x) - 2f(x)\langle \nu_s, f\rangle\Big\}\Big[\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} U(x,y)\nu_s(dy)\Big] \end{aligned} \tag{3.10}$$

is a martingale. Then we apply the Itô formula to compute $\langle \nu_t, f\rangle^2$ from (3.8). We deduce that

$$\begin{aligned} &\langle \nu_t, f\rangle^2 - \langle \nu_0, f\rangle^2 - \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx)\gamma(x) \int_{\mathbb{R}^d} dz D(x,z) 2f(x+z)\langle \nu_s, f\rangle \\ &+ \int_0^t ds \int_{\bar{\mathcal{X}}} \nu_s(dx) 2f(x)\langle \nu_s, f\rangle \Big[\mu(x) + \alpha(x) \int_{\bar{\mathcal{X}}} U(x,y)\nu_s(dy)\Big] - \langle M^f\rangle_t \end{aligned} \tag{3.11}$$

is a martingale. Comparing (3.10) and (3.11) leads to (3.9). The extension to the case where only $E\left[\langle \nu_0, 1\rangle^2\right]$ is finite is straightforward, noting that even in this case, $E[\langle M^f\rangle_t] < \infty$ thanks to (3.1) with $p = 2$. □

4 On the the Bolker-Pacala Moment Equations

Let us now come back to the specific Bolker-Pacala model, and let us give a sense to the mean moment equation given in [2] formula (6). Note that in [2], one may be confused by the notation between the discrete measure ν_t, its expectation $E(\nu_t)$ (defined by $\langle E(\nu_t), f\rangle = E(\langle \nu_t, f\rangle)$), and a density measure $n_t(x)$ of which the definition is not clear (it does not seem to be the one we defined in Proposition 3.3).
In this section we assume that

> Assumption (B): The spatial space is $\bar{\mathcal{X}} = \mathbb{R}^d$, all parameters α, γ, μ, and D of the model are independent of x. Moreover the (bounded) competition kernel $U(x,y)$

has the form $U(x-y)$, and both dispersal and competition kernels are symmetric probability distribution functions, i.e. $D(z) = D(-z)$, $U(x-y) = U(y-x)$, and $\int_{\mathbb{R}^d} D(z)dz = \int_{\mathbb{R}^d} U(z)dz = 1$.

We moreover assume that $E(\langle \nu_0, 1\rangle^2) < \infty$, and that initially, there is at most one plant at each location. So (3.1) with $p = 1$ holds and we can define for each time $t \in [0, T]$

$$n(t) = E(\langle \nu_t, 1\rangle). \tag{4.1}$$

Using Proposition 3.4 (iv) (with $f = 1$), and taking expectations in (3.8), we obtain

$$E(\langle \nu_t, 1\rangle) = E(\langle \nu_0, 1\rangle) + \int_0^t (\gamma - \mu) E(\langle \nu_s, 1\rangle)ds - \alpha \int_0^t E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} U(x-y)\nu_s(dx)\nu_s(dy)\Big)ds \tag{4.2}$$

Hence,

$$\begin{aligned} n(t) &= n(0) + (\gamma-\mu)\int_0^t n(s)ds - \alpha\int_0^t E\Big(\int_{\mathbb{R}^d} U(0)\nu_s(dx)\nu_s(\{x\})\Big)ds \\ &\quad -\alpha\int_0^t E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\{x\neq y\}} U(x-y)\nu_s(dx)\nu_s(dy)\Big)ds \end{aligned} \tag{4.3}$$

But thanks to Proposition 3.2, we know that for all $t \geq 0$, $\int_{\mathbb{R}^d} U(0)\nu_s(dx)\nu_s(\{x\}) = U(0)\langle \nu_s, 1\rangle$. We thus obtain

$$n(t) = n(0) + (\gamma-\mu-\alpha U(0))\int_0^t n(s)ds - \alpha\int_0^t E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\{x\neq y\}} U(x-y)\nu_s(dx)\nu_s(dy)\Big)ds \tag{4.4}$$

Let us now explain the "covariance term" of Bolker and Pacala. Writing

$$\begin{aligned} &\alpha E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\{x\neq y\}} U(x-y)\nu_s(dx)\nu_s(dy)\Big) \\ &= \alpha E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\{x\neq y\}} U(x-y)\nu_s(dx)\Big(\nu_s(dy) - n(s)dy\Big)\Big) + \alpha n^2(s) \end{aligned} \tag{4.5}$$

we obtain from (4.4)

$$\begin{aligned} n(t) &= n(0) + (\gamma-\mu-\alpha U(0))\int_0^t n(s)ds - \alpha\int_0^t n^2(s)ds \\ &\quad -\alpha\int_0^t E\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\{x\neq y\}} U(x-y)\nu_s(dx)\Big(\nu_s(dy) - n(s)dy\Big)\Big)ds \end{aligned} \tag{4.6}$$

That allows us to define, following the terminology of Bolker and Pacala, a covariance measure C_t on $\mathbb{R}^d$ for each time t as

$$C_t(dr) = E\Big(\int_{y\in\mathbb{R}^d} \mathbf{1}_{\{r\neq 0\}}\nu_t\circ\tau_{-y}^{-1}(dr)\otimes\nu_t(dy)\Big) - n^2(t)dr \tag{4.7}$$

defined for each measurable bounded function ϕ with compact support in $\mathbb{R}^d$ by

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(r)C_t(dr) &= E\Big(\int_{\mathbb{R}^d\otimes\mathbb{R}^d} \mathbf{1}_{\{x\neq y\}}\phi(x-y)\nu_t(dx)\nu_t(dy)\Big) - n^2(t)\int_{\mathbb{R}^d}\phi(r)dr \\ &= E\Big(\int_{\mathbb{R}^d\otimes\mathbb{R}^d} \mathbf{1}_{\{x\neq y\}}\phi(x-y)\nu_t(dx)\Big(\nu_t(dy) - n(t)dy\Big)\Big) \end{aligned} \tag{4.8}$$

(The notation τ_{-y} denotes the translation by the vector $-y$). By using these notations, we obtain the mean equation obtained by Bolker and Pacala [2] (formula (6) p 183), with a rigorous sense for the quadratic term.

$$\frac{dn(t)}{dt} = n(t)(\gamma - \mu - \alpha n(t)) - \alpha U(0)n(t) - \alpha \int_{\mathbb{R}^d} \mathbf{1}_{\{r\neq 0\}} U(r) C_t(dr). \tag{4.9}$$

Let us at last remark that, following the same approach, we are able to obtain an evolution equation for the covariance measure, by considering the quantities $\int_{\mathbb{R}^d} \mathbf{1}_{\{r\neq 0\}}\phi(r)C_t(dr)$ for measurable bounded functions ϕ on $\mathbb{R}^d$, but we do not obtain the same equation as in [2] (formula (7) p 184).

5 Infinite particle approximations

Our aim in this section is to observe the effect of different renormalizations on this model. There are essentially two asymptotic behaviours.

The first one consists in considering at initial time an infinite number of particles with infinitely small masses, without changing the parameters of the dynamics, and we will show that in this case the random measure $(\nu_t)_{t\geq 0}$ tends to a deterministic measure solution of a nonlinear partial integro-differential equation. We propose this limiting object as a rigorous interpretation of the "number density".

The second renormalization consists in addition in accelerating the parameters in a convenient way. Then $(\nu_t)_{t\geq 0}$ converges to a sophisticated random measure-valued process, which has been introduced by Etheridge in [5] and called by her the *superprocess version of the Bolker-Pacala model.*

Let us first consider the most general situation.

Notation 5.1 *For each $n \in \mathbb{N}^*$, we consider a set of parameters $(\mu_n, \gamma_n, \alpha_n, U_n, D_n)$ as in Notation 2.1, satisfying for each n Assumption (A), and an initial condition $\nu_0^n \in \mathcal{M}$. Then, we denote by $(\nu_t^n)_{t\geq 0}$ the Bolker-Pacala process (see Definition 2.5) with the corresponding coefficients. We consider the subset $\mathcal{M}^n$ of $M_F(\bar{\mathcal{X}})$ defined by*

$$\mathcal{M}^n = \left\{ \frac{1}{n}\nu,\ \nu \in \mathcal{M} \right\} \tag{5.1}$$

We finally consider the càdlàg $\mathcal{M}^n$-valued Markov process $(X_t^n)_{t\geq 0}$ defined by $X_t^n = \frac{1}{n}\nu_t^n$.

The generator of $(X_t^n)_{t\geq 0}$ is then given, for any measurable map ϕ from $\mathcal{M}^n$ into $\mathbb{R}$ by

$$\begin{aligned} L^n\phi(\nu) &= n\int_{\bar{\mathcal{X}}} \nu(dx) \int_{\mathbb{R}^d} \gamma_n(x) D_n(x,z)dz \left[\phi(\nu + \frac{1}{n}\delta_{x+z}) - \phi(\nu)\right] \\ &\quad + n\int_{\bar{\mathcal{X}}} \nu(dx) \left\{\mu_n(x) + n\alpha_n(x)\int_{\bar{\mathcal{X}}} U_n(x,y)\nu(dy)\right\} \left[\phi(\nu - \frac{1}{n}\delta_x) - \phi(\nu)\right]. \end{aligned} \tag{5.2}$$

Indeed, the generator $\tilde{L}^n$ of $(\nu_t^n)_{t\geq 0}$ is given by (2.3), replacing $(\mu, \gamma, \alpha, U, D)$ by $(\mu_n, \gamma_n, \alpha_n, U_n, D_n)$. Hence,

$$L^n\phi(\nu) = \partial_t E_\nu\left[\phi(X_t^n)\right]_{t=0} = \partial_t E_{n\nu}\left[\phi(\nu_t^n/n)\right]_{t=0} = \tilde{L}^n\phi^n(n\nu) \tag{5.3}$$

where $\phi^n(\mu) = \phi(\mu/n)$. The conclusion follows from a fair computation.

We now deduce the following martingale properties from Lemma 3.4.

Lemma 5.2 *Let $n \geq 1$ be fixed, consider the process $(X_t^n)_{t\geq 0}$ defined in Notation 5.1. Assume that for some $p \geq 2$, $E\left[\langle X_0^n, 1\rangle^p\right] < \infty$.*
(i) For all measurable function ϕ from $\mathcal{M}^n$ into $\mathbb{R}$ such that for some constant C, for all $\nu \in \mathcal{M}^n$, $|\phi(\nu)| + |L^n\phi(\nu)| \leq C(1 + \langle \nu, 1\rangle^p)$, the process

$$\phi(X_t^n) - \phi(X_0^n) - \int_0^t L^n\phi(X_s^n)ds \tag{5.4}$$

is a càdlàg L^1-martingale starting from 0.
(ii) Point (i) applies to any measurable ϕ satisfying $|\phi(\nu)| \leq C(1 + \langle \nu, 1\rangle^{p-2})$.
(iii) Point (i) applies to any function $\phi(\nu) = \langle \nu, f\rangle^q$, with $0 \leq q \leq p-1$ and with f bounded and measurable on $\mathcal{M}$.
(iv) For any f bounded and measurable on $\bar{\mathcal{X}}$, the process

$$\begin{aligned} M_t^{n,f} &= \langle X_t^n, f\rangle - \langle X_0^n, f\rangle - \int_0^t ds \int_{\bar{\mathcal{X}}} X_s^n(dx) \int_{\mathbb{R}^d} \gamma_n(x) D_n(x,z) dz f(x+z) \\ &\quad + \int_0^t ds \int_{\bar{\mathcal{X}}} X_s^n(dx) \left\{ \mu_n(x) + n\alpha_n(x) \int_{\bar{\mathcal{X}}} U_n(x,y) X_s^n(dy) \right\} f(x) \end{aligned} \tag{5.5}$$

is a càdlàg L^2-martingale with (predictable) quadratic variation

$$\begin{aligned} \left\langle M^{n,f}\right\rangle_t &= \frac{1}{n}\int_0^t ds \int_{\bar{\mathcal{X}}} X_s^n(dx) \int_{\mathbb{R}^d} \gamma_n(x) D_n(x,z) dz f^2(x+z) \\ &\quad + \frac{1}{n}\int_0^t ds \int_{\bar{\mathcal{X}}} X_s^n(dx) \left\{ \mu_n(x) + n\alpha_n(x) \int_{\bar{\mathcal{X}}} U_n(x,y) X_s^n(dy) \right\} f^2(x) \end{aligned} \tag{5.6}$$

5.1 Convergence to a nonlinear partial differential equation

Let us now consider the case where the initial number tends to infinity, the parameters of seed production and natural death stay unchanged, whereas the mortality selection parameter tends to zero. We will show that the Bolker-Pacala process can be approximated by a deterministic nonlinear partial differential equation, which might be a better (compared to the moment equations of [2]) deterministic way to account the model, as observed in Section 7. In particular, it allows to deal with space-dependent parameters.

> <u>Assumption $(C1)$:</u>
> 1) The initial conditions X_0^n converge in law and for the vague topology on $M_F(\bar{\mathcal{X}})$ to some deterministic finite measure $\xi_0 \in M_F(\bar{\mathcal{X}})$, and $\sup_n E(\langle X_0^n, 1\rangle^3) < +\infty$.
> 2) There exist some continuous nonnegative functions α, γ, μ on $\bar{\mathcal{X}}$, bounded by $\bar{\alpha}, \bar{\gamma}, \bar{\mu}$, such that $\gamma_n(x) = \gamma(x)$, $\mu_n(x) = \mu(x)$, $\alpha_n(x) = \alpha(x)/n$.
> 3) There exists a bounded nonnegative symmetric continuous function U on $\bar{\mathcal{X}} \times \bar{\mathcal{X}}$ bounded by $\bar{U}$ such that $U_n(x,y) = U(x,y)$.
> 4) There exists a continuous nonnegative function D on $\bar{\mathcal{X}} \times \mathbb{R}^d$, satisfying for each $x \in \bar{\mathcal{X}}$, $\int_{z\in\mathbb{R}^d,\, x+z\in\bar{\mathcal{X}}} D(x,z)dz = 1$, $D(x,z) = 0$ as soon as $x + z \notin \bar{\mathcal{X}}$, and such that $D(x,z) \leq C\tilde{D}(z)$ for a constant $C > 0$ and a probability density $\tilde{D}$ on $\mathbb{R}^d$. We set $D_n(x,z) = D(x,z)$.

The first assertion of Assumption (C1) is satisfied for example if $X_0^n = \frac{1}{n}\sum_{i=1}^n \delta_{Z^i}$ where the random variables Z^i are independent, with law ξ_0. In this case, the number n can be seen as the "volume" of particles at initial time, and the limit of $X_t^n = \frac{1}{n}\nu_t^n$ may then give a rigorous sense to the number density, often introduced by the biologists without definition.

Theorem 5.3 *Assume $(C1)$, and consider the sequence of processes X^n defined in Notation 5.1. Then for all $T>0$, the sequence (X^n) converges in law, in $\mathbb{D}([0,T],M_F(\bar{\mathcal{X}}))$, to a deterministic continuous function $(\xi_t)_{t\geq 0}\in C([0,T],M_F(\bar{\mathcal{X}}))$. This measure-valued function ξ is the unique solution, satisfying $\sup_{[0,T]}\langle\xi_t,1\rangle<\infty$, of the partial differential equation written in its weak form: for all bounded and measurable function f from $\bar{\mathcal{X}}$ into $\mathbb{R}$,*

$$\begin{aligned}\langle\xi_t,f\rangle &= \langle\xi_0,f\rangle+\int_0^t\left\langle\xi_s(dx),\gamma(x)\int_{\mathbb{R}^d}D(x,z)f(x+z)dz\right\rangle ds\\ &\quad -\int_0^t\left\langle\xi_s(dx),f(x)\left\{\mu(x)+\alpha(x)\int_{\bar{\mathcal{X}}}U(x,y)\xi_s(dy)\right\}\right\rangle ds \end{aligned}\tag{5.7}$$

Note that the link between (2.8) and (5.7) is the same as the one between the continuous-time binary Galton-Watson process with birth rate γ and death rate μ and the deterministic differential equation $f'(t)=(\gamma-\mu)f(t)$.

Proof We divide the proof in several steps. Let us fix $T>0$.

Step 1 Let us first show the uniqueness for the equation (5.7). We consider two solutions $(\xi_t)_{t\geq0}$ and $(\bar{\xi}_t)_{t\geq0}$ of (5.7) satisfying $\sup_{[0,T]}\langle\xi_t+\bar{\xi}_t,1\rangle=A_T<+\infty$. We consider the variation norm defined for μ_1 and μ_2 in $M_F(\bar{\mathcal{X}})$ by

$$||\mu_1-\mu_2||=\sup_{f\in L^\infty(\bar{\mathcal{X}}),\ ||f||_\infty\leq 1}|\langle\mu_1-\mu_2,f\rangle|.\tag{5.8}$$

Then, we consider some $f\in L^\infty(\bar{\mathcal{X}})$ such that $||f||_\infty\leq 1$ and obtain

$$\begin{aligned}|\langle\xi_t-\bar{\xi}_t,f\rangle| &\leq \int_0^t\left|\left\langle\xi_s(dx)-\bar{\xi}_s(dx),\gamma(x)\int_{\mathbb{R}^d}D(x,z)f(x+z)dz-\mu(x)f(x)\right\rangle\right|ds\\ &\quad+\int_0^t\left|\left\langle\xi_s(dx)-\bar{\xi}_s(dx),\alpha(x)f(x)\int_{\bar{\mathcal{X}}}\xi_s(dy)U(x,y)\right\rangle\right|ds\\ &\quad+\int_0^t\left|\left\langle\xi_s(dy)-\bar{\xi}_s(dy),\int_{\bar{\mathcal{X}}}\bar{\xi}_s(dx)\alpha(x)f(x)U(x,y)\right\rangle\right|ds \end{aligned}\tag{5.9}$$

But since $||f||_\infty\leq 1$, for all $x\in\bar{\mathcal{X}}$, $\left|\gamma(x)\int_{\mathbb{R}^d}D(x,z)f(x+z)dz-\mu(x)f(x)\right|\leq\bar{\gamma}+\bar{\mu}$ while $\left|\alpha(x)f(x)\int_{\bar{\mathcal{X}}}\xi_s(dy)U(x,y)\right|\leq\bar{\alpha}\bar{U}A_T$, and $\left|\int_{\bar{\mathcal{X}}}\bar{\xi}_s(dx)\alpha(x)f(x)U(x,y)\right|\leq\bar{\alpha}\bar{U}A_T$. We deduce that

$$|\langle\xi_t-\bar{\xi}_t,f\rangle|\leq\left[\bar{\gamma}+\bar{\mu}+2\bar{\alpha}\bar{U}A_T\right]\int_0^t||\xi_s-\bar{\xi}_s||ds\tag{5.10}$$

Taking the supremum over all functions f such that $||f||_\infty\leq 1$, and using then the Gronwall Lemma, we finally deduce that for all $t\leq T$, $||\xi_t-\bar{\xi}_t||=0$. Uniqueness holds.

Step 2 Let us prove some moment estimates. By $(C1)$, it is easy to prove that for all $T>0$,

$$\sup_n E\left(\sup_{[0,T]}\langle X_t^n,1\rangle^3\right)<+\infty\tag{5.11}$$

Indeed, recalling that $X_t^n=\frac{1}{n}\nu_t^n$, one can prove, following line by line the proof of Theorem 3.1 (ii) with $p=3$, that $E[\sup_{[0,T]}\langle\nu_t^n,1\rangle^3]\leq C(T)E[\langle\nu_0^n,1\rangle^3]$, noting that the constant $C(T)$ does not depend on n. One easily concludes, using assumption $(C1)$-1.

Step 3 To show the tightness of the sequence of the laws $Q^n=\mathcal{L}(X^n)$ in $\mathcal{P}(\mathbb{D}([0,T],M_F(\bar{\mathcal{X}})))$, it suffices, following Roelly [11], to show that for any continuous bounded function f on $\bar{\mathcal{X}}$, the sequence of laws of the processes $\langle X^n, f\rangle$ is tight in $\mathbb{D}([0,T],\mathbb{R})$. To this aim, we use the Aldous criterion [1] and the Rebolledo criterion (see [6]). We have to show that

$$\sup_n E(\sup_{[0,T]} |\langle X^n_s, f\rangle|) < \infty, \tag{5.12}$$

and the tightness respectively of the laws of the martingale part and of the drift part of the semimartingales $\langle X^n, f\rangle$.
Clearly, since f is bounded, (5.12) is a consequence of (5.11). Let us thus consider a couple (S, S') satisfying a.s. $0 \le S \le S' \le S+\delta \le T$. Thanks to Doob's inequality, using Lemma 5.2, we get

$$\begin{aligned} E\left[|M^{n,f}_{S'} - M^{n,f}_S|\right] &\le E\left[\langle M^{n,f}\rangle_{S+\delta} - \langle M^{n,f}\rangle_S\right]^{1/2} \\ \le E\left[(\bar\gamma + \bar\mu + \bar\alpha\bar U)\int_S^{S+\delta}\left(\langle X^n_s, 1\rangle + \langle X^n_s, 1\rangle^2\right)ds\right]^{1/2} &\le C\sqrt{\delta} \end{aligned} \tag{5.13}$$

the last inequality coming from (5.11). The finite variation part of $\langle X^n_{S'}, f\rangle - \langle X^n_S, f\rangle$ is bounded by

$$\int_S^{S+\delta}\left[(\bar\gamma + \bar\mu)\langle X^n_s, 1\rangle + \bar\alpha\bar U\langle X^n_s, 1\rangle^2\right]ds \le \delta C\left[1 + \sup_{[0,T]}\langle X^n_s, 1\rangle^2\right] \tag{5.14}$$

Hence, formula (5.11) allows us to conclude that the sequence $Q^n = \mathcal{L}(X^n)$ in tight.
Step 4 Let us now denote by Q the limiting law of a subsequence of Q^n which we still denote by Q^n, in the space of probability measures on $\mathbb{D}([0,T], M_F(\bar{\mathcal{X}}))$, and by $X = (X_t)_{t\ge 0}$ a process with law Q. We remark that by construction, almost surely,

$$\sup_{t\in[0,T]}\ \sup_{f\in L^\infty(\bar{\mathcal{X}}), ||f||_\infty \le 1} |\langle X^n_s, f\rangle - \langle X^n_{s-}, f\rangle| \le 1/n. \tag{5.15}$$

Then it is immediate to conclude that the process X is a.s. strongly continuous.
Step 5 Let us finally identify the limit Q. We will show that for any f in $C_b(\bar{\mathcal{X}})$, equation (5.7) is (almost surely) satisfied by X. Since moreover it is clear from (5.11) that $E(\sup_{[0,T]}\langle X_t, 1\rangle) < +\infty$, that will suffice to conclude the proof by uniqueness of the solution to (5.7).
For $t \le T$ and $\nu \in \mathbb{D}([0,T], M_F(\bar{\mathcal{X}}))$, denote by

$$\begin{aligned} \Psi_t(\nu) = \langle \nu_t, f\rangle - \langle \nu_0, f\rangle - \int_0^t \left\langle \nu_s(dx), \gamma(x)\int_{\mathbb{R}^d} D(x,z) f(x+z)dz\right\rangle ds \\ + \int_0^t \left\langle \nu_s(dx), f(x)\left\{\mu(x) + \alpha(x)\int_{\bar{\mathcal{X}}} U(x,y)\nu_s(dy)\right\}\right\rangle ds. \end{aligned} \tag{5.16}$$

What we have to show is that for any $t \le T$ (which we fix from now on),

$$E_Q\left[|\Psi_t(X)|\right] = 0. \tag{5.17}$$

What we know, from Lemma 5.2 and $(C1)$, is that

$$M^{n,f}_t = \Psi_t(X^n). \tag{5.18}$$

A fair computation using Lemma 5.2, $(C1)$, and (5.11) shows that

$$E\left[|M^{n,f}_t|^2\right] = E\left[\langle M^{n,f}\rangle_t\right] \le \frac{C_f}{n} E\left[\int_0^t \left\{1 + \langle X^n_s, 1\rangle^2\right\}ds\right] \le \frac{C_{f,t}}{n} \tag{5.19}$$

which goes to 0 as n tends to infinity. On the other hand, since X is a.s. strongly continuous, since f is continuous and thanks to $(C1)$, the function Ψ_t is a.s. continuous at X. Furthermore, for any $\nu \in \mathbb{D}([0,T], M_F(\bar{\mathcal{X}}))$,

$$|\Psi_t(\nu)| \leq C_{t,f} \sup_{[0,t]} \left(1 + \langle \nu_s, 1\rangle^2\right). \tag{5.20}$$

Hence using (5.11), we see that the sequence $(\Psi_t(X^n))_n$ is uniformly integrable, and thus

$$\lim_n E\left(|\Psi_t(X^n)|\right) = E\left(|\Psi_t(X)|\right). \tag{5.21}$$

Associating (5.18), (5.19) and (5.21), we conclude that (5.17) holds and thus equation (5.7) is satisfied for any continuous bounded function f on $\bar{\mathcal{X}}$. Then, it is not hard to generalize this equation to measurable bounded functions f using an approximating sequence belonging to $C_b(\bar{\mathcal{X}})$. The proof is finished. $\square$

Proposition 5.4 *Consider an initial condition ξ_0 in $M_F(\bar{\mathcal{X}})$ having a density with respect to the Lebesgue measure, and consider the associated solution $(\xi_t)_{t\geq 0}$ to (5.7). Then for every $t \in [0,T]$, the finite measure ξ_t has a density with respect to the Lebesgue measure.*

Proof The proof is similar as the one of Proposition 3.3. We choose a neglectable Borel set A included in $\bar{\mathcal{X}}$ and apply (5.7) with $f = \mathbf{1}_A$. The RHS expression is equal to 0 since the first term is null by hypothesis, the second one is null since for all x, $\int_{\mathbb{R}^d} \mathbf{1}_{x+z\in A} D(x,z)dz = 0$, and the last term is nonpositive. $\square$

Remark 5.5 *(i) Equation (5.7) is the weak form of the following equation. For all $x \in \bar{\mathcal{X}}$, all $t \geq 0$,*

$$\partial_t \xi_t(x) = \int_{\bar{\mathcal{X}}} \xi_t(y)\gamma(y)D(y, x-y)dy - \mu(x)\xi_t(x) - \alpha(x)\xi_t(x)\int_{\bar{\mathcal{X}}} U(x,y)\xi_t(y)dy \tag{5.22}$$

(ii) To come back to the Bolker-Pacala model, assume now that $\bar{\mathcal{X}} = \mathbb{R}^d$, that the competition kernel is of the form $U(x,y) = U(x-y)$, and that $D(x,z) = D(z)$ does not depend on x. Then (5.7) is the weak form of: for all $x \in \mathbb{R}^d$, all $t \geq 0$,

$$\partial_t \xi_t(x) = \gamma(x)\left[\xi_t \star D\right](x) - \mu(x)\xi_t(x) - \alpha(x)\xi_t(x)\left[\xi_t \star U\right](x) \tag{5.23}$$

where for exemple, $\left[\xi_t \star D\right](x) = \int_{\mathbb{R}^d} D(x-y)\xi_t(dy)$.

5.2 Convergence to a superprocess

We would like in this section to show the relation between the original Bolker-Pacala model (rigorously written in Definition 2.5) and the *superprocess version of the Bolker-Pacala model* introduced by Etheridge [5].

More precisely, we will show in this section that accelerating the rates of production and natural death following the asymptotics n makes the Bolker-Pacala processes converge to a continuous random measure-valued process which generalizes the one studied in [5].

> Assumption $(C2)$:
> 1) The space $\bar{\mathcal{X}} = \mathbb{R}^d$, the initial conditions X_0^n converge in law and for the vague topology on $M_F(\mathbb{R}^d)$ to some deterministic finite measure $X_0 \in M_F(\mathbb{R}^d)$, and $\sup_n E(\langle X_0^n, 1\rangle^3) < +\infty$.

2) There exist some continuous positive functions $\sigma(x), \alpha(x), \gamma(x), \beta(x)$ on $\mathbb{R}^d$ respectively bounded by $\bar{\sigma}, \bar{\alpha}, \bar{\gamma}, \bar{\beta}$, a nonnegative symmetric continuous function $U(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ bounded by $\bar{U}$, such that

$$\begin{aligned} \gamma_n(x) = n\gamma(x) + \beta(x),\ \mu_n(x) = n\gamma(x),\ \alpha_n(x) = \alpha(x)/n, \\ U_n(x, y) = U(x, y) \\ D_n(x, z) = \left(\tfrac{n}{2\pi\sigma(x)}\right)^{d/2} \exp\left(-n|z|^2/2\sigma(x)\right) \end{aligned} \tag{5.24}$$

Note that $D_n(x, z)$ is the density of the distribution of a centered vector of independent Gaussian variables with mean 0 and variance $\sigma(x)$.

With these coefficients and when n tends to infinity, one has more and more seed production and natural death, less competition, and the locations of producted seeds are more and more localized around the "mother plant".

Theorem 5.6 *Assume $(C2)$, and consider the sequence of processes X^n defined in Notation 5.1. Then for all $T > 0$, the sequence (X^n) converges in law, in $\mathbb{D}([0, T], M_F(\mathbb{R}^d))$, to the unique (in law) "superprocess" $X \in C([0, T], M_F(\mathbb{R}^d))$, defined by the following conditions:*

$$\sup_{[0,T]} E\left[\langle X_t, 1\rangle^3\right] < \infty \tag{5.25}$$

and for any $f \in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned} \bar{M}_t^f &= \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \left\langle X_s, \frac{1}{2}\sigma\gamma\Delta f\right\rangle ds \\ &\quad - \int_0^t \left\langle X_s(dx), f(x)\left[\beta(x) - \alpha(x)\int_{\mathbb{R}^d} U(x, y) X_s(dy)\right]\right\rangle ds \end{aligned} \tag{5.26}$$

is a continuous martingale with quadratic variation

$$\langle \bar{M}^f\rangle_t = \int_0^t 2\left\langle X_s, \gamma f^2\right\rangle ds \tag{5.27}$$

Proof We break the proof in several steps, which for many points, are similar to the different steps of the proof of Theorem 5.3.

Step 1 Let us first prove the uniqueness of the solution of the martingale problem defined by (5.25), (5.26) and (5.27), that is the uniqueness of a probability measure P on $C([0, T], M_F(\mathbb{R}^d))$ under which the canonical process X satisfies (5.25), (5.26) and (5.27). This result is well-known for the superBrownian process (defined by a similar martingale problem, but with $\alpha = \beta = 0$, and $\sigma\gamma = 1$). As noted in [5], we may use the version of Dawson's Girsanov transform obtained in Evans-Perkins [4] to deduce the uniqueness in our situation, provided the condition below is satisfied:

$$E_P\left(\int_0^t \left\langle X_s(dx), \left[\beta(x) - \alpha(x)\int U(x, y) X_s(dy)\right]^2\right\rangle ds\right) < +\infty,$$

which is easily obtained from (5.25), since the coefficients are bounded.

Step 2 Next, we would like to obtain uniform three-order moment estimates. Here, the parameters appearing in (5.2) for $\phi(\nu) =< \nu, 1 >^3$ depend on n and we can not neglect the death terms. A fair computation, using Lemma 5.2-(i) with $\phi(\nu) = \langle \nu, 1\rangle^3 \wedge A$, taking into account the birth and natural death terms, and using the fact that for all $x \geq 0$, $\epsilon \in]0, 1]$, $(x+\epsilon)^3 - x^3 \leq C\epsilon(1 + x^2)$ and $|(x+\epsilon)^3 + (x-\epsilon)^3 - 2x^3| \leq C\epsilon^2(1 + x)$ leads to $\sup_{[0,T]} E\left[\langle X_t^n, 1\rangle^3 \wedge A\right] \leq C_T$, the constant

C_T not depending on A, n. On the other hand, we know from Theorem 3.1-(ii) that for each n, $E\left[\sup_{[0,T]} \langle X_t^n, 1\rangle^3\right] < \infty$. We thus can make A tend to infinity and get

$$\sup_n \sup_{[0,T]} E\left[\langle X_t^n, 1\rangle^3\right] < \infty, \tag{5.28}$$

As a consequence, using standard arguments and Lemma 5.2-(iv) (with $f = 1$), we obtain that $\sup_n E(\sup_{[0,T]} < M^{n,1} >_t) < +\infty$ and further that

$$\sup_n E\left(\sup_{[0,T]} \langle X_t^n, 1\rangle\right) < \infty \tag{5.29}$$

Step 3 Now, we prove the tightness of the sequence of the laws $(\mathcal{L}(X^n))_n$ in $\mathcal{P}(\mathbb{D}([0,\infty[, M_F(\mathbb{R}^d)))$, following the same approach as in Theorem 5.3. First, we deduce from Step 2 that $\sup_n E\left[\sup_{[0,T]} |\langle X_s^n, f\rangle|\right] < \infty$, for any bounded f. We thus have to prove that for any $f \in C_b^2(\mathbb{R}^d)$, the sequence $\langle X_t^n, f\rangle$ satisfies the Aldous-Rebolledo criterion. Let us consider a couple (S, S') satisfying a.s. $0 \leq S \leq S' \leq S+\delta \leq T$. Using Lemma 5.2, $(C2)$, Doob's inequality and the fact that $\left|\int_{\mathbb{R}^d} D_n(x,z) f(x+z)dz - f(x)\right| \leq \bar{\sigma}||f''||_\infty/2n$, we deduce the existence of a constant C independent of n such that

$$\begin{aligned}
&\int_S^{S+\delta} ds \int_{\mathbb{R}^d} X_s^n(dx)\bar{\beta}||f||_\infty + \int_S^{S+\delta} ds \int_{\mathbb{R}^d} X_s^n(dx) n\gamma(x) \left|\int_{\mathbb{R}^d} D_n(x,z) f(x+z)dz - f(x)\right| \\
&+ \int_S^{S+\delta} ds \int_{\mathbb{R}^d} X_s^n(dx)\bar{\alpha}\bar{U}||f||_\infty \int_{\mathbb{R}^d} X_s^n(dy) \leq C \int_S^{S+\delta} ds \left(\langle X_s^n, 1\rangle + \langle X_s^n, 1\rangle^2\right)
\end{aligned} \tag{5.30}$$

Using now (5.6) and $(C2)$, we deduce that for some constant C,

$$E\left[\langle M^{n,f}\rangle_{S+\delta} - \langle M^{n,f}\rangle_S\right] \leq CE\left[\int_S^{S+\delta} ds \left(\langle X_s^n, 1\rangle + \langle X_s^n, 1\rangle^2\right)\right] \tag{5.31}$$

Using the moment estimate (5.28), we finally obtain by that the laws of $(M^{n,f})$ and the laws of the drift parts of $\langle X^n, f\rangle$ are tight, and then, by Rebolledo's criterion, the laws of $\langle X^n, f\rangle$ are tight.
Step 4 Let us identify the limit. Let us call $Q^n = \mathcal{L}(X^n)$ and denote by Q a limiting value of the tight sequence Q^n, and by $X = (X_t)_{t\geq 0}$ a process with law Q. Exactly as in the proof of Theorem 5.3, one can show that X belongs a.s. to $C([0,T], M_F(\mathbb{R}^d))$. We have to show that X satisfies the conditions (5.25), (5.26) and (5.27). First note that (5.25) is straightforward from (5.28). Then, we show that for any function f in $C_b^3(\mathbb{R}^d)$, the process $\bar{M}_t^f$ defined by (5.26) is a martingale (the extension to every function in C_b^2 is not hard). We consider $0 \leq s_1 \leq ... \leq s_k < s < t$, some continuous bounded maps $\phi_1, ...\phi_k$ on $\mathcal{M}_F(\mathbb{R}^d)$, and our aim is to prove that, if the function Ψ from $\mathbb{D}([0,T], M_F(\mathbb{R}^d))$ into $\mathbb{R}$ is defined by

$$\begin{aligned}
\Psi(\nu) = \phi_1(\nu_{s_1})...\phi_k(\nu_{s_k}) \Big\{ &\langle \nu_t, f\rangle - \langle \nu_s, f\rangle - \int_s^t \langle \nu_u, \gamma\sigma\Delta f/2\rangle\, du \\
&- \int_s^t \langle \nu_u(dx), f(x)[\beta(x) - \langle \nu_u(dy), \alpha(x)U(x,y)\rangle]\rangle\, du \Big\}
\end{aligned} \tag{5.32}$$

then

$$E(\Psi(X)) = 0 \tag{5.33}$$

We know from Lemma 5.2 that using $(C2)$,

$$0 = E\left[\phi_1(X^n_{s_1})...\phi_k(X^n_{s_k})\left\{M^{n,f}_t - M^{n,f}_s\right\}\right] = E\left[\Psi(X^n)\right] + A_n \tag{5.34}$$

where A_n is defined as

$$\begin{aligned} A_n &= E\Big[\int_s^t du\int_{\mathbb{R}^d} X^n_u(dx)\Big\{\gamma(x)n\left[\int_{\mathbb{R}^d} D_n(x,z)f(x+z)dz - f(x) - \frac{\sigma(x)}{2n}\Delta f(x)\right] \\ &\quad +\beta(x)\left[\int_{\mathbb{R}^d} D_n(x,z)f(x+z)dz - f(x)\right]\Big\}\phi_1(X^n_{s_1})...\phi_k(X^n_{s_k})\Big]. \end{aligned} \tag{5.35}$$

First, an easy computation using $(C2)$, the fact that f is C^3_b, and (5.28) shows that

$$|A_n| \leq \frac{C(f)}{n}\int_s^t E\left[\langle X^n_u, 1\rangle\right] du \longrightarrow 0 \tag{5.36}$$

as n grows to infinity. Next, it is clear from assumption $(C2)$, the fact that f is C^3_b, and that Q only charges the space of continuous processes, that the map Ψ is Q-a.s. continuous. Furthermore, it is bounded up by

$$|\Psi(\nu)| \leq C\left(1 + \langle \nu_s, 1\rangle + \langle \nu_t, 1\rangle + \int_s^t \langle \nu_u, 1\rangle^2 du\right) \tag{5.37}$$

and one easily deduces from (5.28) that the sequence $(|\Psi(X^n)|)_n$ is uniformly integrable. Hence,

$$\lim_n E\left(|\Psi(X^n)|\right) = E_Q\left(|\Psi(X)|\right). \tag{5.38}$$

Associating (5.34), (5.36), (5.38) allows us to conclude that (5.33) holds, and thus $\bar{M}^f$ is a martingale.
We finally have to show that the bracket of $\bar{M}^f$ is given by (5.27). To this aim, we first check that

$$\begin{aligned} \bar{N}^f_t &= \langle X_t, f\rangle^2 - \langle X_0, f\rangle^2 - \int_0^t ds\int_{\bar{\mathcal{X}}} X_s(dx)2\gamma(x)f^2(x) \\ &\quad -\int_0^t 2\langle X_s, f\rangle\, ds\int_{\bar{\mathcal{X}}} X_s(dx)f(x)\left[\beta(x) - \alpha(x)\int_{\bar{\mathcal{X}}} U(x,y)X_s(dy)\right] \\ &\quad -\int_0^t 2\langle X_s, f\rangle\, ds\int_{\bar{\mathcal{X}}} X_s(dx)\frac{1}{2}\sigma(x)\gamma(x)\Delta f(x) \end{aligned} \tag{5.39}$$

is a martingale. This can be done exactly as for $\bar{M}^f_t$, making go to the limit the fact that, thanks to Lemma 5.2-(iii) (with $q=2$),

$$\begin{aligned} N^{n,f}_t &= \langle X^n_t, f\rangle^2 - \langle X^n_0, f\rangle^2 - \int_0^t ds\int_{\bar{\mathcal{X}}} X^n_s(dx)\gamma(x)\left[\int_{\mathbb{R}^d} f^2(x+z)D_n(x,z)dz + f^2(x)\right] \\ &\quad -\int_0^t ds 2\langle X^n_s, f\rangle\int_{\bar{\mathcal{X}}} X^n_s(dx)\left[\beta(x)\int_{\mathbb{R}^d} f(x+z)D_n(x,z)dz - \alpha(x)\int_{\bar{\mathcal{X}}} X^n_s(dy)U(x,y)\right] \\ &\quad -\int_0^t ds 2\langle X^n_s, f\rangle\int_{\bar{\mathcal{X}}} X^n_s(dx)\gamma(x)n\left[\int_{\mathbb{R}^d} f(x+z)D_n(x,z)dz - f(x)\right] \\ &\quad -\frac{1}{n}\int_0^t ds 2\langle X^n_s, f\rangle\int_{\bar{\mathcal{X}}} X^n_s(dx)\beta(x)\int_{\mathbb{R}^d} f^2(x+z)D_n(x,z)dz \\ &\quad -\frac{1}{n}\int_0^t ds 2\langle X^n_s, f\rangle\int_{\bar{\mathcal{X}}} X^n_s(dx)\alpha(x)\int_{\bar{\mathcal{X}}} X^n_s(dy)U(x,y)f^2(x) \end{aligned} \tag{5.40}$$

is a martingale. Next, using the Itô formula in the definition (5.26) of $\bar{M}_t^f$, we deduce that

$$\langle X_t, f\rangle^2 - \langle X_0, f\rangle^2 - \langle \bar{M}^f\rangle_t - \int_0^t 2\langle X_s, f\rangle\, ds \int_{\bar{\mathcal{X}}} X_s(dx) f(x) \left[\beta(x) - \alpha(x)\int_{\bar{\mathcal{X}}} U(x,y) X_s(dy)\right]$$
$$- \int_0^t 2\langle X_s, f\rangle\, ds \int_{\bar{\mathcal{X}}} X_s(dx) \frac{1}{2}\sigma(x)\gamma(x)\Delta f(x) \tag{5.41}$$

is a martingale. Comparing this formula with (5.39) allows to conclude that (5.27) holds. □

6 About extinction and survival

We first of all would like to recall a result of Ethridge, [5]. Consider the superprocess X obtained in Theorem 5.6, assume that σ, γ, β and α are constant on $\mathbb{R}^d$, and that $U(x,y) = h(|x-y|)$, for some nonnegative decreasing function h on $\mathbb{R}_+$ satisfying $\int_0^\infty h(r) r^{d-1} dr < \infty$. Then, if β is sufficiently small, and α is sufficiently large, X does not survive: almost surely, there exists $t \geq 0$ such that for all $s \geq 0$, $X_{t+s} = 0$.
One can also find a complementary result in [5], which shows non-extinction with positive probability for another model, the "stepping-stone version of the Bolker Pacala process". Let us now come back to the Bolker Pacala process, defined as the solution of (2.8). The techniques used in [5] are specific to continuous processes and can not be generalized to the Bolker-Pacala discontinuous process.
Before giving our results, let us point out the following obvious remark.

Remark 6.1 *Assume (A), and that $E[\langle \nu_0, 1\rangle] < \infty$. Consider the Bolker-Pacala process $(\nu_t)_{t\geq 0}$. Assume also that there exist some constants $\gamma_0 \leq \mu_0$ such that for all $x \in \bar{\mathcal{X}}$, $\mu(x) \geq \mu_0$ and $\gamma(x) \leq \gamma_0$. Then $(\nu_t)_{t\geq 0}$ does a.s. not survive, that is $P[\exists s > 0\ \langle \nu_s, 1\rangle = 0] = 1$.*

The proof of this remark is not hard, since in such a case, the process $Z_t = \langle \nu_t, 1\rangle$ can be bounded from above by a standard continuous time binary Galton-Watson process Y_t with death rate μ_0 and birth rate γ_0, of which the extinction probability is one.

In this section, we will first prove almost sure extinction in a case where the state space $\bar{\mathcal{X}}$ is compact. Then, we will show non-extinction in the case of a discrete version of the Bolker-Pacala process, with a specific (and quite not realistic) competition kernel U.

6.1 Extinction in the compact case

We will check a result which essentially says that if the state space $\bar{\mathcal{X}}$ is compact, then the population does almost surely not survive. Let us assume

Assumption (E):
(i) The maps $\alpha(x)$ and $\mu(x) + \alpha(x)U(x,x)$ are bounded below.
(ii) There exists a non-decreasing function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, satisfying $\varphi(0) = 0$, such that $\lim_\infty \varphi(x) = \infty$, such that the map $x\varphi(x)$ is convex on $[0,\infty)$ and such that for all $\nu \in \mathcal{M}$,

$$\langle \nu \otimes \nu, U\rangle \geq \langle \nu, 1\rangle\, \varphi(\langle \nu, 1\rangle) \tag{6.1}$$

Remark 6.2 *Assumption (E)-(ii) holds if $\bar{\mathcal{X}}$ is compact in $\mathbb{R}^d$, and if there exist $\epsilon > 0$ and $\delta > 0$ such that $U(x,y) \geq \epsilon \mathbf{1}_{\{|x-y|\leq\delta\}}$*

Theorem 6.3 *Assume* (A), (E), $\nu_0 \in \mathcal{M}$ *and* $E(\langle \nu_0, 1\rangle) < \infty$. *Consider the corresponding unique Bolker Pacala process* $(\nu_t)_{t\geq 0}$ *obtained in Theorem 3.1. Then there is almost sure extinction, that means that* $P(\ \exists t \geq 0,\ \langle \nu_t, 1\rangle = 0\ ) = 1$.

Proof of Remark 6.2 First of all, we recover $\bar{\mathcal{X}}$ with a family $\{\Delta_l\}_{l\in\{1,\dots,L\}}$ of disjoint cubes of $\mathbb{R}^d$, with side $\delta/\sqrt{d}$. Note that L is clearly finite and that for each l, each $x, y \in \Delta_l$, $|x-y| \leq \delta$. Recall the following consequence of the Cauchy-Schwarz inequality, which says that for all $L \geq 1$, all $\{\alpha_1, \dots, \alpha_L\}$ in $\mathbb{R}$, $\sum_{l=1}^L \alpha_l^2 \geq \frac{1}{L}\left[\sum_{l=1}^L \alpha_l\right]^2$. Hence for all $n \geq 1$, all $x_1, \dots, x_n \in \bar{\mathcal{X}}$,

$$\begin{aligned}\sum_{i,j=1}^n U(x_i,x_j) &\geq \sum_{i,j=1}^n \epsilon \mathbf{1}_{\{|x_i-x_j|\leq\delta\}} \geq \epsilon \sum_{i,j=1}^n \sum_{l=1}^L \mathbf{1}_{\Delta_l}(x_i)\mathbf{1}_{\Delta_l}(x_j) \\ &= \epsilon \sum_{l=1}^L \left[\sum_{i=1}^n \mathbf{1}_{\Delta_l}(x_i)\right]^2 \geq \epsilon \frac{1}{L}\left[\sum_{l=1}^L\sum_{i=1}^n \mathbf{1}_{\Delta_l}(x_i)\right]^2 = \epsilon\frac{1}{L}n^2 \end{aligned} \tag{6.2}$$

One immediately deduces that for any $\nu \in \mathcal{M}$, $\langle \nu\otimes\nu, U\rangle \geq \epsilon\frac{1}{L}\langle \nu, 1\rangle^2$. Hence (E)-(ii) holds with $\varphi(n) = \epsilon\frac{1}{L}n$. □

Proof of Theorem 6.3 We break the proof in several steps.
Step 1 We first of all prove that

$$A = \sup_{t\geq 0} E(\langle \nu_t, 1\rangle) < +\infty. \tag{6.3}$$

To this aim, we set $f(t) = E(\langle \nu_t, 1\rangle)$, and we use Proposition 3.4 with $\phi(\nu) = \langle \nu, 1\rangle$ to obtain

$$f(t) = f(0) + \int_0^t ds E\left[\langle \nu_s, \gamma-\mu\rangle - \langle \nu_s(dx)\otimes\nu_s(dy), \alpha(x)U(x,y)\rangle\right] \tag{6.4}$$

Hence f is differentiable, and if we set $\delta = ||\gamma-\mu||_\infty$ and $\alpha_0 = \inf_{\bar{\mathcal{X}}}\alpha(x)$, we deduce that for any $t\geq 0$,

$$f'(t) \leq \delta f(t) - \alpha_0 E(\langle \nu_t\otimes\nu_t, U\rangle) \tag{6.5}$$

Using then assumption (E) and then the Jensen inequality, we obtain that

$$f'(t) \leq \delta f(t) - \alpha_0 f(t)\varphi(f(t)) \tag{6.6}$$

Let now x_0 be the greatest solution of $\delta x_0 = \alpha_0 x_0 \varphi(x_0)$ (recall that $\varphi(x)$ is non-decreasing, goes to infinity with x, and that $\varphi(0) = 0$). Then one deduces from (6.6) that for any $t\geq 0$, $f(t) \leq f(0)\vee x_0$. This concludes the first step.
Step 2 We now check that almost surely,

$$\liminf_{t\to\infty}\langle \nu_t, 1\rangle \in \{0, \infty\} \tag{6.7}$$

Since $\langle \nu_t, 1\rangle$ is $\mathbb{N}$-valued, it suffices to check that for any $M\in\mathbb{N}^*$, $P\left[\liminf_{t\to\infty}\langle \nu_t, 1\rangle = M\right] = 0$. But this is clear: if $\liminf_{t\to\infty}\langle \nu_t, 1\rangle = M$, then $\langle \nu_t, 1\rangle$ reaches the state M infinitely often, but reaches the state $M-1$ only a finite number of times. This is (a.s.) impossible, because each time $\langle \nu_t, 1\rangle$ reaches the state M, the probability that the next state is $M-1$ is bounded below by ($\epsilon_0 = \inf_{\bar{\mathcal{X}}}[\mu(x) + \alpha(x)U(x,x)] > 0$)

$$\frac{M\epsilon_0}{M\bar{\gamma} + M\bar{\mu} + \bar{\alpha}\bar{U}M^2} > 0 \tag{6.8}$$

Step 3 We immediately deduce from (6.7), the fact that $\langle \nu_t, 1 \rangle$ is $\mathbb{N}$-valued, and that 0 is an absorbing state, that almost surely, $\lim_{t\to\infty} \langle \nu_t, 1 \rangle$ exists and

$$\lim_{t\to\infty} \langle \nu_t, 1 \rangle \in \{0, \infty\} \tag{6.9}$$

Step 4 By Fatou's lemma and Step 1,

$$E\left[\lim_{t\to\infty} \langle \nu_t, 1 \rangle\right] = E\left[\liminf_{t\to\infty} \langle \nu_t, 1 \rangle\right] \le \liminf_{t\to\infty} E\left[\langle \nu_t, 1 \rangle\right] \le A \tag{6.10}$$

Hence $\lim_{t\to\infty} \langle \nu_t, 1 \rangle < \infty$ a.s., and we deduce from (6.9) that $\lim_{t\to\infty} \langle \nu_t, 1 \rangle = 0$ a.s. This concludes the proof. $\square$

6.2 Survival in a simplified case

Next, we would like to show that in some cases, the Bolker-Pacala process survives with positive probability. We are not able to handle a proof in a general case, because the problem seems very difficult. It actually looks much more difficult that the problem of survival for the contact process, which has been studied by many mathematicians (see Ligget [8]). The only result we are able to give is deduced from a comparison with the contact process.

Assumption (S):

(i) The state space $\bar{\mathcal{X}} = \mathbb{Z}^d$.
(ii) The competition kernel U is pointwise, i.e. $U(x, y) = \mathbf{1}_{\{x=y\}}$.
(iii) The dispertion measure $D(x, dz) = D(dz) = \frac{1}{2^d} \sum_{u\in\mathbb{Z}^d, |u|=1} \delta_u(dz)$.
(iv) γ, μ, and α are positive constants satisfying:

$$\frac{\gamma 2^{-d}}{\mu + \alpha} > 2 \tag{6.11}$$

Note that $\bar{\mathcal{X}} = \mathbb{Z}^d$ was not authorized in our construction. The adaptation is however immediate.

Proposition 6.4 *Assume (S), assume that $\nu_0 \in \mathcal{M}$, $\langle \nu_0, 1 \rangle \ge 1$ almost surely, and that $E\left[\langle \nu_0, 1 \rangle\right] < \infty$. Consider the corresponding Bolker-Pacala process $(\nu_t)_{t\ge0}$. This process survives with positive probability. That means that $P(\ \inf_{t\ge0} \langle \nu_t, 1 \rangle \ge 1\ ) > 0$.*

We do not handle a completely rigorous proof. One would have to build a rigorous coupling between the contact process and the Bolker-Pacala process.

Proof We split the proof in two steps.
Step 1 Let us first recall definitions and results about the contact process (see [8] Chapter VI). First, denote by M_F^s the set of nonnegative finite measures η on $\mathbb{Z}^d$ such that for all $x \in \mathbb{Z}^d$, $\eta(\{x\}) \in \{0, 1\}$. The contact process, with parameters $\lambda_d > 0$ and $\lambda_m > 0$ is a Markov process $(\eta_t)_{t\ge0}$, taking its values in M_F^s and with generator K, defined for all ϕ bounded and measurable from $M_F(\mathbb{Z}^d)$ into $\mathbb{R}$, all $\eta \in M_F(\mathbb{Z}^d)$ by

$$\begin{aligned} K\phi(\eta) \quad = \quad & \lambda_d \int_{\mathbb{Z}^d} \eta(dx) \sum_{u\in\mathbb{Z}^d, |u|=1} \mathbf{1}_{\{\eta(\{x+u\})=0\}} \left[\phi(\eta + \delta_{x+u}) - \phi(\eta)\right] \\ & + \lambda_m \int_{\mathbb{Z}^d} \eta(dx) \mathbf{1}_{\{\eta(\{x\})=1\}} \left[\phi(\eta - \delta_x) - \phi(\eta)\right] \end{aligned} \tag{6.12}$$

Consider a (possibly random) initial state η_0 in M_F^s satisfying $\langle \eta_0, 1 \rangle \ge 1$ a.s. Then it is known, (see [8] Chapter VI), that the contact process $(\eta_t)_{t\ge0}$ with parameters $\lambda_d > 0$, $\lambda_m > 0$ and initial

state η_0 exists, is unique (in law), and that under the condition $\lambda_d > 2\lambda_m$, it survives with positive probability.
Step 2 Consider now the Bolker-Pacala process $(\nu_t)_{t\geq 0}$, which takes its values in the integer-valued measures on $\mathbb{Z}^d$, and denote by $\tilde{\eta}_t = \sum_{x\in\mathbb{Z}^d} \mathbf{1}_{\{\nu_t(\{x\})\geq 1\}}\delta_x$. Note that $\tilde{\eta}_t$ is always dominated by ν_t. Then $(\tilde{\eta}_t)_{t\geq 0}$ is a process with values in M_F^s and one can observe that $(\tilde{\eta}_t)_{t\geq 0}$ is a sort of contact process with time and space dependent, random parameters $\lambda_d(t,x,\omega) = \gamma 2^{-d}[1\vee\nu_t(\{x\})]$ and $\lambda_m(t,x,\omega) = \mathbf{1}_{\nu_t(\{x\})\leq 1}(\mu+\alpha)$. Under (S), $\lambda_d(t,x,\omega)$ is uniformly bounded from below by $\underline{\lambda}_d = \gamma 2^{-d}$ while $\lambda_m(t,x,\omega)$ is uniformly bounded from above by $\overline{\lambda}_m = \mu+\alpha$. Hence, the process $(\tilde{\eta}_t)_{t\geq 0}$ is bounded below by a contact process with parameters $\underline{\lambda}_d$ and $\overline{\lambda}_m$. Since under our assumption, $2\overline{\lambda}_m < \underline{\lambda}_d$, the conclusion follows from Step 1. □

Note that the previously described method can not apply to the "continuous-state" Bolker-Pacala process, since we really need the interaction to be strictly local. In fact, the only case we could treat by such a method is the case where the competition kernel is "completely local", and can not propagate: for example, $\mathcal{X} = \mathbb{R}^d$, and $U(x,y) \leq \sum_{p\in\mathbb{Z}^d} \mathbf{1}_{\Delta_p}(x)\mathbf{1}_{\Delta_p}(y)$, where, for $p\in\mathbb{Z}^d$, $\Delta_p = [p_1, p_1+1]\times ... \times [p_d, p_d+1]$.

7 On equilibrium

An interesting question is that of the existence of non trivial equilibria for the Bolker-Pacala process. Since this question seems very complicated, we will first try to give some results about the deterministic equation (5.7). Then, we will exhibit a nontrivial equilibrium for the Bolker Pacala process related to the carrying capacity, under a detailed balance condition which is unfortunately very restrictive. In this specific case, or if the initial condition is closed to the equilibrium, we prove convergence in some sense to this equilibrium. Finally and again under the detailed balance condition, we exhibit a nontrivial equilibrium for the Bolker-Pacala process. We will finally show some simulations. We will assume (B) (see Section 4) in the whole section.

7.1 Equilibrium of the deterministic equation

We first of all point out a trivial remark.

Remark 7.1 *Assume (B) and that $\gamma < \mu$, and consider a nonnegative finite measure ξ_0 on $\mathbb{R}^d$. Consider the corresponding unique solution $(\xi_t)_{t\geq 0} \in C([0,\infty), M_F(\mathbb{R}^d))$ of (5.7). Then ξ_t tends to 0 as t grows to infinity, in the sense that $\langle \xi_t, 1\rangle \leq \langle \xi_0, 1\rangle\, e^{-(\mu-\gamma)t}$.*

This remark follows from a fair application of (5.7) with $f=1$ and of the Gronwall Lemma.
We next generalize the existence of solutions to equation (5.7) to the case of possibly non integrable initial conditions.

Proposition 7.2 *Assume (B), and consider a nonnegative bounded measurable function ξ_0 on $\mathbb{R}^d$.*
1) There exists a unique function $(\xi_t(x))_{t\geq 0, x\in\mathbb{R}^d}$ such that:
(i) for all $t\geq 0$, all $x\in\mathbb{R}^d$, $\xi_t(x)\geq 0$,
(ii) for all $T<\infty$, $\sup_{t\in[0,T], x\in\mathbb{R}^d} \xi_t(x) < \infty$,
(iii) for all $t\geq 0$, all $x\in\mathbb{R}^d$,

$$\xi_t(x) = \xi_0(x) + \int_0^t [\gamma(\xi_s \star D)(x) - \mu\xi_s(x) - \alpha\xi_s(x)(\xi_s\star U)(x)]\, ds \tag{7.1}$$

where, for exemple, $[\xi_t \star D](x) = \int_{\mathbb{R}^d} D(x-y)\xi_t(y)dy$.
2) For all $x\in\mathbb{R}^d$, the map $t\mapsto \xi_t(x)$ is of class C^1 on $[0,\infty)$, and for all $T<\infty$, $|\partial_t\xi_t(x)|$ is

bounded on $[0,T]\times\mathbb{R}^d$.
3) If furthermore $\int_{\mathbb{R}^d}\xi_0(x)dx<\infty$, *then for all* $T<\infty$, $\sup_{t\in[0,T]}\int_{\mathbb{R}^d}\xi_t(x)dx<\infty$, *and the finite measure-valued function* $(\xi_t(x)dx)_{t\geq 0}$ *is the unique solution to (5.7).*

This proposition being quite unsurprising, we only sketch the proof.

Proof First note that point 2 is an immediate consequence of (7.1) and of the boundedness of ξ obtained in (i) and (ii). Point 3 is also easily deduced from point 1. To check the uniqueness part of point 1, it suffices to consider two solutions $(\xi_t(x))_{t\geq 0,x\in\mathbb{R}^d}$ and $(\tilde{\xi}_t(x))_{t\geq 0,x\in\mathbb{R}^d}$ to (i), (ii), (iii), both bounded by some constant A_T on $[0,T]\times\mathbb{R}^d$, and to apply the Gronwall Lemma to the function $\phi(t)=\sup_{s\leq t,x\in\mathbb{R}^d}|\xi_s(x)-\tilde{\xi}_s(x)|$, which can be shown to satisfy, for $t\leq T$,

$$\phi(t)\leq(\gamma+\mu+2\alpha A_T)\int_0^t\phi(s)ds \tag{7.2}$$

The existence part follows from an "implicite" Picard iteration. Define $\xi_t^0(x)=\xi_0(x)$ and construct by induction a sequence of functions $(\xi_t^n)_{t\geq 0}$ such that for each $x\in\mathbb{R}^d$, $t\mapsto\xi_t^n(x)$ is of class C^1 on $\mathbb{R}^+$ and satisfying for $n\geq 1$,

$$\xi_t^{n+1}(x)=\xi_0(x)+\int_0^t\left[\gamma(\xi_s^n\star D)(x)-\mu\xi_s^{n+1}(x)-\alpha\xi_s^{n+1}(x)(\xi_s^n\star U)(x)\right]ds \tag{7.3}$$

One can moreover check at each step that ξ^n is well-defined, nonnegative and bounded on $[0,T]\times\mathbb{R}^d$ for each n, each T. A fair computation shows that for all $t\geq 0$, $\sup_n\sup_{x\in\mathbb{R}^d}\xi_t^n(x)\leq\sup_{x\in\mathbb{R}^d}\xi_0(x)e^{\gamma t}$, and next that for any T, there exists a constant A_T such that for all $t\leq T$,

$$\sup_x|\xi_t^{n+1}(x)-\xi_t^n(x)|\leq A_T\int_0^t\left[\sup_x|\xi_s^{n+1}(x)-\xi_s^n(x)|+\sup_x|\xi_s^n(x)-\xi_s^{n-1}(x)|\right]ds \tag{7.4}$$

It follows that for all T,

$$\sum_{n\geq 1}\sup_{t\in[0,T]\times\mathbb{R}^d}|\xi_t^{n+1}(x)-\xi_t^n(x)|<\infty \tag{7.5}$$

Hence, there exists a function $(\xi_t(x))_{t\geq 0,x\in\mathbb{R}^d}$ such that for any T, $\sup_{t\in[0,T]\times\mathbb{R}^d}|\xi_t(x)-\xi_t^n(x)|$ tends to 0. One easily checks that this function satisfies points (i), (ii), (iii). □

We may now define the equilibria.

Definition 7.3 *Assume* (B). *For* f *a nonnegative bounded continuous function on* $\mathbb{R}^d$, *define the function* Ff *on* $\mathbb{R}^d$ *by*

$$Ff(x)=\frac{\gamma\,[f\star D]\,(x)}{\mu+\alpha\,[f\star U]\,(x)} \tag{7.6}$$

Then equation (7.1) can be rewritten as

$$\xi_t(x)=\xi_0(x)+\int_0^t(\mu+\alpha\,[\xi_s\star U]\,(x))(F\xi_s(x)-\xi_s(x))ds \tag{7.7}$$

This leads us to define the equilbria in the following sense. A continuous bounded nonnegative function c *on* $\mathbb{R}^d$ *is said to be a "reasonnable equilbrium" of equation (7.1) if for all* $x\in\mathbb{R}^d$,

$$c(x)=Fc(x) \tag{7.8}$$

This definition is slightly restrictive, but we may note that if D and U are continuous, then any solution to (7.8) such that $\limsup_{|x|\to\infty}[c\star D](x)/[c\star U](x)<\infty$ will be continuous and bounded.

Remark 7.4 *Assume (B), that $\gamma>\mu$, and that $\alpha>0$. Then the constant function $c_0(x)=(\gamma-\mu)/\alpha$ is a reasonnable equilibrium of (7.1). The constant function $c(x)=0$ is also, of course, a reasonnable equilibrium of (7.1).*

Note that the quantity $(\gamma-\mu)/\alpha$ appears in [2], and is called the *carrying capacity*, which can be understood as a sort of "maximum number of plants per unit of volume". We will use the following estimate.

Lemma 7.5 *Assume (B), that $\gamma>\mu$ and that $\alpha>0$. Define the signed function R on $\mathbb{R}^d$ by $R(x)=D(x)+\frac{\gamma-\mu}{\mu}(D(x)-U(x))$. Then, for all bounded function f, all $x\in\mathbb{R}^d$,*

$$Ff(x)-Fc_0(x)=\frac{\mu}{\mu+\alpha[f\star U](x)}[(f-c_0)\star R](x) \tag{7.9}$$

This result is immediately proved, using simply the expression of F. We now state an assumption which ensures that $R(x)dx$ is a probability measure, and hence that F is a contraction around c_0 in the space of bounded functions.

> Assumption (C): $\gamma>\mu$ and for all $x\in\mathbb{R}^d$, $\gamma D(x)\geq(\gamma-\mu)U(x)$. This implies that $R(x)dx$ is a probability measure on $\mathbb{R}^d$.

Let us now describe a situation for which the constant function c_0 is the unique nontrivial reasonnable equilibrium.

Proposition 7.6 *Assume (B), (C), that $\gamma>2^d\mu$, and that $\alpha>0$. Suppose also that $D(x)=D(|x|)$, where the map D is nonincreasing on $[0,\infty)$. (This hypothesis is physically reasonnable, and appears in [2] where the usual example consists in choosing as dispersing kernel a bilateral exponential). Then any nontrivial reasonnable equilibrium c of equation (7.1) identically equals c_0.*

Proof Let thus c be a nontrivial reasonnable equilibrium for (7.1).
Step 1 Since c is nontrivial, there exists x_0 such that $c(x_0)>0$. Since c is continuous, we deduce that c is bounded below on a neighborhood of x_0. Then (7.8) and the fact that D charges any neighborhood of 0 (since it is nonincreasing) ensure that c does never vanish.
Step 2 We now show that there exists a constant $\epsilon_0>0$ such that for all $x\in\mathbb{R}^d$, $c(x)\geq\epsilon_0$. To this aim, we first consider $\epsilon>0$ such that $\gamma(1/2^d-\epsilon)>\mu$, and then $a>0$ such that $\int_{[0,a]^d}D(x)dx\geq 1/2^d-\epsilon$, which is possible since D is radial.
Consider now any point $x=(x_1,...,x_d)\in\mathbb{R}^d$ and the box $B=[x_1,x_1+a]\times...\times[x_d,x_d+a]$. Denote by $m=\inf_B c$, which is positive since c is continuous and does never vanish. Our aim is to show that $m\geq g(m)$, the C^1 function g being defined on $[0,\infty)$ by

$$g(u)=f[u(1/2^d-\epsilon)] \quad;\quad f(u)=\frac{\gamma u}{\mu+\alpha\frac{\gamma-\mu}{\mu}u} \tag{7.10}$$

This will conclude the proof of Step 2, since one may check that $g'(0)=(1/2^d-\epsilon)\gamma/\mu>1$, so that $m\geq\epsilon_0>0$, ϵ_0 being the smallest positive solution to $u=g(u)$.
We thus check that $m\geq g(m)$. Let $y\in B$. Using (7.8) and (C), we deduce that $c(y)\geq f([c\star D](y))$. But f is nondecreasing, so that $c(y)\geq f\left(m\int_B D(y-z)dz\right)$. Using the symmetry and the nonincreasing properties of D, one easily deduces that since $y\in B$, $\int_B D(y-z)dz\geq\int_{[0,a]^d}D(z)dz\geq$

$1/2^d - \epsilon$. Thus for all $y \in B$, $c(y) \geq f\left(m(1/2^d - \epsilon)\right) = g(m)$, which ends Step 2.
Step 3 The conclusion is immediate: using (7.9), Step 2, and (C), we obtain

$$\sup_{\mathbb{R}^d} |c(x) - c_0| = \sup_{\mathbb{R}^d} |Fc(x) - Fc_0(x)| \leq \frac{\mu}{\mu + \alpha\epsilon_0} \sup_{\mathbb{R}^d} |c(x) - c_0| \tag{7.11}$$

□

Although the above uniqueness result seems quite promising, we are not able to prove for the moment that under the conditions of the previous proposition, any solution $(\xi_t)_{t\geq 0}$ to (7.1) starting from a non trivial initial condition converges to c_0 in some sense. One may however obtain two partial results.

Assumption (DBC): $\alpha > 0$, $\gamma > 0$, $\mu = 0$ and $D = U$.

This assumption is a "detailed balance condition". Indeed, under this condition, the equilibrium $c_0(x) \equiv \gamma/\alpha$ ensures that for any couple of points x and y, the rate of appearance of plants at x due to seed productions at y equals the rate of disappearance of plants at x because of competition of plants at y. In other words, $\gamma D(x-y)c_0(y) = \alpha c_0(x)c_0(y)U(x-y)$. Unfortunately, this condition is very restrictive.

Proposition 7.7 *Assume (B) and (DBC). Let ξ_0 be a positive bounded and measurable function on $\mathbb{R}^d$. Consider the associated unique solution $(\xi_t)_{t\geq 0}$ of (7.1) starting from ξ_0 obtained in Proposition 7.2. Then ξ_t tends to $c_0 = \gamma/\alpha$ as t grows to infinity in the sense that for all x, all t,*

$$[\xi_t(x) - c_0]^2 \leq [\xi_0(x) - c_0]^2 e^{-2\alpha[(\xi_0 \wedge c_0)\star D](x)t} \tag{7.12}$$

We will furthermore see in the proof below that the behaviour of ξ_t is quite simple: if $\xi_0(x) < c_0$, then $\xi_t(x)$ increases to c_0, while if $\xi_0(x) > c_0$, then $\xi_t(x)$ decreases to c_0.
Proof Since in this case, $\partial_t \xi_t(x) = -\alpha\xi_t \star D(x)(\xi_t(x) - c_0)$, we easily show that for all $t \geq 0$, all $x \in \mathbb{R}^d$,

$$\partial_t[\xi_t(x) - c_0]^2 = -2\alpha[\xi_t(x) - c_0]^2[\xi_t \star D](x) \tag{7.13}$$

Since ξ is nonnegative, we deduce that $[\xi_t(x) - c_0]^2$ is nonincreasing in t for each x. Since furthermore $\xi_t(x)$ is continuous in t for each x, we deduce that for any t, x, $\xi_t(x) \geq \xi_0(x) \wedge c_0$. Hence

$$\partial_t[\xi_t(x) - c_0]^2 \leq -2\alpha[\xi_t(x) - c_0]^2[(\xi_0 \wedge c_0) \star D](x) \tag{7.14}$$

from which the conclusion follows. □

We now treat quite a general case of coefficients α, γ, μ, U, D, but we consider an initial condition which is only a "small perturbation" of c_0.

Proposition 7.8 *Assume (B), (C), that $\alpha > 0$, and that U is bounded below by a positive continuous function h on $\mathbb{R}^d$. Consider a nonnegative bounded measurable function ξ_0 on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d}[\xi_0(x) - c_0]^2dx < \infty$. Consider the associated unique solution $(\xi_t)_{t\geq 0}$ of (7.1) starting from ξ_0 obtained in Proposition 7.2. Then ξ_t tends to c_0 as t grows to infinity in the sense that there exists $a > 0$ such that for all t,*

$$\int_{\mathbb{R}^d}[\xi_t(x) - c_0]^2dx \leq e^{-at}\int_{\mathbb{R}^d}[\xi_0(x) - c_0]^2dx \tag{7.15}$$

Proof A fair computation using Proposition 7.2-2, (7.7) and (7.9) shows that for all $t \geq 0$, all $x \in \mathbb{R}^d$,

$$[\xi_t(x) - c_0]^2 = [\xi_0(x) - c_0]^2 - 2\int_0^t \alpha[\xi_s(x) - c_0]^2[\xi_s \star U](x)ds$$
$$-2\int_0^t \mu[\xi_s(x) - c_0]\left\{[\xi_s(x) - c_0] - [(\xi_s(x) - c_0) \star R](x)\right\} ds \tag{7.16}$$

Thanks to (C), R is a probability measure. We furthermore know that ξ_t, and thus $\xi_t \star U$ is bounded on $[0,T] \times \mathbb{R}^d$ for each T. Furthermore an application of the Cauchy-Schwarz and Young inequalities yields to

$$\int_{\mathbb{R}^d} [\xi_t(x) - c_0][(\xi_t(x) - c_0) \star R(x)]dx \leq \int_{\mathbb{R}^d} [\xi_t(x) - c_0]^2 dx. \tag{7.17}$$

One easily deduces that for all $T \geq 0$, $\sup_{[0,T]} \int_{\mathbb{R}^d} [\xi_t(x) - c_0]^2 dx < \infty$. Hence equation (7.16) may be integrated on $x \in \mathbb{R}^d$, and we get that for all $t \geq 0$,

$$\partial_t \int_{\mathbb{R}^d} [\xi_t(x) - c_0]^2 \leq \int_{\mathbb{R}^d} - 2\alpha[\xi_t(x) - c_0]^2[\xi_t \star U](x)dx \tag{7.18}$$

In particular, $\int_{\mathbb{R}^d} [\xi_t(x) - c_0]^2 dx$ is nonincreasing, thus there exists a constant $b < \infty$ such that for all $t \geq 0$, $\int_{\mathbb{R}^d} \mathbf{1}_{\{\xi_t(x) \leq c_0/2\}} dx \leq b$. But since $U(x) \geq h(x)$, for some positive continuous function h, there exists a constant $d > 0$ such that

$$\inf_{A \in \mathcal{B}(\mathbb{R}^d),\ \int_A dx \leq b} \int_{\mathbb{R}^d / A} U(z)dz \geq d \tag{7.19}$$

(indeed, choose any compact subset K of $\mathbb{R}^d$ whose Lebesgue's measure equals $2b$, and set $d = b \inf_{x \in K} h(x)$). Hence, for all x, $[\xi_t \star U](x) \geq dc_0/2$. We thus obtain, using 7.18 again,

$$\partial_t \int_{\mathbb{R}^d} [\xi_t(x) - c_0]^2 \leq -dc_0\alpha \int_{\mathbb{R}^d} [\xi_t(x) - c_0]^2 \tag{7.20}$$

from which the conclusion follows. □

7.2 Equilibrium of the Bolker-Pacala process

We now would like to show that it might be possible to exhibit an equilibrium for the Bolker Pacala processes. This is a first step to study the long time behaviour of the Bolker-Pacala process $(\nu_t)_{t \geq 0}$ defined in Definition 2.5 conditionned on non-extinction. We will unfortunately be able to treat only the case where the detailed balance condition holds. Of course, such an equilibrium will be infinite (that is the number of plants is infinite). One may however state the following rigorous result.

We first of all denote by $\bar{\mathcal{M}}$ the set of nonnegative (possibly infinite) integer-valued measures on $\mathbb{R}^d$. We also denote by $\mathcal{A}$ the set of functions ϕ from $\bar{\mathcal{M}}$ into $\mathbb{R}$ of the form $\phi(\nu) = F(\langle \nu, f \rangle)$, for some bounded measurable function F on $\mathbb{R}$ and some function f with compact support on $\mathbb{R}^d$.

Proposition 7.9 *Assume (B) and (DBC) (see Subsection 7.1), and that $U(0) = 0$. Consider a Poisson measure π on $\mathbb{R}^d$ with intensity measure $c_0 dx$, where $c_0 = \gamma/\alpha$. Then π is a stationnary Bolker Pacala process, in the sense that for all $\phi \in \mathcal{A}$, $L\phi(\pi)$ a.s. exists, belongs to L^1, and $E[L\phi(\pi)] = 0$, where L is defined in (2.3).*

Note that assuming (DBC) and that $U(0) = 0$ implies that there is no "natural death". Remark also that this result is somewhat surprising, since it suggests that at equilibrium, the plants locations are independent. Let us finally mention that a similar result without assumption (DBC) would be much more interesting, however, the stationnary process π does not seem to be Poisson in such a case.
The proof relies on the following lemma, known as Slivnyak's formula in Moller [9] and also obtained from Palm measure considerations (see Kallenberg [7], chap. 10).

Lemma 7.10 *Let ν be a Poisson measure on $\mathbb{R}^d$ with intensity $m(dx)$. Denote by $\{x_i\}_{i\geq 1}$ the points of ν, that is $\nu = \sum_{i\geq 1} \delta_{x_i}$. Then for all measurable function h from $\mathbb{R}^d \times \bar{\mathcal{M}}$ into $\mathbb{R}$ such that $\int_{\mathbb{R}^d} m(dx) E[|h(x, \nu + \delta_x)|] < \infty$,*

$$E\left[\sum_{i\geq 1} h(x_i, \nu)\right] = \int_{\mathbb{R}^d} m(dx) E[h(x, \nu + \delta_x)] \tag{7.21}$$

Proof of Proposition 7.9 Let ϕ belong to $\mathcal{A}$. The fact that $L\phi(\pi)$ a.s. exists and belongs to L^1 for $\phi \in \mathcal{A}$ can be easily checked, using the explicit expression of L, and standard results about Poisson measures. We thus only prove that $E[L\phi(\pi)] = 0$. Denote by $\{x_i\}_{i\geq 1}$ the points of π, that is $\pi = \sum_{i\geq 1} \delta_{x_i}$. Hence, we obtain, using (DBC),

$$\begin{aligned} E[L\phi(\pi)] &= \gamma E\left[\sum_{i\geq 1} \int_{\mathbb{R}^d} D(z)dz \left\{\phi(\pi + \delta_{x_i+z}) - \phi(\pi)\right\}\right] \\ &\quad + \alpha E\left[\sum_{i\geq 1} \left\{\phi(\pi - \delta_{x_i}) - \phi(\pi)\right\} \sum_{j\geq 1} D(x_i - x_j)\right] \\ &= \gamma A_1 + \alpha A_2 \end{aligned} \tag{7.22}$$

where the last equality stands for a definition. We first use Lemma 7.10 with the function $h_1(x, \nu) = \int_{\mathbb{R}^d} D(z)dz \left\{\phi(\nu + \delta_{x+z}) - \phi(\nu)\right\}$.

$$A_1 = E\left[\sum_{i\geq 1} h_1(x_i, \pi)\right] = \int_{\mathbb{R}^d} c_0 dx E\left[\int_{\mathbb{R}^d} D(z)dz \left\{\phi(\pi + \delta_x + \delta_{x+z}) - \phi(\pi + \delta_x)\right\}\right] \tag{7.23}$$

Next, with $h_2(x, \nu) = \{\phi(\nu - \delta_x) - \phi(\nu)\} \langle \nu(dy), D(x - y) \rangle$, we obtain

$$A_2 = E\left(\sum_{i\geq 1} h_2(x_i, \pi)\right) = \int_{\mathbb{R}^d} c_0 dx E(\{\phi(\pi) - \phi(\pi + \delta_x)\} \langle (\pi + \delta_x)(dy), D(x - y) \rangle) \tag{7.24}$$

Since $D(0) = U(0) = 0$, we obtain, setting $h_3^x(y, \nu) = D(x - y) \{\phi(\nu) - \phi(\nu + \delta_x)\}$,

$$A_2 = \int_{\mathbb{R}^d} c_0 dx E\left(\sum_{j\geq 1} h_3^x(x_j, \pi)\right) \tag{7.25}$$

Using Lemma 7.10 again, we obtain

$$\begin{aligned} A_2 &= \int_{\mathbb{R}^d} c_0 dx \int_{\mathbb{R}^d} c_0 dy E\left[D(x - y) \left\{\phi(\pi + \delta_y) - \phi(\pi + \delta_x + \delta_y)\right\}\right] \\ &= c_0^2 \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dz E\left[D(z) \left\{\phi(\pi + \delta_x) - \phi(\pi + \delta_x + \delta_{x+z})\right\}\right] \end{aligned} \tag{7.26}$$

where we have used in the last equality the substitution $(y,x) \mapsto (x, x+z)$. Since $\alpha c_0^2 = \gamma c_0$, we deduce that $\gamma A_1 = -\alpha A_2$, wich ends the proof. $\square$

7.3 Simulations

The previous results suggest that the Bolker-Pacala process, conditionned on non extinction, should converge as time tends to infinity, to a random measure ν_∞, quite well-distributed (not far from the Lebesgue measure), with $(\gamma-\mu)/\alpha$ plants per unit of volume in average. We would like to show simulations about this fact.

We assume that $\bar{\mathcal{X}} = \mathbb{R}$, that $\gamma = 5$, $\mu = 1$, and $\alpha = 1$. We consider the case where $U(x,y) = \mathbf{1}_{\{|x-y|\leq 1/2\}}$ and $D(z) = \frac{1}{6}\mathbf{1}_{\{|z|\leq 3\}}$. Then we we compare the Bolker-Pacala process $(\nu_t)_{t\geq 0}$ with the stationnary solution $c_0(dx) = [(\gamma-\mu)/\alpha]dx$ of (7.1).

On Figure 1, we assume that $\nu_0 = \delta_0$. The boxes represent the empirical density of the Bolker-Pacala process at times $t = 3$ (1a) and then $t = 25$ (1b), obtained with one simulation, while the dotted line is the density of c_0, i.e. $(\gamma-\mu)/\alpha$. One checks that after some time, the Bolker-Pacala process is quite well-approximated by c_0.

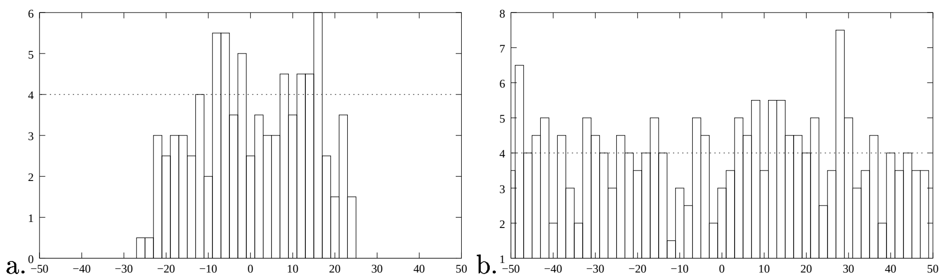

Figure 1: a. $t = 3$, b. $t = 25$

On Figure 2, we show the evolution in time of $\nu_t([-5,5])$ (full line), either starting from $\nu_0 = \delta_0$ (2a) or from $\nu_0 = 60\delta_0$ (2b), and compare it with $c_0([-5,5]) = 10(\gamma-\mu)/\alpha$ (dotted line).

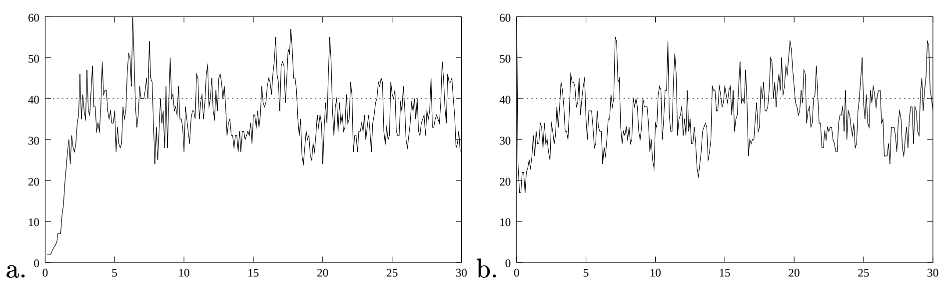

Figure 2: a. $\nu_0 = \delta_0$, b. $\nu_0 = 60\delta_0$

Finally, we would like to measure the power of competition. To this aim, we compare the evolution in time of the rate of interaction of all particles on particles located in a ball, in the case of the Bolker-Pacala process and in the case of c_0.

We assume that $\nu_0 = \delta_0$. On figure 3a, is represented, in full line, the evolution in time of $\langle \nu_t(dx)\nu_t(dy), \mathbf{1}_{|x|\leq 5} U(x,y)\rangle$, obtained by one simulation. The constant value (dotted line) is $\langle c_0(dx)c_0(dy), \mathbf{1}_{|x|\leq 5} U(x,y)\rangle = 2 * 5 * [(\gamma-\mu)/\alpha]^2$. Figure 3b shows the same quantities replacing 5 by 50.

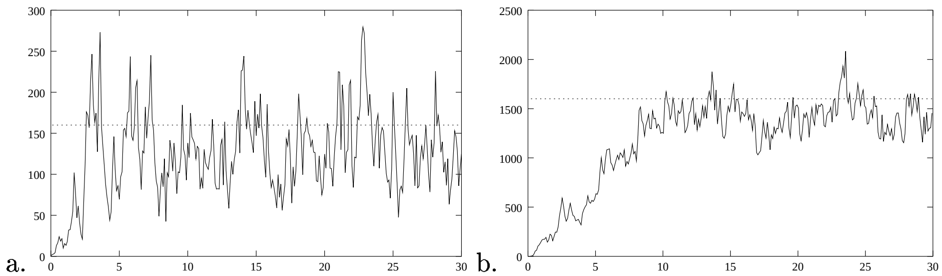

Figure 3: Rate of interaction endured by all particles in $[-5, 5]$ (a) or $[-50, 50]$ (b).

As a conclusion, one can say that on one hand, c_0 seems a good deterministic approximation of the Bolker-Pacala process after a long time, but on the other hand, there are clearly stochastic fluctuations around the deterministic approximation, that it could be interesting to study.

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