SELF-NORMALIZATION IN THE CENTRAL LIMIT THEOREM FOR VECTOR-VALUED RANDOM FIELDS

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Abstract

The classical CLT by Newman for strictly stationary associated real-valued random fields is generalized to quasi-associated vector-valued fields comprising, in particular, positively or negatively associated fields with finite second moments. We also establish a version of CLT with random matrix normalization for the fields under consideration. This main result allows us to construct approximate confidence intervals for the unknown mean vector.

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1. Introduction

The aim of this paper is to prove the CLT with random normalization for strictly stationary random fields defined on a lattice $\mathbb{Z}^d$ and taking values in $\mathbb{R}^k$. The summation regions for the multiindexed random vectors are finite sets $U_n \subset \mathbb{Z}^d$ growing in a certain sense as $n \to \infty$. The dependence structure of the fields under consideration is described in terms of *quasi-association*. This concept was introduced for real-valued random fields in Bulinski and Suquet (2001). Related dependence concepts were proposed for real-valued stochastic processes in Doukhan and Louhichi (1999) and for random fields in Doukhan and Lang (2002).

Let $X = \{X_t, t \in T\}$ be a family of real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and indexed by t in some set T. Recall that this family is called *associated* or *positively dependent* (Esary et al. (1967)) if, for any finite sets $I, J \subset T$ and any coordinatewise nondecreasing functions $f : \mathbb{R}^{|I|} \to \mathbb{R}$, $g : \mathbb{R}^{|J|} \to \mathbb{R}$,

$$\operatorname{cov}(f(X_s, s \in I), g(X_t, t \in J)) \geq 0, \tag{1}$$

whenever the covariance exists (here and in the sequel $|I|$ denotes cardinality of a finite set I). The notation $f(X_s, s \in I)$ means that for $I = \{s_1, \dots, s_m\}$ we consider any $f(x_{u_1}, \dots, x_{u_m})$ where $(u_1, \dots, u_m)$ is an arbitrary permutation of $(s_1, \dots, s_m)$.

There are various modifications of this definition. Under additional requirement $I \cap J = \varnothing$ condition (1) defines *weak association* (Newman (1984)), or *positive association* and the

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counterpart of (1) with opposite inequality defines then *negative association* (Joag-Dev and Proschan (1983)).

Note that any family of independent real-valued random variables is automatically associated and negatively associated.

There are interesting stochastic models in mathematical statistics, reliability theory, percolation theory and statistical mechanics described by families of positively or negatively associated random variables, see, e.g., the references in Bulinski and Suquet (2001).

A collection of real-valued random variables $X = \{X_t, t \in T\}$ with $EX_t^2 < \infty$ $(t \in T)$ is called *quasi-associated* if for all finite disjoint subsets $I, J \subset T$ and any bounded Lipschitz functions $f : \mathbb{R}^{|I|} \to \mathbb{R}$, $g : \mathbb{R}^{|J|} \to \mathbb{R}$ the following inequality holds

$$|\mathrm{cov}(f(X_s, s \in I), g(X_t, t \in J))| \leq Lip(f)Lip(g) \sum_{s \in I} \sum_{t \in J} |\mathrm{cov}(X_s, X_t)|. \tag{2}$$

Here

$$Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_1} < \infty, \tag{3}$$

$\|x\|_1 = \sum_{s=1}^m |x_s|$ for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. Since all norms in $\mathbb{R}^m$ are equivalent, the choice of norm $\| \cdot \|_1$ in (3) is for the sake of convenience only.

In Bulinski and Shabanovich (1998) it was shown that any positively or negatively associated collections of random variables with finite second moment satisfy (2). Consequently, such fields are quasi-associated.

An analogue of (2) for smooth functions f and g was proved by Birkel (1988) for associated random variables (cf. Roussas (1994), Peligrad and Shao (1995), Bulinski (1996)).

Let now $X = \{X_t, t \in T\}$ be a random field with values in $\mathbb{R}^k$. The generalizations of the abovementioned concepts to vector valued families of random variables are considered, e.g., in Burton et al.(1986), Bulinski (2000), Bulinski and Shahkin (2003). The following definition was given in Bulinski (2000). A random field X with values in $\mathbb{R}^k$ is called *quasi-associated* if for any disjoint sets $I, J \subset \mathbb{Z}^d$ and all bounded Lipschitz functions $f : \mathbb{R}^{k|I|} \to \mathbb{R}$, $g : \mathbb{R}^{k|J|} \to \mathbb{R}$ one has

$$|\mathrm{cov}(f(X_s, s \in I), g(X_t, t \in J))| \leq Lip(f)Lip(g) \sum_{s \in I} \sum_{t \in J} \sum_{r,q=1}^{k} |\mathrm{cov}(X_{s,r}, X_{t,q})| \tag{4}$$

where $X_{s,r}$ denotes the r-th component of a vector X_s.

Recently it was proved in Shashkin (2002) that any Gaussian random field $X = \{X_t, t \in T\}$ with values in $\mathbb{R}^k$ is quasi-associated. A real-valued Gaussian random field is associated if and only if its covariance function is nonegative (Pitt(1982)) and negatively associated if and only if $\mathrm{cov}(X_s, X_t) \leq 0$ for all $s \neq t$ (Joag-Dev and Proschan (1983)). Thus the resul by Shashkin shows that the concept of quasi-association is strictly wider than that of positive or negative association for random fields with finite second moments.

To conclude the discussion of dependence conditions note that there are various possibilites to give estimates of the left-hand side of (4) for certain classes of "test functions" f and g (see, e.g., Doukhan and Lang (2002), Bulinski and Shashkin (2003)).

In section 2 we establish a generalization of the classical CLT by Newman (1980) to the vector-valued strictly stationary quasi-associated random fields. In section 3 a statistical variant of the CLT is obtained. Namely, the self-normalized partial sums are studied.

2. CLT for quasi-associated strictly stationary vector-valued random fields

In this section we prove the CLT for partial sums

$$S(U_n) = \sum_{j \in U_n} X_j, \quad n \in \mathbb{N},$$

of multi-indexed quasi-associated random vectors X_j using non-random normalizaton. The summation is carried over finite sets $U_n \subset \mathbb{Z}^d$ growing in a sense.

For $a \in \mathbb{R}^d_+$, $V \subset \mathbb{R}^d$ and $j \in \mathbb{Z}^d$ put

$$\Lambda_0(a) = \{x = (x_1, \ldots, x_d) : 0 < x_p \le a_p,\ p = 1, \ldots, d\}, \tag{5}$$

$$\Lambda_j(a) = \Lambda_0(a) + (j_1 a_1, \ldots, j_d a_d), \tag{6}$$

$$J_a^+(V) = \{j : \Lambda_j(a) \cap V \neq \varnothing\}, \quad N_a^+(V) = |J_a^+(V)|, \tag{7}$$

$$J_a^-(V) = \{j : \Lambda_j(a) \subset V\}, \quad N_a^-(V) = |J_a^-(V)|. \tag{8}$$

One says that $V_n \to \infty$ in the Van Hove sense as $n \to \infty$ if for every $a \in \mathbb{R}^d_+$

$$N_a^-(V_n) \to \infty \ \text{ and } \ \frac{N_a^-(V_n)}{N_a^+(V_n)} \to 1 \ \text{ as } \ n \to \infty. \tag{9}$$

If V_n are bounded measurable subsets of $\mathbb{R}^d$ then (see Ruelle (1964), ch.2, §2.1) $V_n \to \infty$ in the Van Hove sense whenever for any $\varepsilon > 0$ one has

$$\mu((\partial V_n)^\varepsilon)/\mu(V_n) \to 0 \ \text{ as } \ n \to \infty, \tag{10}$$

here G^ε denotes the ε-neighbourhood (in the Euclidean metric) of a set $G \subset \mathbb{R}^d$ and μ is the Lebesgue measure in $\mathbb{R}^d$.

There is a natural discrete analogue of this concept of "regular growth" for sets $U_n \subset \mathbb{Z}^d$ (see, e.g., Bolthausen (1982)). For $U \subset \mathbb{Z}^d$ let

$$\partial U = \{s \in U : \inf_{t \in \mathbb{Z}^d \setminus U} \|s - t\| = 1\}$$

where $\|x\| = \max_{1 \le p \le d} |x_p|$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

One says that a sequence $\{U_n\}_{n \in \mathbb{N}}$ of finite subsets of $\mathbb{Z}^d$ tends to infinity in a *regular manner* (cf. (10)) if

$$|\partial U_n|/|U_n| \to 0 \ \text{ as } \ n \to \infty. \tag{11}$$

The following result is an extension of Theorem 1 by Bulinski and Vronski (1996) which generalized the classical CLT by Newman (1980).

Theorem 1 *Let $X = \{X_j, j \in \mathbb{Z}^d\}$ be a strictly stationary quasi-associated random field with values in $\mathbb{R}^k$. Assume that for all $r, q = 1, \dots, k$ one has*

$$\sigma_{r,q} = \sum_{j \in \mathbb{Z}^d} |\mathrm{cov}(X_{0,r}, X_{j,q})| < \infty. \tag{12}$$

Then for all finite sets $U_n \subset \mathbb{Z}^d$ satisfying condition (11) the following relation holds

$$|U_n|^{-1/2}(S(U_n) - |U_n| E X_0) \xrightarrow{D} N(0, C) \ \ as \ \ n \to \infty. \tag{13}$$

Here C is the matrix with elements

$$c_{r,q} = \sum_{j \in \mathbb{Z}^d} \mathrm{cov}(X_{0,r}, X_{j,q}), \ \ r, q = 1, \dots, k, \tag{14}$$

and " $\xrightarrow{D}$ " means the weak convergence for distributions of random vectors in $\mathbb{R}^k$.

Proof is based on known methods (see Newman (1980), Burton et al.(1986), Bulinski and Vronski (1996)). We begin with preliminary steps.

Lemma 1 *Let $X = \{X_j, j \in \mathbb{Z}^d\}$ be a wide sense stationary random field with values in $\mathbb{R}^k$ such that condition (12) is satisfied. Let $\{U_n\}_{n \in \mathbb{N}}$ be a sequence of regular growing finite subsets of $\mathbb{Z}^d$ (see (11)). Then*

$$|U_n|^{-1} \operatorname{var} S(U_n) \to C \ \ as \ \ n \to \infty \tag{15}$$

where $\operatorname{var} S(U_n)$ is a covariance matrix of $S(U_n)$ and C is a martix defined in (14). The relation (15) means the convergence of all elements of the matrix $|U_n|^{-1} \operatorname{var} S_n$ to the corresponding elements of C as $n \to \infty$.

For a strictly stationary real-valued random field X (i.e. $k = 1$) this statement was established by Bolthausen (1982). The proof in the multidimensional case is straightforward.

The next lemma is an extension of the inequality by Burton et al.(1986).

Lemma 2 *Let $Y_1, \dots, Y_p$ $(p \geq 2)$ be quasi-associated random vectors with values in $\mathbb{R}^k$. Then for any $\lambda \in \mathbb{R}^k$ one has*

$$\left| E \exp\{i \sum_{m=1}^{p} (\lambda, Y_m)\} - \prod_{m=1}^{p} E \exp\{i(\lambda, Y_m)\} \right| \leq \sqrt{2} \|\lambda\|^2 \sum_{m,s=1, m \neq s}^{p} \sum_{r,q=1}^{k} |\mathrm{cov}(Y_{m,r}, Y_{s,q})| \tag{16}$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^k$ and $i = \sqrt{-1}$.

Proof. For any $\lambda \in \mathbb{R}^k$ and $1 \leq v \leq p - 1$ $(v \in \mathbb{N})$ the following identity holds

$$\Delta_{v,p} = E \exp\Big\{ i \sum_{u=v}^{p} (\lambda, Y_u) \Big\} - E \exp\{i(\lambda, Y_v)\} E \exp\Big\{ \sum_{u=v+1}^{p} (\lambda, Y_u) \Big\}$$

$$= \operatorname{cov}\Bigg(\cos(\lambda, Y_v), \cos\Big(\sum_{u=v+1}^{p}(\lambda, Y_u)\Big)\Bigg)\operatorname{cov}\Bigg(\sin(\lambda, Y_v), \sin\Big(\sum_{u=v+1}^{p}(\lambda, Y_u)\Big)\Bigg)$$

$$+i\Bigg[\operatorname{cov}\Big(\cos(\lambda, Y_v), \sin\Big(\sum_{u=v+1}^{p}(\lambda, Y_u)\Big)\Big) + \operatorname{cov}\Big(\sin(\lambda, Y_v), \cos\Big(\sum_{u=v+1}^{p}(\lambda, Y_u)\Big)\Big)\Bigg].$$

Note that $f(x_1,\dots,x_k) = \cos(\sum_{r=1}^{k}\lambda_r x_r)$ and

$$g(y_{1,1},\dots,y_{1,k},\dots,y_{p-v,1},\dots,y_{p-v,k}) = \cos\Big(\sum_{r=1}^{k}\lambda_r\sum_{q=1}^{p-v}y_{q,r}\Big)$$

for each fixed $\lambda \in \mathbb{R}^k$ are the Lipschitz functions such that $Lip(f) \le \|\lambda\|$ and $Lip(g) \le \|\lambda\|$. Using the analogous inequalities for sine-functions and the trivial estimate $|a+ib| \le \sqrt{2}\max\{|a|,|b|\}$ for any $a, b \in \mathbb{R}$ we see that the quasi-association of $Y_1,\dots,Y_p$ implies the inequality

$$|\Delta_{v,p}| \le 2\sqrt{2}\|\lambda\|^2 \sum_{u=v+1}^{p}\sum_{r,t=1}^{k}|\operatorname{cov}(Y_{v,r}, Y_{u,t})|.$$

The left-hand side of (16) can be estimated by $\sum_{v=1}^{p-1}|\Delta_{v,p}|$. This completes the proof of the lemma.

Now we turn to the proof of Theorem 1. For $a = (a_1,\dots,a_d) \in \mathbb{R}_+^d$, $U \subset \mathbb{Z}^d$ and $j \in \mathbb{Z}^d$ set $\widetilde{\Lambda}_0(a) = \Lambda_0(a) \cap \mathbb{Z}^d$, $\widetilde{\Lambda}_j(a) = \Lambda_j(a) \cap \mathbb{Z}^d$ where $\Lambda_0(a)$ and $\Lambda_j(a)$ were introduced in (5) and (6). Then for $U \subset \mathbb{Z}^d$ in a similar way as for $V \subset \mathbb{R}^d$ (i.e. using $\widetilde{\Lambda}_0(a)$ and $\widetilde{\Lambda}_j(a)$ instead of $\Lambda_0(a)$ and $\Lambda_j(a)$ in (7),(8)) define $J_a^+(U)$ and $J_a^-(U)$.

For any fixed $a \in \mathbb{R}_+^d$ consider a set $M_n = J_a^-(U_n)$ and

$$U_n^{(1)} = \bigcup_{j\in M_n}\Lambda_j(a), \quad U_n^{(0)} = U_n \setminus U_n^{(1)} \ (n \in \mathbb{N}). \tag{17}$$

Let $\|\cdot\|_0$ be the Euclidean norm in $\mathbb{R}^k$. Using the estimate

$$E\Big\|\sum_{j\in U_n^{(0)}}(X_j - EX_j)\Big\|_0^2 \le |U_n^{(0)}|\sum_{r=1}^{k}\sigma_{r,r} \tag{18}$$

and relations

$$|U_n^{(0)}|/|U_n| \to 0 \ \text{ and } \ |U_n^{(1)}|/|U_n| \to 1 \ \text{ as } \ n \to \infty \tag{19}$$

it is easy to see that for establishing (13) it is sufficient to prove that

$$|U_n^{(1)}|^{-1/2}(S(U_n^{(1)}) - |U_n^{(1)}|EX_0) \xrightarrow{D} N(0, C(a)) \ \text{ as } \ n \to \infty \tag{20}$$

where $C(a) \to C$ as $a \to \infty$ (i.e. $a_r \to \infty$ for every $r = 1,\dots,k$).

Introduce the random fields $\{Y_j(a), j \in M_n\}$, $n \in \mathbb{N}$ where

$$Y_j(a) = |\widetilde{\Lambda}_j(a)|^{-1/2}(S(\widetilde{\Lambda}_j(a)) - |\widetilde{\Lambda}_j(a)|EX_0). \tag{21}$$

Then

$$|U_n^{(1)}|^{-1/2}(S(U_n^{(1)}) - |U_n^{(1)}|EX_0) = |M_n|^{-1/2} \sum_{j \in M_n} Y_j(a). \tag{22}$$

For any $j \in \mathbb{Z}^d$ due to Lemma 1 one has

$$\operatorname{var} Y_j(a) = \frac{\operatorname{var} S(\widetilde{\Lambda}_0(a))}{|\widetilde{\Lambda}_0(a)|} = C(a) \to C \ \text{ as } \ a \to \infty. \tag{23}$$

Clearly, $\{|M_n|^{-1/2} Y_j(a), j \in M_n\}$ for every $n \in \mathbb{N}$ and $a \in \mathbb{R}_+^k$ is a collection of quasi-associated random vectors. Consequently, Lemma 1 and (23) permit us to reduce the proof of (20) to the CLT for arrys of independent centered random vectors with covariance matrix $C(a)$ whenever for each $a \in \mathbb{R}_+^k$

$$\Delta(n,a) = \frac{1}{|M_n|} \sum_{s,t \in M_n, s \neq t} \sum_{r,q=1}^{k} |\operatorname{cov}(Y_{s,r}(a), Y_{t,q}(a))| \to 0 \ \text{ as } \ n \to \infty.$$

For $\varepsilon \in (0, 1/2)$ consider the sets

$$\Lambda_0^{\varepsilon}(a) = \{s \in \mathbb{Z}^d : \varepsilon a_r < s_r \le (1-\varepsilon) a_r, \ r = 1, \dots, k\}.$$

Let $S_r(U)$ denote the $r-th$ component of a vector $S(U)$, $U \subset \mathbb{Z}^d$. Then for all $n \in \mathbb{N}$ and $a \in \mathbb{R}_+^d$

$$\Delta(n,a) \le |\Lambda_0(a)|^{-1} \sum_{j \neq 0} \sum_{r,q=1}^{k} |\operatorname{cov}(S_r(\Lambda_0(a)), S_q(\Lambda_j(a)))|$$

$$\le \sum_{\|j\| > \varepsilon a_0} \sum_{r,q=1}^{k} |\operatorname{cov}(X_{0,r}, X_{j,q})| + \left(1 - \frac{|\Lambda_0^{\varepsilon}(a)|}{|\Lambda_0(a)|}\right) \sum_{r,q=1}^{k} \sigma_{r,q} \tag{24}$$

where $a_0 = \min_{r=1,\dots,k} |a_r|$. Taking ε small enough and after that taking $a \in \mathbb{R}_+^d$ large enough (i.e. all components of a are large enough) due to condition (12) we obtain from (24) the desired result. The proof of Theorem 1 is complete.

Remark 1. We gave a detailed prof of Theorem 1 to clarify the role of quasi-association condition. Moreover, we see that to prove CLT for strictly stationary vector-valued random fields, it is sufficient to use the estimates for covariances only of cosine and sine type functions of appropriately normalized sums of the initial random vectors taken over certain cubes.

Remark 2. In the same manner as in Newman (1980) we can use the renorm group approach to construct random fields of the type (21) for all $j \in \mathbb{Z}^d$. The reasoning used to prove Theorem 1 shows that finite-dimensional distributions of these fields weakly converge to the ones defined by a Gaussian mean zero vector field with independent values.

3. Self-normalization in the CLT

To construct approximate confidence domains for unknown mean vector of a strictly stationary quasi-associated random field with values in $\mathbb{R}^k$ we need consistent estimates of the covariance matrix C appearing in (13). If C is nondegenerate then (13) implies the following relation

$$(C|U_n|)^{-1/2}(S(U_n) - |U_n|EX_0) \xrightarrow{D} N(0,I) \tag{25}$$

where I is a unit matrix of order k. Thus if we have a sequence of consistent estimates $\widehat{C}(U_n) = (\widehat{c}_{r,q}(U_n))_{r,q=1}^k$ of the matrix $C = (c_{r,q})_{r,q=1}^k$, that is for all $r, q = 1, \dots, k$

$$\widehat{c}_{r,q}(U_n) \xrightarrow{P} c_{r,q} \text{ as } n \to \infty, \tag{26}$$

then due to (25) and (26) we come to the formula

$$(\widehat{C}(U_n)|U_n|)^{-1/2}(S(U_n) - |U_n|EX_0) \xrightarrow{D} N(0,I). \tag{27}$$

Here $\widehat{C}(U_n) = \widehat{C}(X_j, j \in U_n)$, $n \in \mathbb{N}$, and "$\xrightarrow{P}$" means the convergence in probability.

In other words a random normalization is used in the CLT. In this regard one can recall the well-known procedure of studentization for independent summands. For strictly stationary sequences ($d = 1$) posessing either mixing or association properties

and $U_n = \{1, \dots, n\}$, $n \in \mathbb{N}$, two estimates of the variances of partial sums were proposed in Peligrad and Shao (1994) to guarantee the CLT with random normalization. More general families of estimates for variances of partial sums were introduced in Bulinski and Vronski (1996). In the last paper the associated real-valued random fields were studied.

For $j \in U \subset \mathbb{Z}^d$ $(1 \le |U| < \infty)$, $b = b(U) > 0$ and $r, q = 1, \dots, k$ set

$$K_j(b) = \{T \in \mathbb{Z}^d : \|s - t\| \le b\}, \quad Q_j = Q_j(U,b) = U \cap K_j(b), \tag{28}$$

$$\widehat{c}_{r,q}(U) = \frac{1}{|U|}\sum_{j\in U}|Q_j|\left(\frac{S_r(Q_j)}{|Q_j|} - \frac{S_r(U)}{|U|}\right)\left(\frac{S_q(Q_j)}{|Q_j|} - \frac{S_q(U)}{|U|}\right). \tag{29}$$

Note that for dependent summands instead of the traditional estimates of a covariance matrix used for independent observations the averaged variables $S_r(Q_j)/|Q_j|$ have appeared.

Theorem 2 *Let the conditions of Theorem 1 be satisfied. Let U_n be a sequence of regular growing finite sets $U_n \subset \mathbb{Z}^d$ (i.e. satisfying (11)). Assume that $\{b_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers such that*

$$b_n \to \infty \text{ as } n \to \infty, \quad \limsup_{n\in\mathbb{N}} \frac{b_n^{2d}}{|U_n|} < \infty. \tag{30}$$

Then relation (26) holds. Moreover, if C is nondegenerate then (27) takes place.

Proof. The estimates $(\widehat{c}_{r,q}(U))_{r,q=1}^k$ introduced by means of (29) are invariant under transformation $X_j \mapsto X_j - EX_0$ $(j \in U)$. So, without loss of generality we can further on assume that $EX_0 = 0 \in \mathbb{R}^k$. Let $\|\xi\|_L$ stand for the norm of a real-valued random variable $\xi \in L(\Omega, \mathcal{F}, P)$. For any fixed $r, q = 1, \ldots, k$ one has

$$\|\widehat{c}_{r,q}(U_n) - c_{r,q}\|_L \le I_1(U_n) + I_2(U_n) + I_3(U_n)$$

where

$$I_1(U_n) = \frac{1}{|U_n|}\left\|\sum_{j\in U_n}|Q_j|\left(\left(\frac{S_r(Q_j)}{|Q_j|} - \frac{S_r(U_n)}{|U_n|}\right)\left(\frac{S_q(Q_j)}{|Q_j|} - \frac{S_q(U_n)}{|U_n|}\right) - \frac{S_r(Q_j)}{|Q_j|}\frac{S_q(Q_j)}{|Q_j|}\right)\right\|_L,$$

$$I_2(U_n) = \frac{1}{|U_n|}\left\|\sum_{j\in U_n}\frac{1}{|Q_j|}\left(S_r(Q_j)S_q(Q_j) - ES_r(Q_j)S_q(Q_j)\right)\right\|_L,$$

$$I_3(U_n) = \left|\frac{1}{|U_n|}\sum_{j\in U_n}\frac{1}{|Q_j|}ES_r(Q_j)S_q(Q_j) - c_{r,q}\right|.$$

Condition (12) yields

$$I_1(U_n) \le |U_n|^{-3}E|S_r(U_n)S_q(U_n)\sum_{j\in U_n}|Q_j|$$

$$+ |U_n|^{-2}\sum_{j\in U_n}(E|S_r(Q_j)S_q(U_n) + E|S_r(U_n)S_q(Q_j)|)$$

$$\le (\sigma_{r,r}\sigma_{q,q})^{1/2}(|K_0(b_n)||U_n|^{-1} + 2|K_0(b_n)|^{1/2}|U_n|^{-1/2}) \to 0 \ \text{ as } \ n \to \infty. \tag{31}$$

For $c > 0$ introduce the functions

$$h_1(x) = \mathrm{sign(x)}\min\{|x|, c\}, \quad h_2(x) = x - h_1(x), \quad x \in \mathbb{R}. \tag{32}$$

For a nonempty finite set $Q \subset \mathbb{Z}^d$ let

$$\overline{S}_r(Q) = \frac{S_r(Q)}{\sqrt{|Q|}}, \quad r = 1, \ldots, k.$$

Note that

$$I_2(U_n) \le \sum_{p,m=1}^{2} I_2^{(p,m)}(U_n) \tag{33}$$

where

$$I_2^{(p,m)}(U_n) = \frac{1}{|U_n|}\left\|\sum_{j\in U_n}h_p(\overline{S}_r(Q_j))h_m(\overline{S}_q(Q_j)) - Eh_p(\overline{S}_r(Q_j))h_m(\overline{S}_q(Q_j))\right\|_L.$$

For $b, n \in \mathbb{N}$ introduce the sets

$$T_n^{(b)} = \{s \in U_n : \inf_{t\in\partial U_n}\|s - t\| \le b\}.$$

Put $T_n = T_n^{(2b_n)}$ where b_n meet condition (30). Then

$$
\begin{aligned}
& I_2^{(1,2)}(U_n) + I_2^{(2,1)}(U_n) + I_2^{(2,2)}(U_n) \\
& \leq 2|U_n|^{-1} \sum_{j\in U_n} \Big(E|h_1(\overline{S}_r(Q_j))h_2(\overline{S}_q(Q_j))| \\
& +E|h_2(\overline{S}_r(Q_j))h_1(\overline{S}_q(Q_j))| + E|h_2(\overline{S}_r(Q_j))h_2(\overline{S}_q(Q_j))| \Big) \\
& \leq 2\Big(E|h_1(\overline{S}_r(K_0(b_n)))h_2(\overline{S}_q(K_0(b_n)))| \\
& +E|h_2(\overline{S}_r(K_0(b_n)))|h_1(\overline{S}_q(K_0(b_n)))|)| + E|h_2(\overline{S}_r(K_0(b_n)))h_2(\overline{S}_q(K_0(b_n)))| \\
& + 3|T_n||U_n|^{-1}(\sigma_{r,r}\sigma_{q,q})^{1/2} \Big) \\
& \leq 4\Big(\sigma_{r,r} E\big(\overline{S}_q(K_0(b_n))\big)^2 \mathbf{1}\big\{|\overline{S}_q(K_0(b_n))| \geq c\big\}\Big)^{1/2} \\
& + 4\Big(\sigma_{q,q} E\big(\overline{S}_r(K_0(b_n)\big)^2 \mathbf{1}\big\{|\overline{S}_r(K_0(b_n))| \geq c\big\}\Big)^{1/2} \\
& + 6|T_n||U_n|^{-1}(\sigma_{r,r}\sigma_{q,q})^{1/2}
\end{aligned}
$$

where $\mathbf{1}$ is an indicator function.

It is easy to see that for every $r = 1, \ldots, k$ a family $\{\overline{S}_r^2(K_0(b_n))\}_{n=1}^{\infty}$ is uniformly integrable. Consequently for any $\varepsilon > 0$ we can find $c = c(\varepsilon)$ such that for all n large enough

$$
I_2^{(1,2)}(U_n) + I_2^{(2,1)}(U_n) + I_2^{(2,2)}(U_n) < \varepsilon. \tag{34}
$$

Note that

$$
(I_2^{(1,1)}(U_n))^2 \leq |U_n|^{-2} \sum_{j,t\in U_n} |\mathrm{cov}\Big(h_1(\overline{S}_r(Q_j))h_1(\overline{S}_q(Q_j)), h_1(\overline{S}_r(Q_t))h_1(\overline{S}_q(Q_t)) \Big)|.
$$

Using (32) we get

$$
\begin{aligned}
& |U_n|^{-2} \sum_{j,t\in U_n, \|j-t\|\leq 2b_n} \Big|\mathrm{cov}\Big(h_1(\overline{S}_r(Q_j))h_1(\overline{S}_q(Q_j)), h_1(\overline{S}_r(Q_t))h_1(\overline{S}_q(Q_t)) \Big)\Big| \\
& \leq 2c^2|U_n|^{-2} \sum_{j,t\in U_n, \|j-t\|\leq 2b_n} E|h_1(\overline{S}_r(Q_j))h_1(\overline{S}_q(Q_j))| \\
& \leq 2c^2|U_n|^{-1}|K_0(b_n)|(\sigma_{r,r}\sigma_{q,q})^{1/2}. \qquad (35)
\end{aligned}
$$

Now the quasi-association property implies that

$$
|U_n|^{-2} \sum_{j,t\in U_n, \|j-t\|> 2b_n} \Big|\mathrm{cov}\left(h_1(\overline{S}_r(Q_j))h_1(\overline{S}_q(Q_j)), h_1(\overline{S}_r(Q_t))h_1(\overline{S}_q(Q_t)) \right)\Big|
$$

$$\leq c^2|U_n|^{-2} \sum_{j,t\in U_n, \|j-t\|>2b_n} \frac{1}{\sqrt{|Q_j||Q_t|}} \sum_{u\in Q_j, v\in Q_t} (|\mathrm{cov}(X_{u,r},X_{v,q})| + |\mathrm{cov}(X_{u,q},X_{v,r})|$$
$$+|\mathrm{cov}(X_{u,r},X_{v,r})| + |\mathrm{cov}(X_{u,q},X_{v,q})|). \tag{36}$$

Furthermore, we have

$$\sum_{j,t\in U_n, \|j-t\|>2b_n} \frac{1}{\sqrt{|Q_j||Q_t|}} \sum_{u\in Q_j, v\in Q_t} |\mathrm{cov}(X_{u,r},X_{v,q})|$$
$$\leq \sum_{u,v\in U_n} |\mathrm{cov}(X_{u,r},X_{v,q})| \sum_{j\in U_n, Q_j\ni u} \frac{1}{\sqrt{|Q_j|}} \sum_{t\in U_n, Q_t\ni v} \frac{1}{\sqrt{|Q_t|}}$$
$$\leq \frac{1}{2}\sum_{u,v\in U_n} |\mathrm{cov}(X_{u,r},X_{v,q})| \Big[\Big(\sum_{j\in U_n, Q_j\ni u} \frac{1}{\sqrt{|Q_j|}}\Big)^2 + \Big(\sum_{t\in U_n, Q_t\ni v} \frac{1}{\sqrt{|Q_t|}}\Big)^2\Big]$$
$$\leq |K_0(U_n)||U_n|\sigma_{r,q} + |K_0(U_n)|^2|T_n|\sigma_{r,q}.$$

Estimating in a similar way all sums appearing in the right hand side of (36) and using (35) we se that

$$(I_2^{(1,1)}(U_n))^2 \leq c^2[2|U_n|^{-1}|K_0(b_n)|(\sigma_{r,r}\sigma_{q,q})^{1/2}$$
$$+ |K_0(b_n)||U_n|^{-1}(\sigma_{r,q}+\sigma_{q,r}\sigma_{r,r}+\sigma_{q,q}) + |K_0(b_n)|^2|T_n||U_n|^{-1}]. \tag{37}$$

Taking into account (33), (34) and (37) we get that

$$I_2(U_n) \to 0 \ \text{ as } \ n\to\infty. \tag{38}$$

Now observe that

$$|U_n|^{-1}\sum_{j\in U_n}|Q_j|^{-1}ES_r(Q_j)S_q(Q_j) = |U_n|^{-1}|U_n\setminus T_n||K_0(b_n)|ES_r(K_0(b_n))S_q(K_0(b_n))$$
$$+|U_n|^{-1}\sum_{j\in T_n}|Q_J|^{-1}ES_r(Q_j)S_q(Q_j).$$

In view of (13) the following relation is valid

$$|K_0(b_n)|^{-1}ES_r(K_0(b_n))S_q(K_0(b_n)) \to c_{r,q} \ \text{ as } \ n\to\infty.$$

Condition (31) implies that

$$|T_n||U_n|^{-1} \to 0 \ \text{ as } \ n\to\infty.$$

Using the trivial bound

$$|Q_j|^{-1}E|S_r(Q_j)S_q(Q_j)| \leq (\sigma_{r,r}\sigma_{q,q})^{1/2}, \quad j\in\mathbb{Z}^d,$$

we conclude that

$$I_3(U_n) \to 0 \ \text{ as } \ n\to\infty. \tag{39}$$

Relations (31), (38) and (39) yield (26). The proof of Theorem 2 is complete.

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