Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems

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1 Introduction

This paper is a contribution to the study of boundary value problems for systems of elliptic partial differential equations of the form

$$\left\{\begin{array}{rcll} -\Delta u_1 & = & f(x,u_1,u_2) & \text{in} \quad \Omega \\ -\Delta u_2 & = & g(x,u_1,u_2) & \text{in} \quad \Omega \\ u_1 = u_2 & = & 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1}$$

where u_1, u_2 are real-valued functions defined on a smooth bounded domain Ω in $\mathbb{R}^N$, $N \geq 3$, and f and g are Hölder continuous functions defined in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$.

This type of systems has been extensively studied during the last two decades - see for example the survey paper [19] and the references therein. One of the important questions is the existence of *a priori* bounds for positive smooth solutions of these systems.

It is well known that the existence of a priori bounds depends on the growth of the functions f and g as u_1 and u_2 go to infinity. In view of what is known for scalar equations, one expects that some polynomial (subcritical) growth is to be required. In fact such a restriction comes from the Sobolev imbedding theorems in dimension $N \geq 3$. It is also known that a priori bounds are particularly interesting when superlinear equations are considered. In fact, it is classical (see [27], [4], [3], [34]) that establishing a priori bounds for a scalar equation permits, through use of Krasnoselskii's index theory, to obtain existence results for such an equation. Very recently it was shown that systems have an analogous property and definitions of superlinearity for nonvariational systems of two equations were given in [2] ; see also [19] and [40] for applications of index theory to some special types of systems. More general results can be found in [37] where, in addition, systems of many equations are considered.

The simplest case of systems of type (1) — which is the only case in which a priori bounds have been studied up to now — is when the leading parts of f and g involve just pure powers of u_1 and u_2. More precisely, when f and g are such that (1) can be written in the form

$$\left\{ \begin{array}{rcl} -\Delta u_1 &=& a(x)u_1^{\alpha_{11}} + b(x)u_2^{\alpha_{12}} + h_1(x,u_1,u_2) \\ -\Delta u_2 &=& c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}} + h_2(x,u_1,u_2), \end{array} \right. \tag{2}$$

where the exponents α_{ij} are nonnegative real numbers, $a(x), b(x), c(x), d(x)$ are nonnegative continuous functions on $\overline{\Omega}$, and h_1, h_2 are locally bounded functions such that uniformly in $x \in \Omega$

$$\left\{ \begin{array}{rcl} \lim\limits_{|(u_1,u_2)|\to\infty} \left(a(x)u_1^{\alpha_{11}} + b(x)u_2^{\alpha_{12}}\right)^{-1} |h_1(x,u_1,u_2)| &=& 0 \\ \lim\limits_{|(u_1,u_2)|\to\infty} \left(c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}}\right)^{-1} |h_2(x,u_1,u_2)| &=& 0. \end{array} \right. \tag{3}$$

The method used here in order to obtain the a priori bounds, the so-called *blow-up method*, was introduced in [24] to treat the scalar case. The use of this method to treat systems like the one in (2) was first done in [38], and then in [33], [8], [19], [40]. Let us note that the blow-up method itself depends on results of nonexistence of positive solutions of equations and systems in the whole space or in a half-space. Such results are usually referred to as Liouville type theorems – see Section 2.

Our main result, Theorem 1.1, unifies and extends the previous results on a priori bounds for (2). In addition, it allows more general nonlinearities in systems of type (1), namely mixed powers of u_1 and u_2 in the principal part of the nonlinearities f and g.

Specifically, our results will concern the following system

$$\left\{ \begin{array}{rcl} -\Delta u_1 &=& a(x)u_1^{\alpha_{11}} + b(x)u_2^{\alpha_{12}} + f_1(x)u_1^{\gamma_{11}}u_2^{\gamma_{12}} + h_1(x,u_1,u_2) \\ -\Delta u_2 &=& c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}} + f_2(x)u_1^{\gamma_{21}}u_2^{\gamma_{22}} + h_2(x,u_1,u_2), \end{array} \right. \tag{4}$$

where we keep all the hypotheses made before for the system (2).

As for new hypotheses for system (4), we suppose that the continuous functions f_1 and f_2 are nonnegative in Ω, so (2) is a particular case of (4), and

(γ_1) we have $0 \leq \gamma_{ij} \leq \alpha_{ij}$, $i,j = 1,2$,

$$\frac{\gamma_{11}}{\alpha_{11}} + \frac{\gamma_{12}}{\alpha_{12}} = 1 \qquad \text{and} \qquad \frac{\gamma_{21}}{\alpha_{21}} + \frac{\gamma_{22}}{\alpha_{22}} = 1. \tag{5}$$

Hypothesis (5) means that the powers γ_{ij} are such that the terms with coefficients f_i in (4) are exactly of the order of the principal parts in (2) and cannot be included in the functions h_1, h_2, via Young's inequality.

We make the following superlinearity assumptions on the exponents α_{ij} :

$$\text{either} \quad \alpha_{11} > 1, \quad \text{or} \quad \alpha_{22} > 1, \quad \text{or} \quad \alpha_{12}\alpha_{21} > 1. \tag{6}$$

We remark here that $\alpha_{ii} > 1$ means the Emden-Fowler equation $-\Delta u_i = u_i^{\alpha_{ii}}$ is superlinear, while the third inequality in (6) has long been used as a notion of superlinearity for the Lane-Emden system

$$\left\{ \begin{array}{rcl} -\Delta u_1 & = & u_2^{\alpha_{12}} \\ -\Delta u_2 & = & u_1^{\alpha_{21}}. \end{array} \right. \tag{7}$$

Further, we have to specify the maximal growth of f and g in (1) that we can allow. This can best be done by using the following geometric construction. We denote $\vec{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$, and introduce the following lines (see Figure 1 in Section 2)

$$l_1 = \left\{ \vec{\beta} \mid \beta_1 + 2 - \beta_1\alpha_{11} = 0 \right\}, \quad l_2 = \left\{ \vec{\beta} \mid \beta_2 + 2 - \beta_2\alpha_{22} = 0 \right\},$$

$$l_3 = \left\{ \vec{\beta} \mid \beta_1 + 2 - \beta_2\alpha_{12} = 0 \right\}, \quad l_4 = \left\{ \vec{\beta} \mid \beta_2 + 2 - \beta_1\alpha_{21} = 0 \right\}.$$

A thorough explanation of how these lines appear will be given in the next section, when we expound the blow-up method. We just note here that if we set $(\beta_1', \beta_2') = l_1 \cap l_2$ (in case $\alpha_{11}, \alpha_{22} > 1$) and $(\beta_1'', \beta_2'') = l_3 \cap l_4$ (in case $\alpha_{12}\alpha_{21} > 1$), then the Emden-Fowler equations $-\Delta u_i = u_i^{\alpha_{ii}}$ are subcritical provided $\beta_i' > \frac{N-2}{2}$, while system (7) is subcritical provided $\beta_1'' + \beta_2'' > N-2$. Actually, the last condition is equivalent to saying that the exponents α_{12}, α_{21} in (7) are under the so-called "critical hyperbola" — a widely used notion of criticality for (7), introduced in [16] and [26].

We consider points $\vec{\beta} \geq 0$ which are to the left of or on l_1, below or on l_2 (note that both l_1 and l_2 can be empty, and then they introduce no restriction), below or on l_3, and above or on l_4. We call these points **admissible** (this notion will be completely understood in the next section). So given a system of type (2) the set of admissible points is automatically defined.

We divide the systems of type (2) into three classes, determined by the exponents α_{ij}. In each case we make a choice of (β_1, β_2) – which we shall use to state our theorem – and make some assumptions on the coefficients of the system.

Case A. *The intersection of l_1 and l_2 is admissible.* Then we set $(\beta_1, \beta_2) = l_1 \cap l_2$. In this case we shall assume that the functions $a(x)$ and $d(x)$ are bounded below on $\overline{\Omega}$ by a positive constant.

Case B. *The intersection of l_3 and l_4 is admissible.* For this type of systems we take $(\beta_1, \beta_2) = l_3 \cap l_4$. In this case we shall assume that the functions $b(x)$ and $c(x)$ are bounded below on $\overline{\Omega}$ by a positive constant. Further, in case B we have to assume in addition that $\alpha_{12} > 1$ and $\alpha_{21} > 1$ - see Remark 2 below.

Case C. *None of $l_1 \cap l_2$ and $l_3 \cap l_4$ is admissible.* Then either $l_1 \cap l_3$ or $l_2 \cap l_4$ is admissible and we take this intersection point to be our (β_1, β_2). In this case we shall assume that the function $b(x)$ (resp. $c(x)$) is bounded below on $\overline{\Omega}$ by a positive constant.

We stress that nothing prevents (β_1, β_2) from being the intersection of more than two lines. Note that a system can be simultaneously of type A and B, but type C is exclusive of the other two types.

In the special situation in Case B when β is the intersection of more than two lines we shall need in our arguments below a further technical assumption on the γ_{ij} (we believe this hypothesis can be removed) :

$$(\boldsymbol{\gamma_2}) \quad \gamma_{11}, \gamma_{12} \geq 1, \text{ if } \vec{\beta} = l_1 \cap l_3 \cap l_4, \quad ; \quad \gamma_{21}, \gamma_{22} \geq 1, \text{ if } \vec{\beta} = l_2 \cap l_3 \cap l_4.$$

Theorem 1.1 *Assume that system (4) satisfies the conditions stated above, and that the pair (β_1, β_2) which corresponds to the type of the system (A, B or C) satisfies the condition*

$$\min\{\beta_1, \beta_2\} > \frac{N-2}{2}. \tag{8}$$

Then system (4) admits a priori estimates, that is, each couple of positive classical solutions of (4) is bounded in the L^∞-norm by a constant which depends only on L^∞-bounds for the coefficients of the system and on the domain.

Remark 1. In Case C it is actually sufficient to suppose that $\beta_1 > \dfrac{N-2}{2}$ if $l_1 \cap l_3$ is admissible, and that $\beta_2 > \dfrac{N-2}{2}$ if $l_2 \cap l_4$ is admissible.

Remark 2. Note the additional assumption we have made in Case B - that both exponents α_{12} and α_{21} are greater than 1. This is due to the problem (7) in $\mathbb{R}^N_+$. We do not actually know of *any* Liouville-type result on this problem in a half-space, which does not require this hypothesis. See the detailed discussion at the end of the next section.

Remark 3. In [19] the first author presented the blow-up procedure for systems of type (2) and observed that there exist two special types of these systems (called weakly coupled and strongly coupled in [19]), contained in Cases A and B respectively, for which the blow-up procedure leads to Liouville type results for the equation $-\Delta u = u^p$ or for the Lane-Emden system (7).

In [40] Zou considered system (2) and showed that it admits a priori estimates under the supplementary assumptions that all exponents $\alpha_{ij} > 1$, that both coordinates of the point $l_1 \cap l_2$ are larger than $\dfrac{N-2}{2}$, that both coordinates of the point $l_3 \cap l_4$ are larger than $\dfrac{N-2}{2}$ (or one of them is larger than $N-2$), and that the point $l_3 \cap l_4$ lies neither on l_1 nor on l_2. Note that the hypotheses $\alpha_{12} > 1, \alpha_{21} > 1$ are not stated in [40] but they are actually used in the proofs of the results, because of the previous remark.

Remark 4. The question of non-existence of positive *bounded* solutions of (7) in $\mathbb{R}^N$ was completely solved when $N = 3$ by Serrin and Zou, [36]. So, when $N = 3$ one can get more precise results than Theorem 1.1 - see [40], Theorems 1.1, 2.1, and 3.1 for system (2).

As noted before, implementing the blow-up method depends on availability of Liouville type results in $\mathbb{R}^N$ or in a half-space. In a number of situations that we are led to consider the results available in the literature (see Section 2) are not sufficient, so we had to prove Liouville type theorems for systems. Let us stress that our paper differs in that respect from all previous works on a priori bounds for systems, where the conditions on the system were actually chosen so that the limit process in the blow-up method (see Section 2 for details) leads to known Liouville type theorems.

We prove nonexistence results in a half-space for limit systems of (4) by showing that whenever such a system does not admit bounded solutions in $\mathbb{R}^{N-1}$ then it does not possess bounded solutions in any half-space of $\mathbb{R}^N$ either (so actually it is enough to prove the required Liouville results in the whole space). We prove this fact by using a monotonicity result for autonomous systems in a half-space. Both these theorems – of clear independent interest – hold under some supplementary assumptions that we list next.

Suppose we have an autonomous system of the type

$$\left\{ \begin{array}{rcl} \Delta u_1 + f_1(u_1, u_2) & = & 0 \\ \Delta u_2 + f_2(u_1, u_2) & = & 0, \end{array} \right. \tag{9}$$

where $f_i \in C^1(\mathbb{R}^2), i = 1, 2$, and

$$\frac{\partial f_i}{\partial u_j}(\vec{u}) \geq 0 \qquad \text{for all} \;\; i \neq j, \; \vec{u} \in \mathbb{R}^2. \tag{10}$$

Systems satisfying the last property are usually referred to as cooperative (or quasi-monotone). We shall suppose in addition that we can write

$$f_1(u_1, u_2) = u_2^p + g_1(u_1, u_2)u_1, \qquad f_2(u_1, u_2) = u_1^q + g_2(u_1, u_2)u_2, \tag{11}$$

for some $p, q > 1$ and some continuous functions g_1, g_2, which have polynomial growth in u_1, u_2.

Here are the precise statements of the results.

Theorem 1.2 *Suppose we have a nontrivial nonnegative* bounded *classical solution* (u_1, u_2) *of system (9) in* $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x_N > 0\}$*, such that* $u_1 = u_2 = 0$ *on* $\partial\mathbb{R}^N_+$*. Suppose (10) and (11) are satisfied. Then*

$$\frac{\partial u_i}{\partial x_N} > 0 \qquad in \;\; \mathbb{R}^N_+, \quad i = 1, 2. \tag{12}$$

Theorem 1.3 *Suppose we have a system of type (9) which satisfies (10) and (11). If problem (9) with Dirichlet boundary condition has a nontrivial nonnegative bounded solution in* $\mathbb{R}^N_+$*, then the same problem has a positive solution in* $\mathbb{R}^{N-1}$ *(the limit as* $x_N \to \infty$ *of the solution in* $\mathbb{R}^N_+$*).*

We note that Dancer [17] obtained property (12) for the scalar equation $-\Delta u = f(u)$ provided that either $f(0) > 0$ or both $f(0) = 0$ and $f'(0) \geq 0$. Dancer's result was proved to hold for unbounded solutions and globally Lipschitz continuous f with $f(0) \geq 0$ by Berestycki, Caffarelli and Nirenberg ([6]). In this paper the authors also showed that positive solutions of $-\Delta u = f(u)$ are functions of x_N only, provided $f(\sup u) \leq 0$. They obtained even stronger results in dimensions 2 and 3.

The proof of Theorem 1.2 is based on the moving planes method and has two main ingredients. First, we use a maximum principle for cooperative systems in unbounded narrow domains, which follows from a result on scalar equations by Cabre [13]. Second, we make use of a recent Harnack type inequality for nonlinear systems obtained in [12]. By using some ideas from [12] we prove the following Harnack inequality for system (9).

Theorem 1.4 *Let (u_1, u_2) be a positive solution of (9) in some domain G and suppose (10) and (11) hold. Suppose K is a compact set properly included in G and*

$$\max\left\{\inf_{x\in K} u_1, \inf_{x\in K} u_2\right\} \leq 1, \qquad \max\left\{\sup_{x\in G} u_1, \sup_{x\in G} u_2\right\} \leq M.$$

Then

$$\sup_{x\in K} \max\{u_1, u_2\} \leq C \min\left\{\left(\inf_{x\in K} u_1\right)^{\frac{1}{p}}, \left(\inf_{x\in K} u_2\right)^{\frac{1}{q}}\right\}.$$

where C depends only on N and M.

This inequality permits to us to use a technique inspired by the proof of a symmetry result for scalar equations in cylinders in [6]. However, contrary to [6], we avoid using boundary Harnack inequalities. A supplementary difficulty in the argument stems from the fact that we have to use a Harnack inequality on a sequence of systems in which the coupling degenerates.

Finally, we prove some Liouville type results for systems in the whole space by, on one hand, extending to general systems a recent monotonicity result by Busca and Manasevich [10], who obtained a Liouville type theorem for the Lane-Emden system (7), and, on the other hand, by noticing an identity between exponents appearing after a blow-up change of variables and after the passage to polar coordinates used in [10]. For instance, we obtain the following result.

Theorem 1.5 *The system*

$$\left\{\begin{array}{rcl} \Delta u_1 + u_1^{\alpha_1} + u_2^{\alpha_1 \frac{\alpha_2 - 1}{\alpha_1 - 1}} & = & 0 \\ \Delta u_2 + u_1^{\alpha_2 \frac{\alpha_1 - 1}{\alpha_2 - 1}} + u_2^{\alpha_2} & = & 0, \end{array}\right. \tag{13}$$

does not have bounded positive classical solutions in $\mathbb{R}^N$, provided

$$1 < \alpha_1, \alpha_2 < \frac{N+2}{N-2}.$$

The paper is organized as follows. In Section 2 we give some preliminary results and state the known Liouville-type results for scalar equations and Lane-Emden systems in $\mathbb{R}^N$ or $\mathbb{R}^N_+$. In Section 3 we prove Theorem 1.2 and Theorem 1.3. Finally, in Section 4 we prove Theorem 1.5 and Theorem 1.1.

2 Preliminaries

Let us first describe the blow-up procedure. We will only sketch this, a full presentation can be found in [19], where this procedure is explained for system (2) (for the scalar case see Gidas-Spruck [24]). Assume that positive solutions of (4) do not have an a priori bound, that is, there exists a sequence $(u_{1,n}, u_{2,n})$ of positive solutions of (4) such that at least one of the sequences $u_{1,n}$ and $u_{2,n}$ tends to infinity in the L^∞-norm. Let β_1, β_2 be fixed positive constants to be chosen later. We set

$$\lambda_n = ||u_{1,n}||_{L^\infty(\Omega)}^{-\beta_1},$$

if $||u_{1,n}||_{L^\infty(\Omega)}^{\beta_2} \geq ||u_{2,n}||_{L^\infty(\Omega)}^{\beta_1}$ (up to a subsequence), and

$$\lambda_n = ||u_{2,n}||_{L^\infty(\Omega)}^{-\beta_2}$$

otherwise. We shall suppose – without restricting the generality – that we are in the first of these two situations.

Note that we have $\lambda_n \to 0$ as $n \to \infty$. Let $x_n \in \Omega$ be a point where $u_{1,n}$ assumes its maximum. The functions

$$v_{i,n}(x) = \lambda_n^{\beta_i} u_{i,n}(\lambda_n x + x_n),$$

are such that $v_{1,n}(0) = 1$ and $0 \leq v_{i,n} \leq 1$ in Ω. One also verifies that the functions $v_{1,n}$ and $v_{2,n}$ satisfy

$$\begin{cases} -\Delta v_{1,n} = a(\cdot)\lambda_n^{\beta_1+2-\beta_1\alpha_{11}}\, v_{1,n}^{\alpha_{11}} + b(\cdot)\lambda_n^{\beta_1+2-\beta_2\alpha_{12}}\, v_{2,n}^{\alpha_{12}} \\ \qquad + f_1(\cdot)\lambda_n^{\beta_1+2-\beta_1\gamma_{11}-\beta_2\gamma_{12}}\, v_{1,n}^{\gamma_{11}} v_{2,n}^{\gamma_{12}} + \lambda_n^{\beta_1+2}\widetilde{h_{1,n}} \\ \\ -\Delta v_{2,n} = c(\cdot)\lambda_n^{\beta_2+2-\beta_1\alpha_{21}}\, v_{1,n}^{\alpha_{21}} + d(\cdot)\lambda_n^{\beta_2+2-\beta_2\alpha_{22}}\, v_{2,n}^{\alpha_{22}} \\ \qquad + f_2(\cdot)\lambda_n^{\beta_2+2-\beta_1\gamma_{21}-\beta_2\gamma_{22}}\, v_{1,n}^{\gamma_{21}} v_{2,n}^{\gamma_{22}} + \lambda_n^{\beta_2+2}\widetilde{h_{2,n}} \end{cases} \tag{14}$$

in the domain $\Omega_n = \dfrac{1}{\lambda_n}(\Omega - x_n)$, where the dot stands for $\lambda_n x + x_n$, and $\widetilde{h_{i,n}} = h_i\left(\cdot, \lambda_n^{-\beta_1} v_{1,n}, \lambda_n^{-\beta_2} v_{2,n}\right)$. By compactness we can assume that $\{x_n\}$ tends to some point $x_0 \in \overline{\Omega}$.

The idea is to pass to the limit in (14) and obtain a system which can be proven to have only the trivial solution. This would then contradict the fact that the limit of $v_{1,n}$ has value 1 at the origin.

The next lemma deals with the passage to the limit.

Lemma 2.1 *The sequences $v_{1,n}, v_{2,n}$ converge in $W^{2,p}_{\mathrm{loc}}, 2 \le p < \infty$ to functions $v_1, v_2 \in C^2(G) \cap C^0(\overline{G})$, satisfying the limiting system of (14) in $G = \mathbb{R}^N$ or $G = \mathbb{R}^N_+$, provided all the powers of λ_n in (14) are non-negative. This limiting system is obtained by removing the terms in (14) where the powers of λ_n are strictly positive, the terms where the coefficient vanishes at x_0, and the terms containing $h_{i,n}, i = 1, 2$.*

Proof. The argument is standard. The proof of this lemma, except for the last part, can be found in [19]. Note that the passage to the limit in the terms containing products of $v_{1,n}$ and $v_{2,n}$ causes no problem, since $W^{2,p}_{\mathrm{loc}} \cap L^\infty$ is an algebra. To prove that the terms in $h_{i,n}$ tend to zero we distinguish two cases : if the sequences $\lambda_n^{-\beta_1} v_{1,n}$ and $\lambda_n^{-\beta_2} v_{2,n}$ are both bounded this follows from the local boundedness of h_i, if one of these sequences is unbounded, it follows from hypothesis (3). □

Now we can explain the choice of the couple (β_1, β_2) which we made in the introduction (Cases A, B and C). As stated above, in order to be able to make a passage to the limit in (14) all the powers of λ_n in this system have to be nonnegative. Hence, geometrically, we have to pick up a point (β_1, β_2) which is to the left of or on l_1, below or on l_2, below or on l_3, and above or on l_4 (we forget for an instant the mixing terms with coefficients f_i). These are exactly the points which we called admissible - see Figure 1.

Further, it is important to observe that not all admissible points (β_1, β_2) would permit us to prove a priori bounds for positive solutions of (4), as we explain next. Indeed, we have necessarily to make the choice of β_1 and β_2 in such a way that at least two of the powers of λ_n in (14) are zero. Otherwise, after the passage to the limit in (14) we may end up with an uncoupled system in $\mathbb{R}^N$ in which at least one of the equations is the Laplace equation. Hence the corresponding one of the functions v_1 and v_2 can be identically equal to a positive constant, and in this way we do not come to a contradiction.

So (β_1, β_2) has to be chosen on the intersection of at least two of the lines l_1, l_2, l_3, l_4. It can be seen that this is actually possible provided the superlinearity conditions (6) hold. Indeed, the first inequality in (6) implies that the line l_1 is not empty, and similarly the second one implies that l_2 is not empty either. Note also that the slope of l_3 is α_{12}^{-1} and the slope of l_4 is α_{21} ; consequently these two lines meet at a point with positive coordinates provided $\alpha_{12}\alpha_{21} > 1$.

Observe that the weakly coupled case of [19] corresponds to the situation when the intersection of l_1 and l_2 lies strictly below l_3 and strictly above l_4, while the strongly coupled case means l_3 and l_4 meet in Π, where Π denotes the rectangle enclosed by l_1, l_2, and the axes. We remark that l_1 and l_2 can

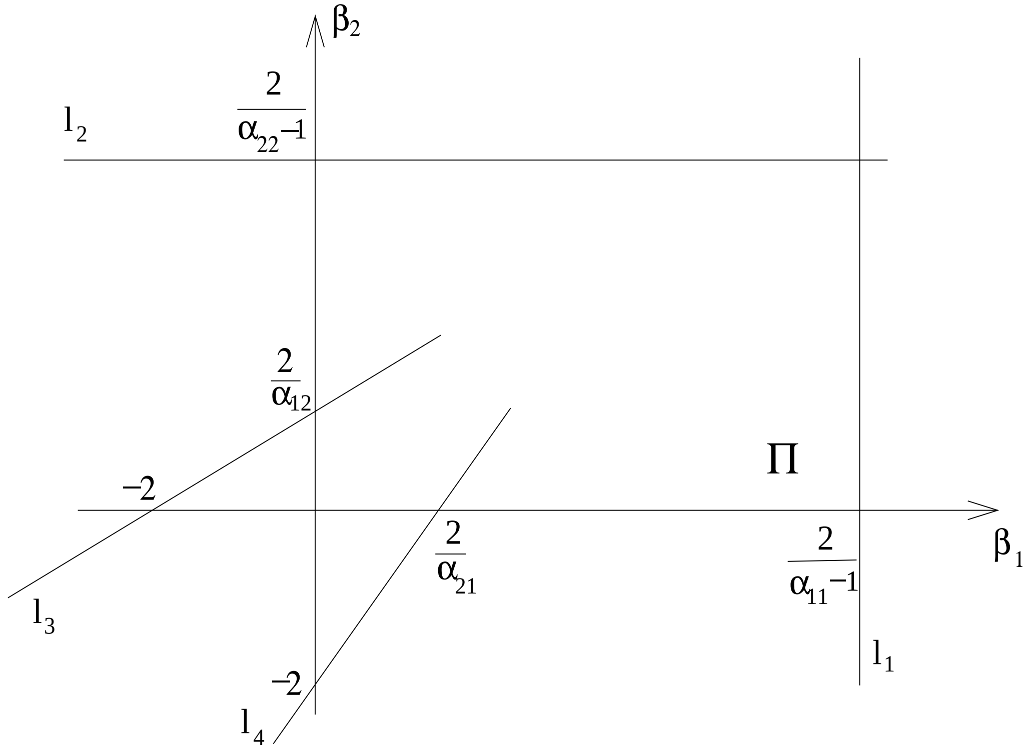

Figure 1: *Admissible couples* (β_1, β_2) *lie to the left of or on* l_1*, below or on* l_2*, below or on* l_3*, and above or on* l_4*.*

be empty, in case $\alpha_{11} \leq 1$ or $\alpha_{22} \leq 1$; in this case Π would be a half strip or the whole positive quadrant of the (β_1, β_2)-plane.

The next lemma shows how the terms in (14) with coefficients f_i transform after the passage to the limit.

Lemma 2.2 *If hypothesis* (γ_1) *holds and* β_1, β_2 *are chosen as in the previous section then*

$$\beta_1 + 2 - \beta_1\gamma_{11} - \beta_2\gamma_{12} \geq 0 \quad \textit{and} \quad \beta_2 + 2 - \beta_1\gamma_{21} - \beta_2\gamma_{22} \geq 0. \tag{15}$$

In Case A the first (resp. the second) inequality in (15) is strict if and only if l_3 *(resp.* l_4*) does not pass through* $l_1 \cap l_2$*. In Case B the first (resp. the second) inequality is strict if and only if* l_1 *(resp.* l_2*) does not pass through* $l_3 \cap l_4$*. In Case C one of the inequalities is always strict and the other is an equality.*

Proof. Suppose first that we are in Case A, that is, we have chosen (β_1, β_2) to be the admissible point $l_1 \cap l_2$. This means that

$$\beta_1 + 2 = \alpha_{11}\beta_1 \geq \alpha_{12}\beta_2 \quad \text{and} \quad \beta_2 + 2 = \alpha_{22}\beta_2 \geq \alpha_{21}\beta_1. \tag{16}$$

Then we have, by (16) and (5),

$$\begin{aligned}\beta_1 + 2 - \beta_1\gamma_{11} - \beta_2\gamma_{12} &= (\alpha_{11} - \gamma_{11})\beta_1 - \gamma_{12}\beta_2 \\ &= \alpha_{11}\left(1 - \frac{\gamma_{11}}{\alpha_{11}}\right)\beta_1 - \gamma_{12}\beta_2 \\ &= \gamma_{12}\left(\frac{\alpha_{11}}{\alpha_{12}}\beta_1 - \beta_2\right) \\ &\geq 0\end{aligned}$$

and the inequality is strict if and only if $\beta_1\alpha_{11} = \beta_1 + 2 > \beta_2\alpha_{12}$, i.e. l_3 does not pass through $l_1 \cap l_2$. The second inequality in (15) is proved in a similar way.

Suppose now we are in Case B, that is, we have picked the admissible point $l_3 \cap l_4$ to be our (β_1, β_2). This means

$$\beta_1 + 2 = \alpha_{12}\beta_2 \geq \alpha_{11}\beta_1 \quad \text{and} \quad \beta_2 + 2 = \alpha_{21}\beta_1 \geq \alpha_{22}\beta_2. \tag{17}$$

Then, as before,

$$\begin{aligned}\beta_1 + 2 - \beta_1\gamma_{11} - \beta_2\gamma_{12} &= -\beta_1\gamma_{11} + (\alpha_{12} - \gamma_{12})\beta_2 \\ &= \gamma_{11}\left(-\beta_1 + \frac{\alpha_{12}}{\alpha_{11}}\beta_2\right) \\ &\geq 0,\end{aligned}$$

and a similar computation proves the second inequality in (15).

Finally, let us consider Case C. Suppose we have chosen $(\beta_1, \beta_2) = l_1 \cap l_3$ (a similar argument can be done when $(\beta_1, \beta_2) = l_2 \cap l_4$). Then

$$\beta_1 + 2 = \alpha_{11}\beta_1 = \alpha_{12}\beta_2 \quad \text{and} \quad \beta_2 + 2 > \max\{\alpha_{21}\beta_1, \alpha_{22}\beta_2\}, \tag{18}$$

so $\beta_1 + 2 - \beta_1\gamma_{11} - \beta_2\gamma_{12} = 0$ as in Case A. To prove the second inequality in (15) we distinguish two cases. First, if $\alpha_{21}\beta_1 \geq \alpha_{22}\beta_2$ we get

$$\begin{aligned}\beta_2 + 2 - \beta_1\gamma_{21} - \beta_2\gamma_{22} &> (\alpha_{21} - \gamma_{21})\beta_1 - \gamma_{22}\beta_2 \\ &= \gamma_{22}\left(\frac{\alpha_{21}}{\alpha_{22}}\beta_1 - \beta_2\right) \\ &\geq 0,\end{aligned}$$

while if $\alpha_{21}\beta_1 \leq \alpha_{22}\beta_2$ we have

$$\begin{aligned}\beta_2 + 2 - \beta_1\gamma_{21} - \beta_2\gamma_{22} &> -\beta_1\gamma_{21} + (\alpha_{22} - \gamma_{22})\beta_2 \\ &= \gamma_{21}\left(-\beta_1 + \frac{\alpha_{22}}{\alpha_{21}}\beta_2\right) \\ &\geq 0.\end{aligned}$$

This finishes the proof of the Lemma. $\square$

We next list the known Liouville type theorems on the equation $-\Delta u = u^p$ and on the Lane-Emden system (7). We shall use some of these in the sequel.

Theorem 2.1 *The problem*

$$\left\{\begin{array}{rcl} -\Delta u & = & u^p \\ u & \geq & 0 \end{array}\right. \qquad \text{in} \quad \mathbb{R}^N \tag{19}$$

has no nontrivial classical solution provided $0 < p < \dfrac{N+2}{N-2}$ *(Gidas-Spruck [24], Chen-Li [15]). The same problem has no nontrivial classical supersolution provided* $1 < p \leq \dfrac{N}{N-2}$ *(Gidas [22], Souto [38], Mitidieri-Pohozaev [32]).*
The problem

$$\left\{\begin{array}{rclll} -\Delta u & = & u^p & \text{in} & \mathbb{R}^N_+ \\ u & \geq & 0 & \text{in} & \mathbb{R}^N_+ \\ u & = & 0 & \text{on} & \partial\mathbb{R}^N_+ \end{array}\right. \tag{20}$$

has no nontrivial classical solution provided $1 < p < \dfrac{N+1}{N-3}$ *(*$1 < p < \infty$ *if* $N = 3$*) - see Gidas-Spruck [24], Dancer [17]. The same problem has no classical supersolution provided* $1 < p \leq \frac{N+1}{N-1}$ *(Bandle-Essen [5], Laptev [29]).*

Theorem 2.2 *Consider the problem*

$$\left\{\begin{array}{rcl} -\Delta u & = & v^p \\ -\Delta v & = & u^q \\ u, v & \geq & 0 \end{array}\right. \qquad \text{in} \quad \mathbb{R}^N \tag{21}$$

If $pq \leq 1$ *(Serrin-Zou [36]) or* $0 < p, q < \dfrac{N+2}{N-2}$ *(de Figueiredo-Felmer [20]) this problem has no nontrivial classical solutions. If* $pq > 1$ *we set*

$$\alpha = \frac{2(p+1)}{pq-1}, \qquad \beta = \frac{2(q+1)}{pq-1}.$$

Then problem (21) has no nontrivial classical solutions provided either $\max\{\alpha, \beta\} \geq N-2$ *(Mitidieri [30], Serrin-Zou [36]), or* $p, q > 1$ *and* $\min\{\alpha, \beta\} > \dfrac{N-2}{2}$ *(Busca-Manasevich [10]). The same problem has no nontrivial supersolutions provided* $pq > 1$ *and* $\max\{\alpha, \beta\} \geq N-2$ *(Mitidieri [31], Laptev [29]).*

The problem

$$\begin{cases} -\Delta u &= v^p \quad \text{in} \quad \mathbb{R}^N_+ \\ -\Delta v &= u^q \quad \text{in} \quad \mathbb{R}^N_+ \\ u, v &\geq 0 \quad \text{in} \quad \mathbb{R}^N_+ \\ u, v &= 0 \quad \text{on} \quad \partial\mathbb{R}^N_+ \end{cases} \tag{22}$$

has no nontrivial solutions provided $p, q > 1$ *and* $\max\{\alpha, \beta\} \geq N - 3$ *(Birindelli-Mitidieri [8]). The same problem does not have nontrivial classical supersolutions provided* $p, q > 1$ *and* $\max\{\alpha, \beta\} \geq N - 1$ *(Laptev [29]).*

The problem of non-existence of solutions of Lane-Emden systems in a half-space (system (22)) deserves some discussion. There are two types of results on this problem. On one hand, it is known that a Lane-Emden system in a half-space does not possess positive supersolutions when $\max\{\alpha, \beta\} \geq N - 1$. This is an exact result and a particular case of a more general theorem about existence of supersolutions in cones. However, it does not permit to obtain more precise results for *solutions*. On the other hand, in the framework of a scalar equation Dancer developed a technique, based on the moving planes method, which gives a monotonicity result for *bounded* solutions of the equation in a half-space. Then one gets as a corollary that the existence of a nontrivial bounded solution in $\mathbb{R}^N_+$ implies the existence of a non-trivial solution in $\mathbb{R}^{N-1}$. However, the moving planes method requires Lipschitz nonlinearities – that is why a result, where this method is directly employed has to require that all powers involved be greater or equal to one.

The next section contains the proofs of Theorems 1.2 and 1.3, as well as another monotonicity lemma, which will play an important role in the proof of Theorem 1.1.

3 Monotonicity results for systems

3.1 A maximum principle in narrow domains

In this section we extend to cooperative systems a maximum principle in narrow domains, proved by Cabre [13] in the case of a scalar equation.

We recall the following definition of [13]. For a given domain $\Omega \subset \mathbb{R}^N$, the quantity $R(\Omega)$ is defined to be the smallest positive constant R such that

$$\text{meas}\,(B_R(x) \setminus \Omega) \geq \frac{1}{2}\,\text{meas}\,(B_R(x))\,, \qquad \text{for all } \; x \in \Omega.$$

If no such radius R exists, we define $R(\Omega) = +\infty$.

It is easy to see that whenever the domain Ω is contained between two parallel hyperplanes at a distance d, we have

$$R(\Omega) \leq \frac{2^N d}{\omega_N}, \tag{23}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$.

In the sequel we shall consider uniformly elliptic second-order operators in the form

$$L = \sum a_{ij}(x)\partial_{ij} + \sum b_i(x)\partial_i + c(x), \tag{24}$$

where $c_0 I \leq (a_{ij}) \leq C_0 I$, $\sup |b_i| \leq b$, for some positive constants c_0, C_0, and some $b \geq 0$.

In [13] (see also [14], Theorem 5.3) Cabre proved the following result.

Proposition 3.1 *Let Ω be a domain such that $R(\Omega) < \infty$ and let L be an operator in the form (24), such that $c \leq 0$ in Ω. Suppose $u \in W^{2,N}_{\mathrm{loc}}(\Omega)$ and $f \in L^\infty(\Omega)$ satisfy $Lu \geq f$ in Ω, $\limsup_{x\to\partial\Omega} u(x) \leq 0$, and $\sup_\Omega u < \infty$. Then*

$$\sup_\Omega u \leq C R(\Omega)^2 ||f||_{L^\infty(\Omega)},$$

where C is a constant depending only on $N, c_0, C_0, bR(\Omega)$.

It is not difficult to deduce from Proposition 3.1 a maximum principle for systems in domains with small $R(\Omega)$.

Theorem 3.1 *Let $L_k, k = 1, \ldots, n$, be uniformly elliptic second-order operators with bounded coefficients and without zero-order term, that is, $L_k = \sum a^{(k)}_{ij}(x)\partial_{ij} + \sum b^{(k)}_i(x)\partial_i$, $c_0 I \leq (a^{(k)}_{ij}) \leq C_0 I$, $\sup |b^{(k)}_i| \leq b$. Let the functions $c_{ij} \in L^\infty(\Omega)$, $|c_{ij}| \leq b$, be such that $c_{ij} \geq 0$ for $i \neq j$. Then there exists a number $\bar{R}$ depending only on N, c_0, C_0 and b, such that $R(\Omega) \leq \bar{R}$ implies that each solution $u_i \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ of*

$$\begin{cases} L_i u_i + \sum_{j=1}^n c_{ij} u_j \geq 0 & \text{in} \quad \Omega, \quad i = 1, \ldots, n \\ u_i(x) \leq 0 & \text{on} \quad \partial\Omega, \quad i = 1, \ldots, n, \end{cases} \tag{25}$$

satisfies $u_i \leq 0$ in Ω, $i = 1, \ldots, n$ (in this case we shall say, with obvious notation, that the matrix operator $\vec{L} + \mathcal{C}(x)$ satisfies the maximum principle in Ω).

Proof. For a function f we denote $f^+ = \max\{f, 0\}$, $f = f^+ - f^-$. Then (25) implies

$$L_i u_i - c_{ii}^- u_i \geq -c_{ii}^+ u_i^+ - \sum_{j \neq i} c_{ij} u_j^+, \quad i = 1 \ldots, n.$$

By applying Proposition 3.1 to these equations we obtain

$$\sup_\Omega u_i^+ \leq C R(\Omega)^2 \sum_{j=1}^n \sup_\Omega |c_{ij}| \sup_\Omega u_j^+.$$

Hence, denoting $\vec{v} = \left(\sup_\Omega u_1^+, \ldots, \sup_\Omega u_n^+\right)^T \geq 0$ we have

$$\vec{v} \leq \bar{R}^2 B \vec{v}$$

where B is a constant matrix whose entries depend only on n, N and on bounds for the coefficients of the elliptic operators. By choosing $\bar{R}$ such that the matrix $I - \bar{R}^2 B$ is positive definite and by multiplying scalarly the last inequality by the nonnegative vector $\vec{v}$, we obtain $\vec{v} \equiv 0$. □

3.2 Harnack type estimates for systems

The moving planes argument in Section 3.3 will require some Harnack estimates which we state in this section. Such results were recently obtained in [12]. The first two theorems below are particular cases of Theorem 3.2 and Propositions 3.1 and 8.1 in [12]. We include them here for the reader's convenience.

In this section G denotes an arbitrary domain in $\mathbb{R}^N$ and Q_l $(l = 1, 2)$ are concentric cubes with side l, properly included in G.

Theorem 3.2 ([12]) *Assume $f_1(u_1, u_2), f_2(u_1, u_2)$ are globally Lipschitz continuous functions, with Lipschitz constant A, which satisfy the cooperativeness assumption (10). Let (u_1, u_2) be a nonnegative solution of (9) in G. We suppose that the system is fully coupled, in the sense that $f_1(0, v) > 0$ for all $v > 0$, and $f_2(u, 0) > 0$ for $u > 0$. Then for any compact subset K of G there exists a function $\Phi(t)$ (depending on A, K and G), continuous on $[0, \infty)$, such that $\Phi(0) = 0$ and*

$$\sup_{x \in K} \max\{u_1, u_2\} \leq \Phi(\inf_{x \in K} \min\{u_1, u_2\}).$$

In particular, if any of u_1, u_2 vanishes at one point in G then both u_1 and u_2 vanish identically in G.

Theorem 3.3 ([12]) *Assume $f_1(u_1,u_2), f_2(u_1,u_2)$ are globally Lipschitz continuous functions, with Lipschitz constant A, which satisfy the cooperativeness assumption (10). Let (u_1,u_2) be a nonnegative* sub*solution of (9) in G. Then for each $p>0$ there exists a constant C depending only on p, N, and A such that*

$$\sup_{x\in Q_1} \max\{u_1,u_2\} \le C\|\max\{u_1,u_2\}\|_{L^p(Q_2)}$$

The same result holds if f_1, f_2 depend also on x and the constant A is uniform in x.

In the sequel we shall need the following two classical results on scalar equations by Krylov and Safonov.

Theorem 3.4 ([25], Theorem 9.22) *Let L be a linear uniformly elliptic operator with bounded coefficients in G, in the form (24). Suppose c_0 is an ellipticity constant for L, and b is an upper bound for the L^∞-norms of the coefficients of L. Let $u\in W^{2,N}_{\mathrm{loc}}(G)$ be a positive function satisfying $Lu\le f$ a.e. in G, for some $f\in L^N_{\mathrm{loc}}(G)$. Then there exist positive constants p_0 and C depending on c_0, b and N such that*

$$\|u\|_{L^{p_0}(Q_2)} \le C\left(\inf_{x\in Q_1} u + \|f\|_{L^N(Q_2)}\right).$$

Theorem 3.5 (Krylov) *Let L be a linear uniformly elliptic operator with bounded coefficients in G, in the form (24). Suppose c_0 is an ellipticity constant for L, and b is an upper bound for the L^∞-norms of the coefficients of L. Let $u\in W^{2,N}_{\mathrm{loc}}(G)$ be a positive function satisfying $Lu\le 0$ a.e. in G and $Lu\le -\rho$ a.e. in a closed subset $\omega\subset Q_2$, for some $\rho>0$. Then there exists a constant $m>0$, depending only on N, c_0, b, and on a positive lower bound on* meas $(\omega)>0$, *such that*

$$\inf_{Q_1} u \ge m\rho. \tag{26}$$

Theorem 3.5 is a consequence of Theorem 12 on p. 129 in [28] - we state it here in the form which was given in [7].

Next we state a partial Harnack inequality for a linear system, which will play a crucial role in the proof of the monotonicity result in Section 3.3.

Theorem 3.6 *Suppose $a,b,c,d\in L^\infty(Q_2)$ are such that $|a|,|d|\le A$, and $0\le b\le A, 0\le c\le A$ in Q_2. Suppose (u_1,u_2) is a positive solution of*

$$\left\{\begin{array}{rcl} \Delta u_1 + a(x)u_1 + b(x)u_2 &=& 0\\ \Delta u_2 + c(x)u_1 + d(x)u_2 &=& 0\end{array}\right.$$

in Q_2. Assume in addition that $b(x)$ is bounded below by a positive constant on Q_1. Then

$$\sup_{x\in Q_1} u_1 \le C \inf_{x\in Q_1} u_1. \tag{27}$$

where the constant C depends on N, A, and on upper bound for $\dfrac{\sup_{Q_2} b}{\inf_{Q_1} b}$.

Proof. Note that inequality (27) was proved in [12] (Theorem 8.2 in that paper) with the constant C depending on N, A, and on constants $\rho > 0, \delta > 0$ such that $b(x) \ge \rho$ on a set with measure δ. Theorem 3.6 follows from this result applied to the system

$$\left\{ \begin{array}{rcl} \Delta u_1 + a(x)u_1 + \widetilde{b}(x)\widetilde{u}_2 & = & 0 \\ \Delta \widetilde{u}_2 + \widetilde{c}(x)u_1 + d(x)\widetilde{u}_2 & = & 0, \end{array} \right. \tag{28}$$

where $\widetilde{u}_2 = (\inf_{Q_1} b)u_2$, $\widetilde{c}(x) = (\inf_{Q_1} b)c(x)$, $\widetilde{d}(x) = (\inf_{Q_1} b)d(x)$, and $\widetilde{b}(x) = b(x)/(\inf_{Q_1} b)$, so that $\widetilde{b} \ge 1$ in Q_1.

We are going to give the argument of the proof of inequality (27), since we shall need it in the sequel. By Theorem 3.3 applied to (28) we have for each $p > 0$

$$\sup_{x\in Q_1} \max\{u_1, \widetilde{u}_2\} \le C\|\max\{u_1, \widetilde{u}_2\}\|_{L^p(Q_2)} \le C\left(\|u_1\|_{L^p(Q_2)} + \|\widetilde{u}_2\|_{L^p(Q_2)}\right).$$

In order to estimate the right-hand side of this inequality we apply Theorem 3.4 to the following two scalar inequalities

$$\left\{ \begin{array}{rcl} \Delta u_1 + a(x)u_1 & \le & 0 \\ \Delta \widetilde{u}_2 + \widetilde{d}(x)\widetilde{u}_2 & \le & 0 \end{array} \right.$$

(these are a consequence of (28)) and obtain

$$\sup_{x\in Q_1} u_1 \le \sup_{x\in Q_1} \max\{u_1, \widetilde{u}_2\} \le C\left(\inf_{x\in Q_1} u_1 + \inf_{x\in Q_1} \widetilde{u}_2\right). \tag{29}$$

Finally, we note that the first equation in (28) implies

$$\Delta u_1 + a(x)u_1 \le 0 \quad \text{in } Q_2 \qquad \text{and} \qquad \Delta u_1 + a(x)u_1 \le -\inf_{x\in Q_1} \widetilde{u}_2 \quad \text{in } Q_1,$$

so Theorem 3.5 gives

$$\inf_{x\in Q_1} u_1 \ge m \inf_{x\in Q_1} \widetilde{u}_2.$$

We finish the proof by combining this inequality and (29). □

We finish this section by giving the proof of the strong Harnack inequality for fully coupled systems satisfying hypothesis (11) - Theorem 1.4.

Proof of Theorem 1.4. By (10) and (11) we have

$$\left\{ \begin{array}{rcl} \Delta u_1 + a(x)u_1 + b(x)u_2 & = & 0 \\ \Delta u_2 + c(x)u_1 + d(x)u_2 & = & 0 \end{array} \right.$$

in Q_2, where the coefficients $a(x) = g_1(u_1(x), u_2(x))$, $b(x) = u_2^{p-1}(x)$, $c(x) = u_1^{q-1}(x)$, $d(x) = g_2(u_1(x), u_2(x))$ are bounded continuous functions such that $b(x), c(x) > 0$ in Q_2.

As in the proof of the previous theorem we have

$$\sup_{x\in Q_1} \max\{u_1, u_2\} \le C \left(\inf_{x\in Q_1} u_1 + \inf_{x\in Q_1} u_2 \right). \tag{30}$$

By applying Theorem 3.5 to

$$\Delta u_1 + a(x)u_1 \le 0 \quad \text{in } Q_2 \quad \text{and} \quad \Delta u_1 + a(x)u_1 \le - \left(\inf_{x\in Q_1} u_2 \right)^p \quad \text{in } Q_1,$$

we get

$$\inf_{x\in Q_1} u_1 \ge m \left(\inf_{x\in Q_1} u_2 \right)^p.$$

In the same way we obtain

$$\inf_{x\in Q_1} u_2 \ge m \left(\inf_{x\in Q_1} u_1 \right)^q.$$

We obtain the statement of Theorem 1.4 by combining (30) with the last two inequalities. □

3.3 Proof of Theorem 1.2

We use the moving planes method of Alexandrov [1], which was subsequently developed in the framework of partial differential equations by Serrin [36], Gidas-Ni-Nirenberg [23], Berestycki-Nirenberg [9].

We shall denote

$$M = \max \left\{ \sup_{\mathbb{R}^N_+} u_1, \sup_{\mathbb{R}^N_+} u_2 \right\}.$$

We can suppose that the functions f_1 and f_2 are globally Lipschitz continuous. Indeed, if they are not, we can replace them by $f_1\varphi$ and $f_2\varphi$, where

φ is a cut-off function such that $\varphi = 1$ on the positive cube with side M, and $\varphi = 0$ outside a cube with side $M+1$, containing properly the previous one.

Hence system (9) satisfies the hypotheses of Theorem 3.2, from which we deduce that either both functions u_1 and u_2 vanish identically on $\mathbb{R}^N_+$ or both u_1 and u_2 are strictly positive on $\mathbb{R}^N_+$. The first case is excluded by hypothesis. So, from now on we shall assume that u_1, u_2 are strictly positive in $\mathbb{R}^N_+$.

For each $\lambda > 0$ we denote

$$T_\lambda = \{x \in \mathbb{R}^N \mid x_N = \lambda\}, \qquad \Sigma_\lambda = \{x \in \mathbb{R}^N \mid 0 < x_N < \lambda\},$$

and introduce the functions

$$v_i^{(\lambda)}(x) = u_i(x', 2\lambda - x_N), \qquad w_i^{(\lambda)}(x) = v_i^{(\lambda)}(x) - u_i(x), \qquad i = 1, 2,$$

defined in Σ_λ. Since both (u_1, u_2) and $(v_1^{(\lambda)}, v_2^{(\lambda)})$ satisfy system (9) we obtain by subtracting the corresponding equations and by Taylor's expansion

$$\left\{ \begin{array}{rcl} \Delta w_1^{(\lambda)} + c_{11}^{(\lambda)}(x) w_1^{(\lambda)} + c_{12}^{(\lambda)}(x) w_2^{(\lambda)} & = & 0 \\ \Delta w_1^{(\lambda)} + c_{21}^{(\lambda)}(x) w_1^{(\lambda)} + c_{22}^{(\lambda)}(x) w_2^{(\lambda)} & = & 0 \end{array} \right. \tag{31}$$

in Σ_λ, where $c_{ij}^\lambda(x)$ is the partial derivative of f_i with respect to u_j, evaluated at some point between $u_j(x)$ and $v_j^{(\lambda)}(x)$. Note that $c_{ij}^{(\lambda)}$ are bounded by a Lipschitz constant of $\vec{f} = (f_1, f_2)$ on $[0, M]^2$, and $c_{12}^{(\lambda)}, c_{21}^{(\lambda)} \geq 0$.

Obviously $\vec{w}^{(\lambda)} = (w_1^{(\lambda)}, w_2^{(\lambda)}) \equiv 0$ on T_λ and $\vec{w}^{(\lambda)} > 0$ on T_0 (recall that $u_i = 0$ on T_0 and $u_i > 0$ on T_λ, $\lambda > 0$). By Theorem 3.1, if λ is small enough, then $\vec{w}^{(\lambda)} \geq 0$ in Σ_λ. Hence

$$\lambda^* = \sup\{\lambda \mid \vec{w}^{(\mu)} \geq 0 \ \text{ in } \ \Sigma_\mu, \ \forall \, \mu < \lambda\} > 0.$$

We see that for each $0 < \lambda \leq \lambda^*$ the function $w_i^{(\lambda)} \geq 0$ satisfies the inequality $\Delta w_i^{(\lambda)} + c_{ii}^{(\lambda)} w_i^{(\lambda)} \leq 0$ in Σ_λ. Hence Hopf's lemma implies $w_i^{(\lambda)} > 0$ and

$$\frac{\partial u_i}{\partial x_N} = -\frac{1}{2} \frac{\partial w_i^{(\lambda)}}{\partial x_N} > 0 \qquad \text{on } \ T_\lambda.$$

Therefore, the theorem is proved if we show that $\lambda^* = +\infty$.

Suppose for contradiction that λ^* is finite.

By Theorem 3.1 we can fix $\varepsilon_0 > 0$ such that the matrix operator $\vec{\Delta} + C_\lambda(x)$ satisfies the maximum principle in the domain $\Sigma_{\lambda^* + \varepsilon_0} \setminus \Sigma_{\lambda^* - \varepsilon_0}$ (here $C_\lambda(x)$ denotes the matrix of the coefficients in (31)). For instance, we can take

$$\varepsilon_0 = \frac{\omega_N}{2^{N+1}} \overline{R}$$

where $\overline{R}$ is the number from Theorem 3.1 (see inequality (23)).

Lemma 3.1 *There exists $\delta_0 \in (0, \varepsilon_0]$, such that for each $\delta \in (0, \delta_0)$ we have*

$$w_i^{(\lambda^*+\delta)} \geq 0 \qquad in\ \ \Sigma_{\lambda^*-\varepsilon_0} \setminus \Sigma_{\varepsilon_0}, \qquad i = 1, 2.$$

Suppose this lemma is proved. Then we can apply Theorem 3.1 to (31) in $\Sigma_{\lambda^*+\delta} \setminus \Sigma_{\lambda^*-\varepsilon_0}$ and in Σ_{ε_0} (these domains are narrow enough) to conclude that $w_i^{(\lambda^*+\delta)} \geq 0$ in $\Sigma_{\lambda^*+\delta}$ for each $\delta \in (0, \delta_0)$. This contradicts the maximal choice of λ^* and proves Theorem 1.2. □

Proof of Lemma 3.1. We denote $y = (x_1, \ldots, x_{N-1})$. Suppose for contradiction that there exist sequences $\delta_m \to 0$ and $x^{(m)} = \left(y^{(m)}, x_N^{(m)}\right) \in \Sigma_{\lambda^*-\varepsilon_0} \setminus \Sigma_{\varepsilon_0}$ such that

$$w_1^{(\lambda^*+\delta_m)}\left(x^{(m)}\right) < 0. \tag{32}$$

We can suppose that $x_N^{(m)} \to x_N^0 \in [\varepsilon_0, \lambda^* - \varepsilon_0]$ as $m \to \infty$.

We define the functions

$$u_i^{(m)}(y, x_N) = u_i(y + y^{(m)}, x_N), \qquad i = 1, 2,$$

and, respectively,

$$w_{i,\lambda}^{(m)}(y, x_N) = u_i^{(m)}(y, 2\lambda - x_N) - u_i^{(m)}(y, x_N), \qquad i = 1, 2.$$

Note that system (9) is autonomous, so $\vec{u}^{(m)}$ satisfies the same system. Since $f_1(u_1^{(m)}, u_2^{(m)})$ and $f_2(u_1^{(m)}, u_2^{(m)})$ are uniformly bounded in m (by a Lipschitz constant of $\vec{f}$ on $[0, M]^2$), standard elliptic theory, applied to (9), then implies

$$||\vec{u}^{(m)}||_{W^{2,p}(K)} \leq CM$$

for each compact set K in the closure of $\mathbb{R}^N_+$, where C depends on a Lipschitz constant of $\vec{f}$ on $[0, M]^2$. It follows, by imbedding theorems and elliptic theory, that $\vec{u}^{(m)}$ converges in $C^{1,\alpha}$ on compact sets to a classical solution $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2)$ of system (9), satisfying the Dirichlet boundary condition on $\{x_N = 0\}$.

It follows from Theorem 3.2 that either both $\widetilde{u}_1$ and $\widetilde{u}_2$ are strictly positive on $\mathbb{R}^N_+$ or both these two functions vanish identically on $\mathbb{R}^N_+$. Suppose that at least one of the functions $\widetilde{u}_1, \widetilde{u}_2$ does not vanish identically in $\mathbb{R}^N_+$ so we are in the first of these two situations. By what we have already shown we know that $w_{i,\lambda}^{(m)}(y, x_N) = w_i^\lambda(y + y^{(m)}, x_N) > 0$ in Σ_λ for all $\lambda \leq \lambda^*$. Hence the limit functions $\widetilde{w}_i^\lambda = \lim\limits_{m\to\infty} w_{i,\lambda}^{(m)}$ are nonnegative in Σ_λ for all $\lambda \leq \lambda^*$. However $\widetilde{w}_i^\lambda$ is to $\widetilde{u}$ what w_i^λ is to u, so by repeating the moving planes argument for $\widetilde{u}$ we see that $\widetilde{\lambda}^* \geq \lambda^*$ ($\widetilde{\lambda}^*$ is the critical value for $\widetilde{u}$), since we can write a system like (31) for $\widetilde{w}_i^\lambda$. Applying the strong maximum principle to this system we get, as before, that $\widetilde{w}_i^\lambda > 0$ in Σ_λ, for $\lambda \le \lambda^*$. On the other hand $\widetilde{w}_1^{\lambda^*}(0, x_N^0) = 0$ and $x_N^0 \in (0, \lambda^* - \varepsilon_0]$, a contradiction.

The argument is considerably more involved in case $\widetilde{u} \equiv 0$ in $\mathbb{R}_+^N$. We fix the rectangular domains

$$Q_0 = \{x \in \mathbb{R}_+^N \mid -1 < x_1 < 1, \ldots, -1 < x_{N-1} < 1,\ \varepsilon_0 < x_N < 2\lambda^* + 1\},$$

$$Q_1 = \{x \in \mathbb{R}_+^N \mid -2 < x_1 < 2, \ldots, -2 < x_{N-1} < 2,\ \frac{\varepsilon_0}{2} < x_N < 2\lambda^* + 2\},$$

$$Q_2 = \{x \in \mathbb{R}_+^N \mid -3 < x_1 < 3, \ldots, -3 < x_{N-1} < 3,\ \frac{\varepsilon_0}{4} < x_N < 2\lambda^* + 3\},$$

and note that all results from the previous section can trivially be applied in Q_1 and Q_2 (one simply has to cover Q_1 and Q_2 with a finite number of cubes and apply Harnack inequalities in these cubes).

Since $\vec{u}^{(m)}$ converges uniformly to zero in Q_2 we can suppose that $u_1^{(m)} \le 1$, $u_2^{(m)} \le 1$ in Q_2. We set

$$\alpha_m = u_1^{(m)}\left(0, x_N^{(m)}\right), \qquad \beta_m = u_2^{(m)}\left(0, x_N^{(m)}\right).$$

By Theorem 1.4 we have

$$\alpha_m \le C\beta_m^{\frac{1}{q}} \qquad \text{and} \qquad \beta_m \le C\alpha_m^{\frac{1}{p}}, \tag{33}$$

where C is independent of m. We also recall inequality (30)

$$\sup_{x\in Q_1} \max\{u_1^{(m)}, u_2^{(m)}\} \le C\left(\inf_{x\in Q_1} u_1^{(m)} + \inf_{x\in Q_1} u_2^{(m)}\right). \tag{34}$$

Next we introduce the functions

$$z_1^{(m)} = \frac{1}{\alpha_m}\, u_1^{(m)}, \quad z_2^{(m)} = \frac{1}{\beta_m}\, u_2^{(m)}, \quad \zeta_1^{(m)} = \frac{1}{\beta_m}\, u_1^{(m)}, \quad \zeta_2^{(m)} = \frac{1}{\alpha_m}\, u_2^{(m)}.$$

Note that $z_1^{(m)}(0, x_N^{(m)}) = z_2^{(m)}(0, x_N^{(m)}) = 1$.

We distinguish two cases : up to a subsequence

$$\inf_{Q_1} u_1^{(m)} \le \inf_{Q_1} u_2^{(m)} \quad \text{(Case 1)} \qquad \text{and} \qquad \inf_{Q_1} u_2^{(m)} \le \inf_{Q_1} u_1^{(m)} \quad \text{(Case 2)}.$$

Suppose we are in Case 1. Then, as in the previous section, $\left(z_1^{(m)}, \zeta_2^{(m)}\right)$ satisfies the linear system

$$\left\{ \begin{array}{rcl} \Delta z_1^{(m)} + a_m(x) z_1^{(m)} + b_m(x)\zeta_2^{(m)} & = & 0 \\ \Delta \zeta_2^{(m)} + c_m(x) z_1^{(m)} + d_m(x)\zeta_2^{(m)} & = & 0 \end{array} \right. \tag{35}$$

in Q_2, where $a_m(x) = g_1(u_1^{(m)}(x), u_2^{(m)}(x))$, $b_m(x) = \left(u_2^{(m)}(x)\right)^{p-1} > 0$, $c_m(x) = \left(u_1^{(m)}(x)\right)^{q-1} > 0$, $d_m(x) = g_2(u_1^{(m)}(x), u_2^{(m)}(x))$ are uniformly bounded in m.

By combining inequality (34) with the hypothesis of Case 1 we obtain

$$\sup_{Q_1} u_2^{(m)} \le C \inf_{Q_1} u_2^{(m)},$$

which implies that the quantity

$$\frac{\sup_{x\in Q_1} b_m(x)}{\inf_{x\in Q_1} b_m(x)}$$

is bounded by a constant independent of m. Hence we can apply Theorem 3.6 to (35), and infer that

$$\sup_{Q_0} z_1^{(m)} \le C_1 \inf_{Q_0} z_1^{(m)} \le C_1,$$

where C_1 does not depend on m.

Next, recall that $w_1^{(\lambda^*)} \ge 0$ in Σ_{λ^*}, which implies

$$z_1^{(m)}(y, x_N) \le z_1^{(m)}(y, 2\lambda^* - x_N) \le C_1$$

for every (y, x_N) in the closure of Σ_{λ^*}. Hence

$$\|z_1^{(m)}\|_{L^\infty(Q)} \le C_1,$$

where

$$Q = \{x \in \mathbb{R}^N_+ \mid -1 < x_1 < 1, \ldots, -1 < x_{N-1} < 1,\ 0 < x_N < 2\lambda^* + 1\}.$$

In Case 2 we write a linear system for $\left(\zeta_1^{(m)}, z_2^{(m)}\right)$ and use an analogous reasoning to infer that

$$\|z_2^{(m)}\|_{L^\infty(Q)} \le C_2,$$

where C_2 does not depend on m.

Next we prove that actually both $z_1^{(m)}$ and $z_2^{(m)}$ are bounded in $L^\infty(Q)$. To this end we remark that $\left(z_1^{(m)}, z_2^{(m)}\right)$ satisfies the system

$$\begin{cases} \Delta z_1^{(m)} + g_1(u_1^{(m)}, u_2^{(m)}) z_1^{(m)} + \dfrac{\beta_m^p}{\alpha_m}\left(z_2^{(m)}\right)^p = 0 \quad \text{in} \quad \mathbb{R}^N_+ \\[2ex] \Delta z_2^{(m)} + \dfrac{\alpha_m^q}{\beta_m}\left(z_1^{(m)}\right)^q + g_1(u_1^{(m)}, u_2^{(m)}) z_2^{(m)} = 0 \quad \text{in} \quad \mathbb{R}^N_+ \\[2ex] z_1^{(m)} = z_2^{(m)} = 0 \quad \text{on} \quad \partial\mathbb{R}^N_+ \end{cases} \tag{36}$$

by (11). We recall that we have already proved that the sequences $\{\beta_m^p \alpha_m^{-1}\}$ and $\{\alpha_m^q \beta_m^{-1}\}$ are bounded (see (33)). It follows from the above argument that one of the sequences $\{z_1^{(m)}\}$, $\{z_2^{(m)}\}$ (say the first) is bounded in $L^\infty(Q)$. Then standard elliptic estimates applied to the second equation in (36), regarded as a scalar equation in $z_2^{(m)}$ with bounded coefficients and bounded right-hand side, imply that $\{z_2^{(m)}\}$ is also bounded in $L^\infty(Q)$ (if necessary, we restrict a little the upper and the lateral boundaries of Q).

Now, since both $\{z_1^{(m)}\}$, $\{z_2^{(m)}\}$ are bounded on Q, elliptic theory and (36) imply that (up to a subsequence) these two sequences converge uniformly to nonnegative functions $z_1, z_2 \in W^{2,p}_{\text{loc}}(Q) \cap C(\overline{Q})$, which satisfy the system

$$\begin{cases} \Delta z_1 + g_1(0,0) z_1 + \beta_0 \, (z_2)^p &= 0 \quad \text{in} \quad Q \\ \\ \Delta z_2 + \alpha_0 \, (z_1)^q + g_1(0,0) z_2 &= 0 \quad \text{in} \quad Q \\ \\ z_1 = z_2 &= 0 \quad \text{on} \quad \{x_N = 0\} \cap \partial Q, \end{cases} \tag{37}$$

for some constants $\alpha_0 \geq 0$, $\beta_0 \geq 0$. Since $z_1 \geq 0$ and $\Delta z_1 + g_1(0,0) z_1 \leq 0$ in Q, the strong maximum principle implies that either z_1 vanishes identically in Q or $z_1 > 0$ in Q. The first possibility is excluded by $z_1(0, x_N^0) = 1$.

Introduce the comparison functions

$$\omega_i^{(\lambda)}(y, x_N) = z_i(y, 2\lambda - x_N) - z_i(y, x_N), \quad , i = 1, 2,$$

defined in $\Sigma_\lambda \cap \overline{Q}$, for all $\lambda \leq \lambda^* + 1/2$. We have, by continuity,

$$\omega_i^{(\lambda^\star)} \geq 0, \; i = 1, 2, \qquad \text{and} \qquad \omega_1^{(\lambda^\star)}(0, x_N^0) = 0$$

(recall (32)). Since $\Delta \omega_1^{(\lambda^\star)} + g_1(0,0) \omega_1^{(\lambda^\star)} \leq 0$ the strong maximum principle implies $\omega_1^{(\lambda^\star)} \equiv 0$ in $\Sigma_{\lambda^*} \cap \overline{Q}$. This contradicts the fact that $z_1 = 0$ on $\{x_N = 0\}$ and $z_1 > 0$ on $\{x_N = 2\lambda^*\}$.

The proof of Theorem 1.2 is finished. □

3.4 Proof of Theorem 1.3

Set

$$x' \in \mathbb{R}^{N-1}, \qquad u_i^{(t)}(x') = u_i(x', t), \quad i = 1, 2.$$

Since the sequence $\{u_i^{(t)}\}_t$ is uniformly bounded and pointwise increasing in t (by Theorem 1.2) the Lebesgue monotone convergence theorem implies that $u_i^{(t)}$ converges as $t \to \infty$ in $L^p_{\text{loc}}\left(\mathbb{R}^{N-1}\right)$, $p < \infty$, to a bounded function $\widetilde{u}_i$. Then the hypotheses we made on f_i imply

$$f_i(u_1^{(t)}, u_2^{(t)}) \longrightarrow f_i(\widetilde{u}_1, \widetilde{u}_2) \qquad \text{as} \;\; t \to \infty,$$

in $L^p_{\mathrm{loc}}\left(\mathbb{R}^{N-1}\right)$, $p < \infty$, $i = 1, 2$. We shall prove that $(\widetilde{u_1}, \widetilde{u_2})$ is a weak solution to the problem in $\mathbb{R}^{N-1}$ (then standard elliptic regularity theory implies that it is a classical solution). Let $\phi(x') \in C_c^\infty(\mathbb{R}^{N-1})$ be an arbitrary function and let

$$\psi(t) = \begin{cases} 0 & \text{if} \quad t \le 0, t \ge 2 \\ t & \text{if} \quad 0 \le t \le 1 \\ 2 - t & \text{if} \quad 1 \le t \le 2. \end{cases}$$

The function $\psi(t)$ is chosen so that $\psi \in C_c(\mathbb{R})$, $\operatorname{supp} \psi = [0, 2]$,

$$\int_0^1 \psi(t)\, dt = \int_1^2 \psi(t)\, dt = \frac{1}{2} \qquad \text{and} \qquad \psi''(t) = \delta(0) - 2\delta(1) + \delta(2),$$

where $\delta(t)$ is the Dirac mass at t. We set $\psi_m(t) = \psi(t - m)$.

We multiply each equation in (9) by $\phi(x')\psi_m(x_N)$ and integrate over $\mathbb{R}^N_+$. Integration by parts and Fubini's theorem then yield

$$\begin{aligned} \int_{\mathbb{R}^N_+} u_i \left(\Delta_{x'} \phi(x') \psi_m(x_N)\right. &+ \left.\phi(x') \psi''_m(x_N)\right)\, dx_N\, dx' \\ &= \int_{\mathbb{R}^N_+} f_i(u_1, u_2) \phi(x') \psi_m(x_N)\, dx_N\, dx' \end{aligned}$$

or

$$\begin{aligned} &\int_{\mathbb{R}^{N-1}} \left(\int_0^2 u_i(x', x_N + m) \psi(x_N)\, dx_N \right) \Delta_{x'} \phi(x')\, dx' \\ &+ \int_{\mathbb{R}^{N-1}} \left(u_i^{(m)}(x') - 2u_i^{(m+1)}(x') + u_i^{(m+2)}(x') \right) \phi(x')\, dx' \\ &= \int_{\mathbb{R}^{N-1}} \left(\int_0^2 f_i\left(u_1(x', x_N + m), u_2(x', x_N + m)\right) \psi(x_N)\, dx_N \right) \phi(x')\, dx'. \end{aligned}$$

Since $u_i^{(m)}$ tends to $\widetilde{u_i}$ in any Lebesgue space, the following lemma finishes the proof of Theorem 1.3. Note that the second integral in the left-hand side of the last equality vanishes at the limit.

Lemma 3.2 *Under the hypotheses we made on f_1, f_2*

$$\int_0^2 f_i\left(u_1(x', x_N + m), u_2(x', x_N + m)\right) \psi(x_N)\, dx_N \tag{38}$$

tends to $f_i\left(\widetilde{u_1}(x'), \widetilde{u_2}(x')\right)$ in $L^1_{\mathrm{loc}}\left(\mathbb{R}^{N-1}\right)$ as $m \to \infty$.

Proof. We split the integral (38) in two and use the fact that ψ is continuous and monotonous in $[0,1]$ and in $[1,2]$. By standard properties of the Riemann integral (38) is equal to

$$\frac{f\left(u_1(x',\xi_1+m),u_2(x',\xi_1+m)\right)+f\left(u_1(x',\xi_2+m),u_2(x',\xi_2+m)\right)}{2}$$

where $\xi_1 \in (0,1)$, $\xi_2 \in (1,2)$. Obviously $u_i(x',\xi_j+m)$ tends to $\widetilde{u_i}(x')$ in $L^p_{\mathrm{loc}}\left(\mathbb{R}^{N-1}\right)$, $p<\infty$, so the lemma follows. □

3.5 A monotonicity lemma

In the proof of Theorem 1.1 we shall need the following monotonicity result, which is an extension of a recent result of Busca and Manasevich [10].

Lemma 3.3 *Let $f_i \in C^1((\mathbb{R}_+)^2)$, $f_i(0,0)=0$, $\nabla f_i(0,0)=0$, $i=1,2$ satisfy (10). Suppose that $f_1(0,v)>0$ for all $v>0$, and $f_2(u,0)>0$ for $u>0$. Let $u_i(t,\theta)\geq 0, i=1,2$ be C^2-functions defined on $\mathbb{R}\times S^{N-1}$ and satisfying*

$$\left\{\begin{array}{rcll} \dfrac{\partial^2 u_i}{\partial t^2}+\Delta_\theta u_i-\delta_i\dfrac{\partial u_i}{\partial t}-\nu_i u_i+f_i(u_1,u_2) & = & 0 & \text{in} \quad \mathbb{R}\times S^{N-1} \\ u_i & \to & 0 & \text{as} \quad t\to-\infty, \end{array}\right. \tag{39}$$

where $\delta_i\geq 0$, $\max\{\delta_1,\delta_2\}>0$, $\nu_i>0$, $i=1,2$ are constants. Suppose also that there exists $t_0\in\mathbb{R}$ such that $\dfrac{\partial u_i}{\partial t}>0$ in $(-\infty,t_0)\times S^{N-1}$, for $i=1,2$. Then $\dfrac{\partial u_i}{\partial t}>0$ in $\mathbb{R}\times S^{N-1}$, for $i=1,2$.

In [10] the authors studied the case $n=2$, $f_1(u_1,u_2)=u_2^p$, $f_2(u_1,u_2)=u_1^q$. Note that the result in [10] was stated without the hypothesis $p,q>1$, but this hypothesis is actually used in their proof.

The proof of Lemma 3.3 is based on the moving planes method and uses both ideas from the proof in [10] and from the proof of the symmetry result for systems in $\mathbb{R}^N$ obtained in [11].

We set

$$h_i(u_1,u_2)=-\nu_i u_i+f_i(u_1,u_2)$$

and remark that all we shall need is the fact that the matrix of the partial derivatives of h_1 and h_2 is negative definite at $(0,0)$. We use this fact as in the proof of Theorem 2 from [11] (see hypotheses (i)–(iii) in Section 2.1 of [11]).

For each $\lambda>0$ we denote

$$T_\lambda=\{(t,\theta)\in\mathbb{R}\times S^{N-1}\mid t=\lambda\}, \qquad \Sigma_\lambda=\{(t,\theta)\in\mathbb{R}\times S^{N-1}\mid t<\lambda\},$$

and introduce the functions

$$w_i^{(\lambda)}(t,\theta) = u_i(2\lambda - t,\theta) - u_i(t,\theta), \qquad i = 1,\dots,n,$$

defined in Σ_λ. By Taylor's expansion $\vec{w}_\lambda$ satisfies the system

$$\frac{\partial^2 \vec{w}^{(\lambda)}}{\partial t^2} + \Delta_\theta \vec{w}^{(\lambda)} - D\frac{\partial \vec{w}^{(\lambda)}}{\partial t} + C_\lambda(x)\vec{w}^{(\lambda)} = -2D\frac{\partial \vec{u}}{\partial t} \qquad \text{in } \Sigma_\lambda, \tag{40}$$

where

$$D = \mathrm{diag}(\delta_1,\delta_2), \quad C_\lambda(x) = (c_{ij}(x))_{i,j=1}^2, \quad c_{ij}(x) = \frac{\partial h_i}{\partial u_j}(\xi_1,\xi_2),$$

and $\xi_{ij} = \xi_{ij}(t,\theta,\lambda)$,

$$\xi_{ij} \in [\min(u_j(t,\theta),u_j(2\lambda-t,\theta)), \max(u_j(t,\theta),u_j(2\lambda-t,\theta))].$$

System (40) is treated in basically the same way as inequality (3)–(4) from [11] - note that the right hand side of (40) is nonpositive in Σ_λ for any $\lambda \le t_0$, by hypothesis. We shall only sketch the argument, since most of the details can be seen in [11] (note only that here the moving planes go "to the right", contrary to the choice made in [11]). First, there exists $\lambda^* < t_0$ such that $\vec{w}^{(\lambda)} \ge 0$ in Σ_λ, for any $\lambda \le \lambda^*$. The proof of this fact goes like the proof of Step 1 in the proof of Theorem 2 in [11]. Second, we define

$$\lambda_0 = \sup\left\{\lambda \in \mathbb{R} \;:\; \vec{w}^{(\mu)} \ge 0 \quad \text{and} \quad \frac{\partial \vec{w}^{(\mu)}}{\partial t} > 0 \;\text{ in } \Sigma_\mu, \quad \text{for all } \mu < \lambda\right\}.$$

If $\lambda_0 = +\infty$ we are done (note that $\dfrac{\partial \vec{w}^{(\lambda)}}{\partial t} = 2\dfrac{\partial \vec{u}}{\partial t}$ on T_λ). If λ_0 is finite we reason like in Step 2 in the proof of Theorem 2 in [11], to infer that either $w_1^{(\lambda_0)}$ or $w_2^{(\lambda_0)}$ vanishes in Σ_{λ_0}. Since the system is fully coupled this implies that $\vec{w}^{(\lambda_0)} \equiv 0$ in Σ_{λ_0}, or, equivalently, $\dfrac{\partial \vec{u}}{\partial t}$ is odd with respect to T_{λ_0}. On the other hand, by (40) at least one of the derivatives with respect to t of u_1 and u_2 (the one which corresponds to a strictly positive δ_i) is even with respect to T_{λ_0}. Hence at least one of the functions u_i is equal to a constant, which contradicts the hypothesis that u_i is strictly increasing in t for $t \in (-\infty, t_0)$. $\Box$

In the next section we prove our main result.

4 Proof of Theorem 1.1

We divide the proof in three parts, according to the case we consider (A, B or C). We recall that the values of the parameters α_{ij} determine which case we are in.

In the sequel G will denote either $\mathbb{R}^N$ or $\mathbb{R}^N_+$.

Proof in Case A. In this case we choose $(\beta_1, \beta_2) = l_1 \cap l_2$, that is

$$\beta_1 = \frac{2}{\alpha_{11} - 1}, \qquad \beta_2 = \frac{2}{\alpha_{22} - 1}. \tag{41}$$

1. First, if none of the lines l_3 and l_4 passes through $l_1 \cap l_2$ we are precisely in the weakly coupled case considered in [19]. By using Lemmas 2.1 and 2.2, and by letting $n \to \infty$ in (14) we obtain (after scaling) the uncoupled system

$$\left\{ \begin{array}{rcl} -\Delta v_1 & = & v_1^{\alpha_{11}} \\ -\Delta v_2 & = & v_2^{\alpha_{22}}, \end{array} \right. \tag{42}$$

in G, which has only the trivial solution because

$$\max\{\alpha_{11}, \alpha_{22}\} < \frac{N+2}{N-2}, \tag{43}$$

which is a consequence of (8) and (41).

2. Next, suppose exactly one of l_3 and l_4 (say l_3) passes through $l_1 \cap l_2$. Then again by Lemmas 2.1 and 2.2, after letting $n \to \infty$ in (14) we obtain the system

$$\left\{ \begin{array}{rcl} -\Delta v_1 & = & a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}} + c_1 v_1^{\gamma_{11}} v_2^{\gamma_{12}} \\ -\Delta v_2 & = & d_0 v_2^{\alpha_{22}} \\ u, v & \geq & 0 \end{array} \right. \quad \text{in} \quad G, \tag{44}$$

where a_0, d_0, b_0, c_1 are constants such that $a_0, d_0 > 0$, $b_0, c_1 \geq 0$. Since (43) holds this system has no nontrivial solution. Indeed, the second equation in (44) implies $v_2 \equiv 0$. Then the first equation becomes a scalar equation, which again has no nontrivial solutions under (43).

3. Finally, suppose all four lines l_1, l_2, l_3, l_4 meet at one point (so we are simultaneously in Cases A and B). Letting $n \to \infty$ in (14), we come to the system

$$\left\{ \begin{array}{rcl} -\Delta v_1 & = & a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}} + c_1 v_1^{\gamma_{11}} v_2^{\gamma_{12}} \\ -\Delta v_2 & = & c_0 v_1^{\alpha_{21}} + d_0 v_2^{\alpha_{22}} + c_2 v_1^{\gamma_{21}} v_2^{\gamma_{22}} \\ u, v & \geq & 0 \end{array} \right. \quad \text{in} \quad G, \tag{45}$$

where $a_0, b_0, c_0, d_0 > 0$, $c_1, c_2 \geq 0$. Note that now

$$\alpha_{12} = \frac{\alpha_{11}(\alpha_{22} - 1)}{\alpha_{11} - 1}, \qquad \alpha_{21} = \frac{\alpha_{22}(\alpha_{11} - 1)}{\alpha_{22} - 1}.$$

Assume first $G = \mathbb{R}^N_+$. Then (45) satisfies the hypotheses of Theorem 1.2 and Theorem 1.3. Hence, by Theorem 1.3, if $(v_1, v_2) \not\equiv (0, 0)$ then there exists a nontrivial solution of (45) in $\mathbb{R}^{N-1}$. So, if we manage to prove that (45) has only the trivial solution in $\mathbb{R}^N$ under (8) (note that a particular case of this result would be Theorem 1.5), then it has no nontrivial solutions in $\mathbb{R}^N_+$ under $\min\{\beta_1, \beta_2\} > \dfrac{N-3}{2}$ which is consequence of (8).

From now on we suppose $G = \mathbb{R}^N$ and distinguish two cases,

$$\max\{\beta_1, \beta_2\} \geq N - 2 \ \ \text{(Case 1)} \qquad \text{and} \qquad \max\{\beta_1, \beta_2\} < N - 2 \ \ \text{(Case 2)}.$$

In Case 1 (say $\beta_1 \geq N - 2$) we have $\alpha_{11} \leq \frac{N}{N-2}$. But the first equality in (45) implies $-\Delta v_1 \geq a_0 v_1^{\alpha_{11}}$ in $\mathbb{R}^N$, so $v_1 \equiv 0$ in $\mathbb{R}^N$, by the results about non-existence of supersolutions (Theorem 2.1). Then the second equation in (45) becomes $-\Delta v_2 = d_0 v_2^{\alpha_{22}}$ in $\mathbb{R}^N$. So $v_2 \equiv 0$ in $\mathbb{R}^N$, because $\alpha_{22} < \frac{N+2}{N-2}$, which is a consequence of (8).

In Case 2 we write system (45) in polar coordinates $(r, \theta) \in \mathbb{R} \times S^{N-1}$ and make the change of variables, as in [10],

$$r = |x|, \qquad t = \ln|x| \in \mathbb{R}, \qquad \theta = \frac{x}{|x|} \in S^{N-1},$$

and set

$$w_i(t, \theta) = e^{\beta_i t} v_i(e^t, \theta).$$

Then system (45) transforms into

$$\left\{ \begin{array}{rcl} -L_1 w_1 & = & a_0 e^{(\beta_1 + 2 - \alpha_{11}\beta_1)t} w_1^{\alpha_{11}} + b_0 e^{(\beta_1 + 2 - \alpha_{12}\beta_2)t} w_2^{\alpha_{12}} \\ & & + c_1 e^{(\beta_1 + 2 - \gamma_{11}\beta_1 - \gamma_{12}\beta_2)t} w_1^{\gamma_{11}} w_2^{\gamma_{12}} \\ & & \\ -L_2 w_2 & = & c_0 e^{(\beta_2 + 2 - \alpha_{21}\beta_1)t} w_1^{\alpha_{21}} + d_0 e^{(\beta_2 + 2 - \alpha_{22}\beta_2)t} w_2^{\alpha_{22}} \\ & & + c_2 e^{(\beta_2 + 2 - \gamma_{21}\beta_1 - \gamma_{22}\beta_2)t} w_1^{\gamma_{21}} w_2^{\gamma_{22}} \end{array} \right. \tag{46}$$

in $\mathbb{R} \times S^{N-1}$, where

$$L_i = \frac{\partial^2}{\partial t^2} + \Delta_\theta - \delta_i \frac{\partial}{\partial t} - \nu_i, \quad i = 1, 2,$$

and

$$\delta_i = 2\beta_i - (N - 2), \qquad \nu_i = \beta_i(N - 2 - \beta_i), \quad i = 1, 2.$$

It is remarkable that after this change of variables one obtains the same powers in the exponential functions in (46) as the powers of the parameter λ_n after the "blow-up" change of variables. So, in the case we consider all these powers are zero, and system (46) is *autonomous*.

Observe that condition (8) implies that $\delta_i > 0$, and that the condition of Case 2 gives $\nu_i > 0$. Further, by by applying Theorem 3.2 to system (45) we see that we can suppose that v_1 and v_2 are strictly positive in $\mathbb{R}^N$. This easily implies that the derivatives of w_1 and w_2 with respect to t are positive for large negative t. So we can now apply Lemma 3.3 to (46), and infer $\dfrac{\partial w_i}{\partial t} > 0$ in $\mathbb{R} \times S^{N-1}$, or

$$\beta_i v_i + r\frac{\partial v_i}{\partial r} > 0\,, \qquad i = 1, 2. \tag{47}$$

Then, using a reasoning from [10], we remark that all the above argument can be carried out for any translation of (v_1, v_2) (since (45) is autonomous), so (47) implies

$$\beta_i v_i(x) + \nabla v_i(x)(x - x_0) \geq 0,$$

for all $x, x_0 \in \mathbb{R}^N$. This easily implies $\nabla v_i(x) \equiv 0$ (write $x_0 = x - \tau e$, $\tau > 0$, $e \in S^{N-1}$, divide by τ, let $\tau \to \infty$ and observe that the resulting inequality holds for any $e \in S^{N-1}$), which contradicts (45).

Proof in Case B. We recall that in this case we choose $(\beta_1, \beta_2) = l_3 \cap l_4$, that is,

$$\beta_1 = \frac{2(1+\alpha_{12})}{\alpha_{12}\alpha_{21} - 1}, \qquad \beta_2 = \frac{2(1+\alpha_{21})}{\alpha_{12}\alpha_{21} - 1}.$$

1. First, suppose none of l_1 and l_2 passes through $l_3 \cap l_4$; this is precisely the strongly coupled case from [19]. Then by using Lemmas 2.1 and 2.2, after the passage to the limit in (14) we obtain (after scaling) the system

$$\left\{\begin{array}{rcl} -\Delta v_1 & = & v_2^{\alpha_{12}} \\ -\Delta v_2 & = & v_1^{\alpha_{21}} \\ u, v & \geq & 0 \end{array}\right. \qquad \text{in} \quad G, \tag{48}$$

Observe that it is known that under (8) system (48) in $\mathbb{R}^N$ has only the trivial solution, see Theorem 2.2. However, as in the beginning of point 3 of the proof of Case A above we see that we can restrict the analysis to $G = \mathbb{R}^N$. Indeed, system (48) satisfies the hypotheses of Theorem 1.2, since by assumption $b_0, c_0 > 0$, $\alpha_{12}, \alpha_{21} > 1$. Let us stress again that this argument requires Lipschitz nonlinearities.

2. Second, if one of l_1 or l_2 (say l_1) passes through $l_3 \cap l_4$, by letting $n \to \infty$ in (14) we obtain, again with the help of Lemmas 2.1 and 2.2, the following system

$$\begin{cases} -\Delta v_1 &= a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}} + c_1 v_1^{\gamma_{11}} v_2^{\gamma_{12}} \\ -\Delta v_2 &= c_0 v_1^{\alpha_{21}} \\ u, v &\geq 0 \end{cases} \quad \text{in} \quad \mathbb{R}^N, \tag{49}$$

where $b_0, c_0 > 0$, $a_0, c_1 \geq 0$. So (49) implies

$$\begin{cases} -\Delta v_1 &\geq b_0 v_2^{\alpha_{12}} \\ -\Delta v_2 &\geq c_0 v_1^{\alpha_{21}} \\ u, v &\geq 0 \end{cases} \quad \text{in} \quad \mathbb{R}^N. \tag{50}$$

We split the argument in two cases once again.

- If $\max\{\beta_1, \beta_2\} \geq N-2$, Theorem 2.2 applied to (50) gives $v_1 = v_2 \equiv 0$ in $\mathbb{R}^N$.
- Finally, if we have $\dfrac{N-2}{2} < \beta_1, \beta_2 < N-2$, we apply Lemma 3.3 to (49), and conclude $v_1 = v_2 \equiv 0$ in $\mathbb{R}^N$, through the same argument as in the last part of the proof of Case A.

Proof in Case C. All that remains to consider are the cases when either l_3 meets l_1 at a point above l_4 or l_4 meets l_2 at a point below l_3. In both cases we take the couple (β_1, β_2) to be this intersection point, that is, for instance in the first of these two situations

$$\beta_1 = \frac{2}{\alpha_{11} - 1}, \qquad \beta_2 = \frac{2\alpha_{11}}{\alpha_{12}(\alpha_{11} - 1)}.$$

Then, after passing to the limit in (14) we obtain a nontrivial *bounded* solution to the system

$$\begin{cases} -\Delta v_1 &= a_0 v_1^{\alpha_{11}} + b_0 v_2^{\alpha_{12}} + c_1 v_1^{\gamma_{11}} v_2^{\gamma_{12}} \\ -\Delta v_2 &= 0 \end{cases} \tag{51}$$

in the whole space or in a half-space, with v_1 and v_2 vanishing on the boundary of the half space. Then Liouville's theorem applied to the second equation in (51) implies that v_2 is identically equal to a constant c. If $c = 0$ (this is the only case if (51) is in a half-space, because of the boundary condition) we replace in the first equation in (51) and obtain a nontrivial solution to the Emden-Fowler equation (19) with $p = \alpha_{11}$, which is known not to have solutions for

$$1 < \alpha_{11} < \frac{N+2}{N-2},$$

and this is a consequence of (8).

If $c > 0$ the first equation in (51) implies that the inequality

$$-\Delta v_1 \geq b_0\, c^{\alpha_{12}} > 0$$

has a bounded solution in $\mathbb{R}^N$, which is well-known to be impossible. $\square$

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