RATES IN THE CLT FOR VECTOR-VALUED RANDOM FIELDS

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The Lindeberg function or the Lyapunov fraction are used to establish convergence rates in the CLT for vector-valued random fields possessing dependence structure more general than positive or negative association. Thus a generalization of the classical Newman CLT is obtained. The Stein method and the Bernstein block techniques are employed. An application to kernel estimates for the density of a stationary random field is provided.

Keywords: dependence conditions, kernel estimates of density, random fields, rates in the CLT.

1. Introduction

The aim of this paper is to establish convergence rates in the CLT for sums of dependent multiindexed random vectors with values in $\mathbb{R}^s$. We develop an approach to description of the dependence structure proposed by Doukhan and Louhichi (1999) for stochastic processes and by Bulinski and Suquet (2001) for random fields.

Let $X = \{X_t, t \in T\}$ be a random field defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ such that X_t takes values in a metric space $(\mathsf{M}, \varkappa)$ for each $t \in T$. The key idea is to measure, for finite disjoint sets $I, J \subset T$ (with cardinalities $|I|, |J|$), the dependence between collections of random variables $X_I = (X_t, t \in I)$ and $X_J = (X_t, t \in J)$ in terms of a functional

$$F(f, g; I, J) = |cov(f(X_I), g(X_J))| \tag{1}$$

where $f : \mathsf{M}^I \to \mathbb{R},\ g : \mathsf{M}^J \to \mathbb{R}$ belong to specified classes of "test functions" (whenever the covariance exists). Some restrictions can be imposed on I and J as well, for instance, $|I| = 1$.

We use here classes of bounded Lipschitz functions f, g. Recall that $G : \mathsf{K} \to \mathsf{L}$ (where (K, τ) and (L, ν) are some metric spaces) is a Lipschitz function if

$$Lip(G) = \sup_{x \neq y} \frac{\nu(G(x), G(y))}{\tau(x,y)} < \infty. \qquad (2)$$

When $\mathsf{K} = \mathsf{M}^I$ we take $\tau(x,y) = \sum_{t \in I} \varkappa(x_t, y_t)$ for $x = (x_t, t \in I)$, $y = (y_t, t \in I)$ in formula (2). Let $BL(\mathsf{K})$ denote the class of *bounded Lipschitz* functions $G : \mathsf{K} \to \mathbb{R}$ (in $\mathbb{R}$ we use the Euclidean distance).

Often it is natural to suppose, when T is endowed with a metric ρ, that the dependence between X_I and X_J is "rather small" if the distance $\rho(I, J) = \inf\{\rho(t, v) : t \in I, v \in J\}$ is "large enough". At the same time the dependence can increase if the distance $\rho(I, J)$ is fixed but I and J are growing in a sense.

To give an exact formulation consider $T = \mathbb{Z}^d$ and introduce a set Θ consisting of functions $\theta(I, J)$ depending on finite disjoint sets $I, J \subset \mathbb{Z}^d$ such that

$$\theta(\tau_n I, \tau_m J) \to 0 \ \text{ as } \ |n - m| \to \infty \ \ (n, m \in \mathbb{Z}^d) \qquad (3)$$

where $\tau_n I = \{t + n : t \in I\}$ is a shift of I, $|n| = \max_{i=1,\dots,d} |n_i|$.

For example, $\theta \in \Theta$ if

$$\theta(I, J) \leq a(|I|, |J|) u(\rho(I, J)) \qquad (4)$$

where a function $a \geq 0$ is nondecreasing in each variable and $u(r) \searrow 0$ as $r \to \infty$.

Definition 1 (Bulinski and Suquet (2001)). A random field $X = \{X_j, j \in \mathbb{Z}^d\}$ with values in a metric space $(\mathsf{M}, \varkappa)$ is called (BL, θ)-dependent if there is a function $\theta \in \Theta$ such that

$$F(f, g; I, J) \leq Lip(f) Lip(g) \theta(I, J) \qquad (5)$$

for all finite disjoint sets $I, J \subset \mathbb{Z}^d$ and any $f \in BL(\mathsf{M}^I)$, $g \in BL(\mathsf{M}^J)$.

The appearance of Lipschitz constants in the right-hand side of (5) is clear since covariance is a bilinear function and $Lip(cf) = |c| Lip(f)$ for every $c \in \mathbb{R}$.

The motivation for the concept of (BL, θ)-dependence is the following. There are a number of interesting models described by means of families of random variables possessing properties of positive or negative association or their modifications. For definitions and examples we refer to the pioneering papers by Harris (1960), Lehmann (1966), Esary et al. (1967), Fortuin et al. (1971), Joag-Dev and Proschan (1983); see also, e.g., Pitt (1982), Newman (1984), Lindqvist (1988), Evans (1990), Lee et al. (1990), Rachev and Xin

(1996), Ebrahimi (2002). Due to Bulinski and Shabanovich (1998) for a positively or negatively associated real-valued random field $X = \{X_j, j \in \mathbb{Z}^d\}$ having finite second moments the inequality (5) holds with

$$\theta(I, J) = \sum_{i \in I} \sum_{j \in J} |cov(X_i, X_j)|. \tag{6}$$

So, for this function θ the bound (4) is valid with $a(|I|, |J|) = \min\{|I|, |J|\}$ and with an analogue of the Cox–Grimmett coefficient

$$u(r) = \sup_{j \in \mathbb{Z}^d} \sum_{q:|q-j| \geq r} |cov(X_j, X_q)|, \quad r \geq 1.$$

Thus, θ appearing in (6) satisfies (3) if $u(1) < \infty$ and $u(r) \to 0$ as $r \to \infty$.

In other words Definition 1 provides a unified approach to studying both families of positively or negatively dependent random variables.

A variant of inequality (5) for smooth functions f and g in associated real-valued random variables was firstly established by Birkel (1988) (related results were proved by Newman (1984), Roussas (1994), Peligrad and Shao (1995), Bulinski (1996)). Some modifications of association for vector-valued processes and random fields leading to (5) were used by Burton et al. (1986), Bulinski (2000), Shashkin (2002).

Note that the choice of indicator functions f and g as the "test-functions" in (1) would lead to the Rosenblatt-type mixing coefficient (see, e.g., Doukhan (1994) and the references therein showing that the calculation or estimation of mixing coefficients is in general a difficult problem whereas using of a covariance function is much more simple). The choice of certain power-type functions f and g in (1) was applied by Bakhtin and Bulinski (1997) to get bounds for absolute moments of partial sums of multiindexed dependent random variables. Linear functions f and g and a correlation coefficient instead of covariance in (1) was recently used by Bradley (2002) to study the boundedness properties for spectral density of weakly stationary random field. The choice of "complex exponential" functions is discussed by Jakubowski (1993), Doukhan and Louhichi (1999).

Remark 1. Following Doukhan and Louhichi (1999) we can define the dependence conditions for a field $X = \{X_t, t \in T\}$ by means of specified "test functions" f and g and inequalities

$$F(f, g; I, J) \leq c(f, g; |I|, |J|)v(\rho(I, J))$$

where I and J are finite disjoint subsets of T, c is a nonnegative function (nondecreasing in $|I|$ and $|J|$) and $v(r) \to 0$ as $r \to \infty$.

Remark 2. In many problems we need not consider the whole random field X on $\mathbb{Z}^d$ but only "a part" $X_U = (X_j, j \in U)$, $U \subset \mathbb{Z}^d$, $|U| < \infty$. Then it is sufficient to use $I, J \subset U$, $I \cap J = \varnothing$ in (5). Moreover, we can introduce

$$\theta_1 = \theta_1(X_U) = \sup F(f, g; \{j\}, U \setminus \{j\}) \tag{7}$$

where the supremum is taken over all $j \in U$ and all $f \in BL(\mathsf{M})$, $g \in BL(\mathsf{M}^{U\setminus\{j\}})$ with $Lip(f) \le 1$, $Lip(g) \le 1$. Note that in (7) a set U need not be a subset of $\mathbb{Z}^d$, that is we can use any finite collection of random variables $X_t, t \in U$, with values in some metric space $(\mathsf{M}, \varkappa)$.

Further on let $T = \mathbb{Z}^d$ and $\mathsf{M} = \mathbb{R}^s$, that is we study a random field $X = \{X_j, j \in \mathbb{Z}^d\}$ with values in $\mathbb{R}^s$. As usual $\mathsf{E}Y$ and $Var(Y)$ denote respectively the mean and covariance matrix of a random vector Y defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

Let U be a finite subset of $\mathbb{Z}^d$. Assume that

$$\mathsf{E}X_j = 0 \in \mathbb{R}^s, \ \ \mathsf{E}\|X_j\|^2 < \infty \ \text{ for all } \ j \in U, \tag{8}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^s$. We shall use $\|z\|_1 = \sum_{i=1}^k |z_i|$ for $z \in \mathbb{R}^k$ as well. Note that $|z| \le \|z\| \le \|z\|_1$ for all $z \in \mathbb{R}^k$. These norms coinside if $k = 1$. Moreover, all norms are equivalent in our finitedimensional case.

Set

$$S = \sum_{j \in U} X_j, \ \ V^2 = \sum_{j \in U} Var(X_j).$$

Suppose $\det V^2 > 0$ and define

$$Y_j = V^{-1} X_j, \ \ W = (W_1, \ldots, W_s) = \sum_{j \in U} Y_j, \ \ R = \|V^{-1}\|_1^2 |U| \theta_1 \tag{9}$$

where V^{-1} is the inverse matrix to the square root of V^2, $\|A\|_1$ is the matrix norm corresponding to the vector norm $\|z\|_1$.

Evidently, V^2, W, R are functions of X_j, $j \in U$, and we use also notation $V^2(X_U)$, $W(X_U)$, $R(X_U)$.

Consider a function $h : \mathbb{R}^s \to \mathbb{R}$ such that for some positive constants M_0, M_1, M_2 and for all $x, x' \in \mathbb{R}^s, k = 1, \ldots, s$, one has

$$|h(x)| \le M_0, \ \ \left|\frac{\partial h(x)}{\partial x_k}\right| \le M_1, \ \ \left|\frac{\partial h(x)}{\partial x_k} - \frac{\partial h(x')}{\partial x_k}\right| \le M_2 \,\|x - x'\|. \tag{10}$$

Using the Stein method (see Stein (1972, 1986)) we provide, for a (BL,θ)-dependent random field $X = \{X_j, j \in \mathbb{Z}^d\}$ with values in $\mathbb{R}^s$, upper estimates of a functional

$$\Delta(h, X_U) = |\mathsf{E}h(W) - \mathsf{E}h(Z)| \tag{11}$$

where h is a function satisfying conditions (10), Z is a standard normal vector in $\mathbb{R}^s$ and U is a finite subset of $\mathbb{Z}^d$.

It is worth remarking that there are various generalizations of the Stein method. We refer to Chen (1975), Tikhomirov (1980, 1983), Barbour (1990), Götze(1991). The approach based on diffusion approximation for positively or negatively dependent random field was used in Bulinski and Shabanovich (1998). Interesting applications of the Stein techniques (with semigroup approach) in the framework of statistical models are discussed in Baddeley (2000).

Our main result (Theorem 1) gives an estimate for $\Delta(h, X_U)$ in terms of the Lindeberg function

$$\mathcal{L}_\varepsilon = \mathcal{L}_\varepsilon(X_U) = \sum_{j\in U} E\|Y_j\|^2 \mathbf{1}\{\|Y_j\| > \varepsilon\}, \quad \varepsilon > 0, \tag{12}$$

of the function R appearing in (9); here random vectors Y_j are defined in (9) and $\mathbf{1}\{\cdot\}$ is an indicator function.

If, moreover, for some $\delta \in (0,1]$

$$\mathsf{E}\|X_j\|^{2+\delta} < \infty, \quad j \in U, \tag{13}$$

then Theorem 2 gives an estimate of $\Delta(h, X_U)$ in terms of the Lyapunov fraction instead of $\mathcal{L}_\varepsilon$.

Using the smoothing techniques we establish (Theorem 3) the upper bound for

$$\Delta(B, X_U) = |\mathsf{P}(W \in B) - \mathsf{P}(Z \in B)| \tag{14}$$

where B is an arbitrary convex set in $\mathbb{R}^s$.

It is also shown that the Bernstein block techniques is useful in combination with the above mentioned theorems (see Theorem 4).

An application to kernel estimates of unknown density of a vector-valued stationary random field is provided as well. Theorem 5 extends some results obtained by Bosq et al. (1999), Roussas (2000, 2001), Veretennikov (2000), Bulinski and Millionshchikov (2002).

2. Results and proofs

Here we keep the notation used in the Introduction.

Theorem 1. *Let $X = \{X_j,\ j \in \mathbb{Z}^d\}$ be a random field with values in $\mathbb{R}^s$ satisfying condition* (8) *where U is a finite subset of $\mathbb{Z}^d$. Assume that for a function h condition* (10) *holds. Then for every $\varepsilon > 0$*

$$\begin{aligned}\Delta(h, X_U) \le s(D_1 + \varepsilon(s+1)D_2)R + 2\varepsilon c(s) D_2 \\ +(2\varepsilon^{-1}D_0 + (6s+1)D_1 + (\varepsilon/2)s(s+1)D_2)\mathcal{L}_\varepsilon\end{aligned} \tag{15}$$

where $c(s) = \sum_{k=1}^{s} k^{3/2} \le s^{5/2}$,

$$\begin{aligned}D_0 = \sqrt{2\pi}M_0, \quad D_1 = \max\{4M_0, \sqrt{2\pi}M_1\}, \\ D_2 = \sqrt{2}\max\{\sqrt{2\pi}M_0 + 2M_1, 4\sqrt{s}M_1, \sqrt{2\pi}M_2\}\end{aligned} \tag{16}$$

and the constants M_0, M_1, M_2 appear in (10).

Proof. For $i = 1, \dots, s$ $(s \ge 1)$ and $x_i, \dots, x_s \in \mathbb{R}$ introduce functions

$$H_i(x_i, \dots, x_s) = \mathsf{E}(h(Z_1, \dots, Z_{i-1}, x_i, \dots, x_s) - h(Z_1, \dots, Z_i, x_{i+1}, \dots x_s)), \tag{17}$$

here $Z = (Z_1, \dots, Z_s)$ is a standard normal vector in $\mathbb{R}^s$ (as usual if $s = 1$ one has $H_1(x_1) = \mathsf{E}(h(x_1) - h(Z_1))$, if $s \ge 2$ then $H_1(x_1, \dots, x_s) = \mathsf{E}(h(x_1, \dots, x_s) - h(Z_1, x_2, \dots, x_s))$ and $H_s(x_s) = \mathsf{E}(h(Z_1, \dots, Z_{s-1}, x_s) - h(Z_1, \dots, Z_s))$).

For $i = 1, \dots, s$ consider a differential equation

$$\frac{\partial f_i}{\partial x_i} - x_i f_i = H_i \tag{18}$$

where functions H_i are defined in (17).

Below we employ the solution of this equation given by the formula

$$f_i = f_i(x_i, \dots, x_s) = e^{x_i^2/2} \int_{-\infty}^{x_i} H_i(u, x_{i+1}, \dots, x_s) e^{-u^2/2} du$$

(for $i = s$ one has $f_s = f_s(x_s) = e^{x_s^2/2} \int_{-\infty}^{x_s} H_s(u) e^{-u^2/2} du$).

Lemma 1. *For all $x = (x_i, \dots, x_s), x' = (x'_i, \dots, x'_s) \in \mathbb{R}^{s-i+1}$ and any $i = 1, \dots, s,\ k = i, \dots, s$ the following inequalities are valid*

$$|f_i(x)| \le D_0, \quad |\partial f_i(x)/\partial x_k| \le D_1, \tag{19}$$

$$|\partial f_i(x)/\partial x_k - \partial f_i(x')/\partial x_k| \le D_2 \|x - x'\| \tag{20}$$

where D_0, D_1, D_2 *are indicated in* (16).

Proof. Note that $\mathsf{E}h(Z_1, \ldots, Z_i, x_{i+1}, \ldots, x_s)$ for a Borel function $h : \mathbb{R}^s \to \mathbb{R}$ is given by the expression

$$(2\pi)^{-i/2} \int\limits_{\mathbb{R}^i} e^{-\frac{u_1^2+\ldots+u_i^2}{2}} h(u_1, \ldots, u_i, x_{i+1}, \ldots, x_s) du_1 \ldots du_i$$

and, for h having bounded partial derivatives in $x_{i+1}, ..., x_s$,

$$\frac{\partial}{\partial x_k} \mathsf{E}h(Z_1, \ldots, Z_i, x_{i+1}, \ldots, x_s) = \mathsf{E} \frac{\partial}{\partial x_k} h(Z_1, \ldots, Z_i, x_{i+1}, \ldots, x_s)$$

for every $k = i+1, \ldots, s$. A simple calculation shows that, if $K : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function, $|K(x)| \le K_0,\ x \in \mathbb{R}$, and $\int_{\mathbb{R}} K(u) e^{-u^2/2} du = 0$, then for all $x \in \mathbb{R}$

$$e^{x^2/2} \left| \int\limits_{-\infty}^{x} K(u) e^{-u^2/2} du \right| \le K_0 \sqrt{\pi/2}, \quad |x| e^{x^2/2} \left| \int\limits_{-\infty}^{x} K(u) e^{-u^2/2} du \right| \le K_0. \tag{21}$$

Thus to establish (19) we use the upper bounds for the absolute values of functions appearing under the signs of integrals in representations for $f_i(x_i, \ldots, x_s)$ and $\partial f_i(x_i, \ldots, x_s)/\partial x_k$.

To obtain (20) consider separately the cases $k > i$ and $k = i$. Moreover, each time we have to consider whether $x_i = x_i'$ or $x_k = x_k'$, $k = i+1, \ldots, s$. As for the case of $x_k = x_k',\ k = i+1, \ldots, s$, let us note the existence of the second partial derivatives $\partial^2 f_i / \partial x_i \partial x_k, k = i, ..., s$. Clearly from the equation (18) we have

$$\partial^2 f_i / \partial x_i \partial x_k = \partial H_i / \partial x_k + x_i \partial f_i / \partial x_k, k = i+1, \ldots, s;$$

$$\partial^2 f_i / \partial x_i^2 = \partial H_i / \partial x_i + (1 + x_i^2) f_i + x_i H_i.$$

The estimate for the second partial derivative in x_i and $x_k,\ k > i$, follows now from (21). To estimate $\partial^2 f_i / \partial x_i^2$ it suffices to integrate by parts in the integral representation for $x_i^2 f_i$. In the case $x_i = x_i'$ we use again the representations for f_i and $\partial f_i / \partial x_k,\ k = i, \ldots, s$. The Lemma is proved.

Continuing the proof of Theorem 1, observe that (17) and (18) imply

$$\sum_{i=1}^{s} \mathsf{E} \left(\frac{\partial}{\partial x_i} - W_i \right) f_i(W_i, \ldots, W_s) = \mathsf{E}h(W) - \mathsf{E}h(Z) \tag{22}$$

where the vector W is defined in (9).

For each fixed $i \in \{1, \dots, s\}$ we estimate the summand

$$\mathsf{E}\left(\frac{\partial}{\partial x_i} - W_i\right) f_i(W_i, \dots, W_s)$$

in the left-hand side of (22). Analogously to Bulinski and Suquet (2001) introduce for a given $\varepsilon > 0$ auxiliary random vectors

$$T_j = (T_{j1}, \dots, T_{js}) = (b(Y_{j1}), \dots, b(Y_{js})), \ \ V_j = (V_{j1}, \dots, V_{js}) = Y_j - T_j$$

where $b(y) = sign(y) \min\{|y|, \varepsilon\}$, $y \in \mathbb{R}$. For the sake of brevity we write $\mathbb{W} = (W_i, \dots, W_s)$ and $\mathbb{T}_j = (T_{ji}, \dots, T_{js})$. Set

$$\mathbb{W}^{(j)} = \mathbb{W} - (Y_{ji}, \dots, Y_{js}), \ \ j \in U.$$

It can be seen that

$$\mathsf{E} W_i f_i(\mathbb{W}) = \sum_{q=1}^{4} R_{iq}$$

where

$$R_{i1} = \sum_{j \in U} \mathsf{E} Y_{ji} f_i(\mathbb{W}^{(j)}),$$

$$R_{i2} = \sum_{j \in U} \mathsf{E} V_{ji}(f_i(\mathbb{W}) - f_i(\mathbb{W}^{(j)})),$$

$$R_{i3} = \sum_{j \in U} \mathsf{E} T_{ji}(f_i(\mathbb{W}) - f_i(\mathbb{W}^{(j)} + \mathbb{T}_j)),$$

$$R_{i4} = \sum_{j \in U} \mathsf{E} T_{ji}(f_i(\mathbb{W}^{(j)} + \mathbb{T}_j) - f_i(\mathbb{W}^{(j)})).$$

Note that for a Lipschitz function $G : \mathbb{R}^m \to \mathbb{R}$ and a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ $(m, n \in \mathbb{N})$ the composition $G(A(\cdot))$ is a Lipschitz function with $Lip(GA) \le Lip(G)\|A\|_1$. Using this fact, definitions (7), (12) and inequalities (19) we derive the following estimates

$$|R_{i1}| \le \sum_{j \in U} |cov(Y_{ji}, f_i(\mathbb{W}^{(j)}))| \le \sum_{j \in U} |cov(T_{ji}, f_i(\mathbb{W}^{(j)}))| + 2D_0 \sum_{j \in U} \mathsf{E}|V_{ji}|$$

$$\le D_1 \|V^{-1}\|_1^2 |U| \theta_1 + 2D_0 \varepsilon^{-1} \sum_{j \in U} \mathsf{E} Y_{ji}^2 \mathbf{1}\{\|Y_j\| > \varepsilon\}$$

$$= D_1 R + 2D_0\varepsilon^{-1}\sum_{j\in U}\mathsf{E}Y_{ji}^2\mathbf{1}\{\|Y_j\| > \varepsilon\},$$

$$|R_{i2}| \le \sum_{j\in U}|\mathsf{E}V_{ji}(f_i(\mathbb{W}) - f_i(\mathbb{W}^{(j)} + \mathbb{T}_j))| + |\mathsf{E}V_{ji}(f_i(\mathbb{W}^{(j)} + \mathbb{T}_j) - f_i(\mathbb{W}^{(j)}))|$$

$$\le D_1\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}(|V_{ji}V_{jk}| + |V_{ji}T_{jk}|)$$

$$\le D_1\sum_{j\in U}\sum_{k=i}^{s}\left(\frac{1}{2}\mathsf{E}(Y_{ji}^2 + Y_{jk}^2)\mathbf{1}\{\|Y_j\| > \varepsilon\} + \varepsilon\mathsf{E}|Y_{ji}|\mathbf{1}\{|Y_{ji}| > \varepsilon\}\right)$$

$$\le \frac{3}{2}D_1(s-i+1)\sum_{j\in U}\mathsf{E}Y_{ji}^2\mathbf{1}\{\|Y_j\| > \varepsilon\} + \frac{1}{2}D_1\mathcal{L}_\varepsilon.$$

In a similar way

$$|R_{i3}| \le D_1\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}|T_{ji}V_{jk}| \le D_1\sum_{j\in U}\sum_{k=i}^{s}\varepsilon\mathsf{E}|Y_{jk}|\mathbf{1}\{\|Y_j\| > \varepsilon\} \le D_1\mathcal{L}_\varepsilon.$$

Due to differentiability of f_i one has

$$\left|f_i(\mathbb{W}^{(j)} + \mathbb{T}_j) - f_i(\mathbb{W}^{(j)}) - \sum_{k=i}^{s}\frac{\partial f_i(\mathbb{W}^{(j)})}{\partial x_k}T_{jk}\right|$$

$$\le \sum_{k=i}^{s}\left|\frac{\partial f_i(\mathbb{W}^{(j)} + \tau_0\mathbb{T}_j)}{\partial x_k} - \frac{\partial f_i(\mathbb{W}^{(j)})}{\partial x_k}\right||T_{jk}| \le D_2(s-i+1)^{1/2}\sum_{k=i}^{s}\max_{i\le p\le s}|T_{jp}||T_{jk}|$$

where $\tau_0 \in [0,1]$. Taking into account the relation $\sum_{j\in U}Var(Y_j) = I$ where I is a unit matrix of order s we conclude that

$$R_{i4} = \sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}T_{ji}T_{jk}\frac{\partial f_i(\mathbb{W}^{(j)})}{\partial x_k} + \triangle_{i1}.$$

One has

$$|\triangle_{i1}| \le \varepsilon D_2(s-i+1)^{1/2}\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}|T_{ji}T_{jk}|$$

$$\le \varepsilon D_2(s+i-1)^{1/2}\sum_{k=i}^{s}\sum_{j\in U}\mathsf{E}(Y_{ji}^2 + Y_{jk}^2)/2 \le \varepsilon D_2(s-i+1)^{3/2}$$

and

$$\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}T_{ji}T_{jk}\frac{\partial f_i(\mathbb{W}^{(j)})}{\partial x_k}=\sum_{j\in U}\sum_{k=i}^{s}cov\left(T_{ji}T_{jk},\frac{\partial f_i(\mathbb{W}^{(j)})}{\partial x_k}\right)$$

$$+\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}T_{ji}T_{jk}\mathsf{E}\left(\frac{\partial f_i(\mathbb{W}^{(j)})}{\partial x_k}-\frac{\partial f_i(\mathbb{W})}{\partial x_k}\right)$$

$$+\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}T_{ji}T_{jk}\mathsf{E}\frac{\partial f_i(\mathbb{W})}{\partial x_k}=\sum_{q=1}^{3}C_{iq}.$$

For any $i,k=1,\ldots,s$ the function $b_{ik}(x)=b(x_i)b(x_k)$, $x=(x_1,\ldots,x_s)\in\mathbb{R}^s$, is a Lipschitz one with $Lip(b_{ik})\le 2\varepsilon$. Consequently

$$|C_{i1}|\le\sum_{j\in U}\sum_{k=i}^{s}\Big|cov\Big(T_{ji}T_{jk},\frac{\partial f_i(\mathbb{W}^{(j)})}{\partial x_k}\Big)\Big|\le 2D_2\varepsilon\|V^{-1}\|_1^2(s-i+1)|U|\theta_1$$

$$\le 2D_2(s-i+1)\varepsilon\, R.$$

$$|C_{i2}|\le\left|\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}T_{ji}T_{jk}\mathsf{E}\left(\frac{\partial f_i(\mathbb{W}^{(j)})}{\partial x_k}-\frac{\partial f_i(\mathbb{W}^{(j)}+\mathbb{T}_j)}{\partial x_k}\right)\right|$$

$$+\left|\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}T_{ji}T_{jk}\mathsf{E}\left(\frac{\partial f_i(\mathbb{W}^{(j)}+\mathbb{T}_j)}{\partial x_k}-\frac{\partial f_i(\mathbb{W})}{\partial x_k}\right)\right|$$

$$\le D_2(s-i+1)^{1/2}\sum_{j\in U}\sum_{k=i}^{s}\mathsf{E}|T_{ji}T_{jk}|\mathsf{E}\max_{p=i,\ldots,s}|T_{jp}|+D_2\sum_{j\in U}\sum_{k=i}^{s}\sum_{p=i}^{s}\mathsf{E}|T_{ji}T_{jk}|\mathsf{E}|V_{jp}|$$

$$\le\varepsilon D_2(s-i+1)^{3/2}+\varepsilon D_2(s-i+1)\mathcal{L}_\varepsilon.$$

To estimate C_{i3} consider at first the case $k=i$. Then

$$\sum_{j\in U}\mathsf{E}T_{ji}^2\mathsf{E}\frac{\partial f_i(\mathbb{W})}{\partial x_i}=\mathsf{E}\frac{\partial f_i(\mathbb{W})}{\partial x_i}\sum_{j\in U}\mathsf{E}\,Y_{ji}^2+\triangle_{i2}=\mathsf{E}\frac{\partial f_i(\mathbb{W})}{\partial x_i}+\triangle_{i2},$$

$$|\triangle_{i2}|=\left|\mathsf{E}\frac{\partial f_i(\mathbb{W})}{\partial x_i}\sum_{j\in U}(\mathsf{E}\,T_{ji}^2-\mathsf{E}\,Y_{ji}^2)\right|\le D_1\sum_{j\in U}\mathsf{E}\,Y_{ji}^2\mathbf{1}\{\|Y_j\|>\varepsilon\}.$$

The bounds for R_{i2} and R_{i3} imply that

$$|\triangle_{i3}| = \Big|\sum_{j\in U}\sum_{k=i+1}^{s} \mathsf{E}T_{ji}T_{jk}\mathsf{E}\frac{\partial f_i(\mathbb{W})}{\partial x_k}\Big|$$

$$= \Big|\sum_{k=i+1}^{s} \mathsf{E}\frac{\partial f_i(\mathbb{W})}{\partial x_k}\sum_{j\in U}\mathsf{E}(Y_{ji}Y_{jk} - T_{ji}V_{jk} - V_{ji}T_{jk} - V_{ji}V_{jk})\Big|$$

$$\leq \sum_{j\in U}\sum_{k=i+1}^{s}\left|\mathsf{E}\frac{\partial f_i(\mathbb{W})}{\partial x_k}\right|(\mathsf{E}|T_{ji}V_{jk}| + \mathsf{E}|V_{ji}T_{jk}| + \mathsf{E}|V_{ji}V_{jk}|)$$

$$\leq D_1((3/2)\mathcal{L}_\varepsilon + (3s/2)\sum_{j\in U}\mathsf{E}\, Y_{ji}^2\mathbf{1}\{\|Y_j\| > \varepsilon\})$$

in view of the relation $\sum_{j\in U} Var(Y_j) = I$, that is $\sum\limits_{j\in U}\mathsf{E}Y_{ji}Y_{jk} = 0, k \neq i.$

Finally we have

$$\Delta(h, X_U) \leq \sum_{i=1}^{s}\left|\mathsf{E}\left(\frac{\partial}{\partial x_i} - W_i\right) f_i(W_i, \ldots, W_s)\right|$$

$$\leq \sum_{i=1}^{s}\left(\sum_{q=1}^{3}(|R_{iq}| + |\triangle_{iq}|) + \sum_{q=1}^{2}|C_{iq}|\right).$$

Observing that $\sum_{i=1}^{s}(s-i+1)^{3/2} = c(s)$ and $\sum_{i=1}^{s}(s-i+1) = s(s+1)/2$ we come to (15). The proof of Theorem 1 is complete.

Corollary 1. *For a family of centered random fields* $X^{(n)} = \{X_j^{(n)}, j \in \mathbb{Z}^d\}$ $(n \in \mathbb{N})$ *with values in* $\mathbb{R}^s$ *and a family of finite subsets* U_n *of* $\mathbb{Z}^d$*, the CLT holds, that is*

$$W(X_{U_n}^{(n)}) \stackrel{Law}{\longrightarrow} Z \ \ as \ \ n \to \infty,$$

whenever, for every $\varepsilon > 0$,

$$\mathcal{L}_\varepsilon(X_{U_n}^{(n)}) \to 0 \ \ and \ \ R(X_{U_n}^{(n)}) \to 0 \ \ as \ \ n \to \infty.$$

If $X_{U_n}^{(n)}$ *consists of independent random vectors then* $R(X_{U_n}^{(n)}) = 0$*. Thus Theorem* 1 *comprises the multidimensional Lindeberg theorem for independent summands. Analogously to Bulinski and Suquet* (2001) *one can also obtain from* (15) *the generalization of the classical Newman CLT for associated random fields.*

Theorem 2. *Assume that conditions of Theorem* 1 *are satisfied and, moreover,* (13) *holds. Then*

$$\begin{aligned}\Delta(h, X_U) \le s(D_1 + D_2(s+1))R \\ +(2D_0 + (6s+1)D_1 + (2c(s) + s(s+1)/2)D_2)L_{2+\delta}\end{aligned} \tag{23}$$

where the Lyapunov fraction

$$L_{2+\delta} = L_{2+\delta}(X_U) = \sum_{j\in U} \mathsf{E}\|Y_j\|^{2+\delta} \le \|V^{-1}\|_1^{2+\delta} \sum_{j\in U} \mathsf{E}\|X_j\|_1^{2+\delta},$$

random vectors Y_j *are defined in* (9) *and* $c(s)$ *appears in* (15).

Proof. Observe that for $\varepsilon = 1$ and $\delta \in (0,1]$ one has $\mathcal{L}_\varepsilon \le L_{2+\delta}$. To estimate Δ_{i1} (and analogously C_{i2}) we use inequalities $|abc| \le (|a|^3 + |b|^3 + |c|^3)/3$ for all $a, b, c \in \mathbb{R}$ and $|a|^3 \le |a|^{2+\delta}$ when $|a| \le 1$. Thus

$$D_2(s-i+1)^{1/2} \sum_{j\in U} \sum_{k=i}^{s} \mathsf{E}|T_{ji} T_{jk} \max_{p=i,\dots,s} T_{jp}|$$

$$\le \frac{1}{3} D_2(s+i-1)^{1/2} \sum_{j\in U} \sum_{k=i}^{s} (\mathsf{E}|T_{ji}|^3 + \mathsf{E}|T_{jk}|^3 + \mathsf{E} \max_{p=i,\dots,s} |T_{jp}|^3)$$

$$\le D_2(s-i+1)^{3/2} \sum_{j\in U} \mathsf{E}\|Y_j\|^{2+\delta} = D_2(s-i+1)^{3/2} L_{2+\delta}.$$

Theorem is proved.

Now we need some new notation. Let $B^{(\gamma)}$ be a γ-neighborhood of a set $B \subset \mathbb{R}^s$ with respect to Euclidean distance (that is $B^{(\gamma)} = \{x \in \mathbb{R}^s : \inf_{y\in B} \|x - y\| < \gamma\}$), ∂B being the boundary of B.

Remark 3. From Theorem 1.4 by Goldstein and Rinott (1996) the estimate for $\Delta(h, X_U)$ (in our notation) can be established when $h \in C_b^3(\mathbb{R}^s)$ and $\mathsf{E}X_j = 0$, $\mathsf{E}\|X_j\|^4 < \infty$, $j \in U$. Using a function $h \in C_b^3$ approximating the indicator function of a convex set $B \subset \mathbb{R}^s$ (more precisely, for a given $\gamma \in (0,1)$ let $h(x) = 1$ for $x \in B$, $h(x) = 0$ for $x \notin B^{(\gamma)}$ and $0 \le h(x) \le 1$ for all $x \in \mathbb{R}^s$) one can derive from the mentioned estimate that

$$\Delta(B, X_U) \le \mathsf{P}(Z \in (\partial B)^{(\gamma)}) + G_\gamma(X_U) \tag{24}$$

where $\Delta(B, X_U)$ is defined in (15), $G_\gamma(\cdot)$ is a certain (nonrandom) functional in X_U. In Theorems 1 and 2 of our paper the estimates of $\Delta(h, X_U)$ are obtained in other terms under lower moment assumptions and for a wider class of functions h satisfying the conditions (10). We have

$$\Delta(B, X_U) \le \mathsf{P}(Z \in (\partial B)^{(\gamma)}) + H_\gamma(X_U) \tag{25}$$

where $H_\gamma(\cdot)$ is a specified nonrandom functional in X_U, as the next theorem shows. For fixed U, ε and s one has $G_\gamma(X_U) = O(\gamma^{-3})$ as $\gamma \to 0$ whereas $H_\gamma(X_U) = O(\gamma^{-2})$ as $\gamma \to 0$.

Write

$$\gamma_0(s) = \min\{1, 3\sqrt{\pi/(2s)}\}.$$

Theorem 3. *Let conditions of Theorem 1 be satisfied and B be a convex set in $\mathbb{R}^s$. Then for any $\gamma \in (0, \gamma_0(s)]$ the estimate (25) holds with*

$$\begin{aligned} H_\gamma(X_U) = \sqrt{2\pi} s\gamma^{-1}(2 + 12\sqrt{2}(s+1)\varepsilon\gamma^{-1})R + 48\sqrt{\pi}c(s)\varepsilon\gamma^{-2} \\ + \sqrt{2\pi}(\varepsilon^{-1} + (12s+2)\gamma^{-1} + 6\sqrt{2}s(s+1)\varepsilon\gamma^{-2})\mathcal{L}_\varepsilon. \end{aligned} \tag{26}$$

If, moreover, conditions of Theorem 2 are satisfied then one can take

$$\begin{aligned} H_\gamma(X_U) = \sqrt{2\pi} s\gamma^{-1}(2 + 12\sqrt{2}(s+1)\gamma^{-1})R \\ + \sqrt{2\pi}(1 + (12s+2)\gamma^{-1} + 12\sqrt{2}(2c(s) + s(s+1)/2)\gamma^{-2})L_{2+\delta}. \end{aligned} \tag{27}$$

Proof. Introduce a function ψ setting

$$\psi(x) = \begin{cases} 1, & x \le 0, \\ 1 - \frac{16x^3}{3\gamma^3}, & x \in (0, \frac{\gamma}{4}], \\ \frac{3}{2} - \frac{2x}{\gamma} - \frac{16}{3\gamma^3}(\frac{\gamma}{2} - x)^3, & x \in (\frac{\gamma}{4}, \frac{3\gamma}{4}], \\ \frac{16}{3\gamma^3}(\gamma - x)^3, & x \in (\frac{3\gamma}{4}, \gamma], \\ 0, & x \ge \gamma. \end{cases} \tag{28}$$

It is easy to verify that the following statement is true.

Lemma 2. *The function $\psi \in C^2(\mathbb{R})$ and for all $u \in \mathbb{R}$ one has*

$$0 \le \psi(u) \le 1, \quad |\psi'(u)| \le 2\gamma^{-1}, \quad |\psi''(u)| \le 8\gamma^{-2}.$$

For a convex set $B \subset \mathbb{R}^s$ define the function

$$h(x) = \psi(\rho(x, B)), \quad x \in \mathbb{R}^s, \tag{29}$$

where ψ is given by (28) and ρ is the Euclidean distance in $\mathbb{R}^s$.

Obviously

$$0 \le h(x) \le 1,\ x \in \mathbb{R}^s, \quad h(x) = 1,\ x \in B, \quad h(x) = 0,\ x \notin B^{(\gamma)}. \qquad (30)$$

Lemma 3. *For the nonnegative function $h \in C^1(\mathbb{R}^s)$ (see (29)) the condition (10) is satisfied with $M_0 = 1$, $M_1 = 2\,\gamma^{-1}$ and $M_2 = 12\,\gamma^{-2}$.*

To prove this result one can use the properties of ψ given in Lemma 2 and take into account that for all $x \notin [B]$ and any $i = 1, \dots, s$ there exists

$$\frac{\partial}{\partial x_i}\rho(x, B) = -\cos(e_i, n).$$

Here e_i is the i-th unit vector of the natural orthonormal basis of $\mathbb{R}^s$, $n = y - x$ where $y \in [B]$ and $\|x - y\| = \rho(x, B)$; $[B]$ is a closure of B in the Euclidean distance.

Lemma 4. *If γ satisfies the conditions of Theorem 3 and the function h is given by (29), the statement of Lemma 1 is valid with*

$$D_0 = \sqrt{\pi/2}, \quad D_1 = 2\sqrt{2\pi}\gamma^{-1}, \quad D_2 = 24\sqrt{\pi}\gamma^{-2}. \qquad (31)$$

Proof. The indicated values for D_0, D_1, D_2 can be easily obtained from the formula (19), analogously to the proof of Lemma 1, using the fact that $|H_i(x)| \le 1$ since $0 \le h(x) \le 1, x \in \mathbb{R}^s$.

Now we proceed with the proof of Theorem 3. Due to Theorem 1 and Lemma 4 we come to the estimate (15) with D_0, D_1, D_2 indicated in (31). In view of (30)

$$\mathsf{P}(W \in B) - \mathsf{P}(Z \in B^{(\gamma)}) \le \mathsf{E}h(W) - \mathsf{E}h(Z) \le \Delta(h, X_U). \qquad (32)$$

In a similar way for the set $B_{(\gamma)} = B \setminus (\partial B)^{(\gamma)}$ one has

$$\mathsf{P}(W \in B) - \mathsf{P}(Z \in B_{(\gamma)}) \ge -\Delta(h, X_U). \qquad (33)$$

Now (32) and (33) imply (25) with $H(X_U)$ given by (26). The second assertion of Theorem 3 follows in the same manner as Theorem 2 was obtained using Theorem 1. Theorem 3 is proved.

The next result gives the possibility to provide an estimate for $\Delta(B, X_U)$ which is uniform on the class $\mathcal{C}_s$ of convex sets of $\mathbb{R}^s$. It is easy to derive immediately the following bound from Corollary 3.2 of Bhattacharia and Ranga Rao (1976).

Lemma 5. *For any $k \in \mathbb{N}$, all $\gamma > 0$ and every convex set $B \subset \mathbb{R}^s$*

$$\mathsf{P}(Z \in (\partial B)^\gamma) \le a(s)\gamma$$

where $a(1) = 2\sqrt{2/\pi}$ and $a(s) = 2^{1/2}(s-1)\Gamma((s-1)/2)/\Gamma(s/2)$ for $s \ge 2$. Thus

$$a(s) \le a_0 s^{1/2}, \quad a_0 = const, \quad s \in \mathbb{N}. \tag{34}$$

Corollary 2. *Let conditions of Theorem 2 be satisfied. Then*

$$\sup_{B \in \mathcal{C}_s} \Delta(B, X_U) \le c(R(X_U) + L_{2+\delta}(X_U))^{1/3}$$

where a factor $c = c(s)$.

Remark 4. We are interested in asymptotical behaviour of random vectors $W = W(X_U)$ as $U \to \infty$ in a sense. In this regard note that in general $R = R(X_U)$ need not tend to zero for growing sets U (if for dependent summands there are points of U which are "rather close" to each other). So, it is natural to use the combination of the obtained results with the Bernstein block techniques. Our next two theorems provide examples of this approach.

Let $X = \{X_j, j \in \mathbb{Z}^d\}$ be a (BL, θ)-dependent random field with values in $\mathbb{R}^s$ such that (4) holds with $a(I, J) = \min\{|I|, |J|\}$ and some function $u(r) \searrow 0$ as $r \to \infty$. Define for a Lipschitz function $\varphi : \mathbb{R}^s \to \mathbb{R}^m$ (in Euclidean spaces we use the norm $\|\cdot\|_1$) a random field $\widetilde{X} = \{\widetilde{X}_j, j \in \mathbb{Z}^d\}$ where $\widetilde{X}_j = \varphi(X_j), j \in \mathbb{Z}^d$.

Lemma 6. *A random field $\widetilde{X}$ is $(BL, \widetilde{\theta})$-dependent where for any finite disjoints sets $I, J \subset \mathbb{Z}^d$*

$$\widetilde{\theta}(I, J) \le \min\{|I|, |J|\}(Lip(\varphi))^2 u(\rho(I, J)). \tag{35}$$

Proof. Note that if $f_k(f_{k-1}(\dots f_1))$ is a composition of Lipschitz functions $f_1, \dots, f_k$ then

$$Lip(f_k(f_{k-1}(\dots f_1))) \le Lip(f_k) \dots Lip(f_1).$$

Now (35) is obvious due to (BL, θ)-dependence of a field X.

Let U be a finite subset of $\mathbb{Z}^d$ such that

$$U = \bigcup_{k=0}^{N} U^{(k)}, \quad N \in \mathbb{N}, \tag{36}$$

where $U^{(0)}, \dots, U^{(N)}$ are disjoint sets and for some positive b and q

$$|U^{(k)}| \le b \ \text{ and } \ \rho(U^{(k)}, U^{(l)}) \ge q, \quad k, l = 1, \dots, N, \quad l \ne k. \tag{37}$$

Assume that

$$\mathsf{E}\widetilde{X}_j = 0 \in \mathbb{R}^m, \quad \mathsf{E}\|\widetilde{X}_j\|^2 < \infty, \quad j \in \mathbb{Z}^d. \tag{38}$$

Set for $k = 0, \dots, N$

$$\overline{X}_k = \sum_{j \in U^{(k)}} \widetilde{X}_j, \quad V_1^2 = \sum_{k=1}^{N} Var(\overline{X}_k), \quad V_0^2 = Var(\overline{X}_0). \tag{39}$$

Using this notation we have $\widetilde{S} = \widetilde{S}(U) = \sum_{j \in U} \widetilde{X}_j = \sum_{k=1}^{N} \overline{X}_k$. Suppose that $\det V_1^2 > 0$ and introduce for random vectors $\overline{Y}_k = V_1^{-1}\overline{X}_k$ ($k = 1, \dots, N$) the Lindeberg function

$$\overline{\mathcal{L}}_\varepsilon = \sum_{k=1}^{N} \mathsf{E}\|\overline{Y}_k\|^2 \mathbf{1}\{\|\overline{Y}_k\| > \varepsilon\}, \quad \varepsilon > 0. \tag{40}$$

Theorem 4. *Let $\widetilde{X}$ be a random field satisfying all the above mentioned conditions and U be a finite set appearing in* (36). *Then for any nonrandom matrix A of order m and all $\varepsilon > 0, \gamma \in (0, \gamma_0(m)]$ one has*

$$\Delta := \sup_{B \in \mathcal{C}_m} |\mathsf{P}(A\widetilde{S} \in B) - \mathsf{P}(Z \in B)|$$

$$\le \ 2a(m)\gamma + \gamma^{-2}\{m\sqrt{2\pi}(2 + 12\sqrt{2}(m+1)\varepsilon)Nb(Lip(\varphi))^2 u(q)\|V_1^{-1}\|_1^2$$
$$+48c(m)\sqrt{\pi}\varepsilon + \sqrt{2\pi}(\varepsilon^{-1} + 12m + 2 + 6\sqrt{2}m(m+1)\varepsilon)\overline{\mathcal{L}}_\varepsilon\} + \gamma^{-2}\overline{\Delta} \tag{41}$$

where

$$\overline{\Delta} = 2m\{\|AV_1 - I\|_1^2(m + \|V_1^{-1}\|_1^2 Nb(Lip(\varphi))^2 u(q)) + m\|A\|_1^2\|V_0^2\|_1\}, \tag{42}$$

$\mathcal{C}_m$ is a class of convex sets in $\mathbb{R}^m$, $a(m)$ appears in (34) *and Z is a Gaussian vector in $\mathbb{R}^m$. If, moreover, for some $\delta \in (0,1]$*

$$\sup_{j \in \mathbb{Z}^d} \mathsf{E}\|\widetilde{X}_j\|^{2+\delta} \le \widetilde{c} < \infty \tag{43}$$

then

$$\Delta \le 2a(m)\gamma + \gamma^{-2}\sqrt{2\pi}\{a_1(m)\|V_1^{-1}\|_1^2 Nb(Lip(\varphi))^2 u(q)$$

$$+ \ \widetilde{c}b^{2+\delta}a_2(m)N\|V_1^{-1}\|_1^{2+\delta}\} + \gamma^{-2}\overline{\Delta} \quad (44)$$

where $a_1(m) = m(2 + 12\sqrt{2}(m+1))$ *and* $a_2(m) = m^{2+(3\delta)/2}(12m + 3 + 12\sqrt{2}(2c(m) + m(m+1)/2))$.

Proof. We need the following two elementary results.

Lemma 7. *Let* ζ_0, ζ_1 *be random vectors with values in* $\mathbb{R}^m$ *such that* $\mathsf{E}\|\zeta_0\|^2 < \infty$. *Then for any value* $\gamma > 0$

$$\sup_{B\in\mathcal{C}_m} |\mathsf{P}(\zeta_0 + \zeta_1 \in B) - \mathsf{P}(Z \in B)| \le \sup_{B\in\mathcal{C}_m} |\mathsf{P}(\zeta_1 \in B) - \mathsf{P}(Z \in B)|$$

$$+ \ \gamma a(m) + \gamma^{-2}\mathsf{E}\|\zeta_0\|^2.$$

Lemma 8. *Let* ζ *be a centered random vector with values in* $\mathbb{R}^m$ *such that* $\mathsf{E}\|\zeta\|^2 < \infty$. *Then for any nonrandom matrix* A *of order* m *one has*

$$\mathsf{E}\|A\zeta\|^2 \le m^2\|A\|_1^2\|Var(\zeta)\|_1.$$

To prove Theorem 4 note that $A\widetilde{S} = \zeta_0 + \zeta_1$ where

$$\zeta_1 = V_1^{-1}\sum_{k=1}^N \overline{X}_k, \quad \zeta_0 = (AV_1 - I)V_1^{-1}\sum_{k=1}^N \overline{X}_k + A\overline{X}_0,$$

here I is a unit matrix of order m.

Using estimate (49) and Lemmas 5,7 we get

$$\Delta \le \sup_{B\in\mathcal{C}_m} |\mathsf{P}(\zeta_1 \in B) - \mathsf{P}(Z \in B)| + \gamma a(m) + \gamma^{-2}\mathsf{E}\|\zeta_0\|^2. \quad (45)$$

Theorem 3 provides a bound

$$\sup_{B\in\mathcal{C}_m} |\mathsf{P}(\zeta_1 \in B) - \mathsf{P}(Z \in B)| \le a(m)\gamma$$

$$+ \ \gamma^{-2}\{m\sqrt{2\pi}(2 + 12\sqrt{2}(m+1)\varepsilon)\overline{R} + 48\sqrt{\pi}c(m)\varepsilon$$

$$+\sqrt{2\pi}(\varepsilon^{-1} + 12m + 2 + 6\sqrt{2}m(m+1)\varepsilon)\overline{\mathcal{L}}_\varepsilon\} \quad (46)$$

where $\overline{\mathcal{L}}_\varepsilon$ appears in (40) and $\overline{R}$ is defined for a collection of random vectors $\overline{X}_1, \dots, \overline{X}_N$ in the same manner as R in (9). Namely

$$\overline{R} = \|V_1^{-1}\|_1^2 N\overline{\theta}_1$$

where $\overline{\theta}_1$ is given for $\overline{X}_U = \{\overline{X}_k, k = 1, \dots, N\}$ accordingly to (7). Hence

$$\overline{R} \le Nb(Lip(\varphi))^2u(q)\|V_1^{-1}\|_1^2. \quad (47)$$

Set $\xi_k = (AV_1 - I)V_1^{-1}\overline{X}_k$. Note that $\sum_{k=1}^N Var(\xi_k) = (AV_1 - I)(AV_1 - I)^*$ and consequently

$$\mathsf{E}\|\sum_{k=1}^N \xi_k\|^2 \le \sum_{k=1}^N \mathsf{E}\|\xi_k\|^2 + \sum_{k=1}^N \sum_{r=1}^m |cov(\xi_{kr}, \sum_{l\neq k} \xi_{lr})|$$

$$\le Tr(AV_1 - I)(AV_1 - I)^* + mN\|AV_1 - I\|_1^2 \|V_1^{-1}\|_1^2 \overline{\theta}_1$$

$$\le m\|AV_1 - I\|_1^2 (m + \overline{R}), \tag{48}$$

here $*$ and Tr stand respectively for conjugation and trace of a matrix. Due to Lemma 8 and (48)

$$\mathsf{E}\|\zeta_0\|^2 \le \overline{\Delta} \tag{49}$$

where $\overline{\Delta}$ is defined in (42). Estimates (45) – (49) imply (41).

To prove (44) we use instead of (46) the relation (27). The proof is complete.

3. Application to the kernel estimate of a density

We consider a stationary random field $X = \{X_j, j \in \mathbb{Z}^d\}$ with values in $\mathbb{R}^s$. Assume that X_0 has a density $f = f(x)$, $x \in \mathbb{R}^s$.

Recall that, given a probability density function K, the Parzen – Rosenblatt (*kernel*) estimators for a density f are defined as follows

$$\widehat{f}_n(x) = \frac{1}{|U_n| h_n^s} \sum_{j \in U_n} K\left(\frac{x - X_j}{h_n}\right), \quad x \in \mathbb{R}^s,$$

where $U_n,\ n \in \mathbb{N}$ are finite subsets of $\mathbb{Z}^d$, $\{h_n\}_{n\ge 1}$ is a sequence of positive numbers such that

$$h_n \to 0 \ \text{ and } \ |U_n| h_n^s \to 0 \ \text{ as } \ n \to \infty.$$

Introduce three types of conditions concerning the properties of a field X, a kernel K and sets U_n ($n \in \mathbb{N}$). Further on C denotes some positive factors (not necessary the same in different expressions) which do not depend on n.

($\mathbf{A_1}$) Let X be a (BL, θ)-dependent random field such that (4) holds with $a(|I|, |J|) = \min\{|I|, |J|\}$ and for some $\lambda > d(s+2)/s$

$$u(r) = O(r^{-\lambda}) \ \text{ as } \ r \to \infty. \tag{50}$$

Assume that the density f is a Lipschitz function and for all $j \in \mathbb{Z}^d$ there exists a joint density $f_j(x,y)$ for random vectors X_0 and X_j such that

$$\sup_{x,y\in\mathbb{R}^s} f_j(x,y) \le c_0 \tag{51}$$

where c_0 does not depend on j.

($\mathbf{A_2}$) Let kernel K be a Lipschitz function and

$$\int\limits_{\mathbb{R}^s} \|x\| K(x)dx < \infty. \tag{52}$$

($\mathbf{A_3}$) Let $U_n,\ n \ge 1$, be regularly growing parallelepipeds in $\mathbb{Z}^d$, that is

$$U_n = \{(a_1(n), a_1(n) + l_1(n)] \times \ldots \times (a_d(n), a_d(n) + l_d(n)]\} \cap \mathbb{Z}^d \tag{53}$$

and for some $C_1 > 0$ and all $n \in \mathbb{N}$, $i,k = 1,\ldots,d$ one has

$$a_i(n) \in \mathbb{Z}, l_i(n) \in \mathbb{N}, i = 1,\ldots,d \ \text{ and } l_i(n)/l_k(n) \le C_1. \tag{54}$$

For fixed $m \in \mathbb{N}$ and different points $x_1,\ldots,x_m \in \mathbb{R}^s$ such that $f(x_i) > 0, i = 1,\ldots,m$, define

$$\sigma_i^2 = f(x_i) \int\limits_{\mathbb{R}^s} K^2(x)dx.$$

Consider a random vector $L(n) = (L_1(n),\ldots,L_m(n))$ with components

$$L_i = \sigma_i^{-1}\sqrt{|U_n|h_n^s}(\widehat{f}_n(x_i) - \mathsf{E}\widehat{f}_n(x_i)), \ \ i = 1,\ldots,m.$$

Theorem 5. *Let X be a random field and U_n be a sequence of subsets of $\mathbb{Z}^d$ satisfying all the conditions mentioned above. Then there exist $\beta > 0$ such that for $h_n = |U_n|^{-\beta}$, $n \in \mathbb{N}$, and some $C_0, \mu > 0$ (independent of U_n) the following inequality holds*

$$\sup_{B\in\mathcal{C}_m} |\mathsf{P}(L(n) \in B) - \mathsf{P}(Z \in B)| \le C_0|U_n|^{-\mu}, \ \ n \in \mathbb{N}. \tag{55}$$

If $s > 2d+1$ and $\lambda \ge \lambda_0 = d(s+4)/(s-2d-1)$, one can choose

$$\beta = \frac{1}{2\Lambda(s,d,\lambda)}, \ \mu = \frac{1}{3\Lambda(s,d,\lambda)}$$

where $\Lambda(s,d,\lambda) = 2 + s/2 + 2d(2\lambda + s + 4)/\lambda$. If $s \le 2d+1$ or $\lambda < \lambda_0$ one can take

$$\beta = \frac{1}{5s}, \ \mu = \frac{2(\lambda s - (s+2)d)}{15s(2\lambda d + 2d + \lambda)}.$$

Proof. Let $\{p_n\}_{n\geq 1}$ and $\{q_n\}_{n\geq 1}$ be nonrandom sequences with values in $\mathbb{N}$ such that

$$q_n \to \infty, \quad q_n/p_n \to 0, \quad p_n^d h_n^s \to 0 \text{ as } n \to \infty.$$

Following the Bernstein method each edge $(a_k(n), a_k(n)+l_k(n)]$ can be represented as a union of disjoint "large" and "small" intervals (open from the left and closed from the right) having the respective lengths $p_n, q_n, p_n, \ldots, q_n, \widetilde{p_{nk}}$ where $p_n \leq \widetilde{p_{nk}} \leq 3p_n$ $(n \in \mathbb{N}, k = 1, \ldots, d)$; see the details in Bulinski and Millionshchikov (2002). Through every end point of these intervals (for each k=1,...,d) draw the hyperplane which is perpendicular to i-th axis. Thus we have a partition of $|U_n|$ into blocks (that is parallelepipeds) of 2^d types. Namely, to each block there corresponds a vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ determining the type such that $\alpha_i = 0$ if the length of block's edge along the i-th axis is equal to q_n and $\alpha_i = 1$ otherwise. Denote by Γ the set of all such vectors α. Numerate as $1, 2, \ldots, m_k$ the intervals of lengths $p_n, q_n, p_n, \ldots, q_n, \widetilde{p_{nk}}$ along the i-th edge of U_n. Then we get a partition of U_n into blocks Π_j where $j \in \Lambda_n = \{(j_1, \ldots, j_d) : 1 \leq j_k \leq m_k, k = 1, \ldots, d\}$, with $m_k = m_k(U_n)$. Let $j \in \Lambda_n^\alpha$ if the block Π_j has type $\alpha = (\alpha_1, \ldots, \alpha_d)$. Below we write Π_j^α instead of Π_j to indicate the type of a parallelepiped.

Keeping the notation of Theorem 4, let $\varphi(y) = (\varphi_1(y), \ldots, \varphi_m(y))$, where

$$\varphi_i(y) = \sigma_i^{-1} h_n^{-s/2} \left(K\Big(\frac{x_i - y}{h_n}\Big) - \mathsf{E}K\Big(\frac{x_i - X_0}{h_n}\Big) \right), \quad i = 1, \ldots, m, \quad y \in \mathbb{R}.$$

We have

$$Lip(\varphi_i) \leq \sigma_i^{-1} Lip(K) h_n^{-s/2-1}. \tag{56}$$

Let $\widetilde{X}_j = \varphi(X_j), j \in \mathbb{Z}^d$. Clearly, the field $\{\widetilde{X}_j, j \in U\}$ satisfies the condition (38). Enumerate the blocks $\Pi_j^{\mathbf{1}}$ of type $\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)$, as $U^{(1)}, \ldots, U^{(N)}$, $N = |\Lambda_n^{\mathbf{1}}|$. Put also $U^{(0)} = U \setminus (\cup_{k=1}^N U^{(k)})$. It is easy to see that condition (37) holds with $b = (3p_n)^d, q = q_n$. For $\alpha \in \Gamma$ we introduce auxiliary random fields with values in $\mathbb{R}^m$

$$\overline{X}^\alpha(n) = \Big\{ \sum_{i \in \Pi_j} \widetilde{X}_i(n), j \in \Lambda_n^\alpha \Big\}, \quad n \in \mathbb{N}.$$

So $\overline{X}^{\mathbf{1}}(n) = \overline{X}(n)$.

Lemma 9. *For each* $\alpha \in \Gamma$

$$\theta_1(\overline{X}^\alpha(n)) \leq 3^d p_n^{\|\alpha\|_1} q_n^{d-\|\alpha\|_1} h_n^{-s-2} u(q_n) \left(\sum_{i=1}^s \sigma_i^{-1} \right)^2 (Lip(K))^2.$$

Proof. Evidently for every Π_j^α one has

$$\rho_0(\Pi_j^\alpha, \bigcup_{i\neq j} \Pi_i^\alpha) \geq q_n,$$

where $\rho_0(B, D) = \min_{s\in B, t\in D}\{\max_{k=1}^d |s_k - t_k|\}$, $B, D \subset \mathbb{Z}^d$, and $|\Pi_j^\alpha| \leq 3^d p_n^{\|\alpha\|_1} q_n^{d-\|\alpha\|_1}$. The Lemma now follows from (35) and (56).

The next Lemma describes some properties of the field $\{\widetilde{X}_j, j \in U\}$ determined by the distributions of order not greater than two.

Lemma 10. *For all $i, j \in U_n, i \neq j$ and $r, t = 1, \ldots, m$*

$$|Var(\widetilde{X}_{ir}(n)) - 1| \leq Ch_n, \tag{57}$$

$$|cov(\widetilde{X}_{ir}(n), \widetilde{X}_{jt}(n))| \leq Ch_n^s. \tag{58}$$

If $r \neq t$ one has

$$|cov(\widetilde{X}_{ir}(n), \widetilde{X}_{it}(n))| \leq Ch_n. \tag{59}$$

Also for all $j \in U_n, r = 1, \ldots, m$

$$\mathsf{E}|\widetilde{X}_{jr}(n)|^3 \leq Ch_n^{-s/2}. \tag{60}$$

Proof. Note that f is a bounded Lipschitz function. Due to (51) and ($\mathbf{A_2}$) simple estimates of the corresponding integrals lead to (57)-(60).

Now we turn to the properties of matrices V_1^2, V_0^2 defined in (39), denoting by I the unit matrix of order m.

Lemma 11. *The following estimate holds*

$$\|V_1^2 - \sum_{k=1}^N |U_n^{(k)}| I\|_1 \leq C \sum_{k=1}^N |U_n^{(k)}|(h_n + p_n^d h_n^s).$$

Proof. By definition of V_1^2, (57) and (58) for any $r = 1, \ldots, m$ we have

$$\Big|\Big(V_1^2 - \Big(\sum_{k=1}^N |U_n^{(k)}|\Big) I\Big)_{rr}\Big| = \Big|\sum_{k=1}^N \sum_{j\in U^{(k)}} Var(\widetilde{X}_{jr}(n))$$

$$+ \sum_{k=1}^N \sum_{\substack{j,q\in U^{(k)} \\ j\neq q}} cov(\widetilde{X}_{jr}(n), \widetilde{X}_{qr}(n)) - \sum_{k=1}^N |U_n^{(k)}|\Big|$$

$$= \left| \sum_{k=1}^{N} \sum_{j \in U^{(k)}} (Var(\widetilde{X}_{jr}(n)) - 1) + \sum_{k=1}^{N} \sum_{\substack{j,q \in U^{(k)} \\ j \neq q}} cov(\widetilde{X}_{jr}(n), \widetilde{X}_{qr}(n)) \right|$$

$$\leq C(h_n + p_n^d h_n^s) \sum_{k=1}^{N} |U_n^{(k)}|.$$

If $t = 1, \dots, m, t \neq r$ then, making use of (58) and (59),

$$\left| \left(V_1^2 - \Big(\sum_{k=1}^{N} |U_n^{(k)}| \Big) I \right)_{rt} \right| = \left| \sum_{k=1}^{N} \sum_{j,q \in U^{(k)}} cov(\widetilde{X}_{jr}(n), \widetilde{X}_{qt}(n)) \right|$$

$$\leq \; C(h_n + p_n^d h_n^s) \sum_{k=1}^{N} |U_n^{(k)}|.$$

Lemma 11 now follows from standard norm estimates.

Thus we can write

$$V_1^2 = (I + M_n) \sum_{k=1}^{N} |U_n^{(k)}| \tag{61}$$

where M_n is a matrix of order m, and $\|M_n\|_1 \leq C(h_n + p_n^d h_n^s)$. Clearly, if $\|M_n\|_1 \leq 1/2$ then $\det V_1^2 > 0$. To apply Theorem 4 we introduce $A = |U_n|^{-1/2} I$. According to (60) we can use the second proposition of Theorem 4 with $\delta = 1$. The next Lemma gives the estimates of matrix expressions occurring in (42).

Lemma 12. *Assume that for M_n in* (61)

$$\|M_n\|_1 \leq 1/2 \text{ and } q_n/p_n \leq 1/2. \tag{62}$$

Then

$$\|V_1^{-1}\|_1 \leq C|U_n|^{-1/2}, \tag{63}$$

$$\|AV_1 - I\|_1^2 \leq C(q_n/p_n + h_n + p_n^d h_n^s)^2, \tag{64}$$

$$\|V_0^2\|_1 \leq C|U_n|(q_n/p_n + p_n^{d-1} q_n h_n^s + (q_n/p_n) h_n^{-s-2} u(q_n)). \tag{65}$$

Proof. To establish (63), (64) we need the simple fact that for every real square matrix T with $\|T\|_1 \leq 1/2$

$$
\begin{aligned}
&\|(I+T)^{-1/2}-I\|_1 \le 2^{1/2}\|T\|_1,\\
&\|(I+T)^{1/2}-I\|_1 \le 2^{-1/2}\|T\|_1.
\end{aligned}
\tag{66}
$$

We shall also use (61) and the estimate $|U_n^{(0)}| \le N p_n^{d-1} q_n \le |U_n| q_n/p_n$. Clearly

$$
V_1^{-1} = (|U_n| - |U_n^{(0)}|)^{-1/2}(I+M_n)^{-1/2} = |U_n|^{-1/2}\Big(1-\frac{|U_n^{(0)}|}{|U_n|}\Big)^{-1/2}(I+M_n)^{-1/2},
$$

$$
AV_1 - I = \Big(1-\frac{|U_n^{(0)}|}{|U_n|}\Big)^{1/2}(I+M_n)^{1/2} - I. \tag{67}
$$

Thus in view of (66), we derive (63) and (64) from (67).

Consider now V_0^2. At first, we have

$$
\|V_0^2\|_1 = \|Var(\sum_{\alpha\ne\mathbf{1}}\sum_{j\in\Lambda_n^\alpha}\overline{X}^{(\alpha)}(n))\|_1 \le (2^d-1)m^{3/2}\sum_{\alpha\ne\mathbf{1}}\|Var(\sum_{j\in\Lambda_n^\alpha}\overline{X}^{(\alpha)}(n))\|_1.
$$

For fixed $\alpha\in\Gamma, \alpha\ne\mathbf{1}$,

$$
\begin{aligned}
\|Var(\sum_{j\in\Lambda_n^\alpha}\overline{X}_j^{(\alpha)}(n))\|_1 &\le \|\sum_{j\in\Lambda_n^\alpha}Var(\overline{X}_j^{(\alpha)}(n))\|_1\\
&+ \sum_{\substack{j,q\in\Lambda_n^\alpha\\ j\ne q}}\sum_{r,t=1}^{m}|cov(\overline{X}_{jr}^{(\alpha)}(n),\overline{X}_{qt}^{(\alpha)}(n))|.
\end{aligned}
$$

Since $|\Pi_j^\alpha| \le p_n^{d-1}q_n$, Lemma 10 provides (analogously to the proof of Lemma 11) the bound

$$
\|Var(\overline{X}_j^{(\alpha)}(n))\|_1 \le p_n^{d-1}q_n(1+C(h_n+p_n^{d-1}q_nh_n^s))
$$

and $p_n^{d-1}q_n|\Lambda_n^\alpha| \le |U_n|q_n/p_n$ follows from the fact that $|\Lambda_n^\alpha|\le N$. Consequently,

$$
\|\sum_{j\in\Lambda_n^\alpha}Var(\overline{X}_j^{(\alpha)}(n))\|_1 \le |U_n|(q_n/p_n)(1+C(h_n+p_n^{d-1}q_nh_n^s)).
$$

Applying Lemma 9 we conclude that for $j\in\Lambda_n^\alpha,\ \ r,t=1,\dots,m$

$$
\sum_{q\in\Lambda_n^\alpha,\, j\ne q}|cov(\overline{X}_{jr}^{(\alpha)}(n),\overline{X}_{qt}^{(\alpha)}(n))| \le \theta_1(\overline{X}^{\alpha}(n),\Lambda_n^\alpha)
$$

$$\leq \quad Cp_n^{d-1} q_n h_n^{-s-2} u(q_n) \leq Cp_n^d (q_n/p_n) h_n^{-s-2} u(q_n).$$

The assertion of Lemma 12 is now evident.

Let us return now to the proof of Theorem 5. From (44), using (50), (60), (63) – (65), we find that, if (62) holds, then for every $\gamma \in (0, \gamma_0(m)]$

$$\begin{aligned} &\sup_{B \in \mathcal{C}_m} |\mathsf{P}(L(n) \in B) - \mathsf{P}(Z \in B)| \leq C(\gamma + \gamma^{-2}(h_n^{-s-2} q_n^{-\lambda} \\ &+ |U_n|^{-1/2} h_n^{-s/2} p_n^{2d} + q_n/p_n + h_n^2 + p_n^{2d} h_n^{2s})). \end{aligned} \tag{68}$$

Let $\tau, \beta, \eta, \mu > 0$ be such positive numbers that $\lambda\tau - (s+2)\beta - \mu > 0$, $\beta - \mu > 0$, $1/2 - s\beta/2 - 2d\eta - 2\mu > 0$, $\eta - \tau - \mu > 0$, $s\beta - d\eta - \mu > 0$. For $n \in \mathbb{N}$ we take $\gamma = \gamma_n = \gamma_0(m)|U_n|^{-\mu}, h_n = |U_n|^{-\beta}, p_n = [|U_n|^\eta], q_n = [|U_n|^\tau]/2$, $[\cdot]$ meaning the integer part of a number. Optimization of (τ, β, η, μ) in (68) leads to (55) for all n such that $|U_n|$ is large enough to guarantee the condition (62). For other n the estimate (55) is obvious. The proof of Theorem 5 is complete.

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