HOW VIOLENT ARE FAST CONTROLS FOR SCHRÖDINGER AND PLATE VIBRATIONS ?

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Abstract. Given a time $T > 0$ and a region Ω on a compact Riemannian manifold M, we consider the best constant, denoted $C_{T,\Omega}$, in the observation inequality for the Schrödinger evolution group of the Laplacian Δ with Dirichlet boundary condition: $\forall f \in L^2(M), \quad \|f\|_{L^2(M)} \leq C_{T,\Omega}\|e^{it\Delta}f\|_{L^2((0,T)\times\Omega)}$. We investigate the influence of the geometry of Ω on the growth of $C_{T,\Omega}$ as T tends to 0.

By duality, $C_{T,\Omega}$ is also the controllability cost of the free Schrödinger equation on M with Dirichlet boundary condition in time T by interior controls on Ω. It relates to hinged vibrating plates as well. We analyze the effects of wavelengths which are greater and lower than the control time T separately. We emphasize a tool of wider scope: *the control transmutation method*.

We prove that $C_{T,\Omega}$ grows at least like $\exp(d^2/8T)$, where d is the largest distance of a point in M from Ω, and at most like $\exp(\alpha_* L_\Omega^2/T)$, where L_Ω is the length of the longest generalized geodesic in M which does not intersect Ω, and $\alpha_* \in]0,4[$ is the best constant in the following inequality for the Schrödinger equation on the segment $[0,L]$ observed from the left end: $\exists C > 0, \forall f \in D(A), \|f\|_{H^1} \leq C\exp(\alpha_* L^2/T)\|\partial_x e^{itA} f_{\rceil x=0}\|_{L^2(0,T)}$, where A is the operator ∂_x^2 with domain $D(A) = \{f \in H^2(0,L) \,|\, Bf(0) = 0 = f(L)\}$ and the inequality holds with $B = 1$ and with $B = \partial_x$. We also deduce such upper bounds on product manifolds for some control regions which are not intersected by all geodesics.

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1. Introduction

1.1. **The problem.** Throughout the paper, (M, g) is a smooth connected compact n-dimensional Riemannian manifold with metric g and smooth boundary ∂M. When $\partial M \neq \emptyset$, M denotes the interior and $\overline{M} = M \cup \partial M$. Let dist : $\overline{M}^2 \to \mathbb{R}_+$ denote the distance function. Let Δ denote the (negative) Dirichlet Laplacian on $L^2(M)$ with domain $H_0^1(M) \cap H^2(M)$. Let $t \mapsto e^{-it\Delta}$ denote the Schrödinger unitary group on $L^2(M)$. The notation $\Omega \Subset M$ means that Ω is an open set of M such that $\overline{\Omega} \subset M$.

Definition **1.** For any $T > 0$ and $\Omega \subset M$, the *controllability cost* from Ω in time T for the Schrödinger equation on M (with Dirichlet boundary condition if $\partial M \neq \emptyset$) is the best constant, denoted $C_{T,\Omega}$, in the observation inequality:

$$\forall u_0 \in L^2(M), \quad \|u_0\|_{L^2(M)} \leq C_{T,\Omega} \|e^{-it\Delta} u_0\|_{L^2(]0,T[\times\Omega)} \ . \tag{1}$$

This observation inequality is a global and quantitative version of unique continuation from the domain $]0, T[\times\Omega$. Let $\mathbf{1}_{]0,T[\times\Omega}$ denote the characteristic function of this space-time control region. By duality (cf. [DR77]), the observation inequality (1) is equivalent to the *exact controllability* of the free Schrödinger equation with Dirichlet boundary conditions in time T by interior controls on Ω, i.e. for all u_0 and u_T in $L^2(M)$ there is a control function $g \in L^2(\mathbb{R} \times M)$ such that the solution $u \in C^0([0,\infty); L^2(M))$ (which can be defined by transposition) of:

$$i\partial_t u - \Delta u = \mathbf{1}_{]0,T[\times\Omega}\, g \quad \text{in }]0, T[\times M, \quad u = 0 \quad \text{on }]0, T[\times \partial M, \tag{2}$$

with Cauchy data $u = u_0$ at $t = 0$, satisfies $u = u_T$ at $t = T$. Moreover, $C_{T,\Omega}$ is also the best constant in the estimate:

$$\|g\|_{L^2(\mathbb{R}\times M)} \leq C_{T,\Omega} \|u_0 - e^{iT\Delta} u_T\|_{L^2(M)}$$

for all data u_0, u_T, and all control g solving this controllability problem.

This paper investigates the influence of the geometry of the control region Ω on the growth of the controllability cost $C_{T,\Omega}$ for the Schrödinger equation as the control time T tends to zero. Fast controls of plate vibrations behave similarly since $\partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta)$ (precise statements for plates can be deduced straightforwardly from our Schrödinger results as in section 5 of [Leb92]).

1.2. **Main results.** In subsection 2.1, we deduce a finer statement of the following theorem (cf. theorem 2.1) from a Gaussian estimate on the heat evolution for complex times (cf. proposition 2.2):

Theorem 1.1. *The controllability cost of the Schrödinger equation on M from a nonempty subset Ω in short times (cf. definition 1) satisfies the following geometric lower bound:*

$$\liminf_{T\to 0} T \ln C_{T,\Omega} \geq \sup_{y\in M} \operatorname{dist}(y, \overline{\Omega})^2/8 \tag{3}$$

Our second result concerns the most simple Schrödinger controllability problem: the Schrödinger equation on a segment controlled at the left end through a Dirichlet condition. Its generalization to Sturm-Liouville operators (cf. theorem 4.1) is proved in section 4 by the analysis of nonharmonic Fourier series. This result is an upper bound of the same type as the lower bound in theorem 1.1, except that the rate 1/8 is replaced by the technical rate (resulting from lemma 4.5):

$$\alpha_* = 4\left(\frac{36}{37}\right)^2 < 4 \ . \tag{4}$$

In its statement below, the notations for Sobolev spaces on the segment $[0, L]$ are:

$$H^1_1(0, L) = \{f \in H^1(0, L) \,|\, f(L) = 0\} \text{ and } H^1_0(0, L) = \{f \in H^1_1(0, L) \,|\, f(0) = 0\} \ .$$

Theorem 1.2. *For any $\alpha > \alpha_*$ defined by (4), there exists $C > 0$ such that, for all $k \in \{0, 1\}$, $L > 0$, $T \in \left]0, \inf(\pi, L)^2\right]$ and $u_0 \in H^1_k(0, L)$ the solution $u \in C^0([0,\infty); H^1_k(0, L))$ of the following Schrödinger equation on $[0, L]$:*

$$i\partial_t u - \partial_s^2 u = 0 \quad \text{in }]0, T[\times]0, L[\ , \quad \partial_s^k u_{\rceil s=0} = 0 = u_{\rceil s=L}\ , \quad u_{\rceil t=0} = u_0\ ,$$

satisfies $\|u_0\|_{H^1} \leq C \exp(\alpha L^2/T)\|\partial_s u_{\rceil s=0}\|_{L^2(0,T)}$.

Our third result, proved in section 5, is an upper bound which is finite only under the *geodesics condition*[1] of C. Bardos, G. Lebeau and J. Rauch, a.k.a. the geometric optics condition (cf. [BLR92]). It is an application of the broader *control transmutation method* (cf. sections 1.3.3 and 5). Here, it consists in writing the control g for the Schrödinger equation as a time integral operator applied to a control f of the wave equation, i.e. $g(t, x) = \int_{\mathbb{R}} v(t, s)f(s, x)\, ds$, where f depends on Ω (not on T) and the compactly supported L^2 kernel v depends on T and L_Ω.

Theorem 1.3. *Let $\Omega \Subset M$ and let L_Ω be the length of the longest generalized geodesic in $\overline{M}$ which does not intersect Ω. If theorem 1.2 holds for some rate α_* then the controllability cost of the Schrödinger equation from Ω in short times (cf. definition 1) satisfies the following geometric upper bound:*

$$\limsup_{T\to 0} T \ln C_{T,\Omega} \leq \alpha_* L^2_\Omega \tag{5}$$

Our last result is that the geodesics condition is not necessary for the controllability cost to grow at most like $\exp(C/T)$ as T tends to 0. In section 6, a remark on the cost in an abstract tensor product setting allows us to deduce from theorem 1.2 and 1.3 similar bounds in some settings violating the geodesics condition: the boundary controllability of cylinders from one end (cf. theorem 6.5) and the following semi-internal controllability on product manifolds (cf. theorem 6.3 for the more abstract form).

Theorem 1.4. *Let $\tilde{M}$ be a smooth complete $\tilde{n}$-dimensional Riemannian manifold and $\tilde{\Delta}$ denote the Laplacian on $L^2(\tilde{M})$ with domain $\{u \in H^1_0(\tilde{M}) \,|\, \tilde{\Delta}u \in L^2(\tilde{M})\}$. For all $T > 0$ and all $\Omega \Subset M$, the controllability cost $C_{T,\omega}$ of the Schrödinger unitary group $t \mapsto e^{-it(\Delta+\tilde{\Delta})}$ on $L^2(M \times \tilde{M})$ from $\omega = \Omega \times \tilde{M}$ in time T is the controllability cost $C_{T,\Omega}$ of $t \mapsto e^{-it\Delta}$ on $L^2(M)$ from Ω in time T (cf. definition 1). In particular, with α_* and L_Ω as in theorem 1.3:* $\limsup_{T\to 0} T \ln C_{T,\omega} \leq \alpha_* L^2_\Omega$.

1.3. **Background.**

[1] In this context, this condition says that all generalized geodesics in $\overline{M}$ intersect the control region Ω (i.e. $L_\Omega < +\infty$ in theorem 1.3). The *generalized geodesics* are continuous trajectories $t \mapsto x(t)$ in $\overline{M}$ which follow geodesic curves at unit speed in M (so that on these intervals $t \mapsto \dot{x}(t)$ is continuous); if they hit ∂M transversely at time t_0, then they reflect as light rays or billiard balls (and $t \mapsto \dot{x}(t)$ is discontinuous at t_0); if they hit ∂M tangentially then either there exists a geodesic in M which continues $t \mapsto (x(t), \dot{x}(t))$ continuously and they branch onto it, or there is no such geodesic curve in M and then they glide at unit speed along the geodesic of ∂M which continues $t \mapsto (x(t), \dot{x}(t))$ continuously until they may branch onto a geodesic in M.

1.3.1. *Controllability for the Schrödinger equation (and the plate equation).* We survey from the geometric point of view the results on the exact controllability of the linear Schrödinger equation in any positive time, without discriminating boundary/interior observability/controllability for the Schrödinger/plate equation. In this respect, the main result (proved by Lebeau in [Leb92]) is that the geodesics condition is sufficient for boundary controllability on a smooth domain of $\mathbb{R}^n$ with a Riemannian metric (cf. [BZ03] for an alternative proof by resolvent estimates). The same strategy applies to interior controllability (cf. the revisited proof in section 3) and the control transmutation method yields yet another proof (cf. theorem 1.3).

Further information on this condition is obtained from the harmonic analysis of several examples (some of them can be generalized and deduced directly from Lebeau's result, cf. section 6). The geodesics condition is not necessary for boundary controllability on a rectangle (cf. [KLS85]) and more generally on cylinders (cf. theorem 6.5), nor for interior controllability on a parallelepiped (cf. [Har89], [Jaf90], [Kom92] in increasing generality), on a torus (by the same proof), and more generally on a product manifolds (cf. theorem 6.3). It is necessary for controllability on the sphere (cf. [Kom92]) except when the control region is an open hemisphere (controllability holds in this case notwithstanding theorem 4.2 of [Kom92]).

Burq also proved a controllability result (for a slightly more regular space of initial data) in the case of convex obstacles where the geodesics condition only fails for some hyperbolic trajectories of the geodesic flow. Allibert studied the boundary control of revolution surfaces when the geodesics condition only fails for a single elliptic trajectory of the geodesic flow: it can be checked that controllability in the natural spaces does not hold (cf. section 2.1 in [All98]). Recently, Burq and Zworski proved in [BZ03] that controllability results for the classical and semiclassical Schrödinger equation can be deduced from resolvent estimates, and give a striking application to the ergodic Bunimovich stadium.

Other results assume geometric conditions which are more restrictive than the geodesics condition of Lebeau (and mostly stick to the Euclidean setting) but require less smoothness than microlocal techniques: they aim at more explicit estimates, nonlinear equations and inverse problems. The radial multiplier was used in [Zua88], [Mac94], [Fab92] and [LT92a]. Carleman estimates are found in [Tat97], [TY99] (Riemannian setting), [Zha01], [LTZ03], [BP02]. Another approach based on local smoothing properties is sketched in [LT92b] and [HL96].

1.3.2. *Controllability cost.* The study of the controllability cost in short times was initiated by Seidman. His first result in [Sei84] concerned the heat equation (see [Mil03] for improvements and other references). For many equations, the controllability on a segment $[0, L]$ from one end can be formulated of as a window problem for series of complex exponentials as in section 4.1 (note that in this case $2L$ is the length of the longest generalized geodesic in $[0, L]$ which does not intersect one of the ends). In [Sei86], Seidman solved the window problem for purely imaginary exponentials corresponding to the Schrödinger equation (he applied it to the plate equation in [KLS85]) therefore proving that in this setting the controllability cost grows at most like $\exp(2(3\pi)^2\beta_* L^2/T)$ where $\beta_* \approx 4.17$ (or rather like $\exp(2\pi^2\beta_* L^2/T)$ if the sketchy remark 1 in section 4 works out). Theorem 1.2 improves on the constant appearing in this bound. An example of Korevaar included in [Sei86] also proves that in this case the controllability cost grows at least like $\exp(L^2/8T)$ (a computational slip in [Sei86] leads to $\exp(L^2/4T)$), which is the exact analogue in this case of our lower bound in theorem 1.1. Seidman and his collaborators later treated the case of finite dimensional linear systems in [Sei88] and [SY96], and generalized the window problem to a larger class of complex exponentials in [SG93] and [SAI00].

Phung's paper [Phu01] prompted our attention to the subject. His theorem 2.3 proves that, under the geodesics condition, the cost of controlling data in $H_0^1(M)$ (one derivative more regular than in theorem 1.3) grows at most as $\exp(C/T^2)$ as T tends to 0 (one power of T more than in theorem 1.3 and no estimate on C). Indeed, his one dimensional theorem 2.2 fell already short of the optimal power of T. It can be checked that the usual Ingham theorem of harmonic analysis (for high frequencies) and the trick introduced by Haraux in [Har89] (for the remaining low frequencies) yield the better (but still short of the optimal dependence in T) upper bound $\exp(C/(-T\ln T))$ (this is the approach followed in [JM01] for the wave equation).

1.3.3. *Transmutation.* The strategy used by Phung to prove theorem 2.3 in [Phu01], is what we have coined the *transmutation control method.* Phung was inspired by [BdM75] and [KS96] where the Schrödinger semigroup on the whole space is written as an integral over the wave group. In fact, the method of transmutation applies between other kinds of equations (cf. [Her75] for a survey), Kannai's formula being probability the best known example (cf. [Mil03] for the corresponding application to heat control).

The most inspiring paper for both our lower and upper bound was [CGT82] which deduces geometric estimates on functions of the Laplace operators from the finite propagation speed of the even homogeneous wave group $W : s \mapsto \cos\left(s\sqrt{-\Delta}\right)$, defined by: $w(s,x) = W(t)w_0(x)$ solves $\partial_s^2 w - \Delta w = 0$ in $\mathbb{R}\times M$ and $w = 0$ on $\mathbb{R}\times\partial M$, with Cauchy data $(w, \partial_s w) = (w_0, 0)$ at $s = 0$. It builds on the following transmutation formula which results from applying a spectral theorem to the Fourier inversion formula for an even function F:

$$F\left(\sqrt{-\Delta}\right) = \int_{-\infty}^{+\infty} \hat{F}(s)W(s)\frac{ds}{2\pi} \text{ , where } \hat{F}(s) = \int_{-\infty}^{+\infty} F(\sigma)\cos(s\sigma)d\sigma \ . \tag{6}$$

When this formula is applied to evolution semigroups (like the Schrödinger group $t \mapsto e^{it\Delta}$), $F(\sigma) = \exp(tG(\sigma))$ where $t \geq 0$ is a time parameter and $\hat{F}$ is a fundamental solution on the line ($\partial_t \hat{F} = \hat{G} *_s \hat{F}$ and $\hat{F} = \delta$ at $t = 0$). The *transmutation control method* consists in replacing this $\hat{F}$ by some *fundamental controlled solution* on the segment $[-L, L]$ controlled at both ends. We use the one dimensional theorem 1.2 to construct this fundamental controlled solution in subsection 5.1. (Phung used a fundamental solution on the whole line controlled outside $[-L, L]$, but it seems harder to optimize α_* in theorem 1.2 for interior control.)

1.4. **The High/Low Frequencies issue.** Throughout the paper, $(\omega_j)_{j\in\mathbb{N}^*}$ is a nondecreasing sequence of nonnegative real numbers and $(e_j)_{j\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(M)$ such that e_j is an eigenvector of $-\Delta$ with eigenvalue ω_j^2, i.e.:

$$-\Delta e_j = \omega_j^2 e_j \quad \text{and} \quad e_j = 0 \text{ on } \partial M \ . \tag{7}$$

The closed linear span of the vector set $\{e_j\}_{j\in J}$ is denoted by $\mathrm{Vect}\,\{\mathrm{e_j}\}_{j\in J}$.

The spectral parameter ω_j can be considered as the frequency of the mode e_j. For any given threshold $\mu > 0$, the space of initial data can be decomposed into $L^2(M) = \mathrm{Vect}\,\{\mathrm{e_j}\}_{\omega_j\leq\mu} \oplus \mathrm{Vect}\,\{\mathrm{e_j}\}_{\omega_j>\mu}$ and this decomposition is invariant under the Schrödinger group $t \mapsto e^{-it\Delta}$. The relevant notion of low (respectively high) frequencies in this paper correspond to wavelengths that are greater (respectively lower) than the order of the control time, i.e. to $\mu \sim d/T$.

Besides the main results already stated, the separate analysis of low and high frequencies presented in sections 2 and 3 give further insight into our initial problem. The cost of controlling low frequencies always grows like $\exp(C/T)$ as $T \to 0$. Under the geodesics condition $L_\Omega < +\infty$, high frequencies are controlled at the much lower cost $C/\sqrt{T}$. Though the upper bounds for low and high frequencies obtained respectively in subsections 2.2 and 3.2 lead to conjecture the finiteness of $\limsup_{T\to 0} T\ln C_{T,\Omega}$ under the geodesics condition, we emphasize that they do not help in proving it: section 5 builds on section 4 but not on sections 2 and 3.

Our high/low frequencies analysis leaves the following problem open: can the controllability of the Schrödinger equation hold at a cost growing faster than $\exp(C/T)$ as T tends to 0 ? In others terms: are there M and $\Omega\subset M$ such that $C_{T,\Omega}<+\infty$ for all $T>0$ and $\liminf_{T\to 0} T\ln C_{T,\Omega}=+\infty$? (n.b. theorem 1.3 proves that violating the geodesics condition is necessary, i.e. Ω must satisfy $L_\Omega=+\infty$.) A positive answer would lead to the investigation of geometric conditions ensuring this ultra-violent behavior (the examples of section 6 prove that violating the geodesics condition is not sufficient).

2. Low frequencies

In this section, we analyze how violent fast controls are for low frequency vibrations (cf. section 1.4).

2.1. Lower bound.

The purpose of this subsection is to prove the following refined version of theorem 1.1:

Theorem 2.1. *For all $\Omega\subset M$ and $d\in]0,\sup_{y\in M}\operatorname{dist}(y,\overline{\Omega})[$:*

$$\liminf_{T\to 0} T\ln \sup_{u_0\in\dot{E}_{d/T}} \frac{\|u_0\|_{L^2(M)}}{\|e^{it\Delta}u_0\|_{L^2(]0,T[\times\Omega)}} \geq \frac{d^2}{8}, \text{ where } \dot{E}_{d/T}=\operatorname*{Vect}_{2\omega_j\leq d/T}\{\mathrm{e_j}\}\setminus\{0\} .$$

This lower bound follows from the construction of a very localized solution of the Schrödinger equation with a large but finite number of modes. For a short control time $T>0$, we consider a Dirac mass as far from Ω as possible, we smooth it out by applying the heat semigroup for a time T and truncate frequencies larger than $d/(2T)$, and finally we take it as initial data in the Schrödinger equation. The proof is similar to the proof of theorem 2.1 in [Mil03] where the main ingredient was Varadhan's formula for the heat kernel in small time. As a substitute here we prove:

Proposition 2.2. *$\forall\varepsilon>0$, $\exists C_\varepsilon>0$, $\forall d>0$, $\forall\Omega\subset M$, $\forall y\in M$ such that $\operatorname{dist}(y,\overline{\Omega})>d+\varepsilon$, $\forall z\in\mathbb{C}$ such that $\operatorname{Re} z>0$:*

$$\|e^{z\Delta}\delta_y\|_{L^2(\Omega)}\leq C_\varepsilon\left(1+(\operatorname{Re} z)^{-N}\left(1+\frac{d^2}{2|z|}\right)^N\right)\exp\left(-\frac{d^2\operatorname{Re} z}{4|z|^2}\right) .$$

Proof. Our proof builds on the finite propagation speed and the boundedness on L^2 of the even homogeneous wave group $W: s\mapsto\cos\left(s\sqrt{-\Delta}\right)$ through the transmutation formula (6). Since $\operatorname{Re} z>0$ implies $e^{z\Delta}\delta_y\in\cap_{k\in\mathbb{N}}\mathrm{D}(\Delta^k)\subset C^\infty(\overline{M})$ and $2N>n/2$ implies $\partial_y\in H^{-2N}_{\mathrm{comp}}(M)\subset\mathrm{D}(\Delta^N)'$ and therefore $W(s)\partial_y\in\mathrm{D}(\Delta^N)'\subset\mathcal{D}'(M)$, the following version of (6) for $F(\sigma)=\exp(z\sigma^2)$ makes sense for all $\varphi\in\mathrm{D}(\Delta^N)$:

$$(8)\qquad \left(e^{z\Delta}\partial_y,\varphi\right)_{L^2(M)}=\int_{-\infty}^{+\infty}e^{-s^2/(4z)}f(s)\frac{ds}{\sqrt{4\pi z}} \text{ , where } f(s)=\langle W(s)\partial_y,\bar{\varphi}\rangle .$$

As usual $\mathrm{D}(\Delta^N)$ denotes the domain of the operator Δ^N, and $\mathrm{D}(\Delta^N)'$ denotes it dual space with respect to the duality product $\langle\cdot,\cdot\rangle$. between distributions $\mathcal{D}'(M)$ and test functions $\mathcal{D}(M)=C^\infty_{\mathrm{comp}}(M)$.

To prove the estimate in the proposition, we may assume $\varphi\in\mathcal{D}(\Omega)$ since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$. We deduce $f\in C^\infty(\mathbb{R})$ with $\operatorname{supp} f\subset\left\{s\in\mathbb{R}|\,|s|\geq\operatorname{dist}(y,\overline{\Omega})\right\}$ since $\partial_s^2 W\partial_y=\Delta W\partial(y)$ and $\operatorname{supp} W(s)\partial_y\subset\{x\in M|\operatorname{dist}(x,y)\leq|s|\}$ (propagation at unit speed). Moreover $f(s)=\langle(1-\Delta)^N W(s)(1-\Delta)^{-N}\partial_y,\varphi\rangle=(1-$

$\partial_s^2)^N g(s)$ where $g(s) = \langle W(s)(1-\Delta)^{-N}\partial_y, \bar{\varphi}\rangle$ satisfies $\|g\|_{L^\infty} \le \|(1-\Delta)^{-N}\partial_y\|_{L^2}\|\varphi\|_{L^2}$ since $W(s)$ is bounded on L^2. Therefore, we may integrate by parts and obtain:

$$\begin{aligned}\int e^{-s^2/(4z)} f(s)ds &= \int e^{-s^2/(4z)}(1-\partial_s^2)^N g(s)ds \\ &= \int g(s)(1-\partial_s^2)^N \left(\chi_\varepsilon(s)e^{-s^2/(4z)}\right) ds\end{aligned} \tag{9}$$

where $\chi_\varepsilon(s) = \chi(\frac{s-d}{\varepsilon})$ is a smooth non-negative cut-off function satisfying $\chi(s) = 1$ on $|s| \ge d+\varepsilon$ and $\chi(s) = 0$ on $|s| \le d$. From the simple estimate: $\forall k \in \mathbb{N}$, $\exists C_k > 0$,

$$\forall z \in \mathbb{C}, \operatorname{Re} z > 0 : \ \left|\partial_s^{2k} e^{-s^2/(4z)}\right| \le \frac{C_k}{|z|^k}\left(1 + \frac{s^2}{4|z|}\right)^k \exp\left(-\frac{s^2 \operatorname{Re} z}{4|z|^2}\right) ,$$

we deduce, setting $\tau = d\sqrt{\operatorname{Re} z}/(2|z|)$:

$$\begin{aligned}\left|\int_d^{+\infty} \partial_s^{2k} e^{-s^2/(4z)} ds\right| &\le \frac{C_k}{|z|^k}\int_\tau^{+\infty}\left(1 + s^2\frac{|z|}{\operatorname{Re} z}\right)^k e^{-s^2}ds \\ &= \frac{C_k e^{-\tau^2}}{|z|^k}\int_0^{+\infty}\left(1 + (s+\tau)^2\frac{|z|}{\operatorname{Re} z}\right)^k e^{-s^2}e^{-2s\tau}ds \\ &\le \frac{C_k e^{-\tau^2}}{|z|^k}\left(1 + \frac{2\tau^2|z|}{\operatorname{Re} z}\right)^k \frac{|z|^k}{(\operatorname{Re} z)^k}\int_0^{+\infty}(1+2s^2)^{2k}e^{-s^2}ds \\ &= C_k'(\operatorname{Re} z)^{-k}\left(1 + \frac{d^2}{2|z|}\right)^k \exp\left(-\frac{d^2 \operatorname{Re} z}{4|z|^2}\right) .\end{aligned} \tag{10}$$

Equations (8), (9) and (10) imply the estimate in proposition 2.2 with a C_ε which only depends on C_k' and on ε through $\sup_{k\le N}\|\partial^k\chi_\varepsilon\|_{L^\infty}$. □

Proof of theorem 2.1. We shall use Weyl's asymptotics for eigenvalues:

$$\exists W > 0, \ \#\{j \in \mathbb{N}^* \,|\, \omega_j \le \omega\} \le W\omega^n \tag{11}$$

and the following consequence of Sobolev's embedding theorem:

$$\exists E > 0, \forall j \in \mathbb{N}^*, \ \|e_j\|_{L^\infty} \le E\omega_j^{n/2} \tag{12}$$

(cf. section 17.5 in [Hör85] for example). The unique continuation property for elliptic operators implies that $Y = \{y \in M \setminus \overline{\Omega} \,|\, e_1(y) \ne 0\}$ is an open dense set in $M \setminus \overline{\Omega}$, so that the supremun in theorem 2.1 can be taken over $y \in Y$ instead of $y \in M$.

Let $y \in Y$ and $D < d < \operatorname{dist}(y, \overline{\Omega})$ be fixed from now on. Applying proposition 2.2 with $\varepsilon = d(y, \overline{\Omega}) - d$ and $z = T - it$ yields a positive constant B such that: $\forall T > 0$, $\forall t \in]0, T]$,

$$\|e^{(T-it)\Delta}\delta_y\|_{L^2(\Omega)} \le C_\varepsilon\left(1 + \frac{1}{T^N}\left(1 + \frac{d^2}{2T}\right)^N\right)e^{-d^2/(8T)} \le Be^{-D^2/(8T)} . \tag{13}$$

Therefore, for all $T > 0$, we take as initial data the following finite modes approximation of $e^{T\Delta}\delta_y$: $u_0^T(x) = \sum_{2T\omega_j\le d}\exp(-T\omega_j^2)e_j(y)e_j(x)$, and we are left with comparing $e^{(T-it)\Delta}\delta_y$ to the corresponding solution

$$u^T(t,x) = \left(e^{-it\Delta}u_0^T\right)(x) = \sum_{2T\omega_j\le d}\exp((it-T)\omega_j^2)e_j(y)e_j(x) .$$

Using the unitarity of the Schrödinger group on $L^2(M)$, Parseval's identity and (12), we obtain

$$\sup_{t\in]0,T]} \|e^{(T-it)\Delta}\delta_y - u^T(t,x)\|_{L^2(M)} \le \|e^{(T-iT)\Delta}\delta_y - u_0^T(x)\|_{L^2(M)}$$
$$= \sum_{2T\omega_j>d} |e^{-T\omega_j^2}e_j(y)|^2 \le E \sum_{2T\omega_j\ge d} e^{-d\omega_j/2}\omega_j^n \le E' \sum_{2T\omega_j\ge d} e^{-D\omega_j/2} ,$$

for some $E'>0$. But, Weyl's law (11) yields, for $c\ge c_0>0$ and $\gamma\ge\gamma_0>0$,

$$\sum_{\omega_j\ge c} e^{-\gamma\omega_j} = \sum_{k\in\mathbb{N}^*}\sum_{kc\le\omega_j<(k+1)c} e^{-\gamma\omega_j} \le W \sum_{k\in\mathbb{N}^*} ((k+1)c)^n e^{-kc\gamma}$$
$$\le W_{\gamma_0}\sum_{k\in\mathbb{N}^*} e^{-kc\gamma}e^{(k+1)c\gamma/4} = W_{\gamma_0}e^{-c\gamma/2}\sum_{k\in\mathbb{N}} e^{-3kc\gamma/4} \le W_{c_0,\gamma_0}e^{-c\gamma/2}$$

where W_{γ_0} and W_{c_0,γ_0} are positive real numbers which depend on their indexes but not on c and γ. Hence, with $c=d(2T)^{-1}>d/2=c_0$ and $\gamma=\gamma_0=D/2$, we obtain:

$$\exists B'>0,\ \forall t\in]0,T]\ \|e^{(T-it)\Delta}\delta_y - u^T(t,x)\|_{L^2(M)} \le B'e^{-dD/(8T)}$$

Together with (13), this estimate yields, setting $B''=B+B'$, for all $T>0$:

$$\|u^T\|_{L^2(]0,T[\times\Omega)} \le \sqrt{T}Be^{-D^2/(8T)} + \sqrt{T}B'e^{-dD/(8T)} \le \sqrt{T}B''e^{-D^2/(8T)} .$$

But using Parseval's identity and $y\in Y$, we have for all $T\in]0,1]$:

$$\|u_0^T\|_{L^2(M)} = \left(\sum_{2T\omega_j\le d} |e^{-T\omega_j^2}e_j(y)|^2\right)^{1/2} \ge e^{-\omega_1^2}|e_1(y)|>0 .$$

Hence, with $A=e^{-\omega_1^2}|e_1(y)|B''$ independent of T, we have

$$\forall T\in]0,1],\ \|u^T\|_{L^2(]0,T[\times\Omega)} \le A\sqrt{T}e^{-D^2/(8T)}\|u_0^T\|_{L^2(M)} .$$

Since $u_0^T\in\dot{E}_{d/T}$ and $D<d$ is arbitrary, this ends the proof of theorem 2.1. □

Remarks 2.3. In the case $M=S^1$ (the unit circle), the transmutation formula (6) is essentially the Poisson summation formula. In this sense, our construction is an extension of Korevaar's one dimensional example in [Sei86].

Following [CGT82], we could also prove point wise Gaussian estimates of the Heat kernel for complex times. Proposition 2.2 is a short path to the estimate required by our construction.

For $z=h+ith$, this proposition is an analogue on the compact manifold M of the localization estimate satisfied by the solution of the semiclassical Schrödinger equation $ih\partial_t u - h^2\Delta u=0$ in $\mathbb{R}^n$ with initial data $u_0(x)=\exp(-(x-y)^2/(4h))$, i.e. a semiclassical coherent state centered at y with no momentum.

2.2. **Upper bound at low frequencies.** Carleman estimates are the most versatile tool to control low frequencies as epitomized by their application in theorem 3 in [LZ98] (also theorem 14.6 in [JL99]):

Theorem 2.4 ([LZ98],[JL99]). *For all non-empty open subset Ω of M:*

$$\exists C>0, \forall v\in\mathbb{C}^{\mathbb{N}^*}, \forall\mu>0, \quad \sum_{\omega_j\le\mu}|v_j|^2 \le Ce^{C\mu}\int_\Omega\left|\sum_{\omega_j\le\mu} v_je_j(x)\right|^2 dx .$$

Applying this observation inequality for fixed time and integrating on $[0,T]$ yields:

$$\exists C, \forall \mu > 0, \forall u_0 \in \underset{\omega_j \le \mu}{\mathrm{Vect}}\{\mathrm{e_j}\}, \quad \|u_0\|_{L^2(M)} \le \frac{C}{\sqrt{T}} e^{C\mu} \|e^{it\Delta} u_0\|_{L^2(]0,T[\times\Omega)} \ .$$

As a counterpart to theorem 2.1, taking $\mu = d/(2T)$ and dividing C by 2, we state:

Corollary 2.5. *For all non-empty open subset Ω of M: $\exists C > 0$, $\forall d > 0$,*

$$\limsup_{T\to 0} T \ln \sup_{u_0 \in \dot{E}_{d/T}} \frac{\|u_0\|_{L^2(M)}}{\|e^{it\Delta} u_0\|_{L^2(]0,T[\times\Omega)}} \le Cd, \text{ where } \dot{E}_{d/T} = \underset{2\omega_j \le d/T}{\mathrm{Vect}}\{\mathrm{e_j}\} \setminus \{0\} \ .$$

3. High frequencies

To analyze how violent fast controls are for high frequency vibrations (cf. section 1.4), we introduce a *wavelength scale* $(h_k)_{k\in\mathbb{N}}$, i.e. a decreasing sequence of positive real numbers converging to 0, and the corresponding *spectral scale* $(E_k)_{k\in\mathbb{N}}$ of subspaces of $L^2(M)$ defined by $E_k = \mathrm{Vect}\{\mathrm{e_j}\}_{a<h_k\omega_j<b}$ for fixed $b > a > 0$ (note that these subspaces may be overlapping), where the spectral data (e_j, ω_j) are defined in (7).

In this section, we take up a strategy of Lebeau in [Leb92]: we reduce the observation of Schrödinger equation on Ω in small time hT to the observation of the semiclassical Schrödinger equation $ih\partial_t\psi - h^2\Delta\psi = 0$, $\psi = \psi_0$ at $t = 0$, on Ω in fixed time $T > 0$ (note that taking $u_0 = \psi_0$, we have $u(t,x) := e^{-it\Delta}u_0 = \psi(t/h, x)$). In the first subsection, we emphasize that the first step of the reduction actually is an equivalence. As in [Bur97a], we perform the semiclassical analysis with a light microlocal tool: the "microlocal measures" introduced independently by P. Gérard, P.-L. Lions and T. Paul, and L. Tartar, and first used by G. Lebeau in control theory (cf. [Bur97b] for a survey). In the second subsection, we keep track of the controllability cost (also note that our presentation avoids estimates in the Besov space $\dot{B}^0_{2,\infty}(\mathbb{R}_t; L^2(\Omega))$ thanks to the lemma 3.7).

3.1. Semiclassical observability.

The relevant notion of observability for the semiclassical Schrödinger equation is:

Definition **2.** *Semiclassical observability* on $\Omega \Subset M$ in time $T > 0$ holds when : for all $\theta \in C^\infty_{\mathrm{comp}}(\mathbb{R}\times M)$ such that $\{\theta \ne 0\} =]0,T[\times\Omega$, for all $b > a \ge 1/2$, there is an observability constant $C_{\mathrm{sc}} > 0$ and a threshold $\hbar > 0$ such that:

$$\forall h \in]0,\hbar], \forall \psi_0 \in \underset{a<h\omega_j<b}{\mathrm{Vect}}\{\mathrm{e_j}\}, \quad \|\psi_0\|_{L^2(M)} \le C_{\mathrm{sc}} \|\theta e^{-ith\Delta}\psi_0\|_{L^2(\mathbb{R}\times M)} \ .$$

The purpose of this subsection is to prove:

Theorem 3.1. *Let $\Omega \Subset M$ and let L_Ω be the length of the longest generalized geodesic in $\overline{M}$ which does not intersect Ω. The geodesics condition $T > L_\Omega$ is necessary and sufficient for semiclassical observability on Ω in time T (cf. definition 2).*

Remarks 3.2. Note that this theorem does not hold with the smooth characteristic function θ replaced by $\mathbf{1}_{]0,T[\times\Omega}$.

If the condition $b > a \ge 1/2$ is replaced by $b > a \ge \underline{a}$ then this becomes the definition of semiclassical observability on Ω in time $2\underline{a}T$. Moreover, the condition " for all $\theta \in C^\infty_{\mathrm{comp}}(\mathbb{R}\times M)$ such that $\{\theta \ne 0\} =]0,T[\times\Omega$, for all $b > a \ge 1/2$ " could be equivalently replaced by " there is a $\theta \in C^\infty_{\mathrm{comp}}(\mathbb{R}\times M)$ such that $\{\theta \ne 0\} =]0,T[\times\Omega$, there is a $b > a = 1/2$ ".

Proof of theorem 3.1. We refer to [Bur97a] and the survey [Bur97b] for the definition and properties of semiclassical measures (a.k.a. Wigner measures) that we use in this proof.

We first prove the sufficiency by contradiction. We assume $T > L_\Omega$ and that semiclassical observability does not hold, i.e. there are real numbers $b > a \geq 2^{-1/2}$, a decreasing sequence $(h_k)_{k\in\mathbb{N}}$ of positive real numbers converging to 0, a sequence $(\psi_0^k)_{k\in\mathbb{N}}$ of initial data in $L^2(M)$ such that:
(14)

$$\forall k \in \mathbb{N}, \quad \psi_0^k \in E_k = \underset{a<h_k\omega_j<b}{\mathrm{Vect}\,\{\mathrm{e_j}\}} \quad \text{and} \quad \|\psi_0^k\|_{L^2(M)} > \frac{1}{k}\|\theta e^{-ith_k\Delta}\psi_0\|_{L^2(\mathbb{R}\times M)} \ .$$

We shall use the more convenient unambiguous abbreviations $h = h_k$, $\psi_0^h = \psi_0^k$ and $h \to 0$ instead of $k \to \infty$. Without loss of generality, we assume $\|\psi_0^h\| = 1$ so that $\psi^h(t,x) = e^{-ith\Delta}\psi_0^h$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}\times M)$, and therefore, without loss of generality again, we assume that (ψ^h) has a semiclassical measure μ. Note that $\mu(t,x,\tau,\xi)$ is a positive Radon measure on $T^*(\mathbb{R}\times M)$ which describes the asymptotic microlocal distribution of the space-time waves density $|\psi^h(t,x)|^2 dtdx$.

The estimate in (14) implies $\|\theta\psi^h\| = o(1)$ so that:

$$\mu\left(\{(t,x) \in]0,T[\times \Omega\}\right) = 0 \ . \tag{15}$$

The first part of (14) says $\psi_0^h \in \mathrm{Vect}\,\{\mathrm{e_j}\}_{a<h\omega_j<b}$ which implies $\|\Delta\psi_0^h\| \leq (b/h)^2\|\psi_0^h\|$, hence $ih\partial_t\psi^h = h^2\Delta\psi^h$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}\times M)$ and, in particular, (ψ^h) is h-oscillating. Therefore:

$$\text{for all non empty interval } I, \quad \mu\left(\{(t,x) \in I \times M\}\right) = |I| > 0 \ . \tag{16}$$

Another consequence of $\psi_0^h \in \mathrm{Vect}\,\{\mathrm{e_j}\}_{a<h\omega_j<b}$ is that $t \mapsto \psi^h$ is a linear combination of semiclassical time exponentials $t \mapsto \exp(it\tau/h)$ with $\tau \in \{-h^2\omega_j^2\}_{a<h\omega_j<b}$ so that $\mathrm{supp}\,\mu \subset \left\{\tau \in \left[a^2, b^2\right]\right\}$. From $ih\partial_t\psi^h - h^2\Delta\psi^h = 0$, it can be deduced by the symbolic calculus that $\mathrm{supp}\,\mu \subset \left\{\tau = |\xi|^2\right\}$ and $\left\{\tau - |\xi|^2, \mu\right\} = \partial_t\mu - 2|\xi|\nabla_x\mu = 0$. Together with the Dirichlet boundary condition, this equation for μ means that, on any surface $\{2|\xi| = v\}$, μ is invariant by the generalized geodesic flow at speed v. But $\mathrm{supp}\,\mu \subset \left\{\tau = |\xi|^2 \in \left[a^2, b^2\right]\right\} \subset \{v = 2|\xi| \geq 2a = 1\}$, hence (15) and the geodesics condition $T > L_\Omega$ imply $\mu = 0$, in contradiction with (16).

Now we prove the necessity by contradiction. We assume that semiclassical observability holds and $T \leq L_\Omega$, i.e. there is a generalized geodesic $x : [0,T] \to \overline{M}$ which does not intersect Ω. Without loss of generality, we assume $x(0) \notin \partial M$.

To construct initial data which concentrate on $x_0 = x(0)$ with initial momentum ξ_0 with $|\xi_0| \in]a,b[$ and with the direction corresponding to $x'(0)$, we introduce a smooth cut-off function χ compactly supported in a chart of M around x_0 such that $\chi = 1$ in a neighborhood of x_0, and define ψ_0^h as the function $x \mapsto \chi(x)\exp(ix.\xi_0/h)\exp(-(x-x_0)^2/h)$ divided by its $L^2(M)$ norm. Then the semiclassical measure of (ψ_0^h) is $\delta(x - x_0, \xi - \xi_0)$ and, thanks to proposition 4.11 in [Bur97a], we may assume without loss of generality that (ψ_0^h) has been projected on $\mathrm{Vect}\,\{\mathrm{e_j}\}_{a<h\omega_j<b}$. As before, we may assume that (ψ^h) has a semiclassical measure μ. Taking the limit $h \to 0$ in the inequality defining semiclassical observability yields

$$0 < \mu\left(\{(t,x) \in]0,T[\times \Omega\}\right) \ . \tag{17}$$

As before μ is invariant by the generalized geodesic flow at speed $v = 2|\xi_0| > 2a = 1$. We may choose ξ_0 with v close enough to 1 so that the support of μ is so close to the image of the generalized geodesic x in $T^*(\mathbb{R}\times M)$ that it does not intersect Ω, in contradiction with (17). $\square$

3.2. **Upper bound at high frequencies under the geodesics condition.** The main result proved in this subsection is that semiclassical observability implies " observability at cost $C/\sqrt{T}$ modulo low frequencies " :

Theorem 3.3. *Semiclassical observability on $\Omega \Subset M$ in time T_Ω (cf. definition 2) implies that: $\exists \underline{k} \in \mathbb{N}$, $\forall d > T_\Omega$, $\exists C_d > 0$, $\forall k \geq \underline{k}$, $\forall T \in [h_k d, h_{\underline{k}} d]$,*

$$\forall u_0 \in L^2(M),\ (1 + O(h_k^2/T))\|u_0\|^2_{L^2(M)} \leq \frac{C_d^2}{T}\|e^{-it\Delta}u_0\|^2_{L^2(]0,T[\times\Omega)} + \|\pi_k u_0\|^2_{L^2(M)} \ ,$$

where $h_k = 2^{-k}$ is the dyadic scale and π_k is the projection on $\mathrm{Vect}\,\{\mathrm{e_j}\}_{h_k\omega_j\leq 1}$.

Taking $T = h_k d$ in this theorem and combining it with theorem 3.1 allow us to state a counterpart to the upper bound at low frequencies of corollary 2.5, i.e. the following upper bound at high frequencies:

Corollary 3.4. *Let $\Omega \Subset M$ and let L_Ω be the length of the longest generalized geodesic in $\overline{M}$ which does not intersect Ω. For all $d > L_\Omega$, there is a constant $C_d > 0$ and a sequence of positive times T converging to 0 such that:*

$$\forall u_0 \in \operatorname*{Vect}_{\omega_j \geq d/T}\{\mathrm{e_j}\}, \quad \|u_0\|_{L^2(M)} \leq \frac{C_d}{\sqrt{T}}\|e^{-it\Delta}u_0\|_{L^2(]0,T[\times\Omega)} \ .$$

Remarks 3.5. As usual, since π_k is a compact operator, we can get rid of the remainder low frequency term in the observability inequality of theorem 3.3 by the unique continuation property of elliptic operators (as in lemma 6 in [Leb92]). Hence, under the geodesics condition, theorems 3.1 and 3.3 imply the exact controllability of Schrödinger equation from Ω in any time, i.e. $C_{T,\Omega} < +\infty$ for all $T > 0$ (this is the analogue for interior controllability of the boundary controllability theorem in [Leb92]).

Note that, by duality, corollary 3.4 proves that any data in $L^2(M)$ can be steered to a low frequency state at cost $C/\sqrt{T}$. Since $\mathrm{Vect}\,\{\mathrm{e_j}\}_{\omega_j\leq d/T} \subset C^\infty(\overline{M})$, this can be regarded as a smoothing result at low control cost.

The first preliminary step in proving theorem 3.3 is to deduce a " high frequency observability inequality " from semiclassical observability:

Lemma 3.6. *Semiclassical observability on $\Omega \Subset M$ in time $T > 0$ with $b = 2 = a^{-1}$ implies that there is an observability constant $C_{hf} > 0$ and a threshold $\underline{k} \in \mathbb{N}^*$ such that: $\forall k \geq \underline{k}$, $\forall S \in [h_k T, h_{\underline{k}-1} T]$,*

$$\forall v_0 \in E_k = \operatorname*{Vect}_{h_{k-1}^{-1} < \omega_j < h_{k+1}^{-1}}\{\mathrm{e_j}\}\ , \quad \|v_0\|_{L^2(M)} \leq \frac{C_{hf}}{\sqrt{S}}\|e^{-it\Delta}v_0\|_{L^2(]0,S[\times M)} \ .$$

Proof. We choose $\theta_\Omega \in C^\infty_{\mathrm{comp}}(M)$ and $\theta_T \in C^\infty_{\mathrm{comp}}(\mathbb{R})$ with values in $[0,1]$ such that $\{\theta_\Omega \neq 0\} = \Omega$, and $\{\theta_T \neq 0\} =]0,T[$. Let C_{sc} and $\hbar$ be the positive constants obtained by applying definition 2 with $\theta(t,x) = \theta_T(t)\theta_\Omega(x)$ and $b = 2 = a^{-1}$. Choosing $\underline{k}$ such that $h_{\underline{k}} < \hbar$, the semiclassical observability inequality of definition 2 implies:

$$\forall k \geq \underline{k}, \forall \psi_0 \in E_k, \quad \|\psi_0\|^2_{L^2(M)} \leq C_{\mathrm{sc}}^2 \int \|\theta_\Omega e^{-ish_k\Delta}\psi_0\|^2_{L^2(M)}|\theta_T(s)|^2 ds \ .$$

The change of variable $t = sh_k$ and the definition of θ_Ω and θ_T yield:

$$\|\psi_0\|^2_{L^2(M)} \leq \frac{C_{\mathrm{sc}}^2}{h_k}\int_0^{h_k T}\|e^{-it\Delta}\psi_0\|^2_{L^2(\Omega)} dt \ .$$

Taking $\psi_0 = e^{-iNh_kT}v_0$ and changing t by a translation yields:

$$\forall N \in \mathbb{N}, \forall k \geq \underline{k}, \forall v_0 \in E_k,\ \|v_0\|^2_{L^2(M)} \leq \frac{C_{\mathrm{sc}}^2}{h_k}\int_{Nh_kT}^{(N+1)h_kT}\|e^{-it\Delta}v_0\|^2_{L^2(\Omega)} dt \ .$$

Let $k \geq m \geq \underline{k}$. Summing up from $N=0$ to $N=h_m h_k^{-1}-1$, multiplying by $h_k h_m^{-1}$ and setting $C_{\rm hf}=C_{\rm sc}\sqrt{2T}$ yield:

$$\forall v_0 \in E_k,\ \|v_0\|^2_{L^2(M)} \leq \frac{C^2_{\rm hf}}{h_{m-1}T}\int_0^{h_m T} \|e^{-it\Delta}v_0\|^2_{L^2(\Omega)}dt\ .$$

This inequality completes the proof of lemma 3.6 since, for all $k \geq \underline{k}$ and $S \in [h_k T, h_{\underline{k}-1}T]$, there is a $m \in [\underline{k}, k]$ such that $S \in [h_m T, h_{m-1}T]$. □

The second preliminary step in proving theorem 3.3 is to introduce a time frequency decomposition which is semiclassically equivalent to the spatial decomposition into the spectral scale (E_k). This provides an easy way to overcome the following difficulty (cf. lemma 3.7): multiplication by θ (which corresponds to observing on $]0,T[\times\Omega)$ does not commute with the projection on E_k. (Note that this difficulty is even greater in boundary observability but can be overcome by the analogue of lemma 3.7).

The Fourier transform of $v \in L^2(\mathbb{R}\times M)$ with respect to t is :

$$\hat{v}(\tau,x) = \int e^{-i\tau t}v(t,x)dt\,.$$

For any $\phi \in L^\infty(\mathbb{R})$, the frequency cut-off $\phi(D_t)$ and the spectral cut-off $\phi(\sqrt{-\Delta})$ are the bounded operators on L^2 defined by:

$$\forall v \in L^2(\mathbb{R}\times M), \qquad (\phi(D_t)v)\,(t,x) = \frac{1}{2\pi}\int e^{i\tau t}\phi(\tau)\hat{v}(\tau,x)d\tau\ ,$$

$$\forall v \in L^2(M), \qquad \phi(\sqrt{-\Delta})v = \sum_{j\in\mathbb{N}^*}\phi(\omega_j)(v|e_j)_{L^2(M)}\ .$$

For instance, the projection on E_k writes $\mathbf{1}_{[h^{-1}_{k-1},h^{-1}_{k+1}]}(\sqrt{-\Delta}) = \mathbf{1}_{[1/2,2]}(h_k\sqrt{-\Delta})$ with this notation. For $\phi \in \mathcal{S}(\mathbb{R})$, these cut-off operators extend to $L^2(M,\mathcal{S}'(\mathbb{R}_t))$ and satisfy the following “ compatibility ” relations :

$$\phi(D_t)\left(e^{it\Delta}e_j\right) = \phi(\omega_j)e^{it\omega_j^2}e_j = e^{it\Delta}\phi(\sqrt{-\Delta})e_j\ .$$

We shall need the following commutator estimate:

Lemma 3.7. *For all $\theta_T \in C^\infty_{comp}(]0,T[)$, $\phi \in \mathcal{S}(\mathbb{R})$, and $v \in L^2(\mathbb{R}\times M)$:*

$$\|[\,\theta_T,\phi(D_t)\,]Wv\|_{L^2(\mathbb{R}\times M)} \leq \|v\|_{L^2(\mathbb{R}\times M)}(1+T)\,\|\partial_t\theta_T\|_{L^\infty}\int(1+|t|)|t\hat{\phi}(t)|\,dt\,,$$

where W denotes the weight multiplication $Wv(t,x)=(1+|t|)v(t,x)$ and $[\,\theta_T,\phi(D_t)\,]$ denotes the commutator $\theta_T\phi(D_t)-\phi(D_t)\theta_T$ between the multiplication by θ_T and the time frequency cut-off $\phi(D_t)$.

Proof. Since the operator does not act in the x variable, we may forget about x and write its kernel as :

$$K(t,s) = (1+|s|)\,(\theta_T(t)-\theta_T(s))\,\hat{\phi}(s-t)\,.$$

By Schur's lemma, the bound sought for this operator will result from the same bound on $\sup_t \int |K(t,s)|\,ds$ and on $\sup_s \int |K(t,s)|\,dt$.

Using $\operatorname{supp}\theta_T \subset]0,T[$ and Taylor's inequality yields :

$$\int |K(t,s)|\,ds = \int (1+|s+t|)\,|\theta_T(t)-\theta_T(s+t)|\,|\hat{\phi}(s)|\,ds$$
$$\leq \int (1+T)(1+|t|)\,|s|\,\|\partial_t\theta_T\|_{L^\infty}\,|\hat{\phi}(s)|\,ds\,,$$

and similarly $\int |K(t,s)|\,dt \leq \int (1+T)(1+|t|)\,|t|\,\|\partial_t\theta_T\|_{L^\infty}|\hat{\phi}(t)|\,dt$. □

We choose a smooth real valued cut-off function $\tau \mapsto \chi(\tau)$ the time frequency parameter $\tau \in \mathbb{R}$, such that:

$$\operatorname{supp} \chi \subset \{1/2 < |\tau| < 2\} \ , \tag{18}$$

$$\sum_{k\geq 0} \chi^2(2^{-k}\tau) \geq 1 \text{ on } \{|\tau| > 1\} \ . \tag{19}$$

For any $m \in \mathbb{N}^*$, since $h_m|\omega_j| > 1$ and $h_k|\omega_j| < 2$ imply $k \geq m$, (18) and (19) imply:

$$\forall v \in L^2(M), \|v - \pi_m v\|^2_{L^2(M)} \leq \sum_{k\geq m} \|\chi(h_k\sqrt{-\Delta})v\|^2_{L^2(M)} \ . \tag{20}$$

Setting $C_\chi = 2\|\chi\|^2_{L^\infty}$, (18) implies $\sum_{k\in\mathbb{N}} \chi^2(2^{-k}\tau) \leq C_\chi$ (for each τ there are at most two nonzero terms in this sum), so that:

$$\forall v \in L^2(M\times M), \sum_{k\in\mathbb{N}} \|\chi(h_k D_t)v\|^2_{L^2(\mathbb{R}\times M)} \leq C_\chi \|v\|^2_{L^2(\mathbb{R}\times M)} \ . \tag{21}$$

Proof of theorem 3.3. We assume semiclassical observability on $\Omega \Subset M$ in time T_Ω and apply lemma 3.6. Let $d > T_\Omega$, $k \geq \underline{k}$, and $T \in [h_k d, h_{\underline{k}} d]$. Applying the high frequency observability inequality of lemma 3.6 with $S = T/2$ and $v_0 = e^{-iS\Delta}\chi(h_m\sqrt{-\Delta})u_0$, choosing $\theta_T \in C^\infty_{\text{comp}}(\mathbb{R})$ with values in $[0,1]$ such that $\{\theta_T \neq 0\} =]0,T[$ and $\{\theta_T = 1\} =]S/2, 3S/2[$, and setting $C'_d = \sqrt{d/T_\Omega}C_{\text{hf}}$, we obtain:

$$\begin{aligned}\forall m \geq k, \forall u_0 \in L^2(M), &\|\chi(h_m\sqrt{-\Delta})u_0\|_{L^2(M)} \\ &\leq \frac{C_{\text{hf}}}{\sqrt{S}} \|\chi(h_m\sqrt{-\Delta})u\|_{L^2(]\frac{S}{2},\frac{3S}{2}[\times\Omega)} \leq \frac{C'_d}{\sqrt{T}} \|\theta_T\chi(h_m\sqrt{-\Delta})u\|_{L^2(\mathbb{R}\times\Omega)} \ .\end{aligned} \tag{22}$$

Since $t \mapsto e^{-it\Delta}$ is unitary:

$$\int_{\mathbb{R}\times M} |(1+|t|^{-1})u(t,x)|^2 dxdt \leq \int_M |u(t,x)|^2 dx \int_{\mathbb{R}} |1+|t|^{-1}|^2 dt \ .$$

Applying lemma 3.7 with $\psi\tau = \chi(h_m\tau)$ and $v(t,x) = (1+|t|^{-1})\,\mathbf{1}_\Omega(x)u(t,x)$, and setting $C = \|1+|t|^{-1}\|_{L^2}(1+T)\,\|\partial_t\theta_T\|_{L^\infty}\int(1+|t|)|t\hat{\phi}(t)|\,dt$, we obtain:

$$\|\theta_T\chi(h_m\sqrt{-\Delta})u - \chi(h_m D_t)\theta_T u\|^2_{L^2(\mathbb{R}\times\Omega)} \leq C^2 h_m^2 \|u_0\|^2_{L^2(\Omega)} \ .$$

Since $|\theta_T| \leq 1$, this combines with (22) into: $\forall m \geq k$, $\forall u_0 \in L^2(M)$,

$$\|\chi(h_m\sqrt{-\Delta})u_0\|^2_{L^2(M)} \leq \frac{2{C'_d}^2}{T}\left(\|\chi(h_m D_t)u\|^2_{L^2(]0,T[\times\Omega)} + C^2 h_m^2 \|u_0\|^2_{L^2(\Omega)}\right) \ .$$

Summing up this inequality for $m \geq k$, we obtain, thanks to (20), (21) and $\sum_{m\geq k} h_m^2 = 4h_k^2/3$:

$$\|u_0 - \pi_k u_0\|^2_{L^2(M)} \leq \frac{2{C_d}^2}{T}\left(C_\chi\|u\|^2_{L^2(]0,T[\times\Omega)} + \frac{4}{3}C^2 h_k^2 \|u_0\|^2_{L^2(\Omega)}\right) \ .$$

Adding $\|\pi_k u_0\|^2_{L^2(M)} - \frac{8}{3}{C'_d}^2 C^2 h_k^2 \|u_0\|^2_{L^2(\Omega)}$ on both sides completes the proof of theorem 3.3 with $C_d = \sqrt{2d/T_\Omega}C_{\text{hf}}C_\chi$. □

4. A window problem for nonharmonic Fourier series

In this section we prove theorem 4.1 which generalizes theorem 1.2 to Sturm-Liouville operators (in particular to a segment with any Riemannian metric). By spectral analysis, it reduces to a refinement (cf. proposition 4.2) of the well studied window problem for nonharmonic Fourier series (cf. [SAI00]) which is solved in subsections 4.2 and 4.3 following [Mil03] quite closely.

4.1. **Boundary control of a segment.** Let $X > 0$. We consider the Sturm-Liouville operator A on $L^2(0,X)$ with domain $D(A)$ defined by

$$(Af)(x) = (p(x)f'(x))' + q(x)f(x) \quad \text{for } x \in [0,X]$$
$$D(A) = \{f \in H^2(0,X) \mid (a_0 + b_0 f')(0) = 0 = (a_1 + b_1 f')(X)\}$$

where all the coefficients are real and satisfy:

$$a_0^2 + b_0^2 = a_1^2 + b_1^2 = 1\,, \quad 0 < p \in C^2([0,X])\,, \quad q \in C^0([0,X])\,, \tag{23}$$

Under these assumptions, $-A$ is self-adjoint and has a sequence $\{\lambda_n\}_{n\in\mathbb{N}^*}$ of increasing eigenvalues and an orthonormal Hilbert basis $\{e_n\}_{n\in\mathbb{N}^*}$ in $L^2(0,X)$ of corresponding eigenfunctions, i.e.:

$$\forall n \in \mathbb{N}^*, \quad -Ae_n = \lambda_n e_n \quad \text{and} \quad \lambda_n < \lambda_{n+1}\,.$$

Moreover, (23) ensures the following eigenvalues asymptotics:

$$\exists \nu \in \mathbb{R},\ \lambda_n = \frac{\pi^2}{L^2}(n+\nu)^2 + O(1) \text{ as } n \to \infty\,, \quad \text{where} \quad L = \int_0^X \sqrt{p(x)}\,dx\,. \tag{24}$$

We use the following notations for the Sobolev spaces based on A:

$$H_A^0(0,X) = L^2(0,X) \quad \text{and} \quad H_A^1(0,X) = \overline{D(A)}^{H^1}\,.$$

Theorem 4.1. *For any $\alpha > \alpha_*$ defined by* (4)*, there exists $C > 0$ such that, for any coefficients* (23)*, setting $k = 1$ if $b_1 = 0$ and $k = 0$ otherwise, for all $T \in]0, \inf(\pi, L)^2]$ and $u_0 \in H_A^k(0,X)$ the solution $u \in C^0([0,\infty); H_A^k(0,X))$ of*

$$i\partial_t u = \partial_x (p(x)\partial_x u) + q(x)u \quad \textit{for } (t,x) \in]0,T[\times]0,X[\,,$$
$$(a_0 + b_0\partial_x u)_{\rceil x=0} = 0 = (a_1 + b_1\partial_x u)_{\rceil x=X} \quad \textit{and} \quad u_{\rceil t=0} = u_0\,,$$

satisfies $\|u_0\|_{H_A^k(0,X)} \le C\exp(\alpha L^2/T)\|\partial_x^k u_{\rceil x=X}\|_{L^2(0,T)}$.

4.2. **Reduction of the window problem to a problem on entire functions.** First note that the theorem 4.1 can be reduced to the case $\lambda_1 > 0$ by the multiplier $t \mapsto \exp(i\lambda t)$, to the case $L = \pi$ by the time rescaling $t \mapsto \sigma t$ with $\sigma = (\pi/L)^2$, and to the time interval $[-T/2, T/2]$ by the time translation $t \mapsto t - T/2$.

From now on we assume $\lambda_1 > 0$ and $L = \pi$. Making a weaker assumption on the remainder term in (24), we shall only use the following spectral assumption:

$$\forall n \in \mathbb{N}^*,\ 0 < \lambda_n < \lambda_{n+1} \quad \text{and} \quad \lambda_n = n^2 + O(n) \text{ as } n \to \infty\,. \tag{25}$$

In terms of the coordinates $c = (c_k)_{k\in\mathbb{N}^*}$ of $A^{k/2}u_0$ in the Hilbert basis $(e_k)_{k\in\mathbb{N}^*}$, we have to solve the following window problem:

Proposition 4.2. *For any $\alpha > \alpha_*$, there exists $C > 0$ such that, for all $(\lambda_n)_{n\in\mathbb{N}^*}$ satisfying (25), for all $T \in]0,\pi]$:*

$$\forall c \in l^2(\mathbb{N}^*), \quad \|c\|_{l^2} \le Ce^{\alpha\pi^2/T}\|f\|_{L^2(-\frac{T}{2},\frac{T}{2})} \textit{ where } f(t) = \sum_{n=1}^{\infty} c_n e^{i\lambda_n t}\,. \tag{26}$$

The well-known method to study the nonharmonic Fourier series f is to construct a sequence $(g_n)_{n\in\mathbb{N}^*}$ in $L^2(-T/2, T/2)$ which is bi-orthogonal to the sequence $\{\exp(-\lambda_n t)\}_{n\in\mathbb{N}^*}$, i.e.

$$\int_{-T/2}^{T/2} g_n(t)e^{-\lambda_n t}\,dt = 1 \quad \text{and} \quad \forall k \in \mathbb{N}^*,\ k \neq n,\ \int_{-T/2}^{T/2} g_n(t)e^{-\lambda_k t}\,dt = 0\,. \tag{27}$$

Then: $\|c\|_{l^2}^2 = \sum_n (f, g_n)\overline{c_n} = (f, \sum_n g_n c_n) \le \|f\|_{L^2}\|\sum_n g_n c_n\|_{L^2}$. Hence, introducing the Gramm operator G on $l^2(\mathbb{N}^*)$ defined by the coefficients $(g_n, g_k)_{L^2}$ for n and k in $\mathbb{N}^*$, (26) results from $\|G\| \le Ce^{\alpha/T}$. But, applying Schur's lemma to the self-adjoint operator G yields: $\|G\|^2 \leq \sup_n \sum_k |(g_n, g_k)_{L^2}|$. Thus, to prove proposition 4.2 it is enough to construct bi-orthogonal functions g_n with good growth estimates of their scalar products as T tends to zero. We shall readily explain how the following proposition on entire functions yields this construction, and postpone its proof to subsection 4.3.

Proposition 4.3. *Let α_* be defined by* (4). *Let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be a sequence of real numbers satisfying* (25). *For all $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that, for all $\tau \in]0,1]$, there is a sequence of entire functions $\{G_n\}_{n\in\mathbb{N}^*}$ satisfying, for all n and k in $\mathbb{N}^*$:*

(28) $\quad G_n$ *is of exponential type* τ, *i.e.* $\limsup_{r\to+\infty} r^{-1} \sup_{|z|=r} \ln |G_n(z)| \leq \tau$,

(29) $\quad G_n(\lambda_n) = 1 \quad$ *and* $\quad G_n(\lambda_k) = 0 \;$ *if* $\; k \neq n$,

(30) $\quad |(G_n, G_k)_{L^2}| \leq C_\varepsilon e^{-\varepsilon\sqrt{|\lambda_n-\lambda_k|/2}} e^{\alpha_*(\pi+\sqrt{2}\varepsilon)^2/\tau}$.

According to the Paley-Wiener theorem (1934), (28) implies that the function $x \mapsto G_n(x)$ is the unitary Fourier transform of a function $t \mapsto g_n(t)$ in $L^2(\mathbb{R})$ supported in $[-\tau,\tau]$. With $\tau = T/2$, this yields:

$$G_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} g_n(t) e^{-itx}\, dt \quad \text{and} \quad \|g_n\|_{L^2} = \|G_n\|_{L^2} \; . \tag{31}$$

Hence (29) implies (27) and (30) implies that:

$$\|G\|^2 \leq \sup_n \sum_k |(g_n, g_k)_{L^2}| \leq C_\varepsilon e^{2\alpha_*(\pi+\sqrt{2}\varepsilon)^2/T} \sup_n \left(1 + \sum_{k\neq n} e^{-\frac{\varepsilon}{\sqrt{2}}\sqrt{|\lambda_n-\lambda_k|}} \right) .$$

To complete the proof of proposition 4.2, we just have to estimate the last sum uniformly with respect to n. For this purpose, we introduce the counting function of the sequence $(|\lambda_k - \lambda_n|)_{k\in\mathbb{N}^*\setminus\{n\}}$ for every $n \in \mathbb{N}^*$:

$$N_n(r) = \#\{k \in \mathbb{N}^* \mid 0 < |\lambda_k - \lambda_n| \leq r\} \; .$$

From the spectral asymptotics (25), we deduce that:

$$\exists \underline{r} > 0, \forall r \in]0, \underline{r}[, N_n(r) = 0 \; , \quad \exists A > 0, \; \forall r, \; |\sqrt{r} - N_n(r)| \leq A \; . \tag{32}$$

Indeed, (25) implies $|\sqrt{\lambda_n} - n| \leq C$ for some $C > 0$. If $\lambda_n \leq r$ then $\sqrt{\lambda_n + r} \leq \sqrt{2r}$ so that $N_n(r) \leq \sqrt{2r} + C$. If $\lambda_n > r$ then $N_n(r) \leq \sqrt{\lambda_n + r} - \sqrt{\lambda_n - r} + 2C \leq \sqrt{2r} + 2C$. Now the last sum writes:

$$\sum_{k\neq n} e^{-\frac{\varepsilon}{\sqrt{2}}\sqrt{|\lambda_n-\lambda_k|}} = \int_0^\infty \exp\left(-\frac{\varepsilon}{\sqrt{2}}\sqrt{r}\right) dN_n(r)$$
$$= \frac{\varepsilon}{2\sqrt{2}} \int_0^\infty \frac{N_n(r)}{\sqrt{r}} \exp\left(-\frac{\varepsilon}{\sqrt{2}}\sqrt{r}\right) dr \leq \varepsilon \int_0^\infty (2s + A) e^{-\varepsilon s}\, ds \; .$$

This completes the proof that proposition 4.3 implies proposition 4.2 which implies theorem 4.1.

4.3. **Entire functions construction.** In this subsection, we prove proposition 4.3. We follow a classical method in complex analysis: for all $n \in \mathbb{N}^*$ and small $\tau > 0$, we shall form, in a first lemma, an infinite product F_n normalized by $F_n(\lambda_n) = 1$ with zeros at λ_k for every positive integer $k \neq n$, and construct, in a second lemma, a multiplier M_n of exponential type τ with fast decay at infinity on the real axis so that $G_n = M_n F_n$ is in L^2 on the real axis.

Lemma 4.4. *Let $\{\lambda_n\}_{n\in\mathbb{N}^*}$ be a sequence of real numbers satisfying* (25)*. For all $\varepsilon > 0$ there is a $A_\varepsilon > 0$ such that, for all $n \in \mathbb{N}^*$, the entire function F_n defined by $F_n(z) = \prod_{k\neq n}\left(1 - \dfrac{z-\lambda_n}{\lambda_k - \lambda_n}\right)$ satisfies*

$$\ln|F_n(z-\lambda_n)| \leq (\sqrt{2}\pi + \varepsilon)\sqrt{|z|} + A_\varepsilon \tag{33}$$

Proof. To prove (33), we estimate the left hand side in terms of N_n:

$$\ln|F_n(z+\lambda_n)| \leq \sum_{k\neq n} \ln\left(1 + \frac{|z|}{|\lambda_k - \lambda_n|}\right) = \int_0^\infty \ln\left(1+\frac{|z|}{r}\right) dN_n(r)$$
$$= \int_0^\infty N_n(r)\frac{|z|}{|z|+r}\frac{dr}{r} = \int_0^\infty \frac{N_n(|z|s)}{1+s}\frac{ds}{s}$$

To estimate this last integral we use (32) and the integral computations:

$$\int_0^\infty \frac{\sqrt{s}}{1+s}\frac{ds}{s} = \int_0^\infty \frac{2dr}{1+r^2} = \pi \ , \quad \int_{\frac{r}{|z|}}^\infty \frac{ds}{s(1+s)} = \left[\ln\left|\frac{s}{1+s}\right|\right]_{\frac{r}{|z|}}^\infty = \ln(1+\frac{|z|}{r})$$

Thus we obtain $\ln|f_n(z)| \leq \pi\sqrt{2|z|} + A\ln(1+\frac{|z|}{r})$, so that, for all $\varepsilon > 0$ there is a $A_\varepsilon > 0$ such that $\ln|f_n(z)| \leq (\sqrt{2}\pi+\varepsilon)\sqrt{|z|} + A_\varepsilon$. □

We quote the following lemma from [Mil03]:

Lemma 4.5. *Let α_* be defined by* (4)*. For all $d > 0$ there is a $D > 0$ such that for all $\tau > 0$, there is an even entire function M of exponential type (lower or equal to) τ satisfying: $M(0) = 1$ and*

$$\forall x > 0, \quad \ln|M(x)| \leq \frac{\alpha_* d^2}{4\tau} + D - d\sqrt{x}\ . \tag{34}$$

To prove proposition 4.3, we use lemmas 4.4 and 4.5 with $d = \sqrt{2}\pi + 2\varepsilon$ and define: $G_n = F_n M_n$ with $M_n(z) = M(z-\lambda_n)$. The entire function G_n has the same exponential type as M since (33) implies that the exponential type of F_n is 0. Hence (28) holds. Equation (29) is an obvious consequence of $M_n(\lambda_n) = M(0) = 1$ and the definition of F_n. Since $d = \pi + 2\varepsilon$ and M is even, (33) and (34) imply

$$\forall x \in \mathbb{R}, \quad \ln|G_n(x+\lambda_n)| \leq D_\varepsilon + A_\varepsilon - \varepsilon\sqrt{|x|} + \frac{\alpha_* d^2}{4\tau}\ .$$

Setting $C_\varepsilon = 2e^{2(D_\varepsilon + A_\varepsilon)}\int_0^{+\infty} e^{-\varepsilon\sqrt{s}}\,ds$ and $\Delta = |\lambda_n - \lambda_k|/2$, this yields (30):

$$|(G_n, G_k)_{L^2}| \leq e^{2(D_\varepsilon+A_\varepsilon)+\frac{\alpha_* d^2}{2\tau}} \int_{-\infty}^{+\infty} e^{-\varepsilon\sqrt{|x+\lambda_n|}-\varepsilon\sqrt{|x-\lambda_n|}}\,dx$$
$$\leq e^{2(D_\varepsilon+A_\varepsilon)+\frac{\alpha_* d^2}{2\tau}} \int_{-\infty}^{+\infty} e^{-\varepsilon\sqrt{|s+\Delta|}-\varepsilon\sqrt{|s-\Delta|}}\,ds \leq C_\varepsilon e^{-\varepsilon\sqrt{\Delta}} e^{\alpha_*(\pi+\sqrt{2}\varepsilon)^2/\tau}$$

Thus proposition 4.3 is proved.

Remarks 4.6. Under the assumption (25), lemma 3 in [SAI00] (which applies to more general sequences) proves that the function $F_n(z) = \prod_{k\neq n}\left[1 - \left(\dfrac{z-\lambda_n}{\lambda_k-\lambda_n}\right)^2\right]$ satisfies $\ln|F_n(\lambda_n + z)| \leq 2\pi\sqrt{|z|}$. In lemma 4.4, the constant 2π improves to $\sqrt{2}\pi$. We do not know if the optimal constant is π as in lemma 4.3 in [Mil03].

Seidman obtained lemma 4.5 for $\alpha_* = 2\beta_*$ with $\beta_* \approx 42.86$ in the proof of Theorem 3.1 in [Sei84]. His later Theorem 1 in [Sei86] improves the rate to $\alpha_* = 4\beta_*$ with $\beta_* \approx 4.17$. Theorem 2 in [SAI00], which applies to much more general spectral sequences, yields lemma 4.5 for $\alpha_* = 48$. As explained in [Mil03], lemma 4.5 does not hold for $\alpha_* < 1/2$ and it is an interesting problem of entire function analysis to determine the smallest value of α_* for which it does.

5. Upper bound under the geodesics condition

In this section we prove theorem 1.3. $\mathcal{D}'(\mathcal{O})$ denotes the space of distributions on the open set $\mathcal{O}$ endowed with the weak topology and $\mathcal{M}(\mathcal{O})$ denotes the subspace of Radon measures on $\mathcal{O}$. When $\mathcal{O}$ is a vector space, δ denotes the Dirac measure at the origin.

5.1. **The fundamental controlled solution.** In this subsection we construct a "fundamental controlled solution" v of the Schrödinger equation on a segment controlled by Dirichlet conditions at both ends. The precise definition is the following.

Definition **3.** The distribution $v \in C^0([0,T];\mathcal{M}(]-L,L[))$ is a fundamental controlled solution for the Schrödinger equation on $]0,T[\times]-L,L[$ at cost (A,α) if

$$i\partial_t v - \partial_s^2 v = 0 \quad \text{in } \mathcal{D}'(]0,T[\times]-L,L[) \ , \tag{35}$$

$$v_{\rceil t=0} = \delta \quad \text{and} \quad v_{\rceil t=T} = 0 \ , \tag{36}$$

$$\|v\|_{L^2(]0,T[\times]-L,L[)} \leq Ae^{\alpha L^2/T} \ . \tag{37}$$

Theorem 1.2 allows us to construct a family of fundamental controlled solutions depending on $L > 0$ and $T > 0$ with a good cost estimate thanks to the following proposition which shows that the upper bound for the controllability cost of the Schrödinger equation on the segment $[0,L]$ controlled at one end is the same as the controllability cost of the Schrödinger equation on the twofold segment $[-L,L]$ controlled at both ends.

Proposition 5.1. *If theorem 1.2 holds for some rate α_*, then for any $\alpha > \alpha_*$, there exists $A > 0$ such that, for all $L > 0$, $T \in]0, \inf(\pi/2, L)^2]$ and $v_0 \in H^{-1}(-L,L)$, there are g_- and g_+ in $L^2(0,T)$ such that the solution $v \in C^0([0,\infty); H^{-1}(-L,L))$ of the following Schrödinger equation on $[-L,L]$ controlled by g_- and g_+:*

$$i\partial_t v - \partial_s^2 v = 0 \quad \text{in }]0,T[\times]-L,L[, \quad v_{\rceil s=\pm L} = g_\pm, \quad v_{\rceil t=0} = v_0 \tag{38}$$

satisfies $v = 0$ at $t = T$ and $\|g_\pm\|_{L^2(0,T)} \leq Ae^{\alpha L^2/T}\|v_0\|_{H^{-1}(-L,L)}$.

Proof. By duality (cf. [DR77]), it is enough to prove the observation inequality: $\exists C > 0, \forall v_0 \in H^1_0(-L,L), \|v_0\|_{H^1} \leq Ce^{\alpha L^2/T}\|\partial_s e^{it\Delta}v_{0\rceil s=\pm L}\|_{L^2(0,T)^2}$. Applying theorem 1.2 to the odd and even parts of v_0 completes the proof (as in the proof of proposition 5.1 in [Mil03]). □

Applying proposition 5.1 with $v_0 = \delta \in H^{-1}(-L,L)$, and using Duhamel's formula to estimate v in terms of $g_\pm = v_{\rceil s=\pm L}$, we obtain:

Corollary 5.2. *If theorem 1.2 holds for some rate α_*, then for any $\alpha > \alpha_*$, there exists $A > 0$ such that for all $L > 0$ and $T \in]0, \inf(\pi/2, L)^2]$ there is a fundamental controlled solution for the Schrödinger equation on $]0,T[\times]-L,L[$ at cost (A,α).*

5.2. **The transmutation of waves controls into Schrödinger controls.** In this subsection we perform a transmutation of a control for the wave equation into a control for the Schrödinger equation. Our transmutation formula (cf. (44)) can be regarded as the analogue of the formula (6) with $F(\sigma) = \exp(it\sigma^2)$ where the kernel $e^{-i\pi/4}e^{is^2/(4t)}/\sqrt{4\pi t}$, which is the fundamental solution of the Schrödinger equation on the line, is replaced by the fundamental controlled solution that we have constructed in the previous subsection. To ensure existence of an exact control for the wave equation we use the geodesics condition (cf. the footnote on page 3):

Theorem 5.3 ([BLR92]). *Let $\Omega \Subset M$. Let L_Ω be the length of the longest generalized geodesic in $\overline{M}$ which does not intersect Ω. If $L > L_\Omega$ then for all (w_0, w_1) and (w_2, w_3) in $L^2(M) \times H^{-1}(M)$ there is a control function $f \in L^2([0,\infty) \times M)$ such that the solution $w \in C^0([0,\infty); L^2(M)) \cap C^1([0,\infty); H^{-1}(M))$ of the mixed Dirichlet-Cauchy problem (n.b. the time variable is denoted by s here):*

$$\partial_s^2 w - \Delta w = \mathbf{1}_{]0,L[\times\Omega} f \quad \text{in } [0,\infty) \times M, \quad w = 0 \quad \text{on } [0,\infty) \times \partial M, \tag{39}$$

with Cauchy data $(w, \partial_s w) = (w_0, w_1)$ at $s = 0$, satisfies $(w, \partial_s w) = (w_2, w_3)$ at $s = L$. Moreover, the operator $S_W : \left(L^2(M) \times H^{-1}(M)\right)^2 \to L^2([0,\infty) \times M)$ defined by $S_W((w_0, w_1), (w_2, w_3)) = f$ is continuous.

Proof of theorem 1.3. We assume that theorem 1.2 holds for some rate α_*. Let $\alpha > \alpha_*$, $T \in]0, \inf(1, L_\Omega^2)[$ and $L > L_\Omega$ be fixed from now on. Let $A > 0$ and $v \in L^2(]0,T[\times]-L,L[)$ be the corresponding constant and fundamental controlled solution given by corollary 5.2. We define $\underline{v} \in L^2(\mathbb{R}^2)$ as the extension of v by zero, i.e. $\underline{v}(t,s) = v(t,s)$ on $]0,T[\times]-L,L[$ and $\underline{v}$ is zero everywhere else. It inherits from v the following properties

$$i\partial_t \underline{v} - \partial_s^2 \underline{v} = 0 \quad \text{in } \mathcal{D}'(]0,\infty)\times]-L,L[) \ , \tag{40}$$

$$\underline{v} \in C^0([0,\infty); \mathcal{M}(\mathbb{R})) \quad \text{and} \quad \underline{v}_{\rceil t=0} = \delta \ , \tag{41}$$

$$\|\underline{v}\|_{L^2(]0,\infty)\times\mathbb{R})} \le A e^{\alpha L^2/T} \ . \tag{42}$$

Let $u_0 \in L^2(M)$ be an initial data for the Schrödinger equation (2). Let w and f be the corresponding solution and control function for the wave equation obtained by applying theorem 5.3 with $w_0 = u_0$ and $w_1 = w_2 = w_3 = 0$. We define $\underline{w} \in L^2(\mathbb{R} \times M)$ and $\underline{f} \in L^2(\mathbb{R} \times M)$ as the extensions of w and f by reflection with respect to $s = 0$, i.e. $\underline{w}(s,x) = w(s,x) = \underline{w}(-s,x)$ and $\underline{f}(s,x) = f(s,x) = \underline{f}(-s,x)$ on $\mathbb{R}_+ \times M$. Since $w_1 = 0$, equation (39) imply

$$\partial_s^2 \underline{w} - \Delta \underline{w} = \mathbf{1}_{]-L,L[\times\Omega} \underline{f} \quad \text{in } \mathcal{D}'(\mathbb{R} \times M), \quad \underline{w} = 0 \quad \text{on } \mathbb{R} \times \partial M, \tag{43}$$

The main idea of our proof is to use $\underline{v}$ as a kernel to transmute $\underline{w}$ and $\underline{f}$ into a solution u and a control g for (2). Since $\underline{v} \in L^2(\mathbb{R}^2)$, $\underline{w} \in L^2(\mathbb{R} \times M)$ and $\underline{f} \in L^2(\mathbb{R} \times M)$, the transmutation formulas

$$u(t,x) = \int_{\mathbb{R}} \underline{v}(t,s) \underline{w}(s,x)\, ds \quad \text{and} \quad g(t,x) = \int_{\mathbb{R}} \underline{v}(t,s) \underline{f}(s,x)\, ds \ , \tag{44}$$

define functions u and g in $L^2(\mathbb{R} \times M)$. Since $\underline{w}(s,x) = \partial_s \underline{w}(s,x) = 0$ for $|s| = L$, equations (43) and (40) imply

$$i\partial_t u - \Delta u = \mathbf{1}_{]0,T[\times\Omega}\, g \quad \text{in } \mathcal{D}'(]0,\infty) \times M) \quad \text{and} \quad u = 0 \quad \text{on }]0,T[\times\partial M, \tag{45}$$

Since $\underline{w} \in C^0(\mathbb{R}, L^2(M))$, the property (41) of $\underline{v}$ implies

$$u \in C^0([0,\infty); L^2(M)) \quad \text{and} \quad u_{\rceil t=0} = u_0 \ . \tag{46}$$

Since $\underline{v}_{\rceil t=T} = 0$, we also have

$$u_{\rceil t=T} = 0 \ . \tag{47}$$

Setting $C = \sqrt{2} A \|S_W\|$, Cauchy-Schwartz inequality with respect to s, the estimate (42) and $\|\underline{f}\|^2_{L^2(\mathbb{R}\times M)} = 2\|S_W((u_0,0),(0,0))\|^2_{L^2([0,\infty)\times M)}$ imply

$$\|g\|_{L^2(\mathbb{R}\times M)} \le \|\underline{v}\|_{L^2(\mathbb{R}^2)} \|\underline{f}\|_{L^2(\mathbb{R}\times M)} \le C e^{\alpha L^2/T} \|u_0\|_{L^2(M)} \ . \tag{48}$$

We have proved that for all $\alpha > \alpha_*$ there is a $C > 0$ such that for all $u_0 \in L^2(M)$, $T \in]0, \min\{1, L_\Omega^2\}[$ and $L > L_\Omega$, there is a control g which solves the controllability problem (45), (46), (47) at a cost so estimated in (48). Therefore, using the dual definition of $C_{T,\Omega}$ given after definition 1: $\limsup_{T\to 0} T \ln C_{T,\Omega} \leq \alpha L^2$. Letting α and L tend to α_* and L_Ω in this estimate completes the proof of (5). $\square$

6. Upper bound for some examples violating the geodesics condition

In this section, we deduce from theorem 1.2 and 1.3 that the same upper bounds are satisfied for some Schrödinger evolution groups of product type violating the geodesics condition. The proof elaborates on the yet unpublished remark of Burq (back in 1992, cf [BZ03]) that the result of [Har89] can be extended to product manifolds with a much simpler proof: the point here is that the controllability cost is tracked.

The following lemma generalizes this remark to the abstract setting for the theory of observation and control (cf. [DR77]).

Lemma 6.1. *Let X, Y and Z be Hilbert spaces and I denote the identity operator on each of them. Let $A : D(A) \to X$ and $B : D(B) \to Y$ be generators of strongly continuous semigroups of bounded operators on X and Y. Let $C : D(C) \to Z$ be a densely defined operator on X such that $e^{tA}D(C) \subset D(C)$ for all $t > 0$. Let $X\overline{\otimes}Y$ and $X\overline{\otimes}Z$ denote the closure of the algebraic tensor products $X \otimes Y$ and $X \otimes Z$ for the natural Hilbert norms.*

i) The operator $A\otimes I+I\otimes B$ defined on the algebraic $D(A)\otimes D(B)$ is closable and its closure, denoted $A + B$, generates a strongly continuous semigroup of bounded operators on $X\overline{\otimes}Y$ satisfying:

$$\forall t \geq 0, \forall (x, y) \in D(C) \times Y, \quad \|(C \otimes I)e^{t(A+B)}(x \otimes y)\| = \|Ce^{tA}x\| \, \|e^{tB}y\| \tag{49}$$

ii) If iB is self-adjoint, then for all $T \geq 0$:

$$\inf_{\psi\in X\overline{\otimes}Y, \|\psi\|=1} \int_0^T \|(C \otimes I)e^{t(A+B)}\psi\|^2 dt = \inf_{x\in X, \|x\|=1} \int_0^T \|Ce^{tA}x\|^2 dt \ . \tag{50}$$

Proof. Let G denote the generator of the strongly continuous semigroup $t \mapsto e^{tA} \otimes e^{tB}$ (defined since the natural Hilbert norm is a uniform cross norm, cf. [Sch50]). Since $D(A) \otimes D(B)$ is dense in $X \otimes Y$ and invariant by $t \mapsto e^{tG}$, it is a core for G (cf. theorem X.49 in [RS79]). Since $A \otimes I + I \otimes B = G_{\rceil D(A)\otimes D(B)}$, it is closable and $A+B = G$. Therefore $e^{t(A+B)} = e^{tA} \otimes e^{tB}$ and (49) follows (by the cross norm property).

To prove point ii), we denote the left and right hand sides of (50) by $\mathcal{I}_{A+B}$ and $\mathcal{I}_A$. Taking $\psi = x \otimes y$ with $\|y\| = 1$, $\mathcal{I}_{A+B} \leq \mathcal{I}_A$ results from (49). To prove $\mathcal{I}_{A+B} \geq \mathcal{I}_A$, we only consider the case in which both X and Y are infinite dimensional and separable (this simplifies the notation and the other cases are similar). Let $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ be orthonormal bases for X and Y. Since $(e_n \otimes f_m)_{n,m\in\mathbb{N}}$ is an orthonormal base for $X\overline{\otimes}Y$, any $\psi \in X\overline{\otimes}Y$ writes:

$$\psi = \sum_m x_m \otimes f_m \quad \text{with} \quad x_m = \sum_n c_{n,m}e_n \quad \text{and} \quad \|\psi\|^2 = \sum_{n,m} |c_{n,m}|^2 = \sum_m \|x_m\|^2 \ .$$

Since iB is self-adjoint, $t \mapsto e^{tB}$ is unitary for all $t \geq 0$ so that $(e^{tB}f_n)_{n\in\mathbb{N}}$ is orthonormal. Therefore, using (49):

$$\|Ce^{t(A+B)}\psi\|^2 = \|\sum_m (Ce^{tA}x_m) \otimes (e^{tB}f_m)\|^2 = \sum_m \|Ce^{tA}x_m\|^2 \ .$$

By definition, $\int_0^T \|Ce^{tA}x_m\|^2 dt \geq \mathcal{I}_A\|x_m\|^2$. Summing up over $m \in \mathbb{N}$, we obtain:

$$\int_0^T \|(C \otimes I)e^{t(A+B)}\psi\|^2 dt = \int_0^T \sum_m \|Ce^{tA}x_m\|^2 \geq \mathcal{I}_A \sum_m \|x_m\|^2 = \mathcal{I}_A\|\psi\|^2 \ .$$

This proves $\mathcal{I}_{A+B} \geq \mathcal{I}_A$ and completes the proof of lemma 6.1. $\square$

Remarks 6.2. When C is an admissible observation operator, (50) says that the cost of observing $t \mapsto e^{t(A+B)}$ through $C \otimes I$ in time T is exactly the cost of observing $t \mapsto e^{tA}$ through C in time T.

If A and B are self-adjoint, then $A + B$ defined in lemma 6.1 is self-adjoint (cf. theorem VIII.33 in [RS79]).

The proof of part i) of lemma 6.1 is still valid if X, Y and Z are Banach spaces and $X\overline{\otimes}Y$ and $X\overline{\otimes}Z$ are closures with respect to some uniform cross norms (cf. [Sch50]).

Theorem 1.4 is a particular case of the following direct consequence of lemma 6.1 and theorem 1.2 (with $X = Z = L^2(M)$, $Y = \mathcal{B}$, $A = i\Delta$ and bounded $C = \mathbf{1}_\Omega$).

Theorem 6.3. *Let B be a self-adjoint operator on a Hilbert space $\mathcal{B}$. The operator $H = \Delta \otimes \mathrm{id}_\mathcal{B} + \mathrm{id}_{L^2(M)} \otimes B$ is essentially self-adjoint on $\mathcal{H} = L^2(M) \otimes \mathcal{B}$. For all $T > 0$ and $\Omega \Subset M$, $C_{T,\Omega}$ (cf. definition 1) is also the cost of controlling the Schrödinger group $t \mapsto e^{itH}$ on $\mathcal{H}$ with controls in $L^2(\Omega) \otimes \mathcal{B}$, i.e. $C_{T,\Omega}$ is the best constant in the observability inequality: $\forall v \in \mathcal{H}$, $\|v\|_\mathcal{H} \leq C_{T,\Omega}\|\mathbf{1}_\Omega\, e^{itH}v\|_{L^2(]0,T[;\mathcal{H})}$. In particular:*

$$\limsup_{T\to 0} T \ln \sup_{v\in\mathcal{H}\setminus\{0\}} \frac{\|v\|_\mathcal{H}}{\|\mathbf{1}_\Omega\, e^{itH}v\|_{L^2(]0,T[;\mathcal{H})}} \leq \alpha_* L_\Omega^2 \ ,$$

where α_* is defined in (4) and L_Ω is the length of the longest generalized geodesic in $\overline{M}$ which does not intersect Ω.

Remarks 6.4. Note that $\mathcal{H} = L^2(\Omega;\mathcal{B})$ and $L^2(0,T;\mathcal{H}) = L^2(]0,T[\times M;\mathcal{B})$ when $\mathcal{B}$ is separable (cf. theorem II.10 in [RS79]). With $\mathcal{B} = L^2(\tilde{M})$, $B = \tilde{\Delta}$, $\mathcal{H} = L^2(M \times \tilde{M})$, theorem 6.3 proves theorem 1.4.

The semi-internal controllability of a rectangular plate proved in [Har89] corresponds to the setting $M = [0,X]$, $\mathcal{B} = L^2([0,Y])$, $B = \partial_y^2$ with Dirichlet condition, $\mathcal{H} = L^2([0,X] \times [0,Y])$. Note that our theorem still applies to an infinite strip $[0,X] \times \mathbb{R}$ controlled from any infinite strip $[a,b] \times \mathbb{R}$ with $[a,b] \subset]0,X[$.

The resolvent method introduced in [BZ03] also yields the controllability in theorem 6.3 for some control time (and for any positive control time by a temporal black box), but it does not keep track of the cost.

The following analogue of theorem 6.3 for the boundary controllability of cylinders from one end is a direct consequence of lemma 6.1 and theorem 1.2 (with $X = H_A^k(0,X)$, $Y = \mathcal{B}$, $Z = \mathbb{R}$, $D(C) = D(A)$ and $Cu = \partial_x u_{\rceil x=X}$).

Theorem 6.5. *Let B be a self-adjoint operator on a Hilbert space $\mathcal{B}$. Let A be the Sturm-Liouville operator on $L^2(0,X)$ and L be the length of $[0,X]$ defined in subsection 4.1. The operator $H = A \otimes \mathrm{id}_\mathcal{B} + \mathrm{id}_{L^2(0,X)} \otimes B$ is essentially self-adjoint on $\mathcal{H} = L^2(0,X) \otimes \mathcal{B}$. For any $\alpha > \alpha_*$ defined by (4), there exists $C > 0$ such that, for any coefficients (23), setting $k = 1$ if $b_1 = 0$ and $k = 0$ otherwise, for all $T \in]0, \inf(\pi,L)^2]$:*

$$\forall v \in \mathcal{H}^k = H_A^k(0,X) \otimes \mathcal{B}, \quad \|v\|_{\mathcal{H}^k} \leq C \exp(\alpha L^2/T)\|\partial_x^k e^{itH} v_{\rceil x=X}\|_{L^2(0,T;\mathcal{B})} \ .$$

Remarks 6.6. With $\mathcal{B} = L^2(\tilde{M})$, $B = \tilde{\Delta}$, $\mathcal{H} = L^2(C)$, $A = \partial_x^2$, $k = 1$, this theorem applies to the Schrödinger equation on the cylinder $C = [0,X] \times \tilde{M}$ controlled at the end $\Gamma = \{X\} \times \tilde{M}$, with a base $\tilde{M}$ as in theorem 6.3. The segment $S = [0,X]$ is endowed with a Riemannian metric, L denotes the total length of $[0,X]$ and Δ_S denotes the Dirichlet Laplacian on $[0,X]$, so that the Laplacian on the $(n+1)$-dimensional product manifold C is $\Delta_C = \Delta_S + \tilde{\Delta}$. In this setting, the controllability cost is the best constant, denoted $C_{T,\Gamma}$, in the observation inequality:

$$\forall u_0 \in \mathcal{H}^1 = H_0^1(S; L^2(\tilde{M})), \quad \|u_0\|_{\mathcal{H}^1} \leq C_{T,\Gamma}\|\partial_s e^{it\Delta_C} u_{0\rceil\Gamma}\|_{L^2(]0,T[\times M)} \ . \tag{51}$$

Although the geodesics condition is not satisfies for Γ in C, theorem 6.5 proves that the controllability cost $C_{T,\Gamma}$ satisfies, as in theorem 1.2, an upper bound of the same type as the lower bound in theorem 1.1: $\forall\alpha > \alpha_*$, $\exists\beta > 0$, $C_{T,\Gamma} \leq \beta \exp(\alpha L^2/T)$.

Note that the observability inequality (51) does not hold in the " energy space ", i.e. the space $\mathcal{H}^1 = H_0^1(S, L^2(\tilde{M}))$ cannot be replaced by $H_0^1(C)$.

The boundary controllability of a rectangular plate from one side was proved in [KLS85] (theorem 2).

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