SMALL TIME FLUCTUATIONS OF THE BROWNIAN BRIDGE

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ABSTRACT. We consider the small-time asymptotics of the diffusion process associated to an elliptic or sub-elliptic second-order operator, conditioned by its initial and final positions. When these points lie outside the cut-locus of the operator, we establish convergence, to a Gaussian limit, of the fluctuations of the process about the unique path of minimal energy. The Gaussian limit is characterized in terms of the second variation of the energy functional on paths at a minimum, the formulation of which is new in the sub-elliptic case. In the elliptic case our result agrees with one derived by Molchanov. The methods of stochastic differential equations and Malliavin calculus allow us to give a complete proof of Molchanov's result and to extend it to sub-elliptic operators.

1. INTRODUCTION

Let M be a compact, connected C^∞ manifold and let $\mathcal{L}$ be a second order differential operator on M with C^∞ coefficients. Assume that the principal symbol a of $\mathcal{L}$ is non-negative definite and that $\mathcal{L}1 = 0$. In local coordinates $\mathcal{L}$ has the form

$$\mathcal{L} = \tfrac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x^i}.$$

In the case where a is positive definite, $\mathcal{L}$ is elliptic and can be written in the form

$$\mathcal{L} = \tfrac{1}{2}\Delta + \beta$$

where Δ is the Laplace–Beltrami operator corresponding to the Riemannian metric a^{-1} and where β is some vector field. For the main results of this paper, we do not require a to be positive definite. However, we do assume that the following weaker non-degeneracy condition holds: there exist $m \in \mathbb{N}$ and C^∞ vector fields $X_0, X_1, \dots, X_m$ such that

$$\mathcal{L} = \tfrac{1}{2}\sum_{l=1}^{m} X_l^2 + X_0$$

and such that the Lie algebra of commutators generated by $X_1, \dots, X_m$ spans TM at every point. This is called the *bracket condition*. Under this condition, the principal symbol a defines a sub-Riemannian structure on M. A detailed study of this structure is made in §2, extending some standard notions of Riemannian geometry, as a precursor to our main result. On the other hand, since we judge that the simpler, elliptic, case may be

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of more general interest, we have tried to arrange the paper so that the complexities of sub-Riemannian geometry can be ignored easily when appropriate.

Under the bracket condition, it is known that the heat semigroup $(P_t : t \geq 0)$ associated with $\mathfrak{L}$ is given by a positive C^∞ density function

$$p : (0, \infty) \times M \times M \to (0, \infty)$$

with respect to any positive C^∞ reference measure μ on M. The existence and smoothness of p follow from Hörmander's criterion for hypoellipticity [Hör67], applied to the heat operator $\partial/\partial t - \mathfrak{L}$. Positivity is established in [AKS90].

We shall be concerned here, for a given pair of points $x, y \in M$, with the asymptotic behaviour as $t \downarrow 0$ of the diffusion process $(x_s : 0 \leq s \leq t)$ associated with $\mathfrak{L}$, conditioned by $x_0 = x$ and $x_t = y$. Denote by $\Omega^{x,y,t}$ the set of continuous paths $s \mapsto \omega_s : [0, t] \to M$ such that $\omega_0 = x$ and $\omega_t = y$. Then there exists a unique probability measure $\mu^{x,y,t}$ on $\Omega^{x,y,t}$ such that, for all $k \in \mathbb{N}$, for all $0 < t_1 < t_2 < \ldots < t_k < t$ and all $x_1, \ldots, x_k \in M$,

$$\begin{aligned} &\mu^{x,y,t}(\{\omega : \omega_{t_1} \in dx_1, \ldots, \omega_{t_k} \in dx_k\}) \\ &\quad = \frac{p(t_1, x, x_1)p(t_2 - t_1, x_1, x_2) \ldots p(t - t_k, x_k, y)}{p(t, x, y)} \mu(dx_1) \ldots \mu(dx_k). \end{aligned}$$

This measure $\mu^{x,y,t}$ does not depend on the choice of reference measure μ.

In order to describe the asymptotic behaviour of $\mu^{x,y,t}$ as $t \downarrow 0$, it is convenient to rescale time so that all measures considered are defined on the same space $\Omega^{x,y} = \Omega^{x,y,1}$. So define $\tau_\varepsilon : \Omega^{x,y,\varepsilon} \to \Omega^{x,y}$ by $(\tau_\varepsilon \omega)_t = \omega_{\varepsilon t}$ and set

$$\mu_\varepsilon^{x,y} = \mu^{x,y,\varepsilon} \circ (\tau_\varepsilon)^{-1}.$$

Then $\mu_\varepsilon^{x,y}$ is the law of the diffusion process $(x_t^\varepsilon : 0 \leq t \leq 1)$ associated with $\varepsilon\mathfrak{L}$ and conditioned by $x_0^\varepsilon = x, x_1^\varepsilon = y$.

Given a locally Lipschitz path $\omega \in \Omega^{x,y}$, there may exist a measurable path $\xi : [0, 1] \to T^*M$ over ω such that, for almost all t,

$$\dot{\omega}_t = a(\xi_t). \tag{1.1}$$

In that case we define the *energy* of ω by

$$I(\omega) = \int_0^1 a(\xi_t, \xi_t)\, dt,$$

where $a(\xi, \xi) = \langle \xi, a(\xi) \rangle$. This quantity does not depend on the choice of ξ satisfying (1.1). We extend I to the whole of $\Omega^{x,y}$ by setting $I(\omega) = \infty$ in all other cases. The subset of $\Omega^{x,y}$ where I is finite is denoted $H^{x,y}$.

We make the following assumption on x and y throughout: there exists $\gamma \in H^{x,y}$ such that $I(\gamma) < I(\omega)$ for all $\omega \in H^{x,y} \setminus \{\gamma\}$. To set this assumption in context, we note that the bracket condition ensures that $I(\gamma) < \infty$ for some $\gamma \in \Omega^{x,y}$, see [Bis84, Str86]. Moreover, a standard weak compactness argument shows that there exists at least one $\gamma \in H^{x,y}$ of minimal energy. The real content of our assumption is thus that there is not more than one path of minimal energy. The case where there are a finite number is not substantially more difficult, but we leave it aside.

The first order asymptotics of $\mu_\varepsilon^{x,y}$ are given by the following well known result.

Proposition 1.1.

$$\mu_\varepsilon^{x,y} \Rightarrow \delta_\gamma \quad as \quad \varepsilon \downarrow 0.$$

Here $\Rightarrow$ denotes weak convergence of measures on $\Omega^{x,y}$, that is with respect to the class of continuous bounded functions, and δ_γ denotes the unit mass at γ. The topology on $\Omega^{x,y}$ is that of uniform convergence.

The aim of this paper is to identify the second order asymptotics of $\mu_\varepsilon^{x,y}$. We seek to refine the 'law of large numbers' of Proposition 1.1 by a central limit theorem, where the deviation of the process $(x_t^\varepsilon : 0 \le t \le 1)$ from its deterministic limit γ, suitably renormalized, is shown to converge to an explicit Gaussian limit.

Let us discuss now, briefly, two obvious questions about the generality of the class of measures $\mu_\varepsilon^{x,y}$. The first concerns time-reversal. Since M is compact, $\mathcal{L}$ has a smooth, positive invariant measure μ. If we use this as our reference measure in defining the heat kernel $p(t,x,y)$, then $\hat{p}(t,x,y) = p(t,y,x)$ is also a heat kernel, for another operator $\hat{\mathcal{L}}$, which satisfies the same conditions as $\mathcal{L}$. Hence the class of measures $\mu_\varepsilon^{x,y}$ is preserved under time-reversal. The second question concerns the role of compactness. Compactness plays an obvious role in guaranteeing the existence of minimal paths in $\Omega^{x,y}$, but all we really need is the local condition that there exists a compact subset of M which contains all nearly minimal paths in $\Omega^{x,y}$. Without compactness, we have to decide on boundary conditions for the heat flow. Of course it is always possible to choose Dirichlet conditions, for which certainly the measures $\mu_\varepsilon^{x,y}$ are still well defined. The time reversal argument, just given, now runs into some further difficulties. However, by standard estimates, the asymptotics of $\mu_\varepsilon^{x,y}$ as $\varepsilon \downarrow 0$ are unaffected by modifications to $\mathcal{L}$ away from a relatively compact neighbourhood of the unique minimal path in $\Omega^{x,y}$. Thus, our results for the compact case are also informative for diffusion in non-compact manifolds. By sticking to the compact case, we simply avoid some difficulties in the general formulation of the problem, which are irrelevant in any case to the considered asymptotics.

The second order results that we have rely on a further geometric condition, which expresses that I has a non-degenerate minimum at $\gamma \in H^{x,y}$. In the elliptic case, this is equivalent to the standard condition that x and y are non-conjugate along γ. Some care is needed in formulating the condition precisely when a is not positive definite, as $H^{x,y}$ is then not guaranteed to have any reasonable differentiable structure. Instead, for now, following [Bis84], the condition will be given in terms of the bicharacteristic flow of $\mathcal{L}$, which is the Hamiltonian flow on T^*M associated with the principal symbol a. Define $H : T^*M \to \mathbb{R}$ by

$$H(\xi) = a(\xi, \xi)$$

and let Y denote the C^∞ vector field on T^*M given by

$$\beta(Y, .) = dH,$$

where β is the canonical symplectic two-form on T^*M. Then let $(\psi_t : t \in \mathbb{R})$ be the flow of diffeomorphisms $\psi_t : T^*M \to T^*M$ given by

$$\dot{\psi}_t(\xi_0) = Y(\psi_t(\xi_0)), \quad \psi_0(\xi_0) = \xi_0.$$

In local coordinates $\psi_t(\xi_0) = (x_t, p_t)$, we have

$$\begin{aligned} \dot{x}_t &= a(x_t)p_t, \\ \dot{p}_t &= -\nabla a(x_t)(p_t, p_t). \end{aligned}$$

The integral curves of Y are called *bicharacteristics*. We assume that the unique minimal path $\gamma \in H^{x,y}$ is the projection of a bicharacteristic:

$$\gamma_t = \pi\xi_t, \quad \xi_t = \psi_t(\xi_0), \quad \xi_0 \in T_x^*M.$$

This is always the case if a is positive definite. More generally, a useful sufficient condition was obtained by Bismut [Bis84], in terms of the deterministic Malliavin covariance matrix. Under this assumption, we define linear maps

$$J_t : T_x^*M \to T_{\gamma_t}M, \quad K_t : T_y^*M \to T_{\gamma_t}M$$

by

$$J_t\eta_0 = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \pi\psi_t(\xi_0 + \varepsilon\eta_0), \quad K_t\zeta_1 = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \pi\psi_{-(1-t)}(\xi_1 + \varepsilon\zeta_1).$$

In the elliptic case $J_t\eta_0$ and $K_t\zeta_1$ are Jacobi fields along γ.

Proposition 1.2.

$$J_1 = K_0^*.$$

Proof. Since Y is Hamiltonian, its flow preserves the symplectic form β. See for example [Mar74]. For $\eta \in T^*M$ let us write $\tilde{\eta}$ for the corresponding vertical vector in TT^*M and write ψ_t^* for the action of ψ_t on TT^*M. Then

$$\langle J_1\eta, \zeta\rangle = \langle \pi^*\psi_1^*\tilde{\eta}, \zeta\rangle = \beta(\psi_1^*\tilde{\eta}, \tilde{\zeta}) = \beta(\tilde{\eta}, \psi_{-1}^*\tilde{\zeta}) = \langle \eta, \pi^*\psi_{-1}^*\tilde{\zeta}\rangle = \langle \eta, K_0\zeta\rangle.$$

□

If J_1 is invertible we say that x and y are *non-conjugate* along γ. We follow [Bis84], [BA88] in defining the *cut locus* $\mathrm{Cut}(a) \subseteq M \times M$. In fact we define its complement. We say that $(x, y) \notin \mathrm{Cut}(a)$ if:

(i) there exists a unique path of minimal energy $\gamma \in H^{x,y}$;
(ii) there exists $\xi_0 \in T_x^*M$ such that $\gamma_t = \pi\psi_t(\xi_0)$ for all $t \in [0, 1]$;
(iii) x and y are non-conjugate along γ.

This notion coincides with the usual cut locus of Riemannian geometry when a is positive definite.

We now describe a class of rescaling procedures for $\mu_\varepsilon^{x,y}$, which we use to express our main result. Take any C^∞ map $\theta : [0, 1] \times M \to TM$ such that, for $0 \le t \le 1$ and $z \in M$,

$$\theta(t, z) \in T_{\gamma_t}M, \quad \theta(t, \gamma_t) = 0, \quad \frac{\partial\theta}{\partial x^i}(t, \gamma_t) = \frac{\partial}{\partial x^i}\bigg|_{x=\gamma_t}.$$

In the case where a is positive definite, a natural choice is provided by the exponential map $\theta(t, z) = \exp_{\gamma_t}^{-1}(z)$, at least when z is close to γ_t. However our result is insensitive to the choice of θ. Denote by $T_\gamma\Omega^{x,y}$ the set of continuous paths $y : [0, 1] \to TM$ such that $y_t \in T_{\gamma_t}M$ for all t and $y_0 = 0, y_1 = 0$. Define $\sigma_\varepsilon : \Omega^{x,y} \to T_\gamma\Omega^{x,y}$ by

$$(\sigma_\varepsilon\omega)_t = \theta(t, \omega_t)/\sqrt{\varepsilon}, \quad 0 \le t \le 1$$

and set

$$\tilde{\mu}_\varepsilon^{x,y} = \mu_\varepsilon^{x,y} \circ \sigma_\varepsilon^{-1}. \tag{1.2}$$

Proposition 1.3. *Assume that* $(x,y) \notin \mathrm{Cut}(a)$*. Then there exists a unique zero-mean Gaussian measure* μ_γ *on* $T_\gamma\Omega^{x,y}$ *such that, for all* $0 \le s \le t \le 1$

$$\int_{T_\gamma\Omega^{x,y}} y_s \otimes y_t \; \mu_\gamma(dy) = J_s J_1^{-1} K_t^*.$$

Our main result is the following, generalizing a result of Molchanov [Mol75], which is restricted to the elliptic case:

Theorem 1.4. *Assume that* $(x,y) \notin \mathrm{Cut}(a)$*. Then*

$$\tilde{\mu}_\varepsilon^{x,y} \Rightarrow \mu_\gamma \quad as \quad \varepsilon \downarrow 0.$$

The proof of Proposition 1.3 is given in §2, along with a reformulation of the cut-locus and an alternative characterization of μ_γ, both in terms of the second variation of the energy near γ. The analysis needed for Theorem 1.4 is done in §3. The techniques used there enable us to give a more complete proof than that offered by Molchanov.

For the remainder of the introduction we focus on the case where a is positive definite. The Gaussian measure μ_γ then has a number of alternative characterizations, which we now review, making use of some standard notions of Riemannian geometry. We write ∇ for the Levi–Civita connection and R for the Riemannian curvature tensor. Let us fix a minimal path $\gamma \in H^{x,y}$. Define $R_t \in \mathrm{End}\, T_{\gamma_t}M$ by $R_t = R(.,\dot\gamma_t)\dot\gamma_t$. Let $T_\gamma H^{x,y}$ denote the set of paths $v : [0,1] \to TM$ over γ with $v_0 = 0$ and $v_1 = 0$ such that

$$\int_0^1 |\nabla v_t|^2 \, dt < \infty.$$

Then the second variation of the energy I at γ defines a quadratic form on $T_\gamma H^{x,y}$, given by

$$Q(v) = \int_0^1 |\nabla v_t|^2 \, dt - \int_0^1 \langle v_t, R_t v_t \rangle \, dt. \tag{1.3}$$

See, for example, [Kli82]. The map $J_t : T_x^*M \to T_{\gamma_t}M$, defined above, satisfies the Jacobi equation

$$\nabla^2 J_t + R_t J_t = 0, \quad J_0 = 0, \quad \nabla J_0 = a(\gamma_0).$$

Let $(b_t : 0 \le t \le 1)$ denote a Brownian motion in T_xM, starting from 0, and set $z_t = b_t - t b_1$, so $(z_t : 0 \le t \le 1)$ is a Brownian bridge from 0 to 0 in time 1. For $0 \le t \le 1$, write τ_t for the parallel translation $T_xM \to T_{\gamma_t}M$ along γ. Let ν denote the law of $(\tau_t z_t : 0 \le t \le 1)$ on $T_\gamma\Omega^{x,y}$. The property that x and y are non-conjugate along γ can be expressed in a number of different ways:

Proposition 1.5. *Assume that a is positive definite. Let* $\gamma \in H^{x,y}$ *be minimal. Then the following are equivalent:*

(i) J_1 *is invertible;*
(ii) Q *is positive definite on* $T_\gamma H^{x,y}$*;*

(iii) *there exists a unique path* $A \in C^1([0,1) : \operatorname{End} TM)$ *over* γ *solving the Riccati equation:*

$$\nabla A_t + A_t^2 + R_t = 0, \quad (1-t)A_t \to -I \quad \textit{as} \quad t \uparrow 1;$$

(iv) *we have*

$$\int_{T_\gamma \Omega^{x,y}} \exp\left\{\int_0^1 \langle y_t, R_t y_t\rangle\, dt\right\} \nu(dy) < \infty.$$

Equivalence of (i),(ii) and (iii) is standard in Riemannian geometry. In §2, we establish a generalization of the equivalence of (i) and (ii) to sub-Riemannian manifolds. For (iv), we refer to [Bis84], Theorem 4.17.

Theorem 1.6. *Assume that* a *is positive definite. Let* $\gamma \in H^{x,y}$ *be minimal and suppose that* x *and* y *are non-conjugate along* γ*. Let* μ *be a zero-mean Gaussian measure on* $T_\gamma \Omega^{x,y}$*. Then the following are equivalent:*

(i) *for all* $0 \le s \le t \le 1$,

$$\int_{T_\gamma \Omega^{x,y}} y_s \otimes y_t \; \mu(dy) = J_s J_1^{-1} K_t^*;$$

(ii) μ *has reproducing-kernel Hilbert space* $(T_\gamma H^{x,y}, Q)$*;*

(iii) *under* μ*, the coordinate process* y *on* $T_\gamma \Omega^{x,y}$ *satisfies a covariant linear stochastic differential equation over* γ *of the form*

$$D y_t = \tau_t db_t + A_t y_t dt, \quad y_0 = 0;$$

(iv) μ *is absolutely continuous with respect to* ν*, with Radon–Nikodym derivative*

$$\frac{d\mu}{d\nu}(y) \propto \exp\left\{\int_0^1 \langle y_t, R_t y_t\rangle\, dt\right\}. \tag{1.4}$$

Proof. There is exactly one zero-mean Gaussian measure satisfying (ii). The same is true for (iii) and (iv). There is at most one zero-mean Gaussian measure satisfying (i). The equivalence of (i) and (ii) is established in a more general context in §2.

Suppose that μ satisfies (iii). Note that $A_t K_t + \nabla K_t = 0$ and K_t is invertible for $0 \le t < 1$. Set $w_t = K_t^{-1} y_t$, then $K_t dw_t = \tau_t db_t$, so

$$w_t = \int_0^t K_s^{-1} \tau_s db_s.$$

Hence, for $0 \le s \le t \le 1$,

$$\mathbb{E}(y_s \otimes y_t) = K_s \left(\int_0^s K_r^{-1}(K_r^{-1})^* dr\right) K_t^* = J_s J_1^{-1} K_t^*,$$

the last equality obtained by verifying that $K_s(\int_0^s K_r^{-1}(K_r^{-1})^* dr) J_1$ satisfies the Jacobi equation. Hence (i) is equivalent to (iii).

Given (1.3), it is a routine exercise in Gaussian processes to establish the equivalence of (ii) and (iv). See for example [?]. □

We examine now how our result specializes in some simple cases. When $\mathcal{L}$ is the Laplacian on $\mathbb{R}^n$, the analysis is trivial, because $\tilde{\mu}_\varepsilon^{x,y} = \mu_\gamma$ for all $\varepsilon > 0$. We have $J_s = sI, K_t = tI$ and $A_t = -(1-t)^{-1}I$, so the alternatives in Theorem 1.6 recover some of the standard descriptions of the Brownian bridge in $\mathbb{R}^n$.

In the case where $\mathcal{L}$ is the Laplace–Beltrami operator on a sphere or on hyperbolic space, we can rewrite (1.4) in the form

$$\frac{d\mu_\gamma}{d\nu}(y) \propto \exp\left\{\frac{Kd(x,y)^2}{2}\int_0^1 |y_t|^2\,dt\right\},$$

where K, the sectional curvature, is 1 for the sphere and -1 for hyperbolic space. Thus, on a sphere, the variance of the fluctuations is larger than in $\mathbb{R}^n$, whereas, in hyperbolic space it is less. This does not contradict the tendency of Brownian paths to separate quickly in hyperbolic space because we are conditioning on the endpoint. Thus we tend to see those paths which have never deviated far from the geodesic.

2. Second variation of the energy for sub-Riemannian manifolds

In this section we develop the geometric notions needed to express our central limit theorem for conditioned diffusions. We shall show that, for a certain class of paths ω in $H^{x,y}$, there is a well-defined tangent space $T_\omega H^{x,y}$ and, when ω is minimizing, there is a non-negative quadratic form Q on $T_\omega H^{x,y}$, which arises as the second variation of the energy function. The form Q is positive when x and y are non-conjugate along ω. This is standard for Riemannian geometry, see for example [Kli82]. The essentials of the sub-Riemannian case were worked out by Bismut [Bis84]. Our contribution here is to demonstrate that certain objects introduced by Bismut are intrinsic to the principal symbol a and do not depend on the choice of vector fields $X_1, \dots, X_m$ in the representation

$$a(x) = \sum_{l=1}^m X_l(x) \otimes X_l(x). \tag{2.1}$$

The section finishes with a number of equivalent formulations of the Gaussian process which appears as a limit in our main theorem.

Denote by Ω the set of continuous paths $\omega : [0,1] \to M$ and by H the set of finite-energy paths. Recall that ω is of *finite energy* if ω is absolutely continuous and there exists a measurable path $\xi : [0,1] \to T^*M$ over ω such that, for almost all t,

$$\dot{\omega}_t = a(\xi_t) \tag{2.2}$$

and such that

$$I(\omega) = \int_0^1 a(\xi_t, \xi_t)\,dt < \infty.$$

We say that ξ is a *regular lifting* of ω if (2.2) holds and, for some smooth Riemannian metric on M,

$$\int_0^1 |\xi_t|^2\,dt < \infty.$$

Obviously this condition does not depend on the choice of metric. We say that a finite-energy path ω is *regular* if it has a regular lifting. If a is not of constant rank, then not all finite-energy paths are regular. We write H^x for the set of finite-energy paths starting at

x and $H^{x,y}$ for the set of such paths terminating at y. We say that $\omega \in H^{x,y}$ is *minimal* if $I(\omega) \leq I(\omega')$ for all $\omega' \in H^{x,y}$. Write $H^0(\mathbb{R}^m)$ for the space of absolutely continuous paths $h : [0,1] \to \mathbb{R}^m$ starting from 0 such that

$$\|h\|^2 = \int_0^1 |\dot{h}_t|^2 dt < \infty.$$

Given the representation (2.1) and $\omega \in H^x$, we denote by $h(\omega)$ the unique element in $H^0(\mathbb{R}^m)$ such that

$$\dot{h}_t(\omega) = X(\omega_t)^* \xi_t$$

whenever ξ satisfies (2.2). Here $X(x) : \mathbb{R}^m \to T_x M$ is given by

$$X(x) = (X_1(x), \dots, X_m(x)).$$

We can define a map $\phi : M \times H^0(\mathbb{R}^m) \to \Omega$ by the differential equation

$$\dot{\phi}_t(x,h) = X_l(\phi_t(x,h))\dot{h}_t^l, \quad \phi_0(x,h) = x.$$

Here and throughout, the repeated index l is summed from 1 to m. Denote by $p_X(x)$ the orthogonal projection $\mathbb{R}^m \to (\ker X(x))^\perp$. Given $k \in H^0(\mathbb{R}^m)$, define $\pi(x,h)k \in H^0(\mathbb{R}^m)$ by

$$\dot{\pi}_t(x,h)k = p_X(\phi_t(x,h))\dot{k}_t, \quad \pi_0(x,h)k = 0.$$

The following result gives the basic parameterization of H^x used by Bismut.

Proposition 2.1. *For all $x \in M$ and $h \in H^0(\mathbb{R}^m)$,*

(i) $\operatorname{im} \phi(x,.) = H^x$,
(ii) $\phi(x,h) = \phi(x, \pi(x,h)h)$,
(iii) $I(\phi(x,h)) = \|\pi(x,h)h\|^2 \leq \|h\|^2$,
(iv) $h(\phi(x,h)) = \pi(x,h)h$.

Proof. Claim (ii) and the inequality in (iii) are obvious. Fix $x \in M$ and $h \in H^0(\mathbb{R}^m)$. Set $\omega_t = \phi_t(x,h)$. Since $(\ker X(x))^\perp = \operatorname{im} X(x)^*$, there is a measurable map $\xi : [0,1] \to T^*M$ over ω such that $\dot{\pi}_t(x,h)h = X(\omega_t)^*\xi_t$. Then $\dot{\omega}_t = X(\omega_t)X(\omega_t)^*\xi_t = a(\xi_t)$ and

$$I(\omega) = \int_0^1 a(\xi_t, \xi_t)dt = \int_0^1 |X(\omega_t)^*\xi_t|^2 dt = \|\pi(x,h)h\|^2.$$

Hence $\operatorname{im} \phi(x,.) \subseteq H^x$ and (iii) holds.

On the other hand, given $\omega \in H^x$, we have $\dot{\omega}_t = X(\omega_t)\dot{h}_t(\omega)$, so $\phi(x, h(\omega)) = \omega$. Hence $H^x \subseteq \operatorname{im} \phi(x,.)$. □

We now define the space of *finite-energy variations* of a regular finite-energy path $\omega \in H^x$. Fix a regular lifting $\xi : [0,1] \to T^*M$ of ω and denote by $T_\omega H^x$ the set of absolutely continuous maps $v : [0,1] \to TM$ over ω such that

$$\dot{v}_t = \nabla a(\xi_t)v_t + a(\eta_t), \quad v_0 = 0 \tag{2.3}$$

for some $\eta : [0,1] \to T^*M$ over ω, with

$$\|v\|_\xi^2 = \int_0^1 a(\eta_t, \eta_t)dt < \infty.$$

The differential equation for v is independent of the coordinate system in which it is written. It is a consequence of the next proposition that the space $T_\omega H^x$ does not depend on the choice of ξ, that the norms $\|\dots\|_\xi$ are all equivalent, and that all make $T_\omega H^x$ into a Hilbert space.

By standard arguments, the map $\phi : M \times H^0(\mathbb{R}^m) \to H$, defined above, is differentiable. Set $\omega_t = \phi_t(x, h)$. Also set

$$u_t = \frac{\partial}{\partial x}\phi_t(x, h) \in T_{\omega_t}M \otimes T_x^*M$$

and, fixing $k \in H^0(\mathbb{R}^m)$, set

$$v_t = \left(\frac{\partial}{\partial h}\phi_t(x, h)\right) k \in T_{\omega_t}M.$$

Then u and v satisfy the differential equations

$$\dot{u}_t = \nabla X_l(\omega_t)u_t\dot{h}_t^l, \quad u_0 = I,$$
$$\dot{v}_t = \nabla X_l(\omega_t)v_t\dot{h}_t^l + X_l(\omega_t)\dot{k}_t^l, \quad v_0 = 0.$$

Note that

$$v_t = u_t \int_0^t u_s^{-1} X_l(\omega_s)\dot{k}_s^l ds. \tag{2.4}$$

Proposition 2.2. *Let $\omega \in H^x$ be regular and set $h = h(\omega)$. Then*

$$\operatorname{im} \frac{\partial}{\partial h}\phi(x, h) = T_\omega H^x.$$

Moreover, given $v \in T_\omega H^x$, a regular lifting ξ of ω, and some η satisfying (2.3)*, let us define $k = k(\xi, v)$ by*

$$\dot{k}_t^l = \langle \xi_t, \nabla X_l(\omega_t)v_t\rangle + \langle \eta_t, X_l(\omega_t)\rangle, \quad k_0 = 0.$$

Then $k \in H^0(\mathbb{R}^m)$ and

$$v = \frac{\partial}{\partial h}\phi(x, h)k.$$

Moreover there is a constant $C < \infty$, depending only on ξ, such that

$$C^{-1}\|k\| \le \|v\|_\xi \le C\|\pi(x, h)k\|.$$

Proof. Fix a Riemannian metric on M and a regular lifting ξ of ω. Given $k \in H^0(\mathbb{R}^m)$, set $v = (\partial\phi/\partial h)(x, h)k$. Then v satisfies

$$\dot{v}_t = \nabla X_l(\omega_t)v_t\dot{h}_t^l + X_l(\omega_t)\dot{k}_t^l, \quad v_0 = 0, \tag{2.5}$$

so we have an estimate

$$\|v\|_\infty \le C\|\pi(x, h)k\|.$$

Define g by

$$\dot{g}_t^l = \langle \xi_t, \nabla X_l(\omega_t)v_t\rangle, \quad g_0 = 0.$$

Since ξ is regular we have an estimate $\|g\| \le C\|v\|_\infty$. We can find a measurable map $\eta : [0, 1] \to T^*M$ over ω, such that

$$a(\eta_t) = X_l(\omega_t)(\dot{k}_t^l - \dot{g}_t^l). \tag{2.6}$$

Note that

$$\nabla a(x) = \nabla X_l(x)X_l(x)^* + X_l(x)\nabla X_l(x)^*.$$

Then v satisfies the equation

$$\dot{v}_t = \nabla a(\xi_t)v_t + a(\eta_t), \quad v_0 = 0 \tag{2.7}$$

and, moreover

$$\|v\|_\xi^2 = \int_0^1 a(\eta_t, \eta_t)dt = \|\pi(x,h)(k-g)\|^2 \leq C\|\pi(x,h)k\|^2.$$

Hence $v \in T_\omega H^x$.

On the other hand, given $v \in T_\omega H^x$, from (2.7) we obtain the estimate $\|v\|_\infty \leq C\|v\|_\xi$. Now $k = k(\xi, v)$ satisfies

$$\dot{k}_t^l = \dot{g}_t^l + \langle \eta_t, X_l(\omega_t)\rangle$$

so v satisfies (2.5) and $\|k\| \leq C\|v\|_\xi$. □

We say that a regular path $\omega \in H^{x,y}$ is *elliptic* if

$$\{v_1 : v \in T_\omega H^x\} = T_y M.$$

This condition is obviously independent of the representation (2.1). Moreover, it is easy to see that ω is elliptic if and only if its time-reversal is elliptic. When a is non-degenerate, every $\omega \in H^{x,y}$ is elliptic. By Proposition 2.2, a regular finite-energy path ω is elliptic if and only if the linear map

$$\frac{\partial}{\partial h}\phi_1(x, h(\omega)) : H_\omega^0(\mathbb{R}^m) \to T_y M$$

is onto. By (2.4), this in turn is equivalent to Bismut's condition that the *deterministic Malliavin covariance matrix*

$$C(\omega) = \int_0^1 \left(u_t^{-1}X_l(\omega_t)\right)\left(u_t^{-1}X_l(\omega_t)\right)^* dt \tag{2.8}$$

is invertible.

Suppose now that $\omega \in H^{x,y}$ is elliptic. Fix a representation (2.1) and set $h = h(\omega)$. Let

$$K = \ker \frac{\partial}{\partial h}\phi_1(x,h).$$

Then

$$K^\perp = \{k \in H^0(\mathbb{R}^m) : \dot{k}_t^l = \langle \eta_0, u_t^{-1}X_l(\omega_t)\rangle, \eta_0 \in T_x^* M\}$$

and $(\partial/\partial h)\phi_1(x,h)|_{K^\perp}$ is invertible. So, by the implicit function theorem, there exist $\delta > 0$ and a C^∞ map

$$\theta : \{k \in K : \|k\| < \delta\} \to K^\perp$$

such that, for $k \in K, k' \in K^\perp$ with $\|k + k'\| < \delta$,

$$\phi_1(x, h + k + k') = y \quad \text{if and only if} \quad k' = \theta(k).$$

Note that $\theta(0) = 0$. For $k \in K$ sufficiently small, we have

$$\phi_1(x, h + k + \theta(k)) = y \tag{2.9}$$

so

$$\frac{\partial}{\partial h}\phi_1(x,h)(k+\theta'(0)k)=0.$$

This forces $\theta'(0)=0$. On differentiating (2.9) twice, we obtain the following formula which determines $\theta''(0)$:

$$\frac{\partial^2}{\partial h^2}\phi_1(x,h)(k,k)+\frac{\partial}{\partial h}\phi_1(x,h)\theta''(0)(k,k)=0. \tag{2.10}$$

We note also the following useful identity: for $k,k'\in K^\perp$ with

$$\dot{k}_t^l=\langle\eta_0,u_t^{-1}X_l(\omega_t)\rangle,\quad \eta_0\in T_x^*M$$

and

$$\frac{\partial}{\partial h}\phi_1(x,h)k'=v_1$$

we have

$$\langle k,k'\rangle=\int_0^1\langle\eta_0,u_t^{-1}X_l(\omega_t)\rangle(\dot{k}'_t)^l\,dt=\langle\eta_0,u_1^{-1}v_1\rangle. \tag{2.11}$$

Theorem 2.3. *Let $\omega\in H^{x,y}$ be elliptic and let ξ be a regular lifting of ω. Let $v\in T_\omega H^{x,y}$ and suppose that η in* (2.7) *can be chosen so that*

$$\int_0^1|\eta_t|^2dt<\infty. \tag{2.12}$$

Then there exists a measurable map

$$(\varepsilon,t)\mapsto\xi_t^\varepsilon:(-1,1)\times[0,1]\to T^*M$$

such that $\xi^0=\xi$ and:

(i) *$\omega^\varepsilon=\pi\xi^\varepsilon\in H^{x,y}$ for all ε, with $\dot\omega_t^\varepsilon=a(\xi_t^\varepsilon)$;*
(ii) *in any system of coordinates along ω, there is a constant $C<\infty$ such that, for all ε,*

$$\sup_t|\omega_t^\varepsilon-\omega_t-\varepsilon v_t|\le C\varepsilon^2,$$

and, writing $\xi^\varepsilon=(\omega^\varepsilon,p^\varepsilon)$ and $\eta=(\omega,q)$,

$$\int_0^1 a(\omega_t^\varepsilon)(p_t^\varepsilon-p_t-\varepsilon q_t,p_t^\varepsilon-p_t-\varepsilon q_t)dt\le C\varepsilon^4.$$

For any such map, the map $\varepsilon\mapsto I(\omega^\varepsilon)$ is differentiable at $\varepsilon=0$ and

$$L(v)=\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0}I(\omega^\varepsilon)$$

is a densely defined, bounded linear form on $T_\omega H^{x,y}$. In the case where ω is minimal, the map $\varepsilon\mapsto I(\omega^\varepsilon)$ is twice differentiable at $\varepsilon=0$ and

$$Q_\xi(v)=\frac{1}{2}\left.\frac{\partial^2}{\partial\varepsilon^2}\right|_{\varepsilon=0}I(\omega^\varepsilon) \tag{2.13}$$

is a densely defined, bounded quadratic form on $T_\omega H^{x,y}$. Finally, given a representation (2.1)*, for $h=h(\omega)$ and $k=k(\xi,v)$, we have*

$$L(v)=2\langle h,k\rangle$$

and, when ω is minimal, then $h \in K^\perp$ and
$$Q_\xi(v) = \|k\|^2 + \langle h, \theta''(0)(k,k)\rangle.$$

Proof. We will first use the representation (2.1) to construct a map ξ^ε having the claimed properties. Set $h = h(\omega)$ and $k = k(\xi, v)$. Since $v_1 = 0$, we have $k \in K$. Set $\omega_t^\varepsilon = \phi_t(x, h^\varepsilon)$, where $h^\varepsilon = h+\varepsilon k+\theta(\varepsilon k)$ and where θ is given by (2.9). Choose a system of local coordinates along ω and write $\xi = (\omega, p)$ and $\eta = (\omega, q)$. Then, for some $C < \infty$, for ε sufficiently small, we have $\omega^\varepsilon \in H^{x,y}$ and
$$\sup_t |\omega_t^\varepsilon - \omega_t - \varepsilon v_t| \le C\varepsilon^2.$$
Moreover
$$\dot\omega_t^\varepsilon - a(\omega_t^\varepsilon)(p_t + \varepsilon q_t) = X_l(\omega_t^\varepsilon)\dot g_t^{\varepsilon,l},$$
where
$$\begin{aligned}\dot g_t^{\varepsilon,l} &= \dot h_t^l + \varepsilon \dot k_t^l + \dot\theta_t^l(\varepsilon k) - \langle p_t + \varepsilon q_t, X_l(\omega_t^\varepsilon)\rangle \\ &= -\langle p_t, X_l(\omega_t^\varepsilon) - X_l(\omega_t) - \varepsilon\nabla X_l(\omega_t)v_t\rangle \\ &\quad - \varepsilon\langle q_t, X_l(\omega_t^\varepsilon) - X_l(\omega_t)\rangle + \dot\theta_t^l(\varepsilon k).\end{aligned}$$
For some $C < \infty$, for ε sufficiently small, we have
$$\int_0^1 |\dot g_t^\varepsilon|^2 dt \le C\varepsilon^4$$
so we can find a measurable map r_t^ε such that
$$X_l(\omega_t^\varepsilon)\dot g_t^{\varepsilon,l} = a(\omega_t^\varepsilon) r_t^\varepsilon$$
and
$$\int_0^1 a(\omega_t^\varepsilon)(r_t^\varepsilon, r_t^\varepsilon)dt \le C\varepsilon^4.$$
If we now set $p_t^\varepsilon = p_t + \varepsilon q_t + r_t^\varepsilon$, then $\xi_t^\varepsilon = (\omega_t^\varepsilon, p_t^\varepsilon)$ has the required properties.

Suppose now, more generally, that $\xi^\varepsilon = (\omega^\varepsilon, p^\varepsilon)$ and $\eta = (\omega, q)$ denote maps having the properties described in the statement. Define h^ε by
$$\dot h_t^{\varepsilon,l} = \langle p_t^\varepsilon, X_l(\omega_t^\varepsilon)\rangle, \quad h_0^\varepsilon = 0.$$
Then
$$\begin{aligned}&\dot h_t^{\varepsilon,l} - \dot h_t^l - \varepsilon \dot k_t^l \\ &\quad = \langle p_t^\varepsilon - p_t - \varepsilon q_t, X_l(\omega_t^\varepsilon)\rangle + \varepsilon\langle q_t, X_l(\omega_t^\varepsilon) - X_l(\omega_t)\rangle \\ &\qquad + \langle p_t, X_l(\omega_t^\varepsilon) - X_l(\omega_t) - \varepsilon\nabla X_l(\omega_t)v_t\rangle\end{aligned}$$
so
$$\|h^\varepsilon - h - \varepsilon k\| \le C\varepsilon^2.$$
Now $\phi_1(x, h^\varepsilon) = y$, so, for ε sufficiently small,
$$h^\varepsilon = h + \varepsilon k^\varepsilon + \theta(\varepsilon k^\varepsilon)$$
where k^ε denotes the orthogonal projection of $\varepsilon^{-1}(h^\varepsilon - h)$ onto K. Since $k \in K$, we have $\|k^\varepsilon - k\| \le C\varepsilon$. Hence, as $\varepsilon \to 0$,
$$I(\omega^\varepsilon) - I(\omega) = \|h + \varepsilon k^\varepsilon + \theta(\varepsilon k^\varepsilon)\|^2 - \|h\|^2 = 2\varepsilon\langle h, k\rangle + O(\varepsilon^2).$$
Hence $\varepsilon \mapsto I(\omega^\varepsilon)$ is differentiable at $\varepsilon = 0$ with the claimed derivative.

Now, if ω is minimal, we must have $L(v) = 0$ for all $v \in T_\omega H^{x,y}$. For any $k' \in K$, if $v = (\partial\phi/\partial h)(x,h)k'$, then $\langle k' - k(\xi, v), h\rangle = 0$, so $L(v) = 0$ forces $h \in K^\perp$. Hence, for ω minimal, we have, as $\varepsilon \to 0$,

$$I(\omega^\varepsilon) - I(\omega) = \varepsilon^2\|k^\varepsilon\|^2 + 2\langle h, \theta(\varepsilon k^\varepsilon)\rangle + \|\theta(\varepsilon k^\varepsilon)\|^2 = \varepsilon^2\{\|k\|^2 + \langle h, \theta''(0)(k,k)\rangle\} + O(\varepsilon^3).$$

This shows that $\varepsilon \mapsto I(\omega^\varepsilon)$ is twice differentiable at $\varepsilon = 0$ with the claimed second derivative.

Finally, we note that L and Q are bounded, by the estimates of Proposition 2.2, and that (2.12) can be satisfied on at least a dense subspace in $T_\omega H^{x,y}$. □

We now investigate the relationship between minimal paths, the quadratic forms Q_ξ, and the bicharacteristic flow, introduced in §1.

We recall that

$$K^\perp = \{k \in H^0(\mathbb{R}^m) : \dot{k}^l_t = \langle \eta_0, u_t^{-1}X_l(\omega_t)\rangle, \eta_0 \in T^*_x M\}.$$

By Theorem 2.3, if ω is elliptic and minimal, then $h = h(\omega) \in K^\perp$. So we can write $\dot{h}^l_t = \langle \xi_0, u_t^{-1}X_l(\omega_t)\rangle$ for some $\xi_0 \in T^*_x M$. Write, in some system of local coordinates along ω, $\xi_0 = (\omega_0, p_0)$, and set $\xi_t = (\omega_t, p_t)$, where $p_t = (u_t^{-1})^* p_0$. Then

$$\begin{aligned}\dot\omega_t &= X_l(\omega_t)\dot{h}^l_t,\\ \dot{p}_t &= -\langle p_t, \nabla X_l(\omega_t)\rangle \dot{h}^l_t,\\ \dot{h}^l_t &= \langle p_t, X_l(\omega_t)\rangle.\end{aligned}$$

It follows by uniqueness of solutions that $\xi_t = \psi_t(\xi_0)$. In particular, ω is the projection of a bicharacteristic.

The preceding argument is of a standard type and is copied from [Bis84]. The crucial observation is that, when ω is elliptic and minimal, then, for some representation (2.1), $\omega = \phi(x,h)$ with $h \in K^\perp$. This observation in fact holds under the weaker conditions that $\omega = \phi(x,h)$ is minimal and $(\partial/\partial h)\phi_1(x,h)$ is of maximal rank. See [Bis84] or the proof of Theorem 2.3. In particular, if we make these, representation-specific, conditions then no *a priori* assumption of regularity is needed – but regularity comes out as a conclusion, as bicharacteristics are smooth.

Proposition 2.4. *Let $\omega \in H^{x,y}$ be elliptic and minimal. Let ξ denote some bicharacteristic over ω and let $v \in T_\omega H^{x,y}$. Fix some representation* (2.1) *and set $h = h(\omega)$ and, for $k \in K$, set*

$$q(k) = \|k\|^2 - \langle \xi_1, (\partial/\partial h)^2\phi_1(x,h)(k,k)\rangle.$$

Then

$$Q_\xi(v) = \inf\left\{q(k) : \frac{\partial}{\partial h}\phi(x,h)k = v, k \in H^0(\mathbb{R}^m)\right\}.$$

Proof. Note that $\dot{h}^l_t = \langle \xi_0, u_t^{-1}X_l(\omega_t)\rangle$ and $\xi_1 = (u_1^{-1})^*\xi_0$, so by (2.10) and (2.11), for $k \in K$,

$$q(k) = \|k\|^2 + \langle h, \theta''(0)(k,k)\rangle. \tag{2.14}$$

So by Theorem 2.3, we have $Q_\xi(v) = q(k(\xi, v))$. Let $k' \in H^0(\mathbb{R}^m)$ with

$$\frac{\partial}{\partial h}\phi(x,h)k' = 0.$$

Then, since ω is minimal and $\|h\|^2 = I(\omega)$,

$$q(k') = \frac{1}{2}\,\frac{\partial^2}{\partial\varepsilon^2}\bigg|_{\varepsilon=0} \|h + \varepsilon k' + \theta(\varepsilon k')\|^2 \geq 0.$$

So it suffices to show that, for such k' and for $k = k(\xi, v)$, we have $q(k, k') = 0$.

Note that $X_l(\omega_t)(\dot{k}'_t)^l = 0$ for all t. We differentiate the identity

$$\left(\frac{\partial}{\partial x}\phi_1(x,h)\right)^{-1} \frac{\partial}{\partial h}\phi_1(x,h)k' = \int_0^1 \left(\frac{\partial}{\partial x}\phi_t(x,h)\right)^{-1} X_l(\phi_t(x,h))(\dot{k}'_t)^l\,dt \tag{2.15}$$

in h, in the direction k, to obtain

$$u_1^{-1}\frac{\partial^2}{\partial h^2}\phi_1(x,h)(k,k') = \int_0^1 u_t^{-1}\nabla X_l(\omega_t)v_t(\dot{k}'_t)^l\,dt.$$

Hence

$$\begin{aligned}\langle \xi_1, \frac{\partial^2}{\partial h^2}\phi_1(x,h)(k,k')\rangle &= \langle \xi_0, \int_0^1 u_t^{-1}\nabla X_l(\omega_t)v_t(\dot{k}'_t)^l\,dt\rangle \\ &= \int_0^1 \langle \xi_t, \nabla X_l(\omega_t)v_t\rangle(\dot{k}'_t)^l\,dt = \langle k, k'\rangle.\end{aligned}$$

This shows that $q(k, k') = 0$ as required. □

Proposition 2.4, together with (2.14), shows that the quadratic form $Q = Q_\xi$ does not depend on the choice of bicharacteristic ξ over ω. So from now on we shall drop the subscript ξ.

For $s \leq t$, define $J_{ts} : T^*_{\gamma_s}M \to T_{\gamma_t}M$ by

$$J_{ts}\eta_s = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \pi\psi_{t-s}(\xi_s + \varepsilon\eta_s).$$

Set $J_t = J_{t0}$, as in §1, and set $v_t = J_t\eta_0$. By differentiating the equations

$$\begin{aligned}\omega_t &= \phi_t(x,h),\\ \dot{h}^l_t &= \langle \xi_0, \left(\frac{\partial}{\partial x}\phi_t(x,h)\right)^{-1} X_l(\phi_t(x,h))\rangle,\end{aligned}$$

in ξ_0, in the direction η_0, we find that $v = (\partial/\partial x)\phi(x,h)k$, where k satisfies

$$\dot{k}^l_t = \langle \eta_0, u_t^{-1}X_l(\omega_t)\rangle + A^l_{0t}(\xi, k) \tag{2.16}$$

and

$$A^l_{0t}(\xi,k) = \langle \xi_0, \frac{\partial}{\partial h}\left[\left(\frac{\partial}{\partial x}\phi_t(x,h)\right)^{-1} X_l(\phi_t(x,h))\right] k\rangle.$$

By writing a differential equation for $A^l_{0t}(\xi, k)$, we obtain the estimate, for $0 \leq t \leq 1$,

$$|A_{0t}(\xi,k)|^2 \leq C\int_0^t |\dot{k}_s|^2\,ds.$$

It follows that the equation (2.16) uniquely determines k.

Proposition 2.5. *Let $\omega \in H^{x,y}$ be elliptic and minimal and let $v \in T_\omega H^{x,y}$. Then $Q(v) = 0$ if and only if $v_t = J_t\eta_0$, for some $\eta_0 \in T^*_xM$.*

Proof. Suppose that $Q(v) = 0$. Set $k = k(\xi, v)$, where ξ is some bicharacteristic over ω. Then $q(k) = 0$. Since ω is minimal, q is non-negative, so $q(k, k') = 0$ for all $k' \in K$. On differentiating the identity (2.15) in h, in the direction k, we obtain

$$\begin{aligned} &u_1^{-1} \frac{\partial^2}{\partial h^2} \phi_1(x,h)(k,k') \\ &\quad = \int_0^1 (\dot{k}'_t)^l \left\{ u_t^{-1} X_l(\omega_t) v_t + \frac{\partial}{\partial h} \left[\left(\frac{\partial}{\partial x} \phi_t(x,h) \right)^{-1} X_l(\phi_t(x,h)) \right] k \right\} dt \end{aligned}$$

so

$$0 = q(k,k') = \int_0^1 (\dot{k}'_t)^l \left\{ \dot{k}_t^l - \langle \xi_0, \frac{\partial}{\partial h} \left[\left(\frac{\partial}{\partial x} \phi_t(x,h) \right)^{-1} X_l(\phi_t(x,h)) \right] k \rangle \right\} dt.$$

Hence there exists an $\eta_0 \in T_x^* M$ such that

$$\dot{k}_t^l = \langle \eta_0, u_t^{-1} X_l(\omega_t) \rangle + A_{0t}^l(\xi_0, k).$$

As we argued above, this forces $v_t = J_t \eta_0$ for all t.

The reverse implication is now obvious. □

Proposition 2.6. *Let $\omega \in H^{x,y}$ be elliptic and minimal. Then, for $\eta_0 \in T_x^* M$, $J_t \eta_0 = 0$ for all t implies $\eta_0 = 0$.*

Proof. Fix a representation (2.1) and set $k = k(\xi, v)$, where ξ is some bicharacteristic over ω and $v_t = J_t \eta_0$. Then $v = 0$ implies $k = 0$, by Proposition 2.2, which implies

$$\langle \eta_0, u_t^{-1} X_l(\omega_t) \rangle = 0$$

for all t, by (2.16), which implies $\eta_0 = 0$ since ω is elliptic. □

Theorem 2.7. *Let $\omega \in H^{x,y}$ be elliptic and minimal. Then Q is positive-definite on $T_\omega H^{x,y}$ if and only if J_1 is invertible.*

Proof. Suppose $J_1 \eta_0 = 0$, then $v_t = J_t \eta_0 \in T_\omega H^{x,y}$ and $Q(v) = 0$. Hence, if Q is positive-definite, then $J_t \eta_0 = 0$ for all t, so $\eta_0 = 0$, by Proposition 2.6.

On the other hand, if $Q(v) = 0$ for some $v \in T_\omega H^{x,y}$, then $v_t = J_t \eta_0$ for some $\eta_0 \in T_x^* M$ by Proposition 2.5, and $J_1 \eta_0 = 0$. So, if J_1 is invertible, then $\eta_0 = 0$, so $v = 0$. □

Recall from §1 that we define $K_t : T_y^* M \to T_{\omega_t} M$ by

$$K_t \zeta_1 = \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon = 0} \pi \psi_{-(1-t)}(\xi_1 + \varepsilon \zeta_1)$$

and, for $0 \le s \le t \le 1$, set

$$C(s,t) = C(t,s)^* = J_s J_1^{-1} K_t^* \in T_{\omega_s} M \otimes T_{\omega_t} M.$$

Proposition 2.8. *Let $\omega \in H^{x,y}$ be elliptic and minimal. Suppose that x and y are non-conjugate along ω. Given $\beta \in T_{\omega_t}^* M$, set $v_s^{\beta,t} = C(s,t)\beta$. Then, for all $v \in T_\omega H^{x,y}$,*

$$Q(v, v^{\beta,t}) = \langle \beta, v_t \rangle.$$

Proof. Set $\eta_0 = J_1^{-1} J_{t1}\beta$ and set

$$w = J_s\eta_0 - J_{st}\beta 1_{s>t}$$

so that $w_1 = 0$ and $w \in T_\omega H^{x,y}$. Then $k = k(\xi, w) = k^1 + k^2$, where

$$(\dot{k}_s^1)^l = \langle \eta_0, u_s^{-1} X_l(\omega_s)\rangle + A_{0s}^l(\xi, k^1)$$

and $(\dot{k}_s^2)^l = 0$ for $s \le t$ and, for $s > t$,

$$(\dot{k}_s^2)^l = \langle \beta, u_{st}^{-1} X_l(\omega_s)\rangle + A_{ts}^l(\xi, k^2) = \langle u_t^*\beta, u_s^{-1} X_l(\omega_s)\rangle + A_{0s}^l(\xi, k^2).$$

Now, for $k' \in K$ and for $v' = (\partial/\partial h)\phi(x, h)k'$ we have

$$\begin{aligned} Q(w, v') = q(k, k') &= \int_0^1 (\dot{k}_s')^l \left\{ \dot{k}_s^l - A_{0s}^l(\xi, k) \right\} ds \\ &= \int_0^1 \langle \eta_0 + u_t^*\beta, u_s^{-1} X_l(\omega_s)\rangle (\dot{k}_s')^l \, ds \\ &= \langle \beta, v' \rangle. \end{aligned}$$

By uniqueness in (2.16), w is the unique element of $T_\omega H^{x,y}$ having this property. For $s \le t$ we have

$$w_s = J_{s0} J_{10}^{-1} J_{1t}\beta = J_s J_1^{-1} K_t^*\beta = C(s, t)\beta.$$

By time symmetry, for $s > t$, we have

$$w_s = K_{s1} K_{01}^{-1} K_{0t}\beta = K_s (J_1^{-1})^* J_t^*\beta = C(t, s)^*\beta.$$

Hence $w = v^{\beta,t}$, and $v^{\beta,t}$ has the claimed property. ☐

We can now prove a strengthening of Proposition 1.3:

Theorem 2.9. *Let $\omega \in H^{x,y}$ be elliptic and minimal. Suppose that x and y are non-conjugate along ω. Then there exists a unique zero-mean Gaussian measure μ_ω on $T_\omega \Omega^{x,y}$ such that, for all $0 \le s \le t \le 1$*

$$\int_{T_\omega \Omega^{x,y}} y_s \otimes y_t \; \mu_\omega(dy) = J_s J_1^{-1} K_t^*.$$

Moreover, μ_ω has reproducing-kernel Hilbert space $(T_\omega H^{x,y}, Q)$.

Furthermore, given a representation (2.1), we can construct μ_ω as follows. Let ν denote the unique Gaussian measure on $\Omega^0(\mathbb{R}^m)$ such that

$$\int_{\Omega^0(\mathbb{R}^m)} e^{i\langle w,k\rangle} \nu(dw) = e^{-\|Pk\|^2/2}$$

for all $k \in H^0(\mathbb{R}^m)$, where P denotes the orthogonal projection $H \to K$. Set

$$y_t(w) = (\partial/\partial h)\phi_t(x, h)w,$$

$$S(w) = \langle \xi_1, (\partial^2/\partial h^2)\phi_1(x, h)(w, w)\rangle$$

where ξ is some bicharacteristic over ω and $h = h(\omega)$. Then we can define a new probability measure $\tilde{\nu}$ on $\Omega^0(\mathbb{R}^m)$ by

$$d\tilde{\nu}/d\nu \propto e^{-S/2}$$

and μ_γ is the law of $(y_t : 0 \le t \le 1)$ under $\tilde{\nu}$.

Proof. Choose an orthonormal basis $\{k_i : i \in I\}$ in K to diagonalize S, considered as a quadratic form on K:

$$S(k) = \sum_i \lambda_i \langle k, k_i \rangle^2.$$

Note that S is Hilbert–Schmidt on K and, since x and y are non-conjugate along ω, for all i,

$$1 + \lambda_i = \|k_i\|^2 + S(k_i) = q(k_i) > 0.$$

Note also that, for the standard Gaussian distribution γ on $\mathbb{R}$,

$$\int_{\mathbb{R}} e^{i(1+\lambda)ux} e^{-\lambda x^2/2} \gamma(dx) = \frac{1}{\sqrt{1+\lambda}} e^{-(1+\lambda)u^2/2}.$$

Hence

$$\int_{\Omega^0(\mathbb{R}^m)} e^{-S(w)/2} \nu(dw) = \prod_i \frac{1}{\sqrt{1+\lambda_i}} < \infty$$

so $\tilde{\nu}$ is well-defined, and

$$\int_{\Omega^0(\mathbb{R}^m)} e^{iq(k,w)} \tilde{\nu}(dw) = e^{-q(k)/2}.$$

Hence, if μ_ω denotes the law of $(y_t : 0 \leq t \leq 1)$ under $\tilde{\nu}$, then, by Proposition 2.4, for all $v \in T_\omega H^{x,y}$,

$$\int_{T_\omega \Omega^{x,y}} e^{iQ(v,y)} \mu_\omega(dy) = e^{-Q(v)/2}.$$

It remains to check that μ_ω has the claimed covariance. For $\alpha \in T^*_{\omega_s} M$ and $\beta \in T^*_{\omega_t} M$, we have

$$\begin{aligned}
&\int_{T_\omega \Omega^{x,y}} \langle \alpha, y_s \rangle \langle \beta, y_t \rangle \mu_\omega(dy) \\
&\quad = \int_{T_\omega \Omega^{x,y}} Q(v^{\alpha,s}, y) Q(v^{\beta,t}, y) \mu_\omega(dy) = Q(v^{\alpha,s}, v^{\beta,t}) = \langle \alpha, C(s,t)\beta \rangle.
\end{aligned}$$

□

3. PROOF OF THEOREM 1.4

3.1. **Sketch of the proof.** Throughout this section, $(x, y) \notin \mathrm{Cut}(a)$. Without loss of generality, we assume that M is an embedded submanifold of $\mathbb{R}^n$ for a suitably large $n \in \mathbb{N}$ and that the vector fields $\tilde{X}_0$, ..., $\tilde{X}_m$ are the restrictions of the vector fields X_0, ..., X_m with coefficients in $\mathcal{C}^\infty_b(\mathbb{R}^n, \mathbb{R}^n)$ (i.e bounded derivatives of all orders).

Our problem should then be reformulated as follows : Given $\varepsilon > 0$, let $(z^\varepsilon_t : 0 \leq t \leq 1)$ be defined as

$$\forall t \in [0,1], \quad z^\varepsilon_t = \frac{x^\varepsilon_t - \gamma_t}{\sqrt{\varepsilon}}$$

where $(x^\varepsilon_t : 0 \leq t \leq 1)$ denotes the unique solution of the SDE

$$dx^\varepsilon_t = \sqrt{\varepsilon} \tilde{X}_l(x^\varepsilon_t) \circ dB^l_t + \varepsilon \tilde{X}_0(x^\varepsilon_t) dt, \quad x^\varepsilon_0 = x,$$

and $(\gamma_t : 0 \leq t \leq 1)$ stands for the unique solution of the ODE

$$d\gamma_t = \tilde{X}_l(x^\varepsilon_t) \dot{h}^l_t dt, \quad \gamma_0 = x.$$

In the first equation, B^1, ..., B^m are n-dimensional independent Brownian motions defined on some underlying probability space $(\Omega, \mathcal{F}, \mu)$. In the second equation, h is the unique element in $H^0(\mathbb{R}^m)$ such that the unique minimizer γ of the energy may be written $\gamma = \phi(x, h)$. In both equations, we have omitted the symbol $\sum_{l=1}^{m}$ and $\circ d$ denotes Stratonovich integration. Denote by $\tilde{\mu}_\varepsilon^{x,y}$ the law of $(z_t^\varepsilon : 0 \leq t \leq 1)$ conditioned by the terminal condition $x_1^\varepsilon = y$. Then, according to the statements of Theorem 1.4, one has to show that

$$\tilde{\mu}_\varepsilon^{x,y} \Longrightarrow \mu_\gamma. \tag{3.1}$$

In order to describe μ_γ, we adopt the notation of Theorem 2.9. Set $B = (B^1, ..., B^m)$ and

$$\begin{aligned} y_t(\omega) &= \frac{\partial}{\partial h}\phi_t(x,h)B(\omega), \\ S(\omega) &= \langle \xi_1, \frac{\partial^2}{\partial h^2}\phi_1(x,h)(B(\omega), B(\omega))\rangle \end{aligned} .$$

Denote by ν the law of $(y_t(\omega) : 0 \leq t \leq 1)$ under μ and define a new probability measure by

$$\frac{d\tilde{\nu}}{d\nu}(\omega) = \frac{e^{-S(\omega)/2}}{\displaystyle\int_\Omega e^{-S(\omega)/2}\,\nu(d\omega)}.$$

Then μ_γ is the law of $(y_t : 0 \leq t \leq 1)$ under $\tilde{\nu}$. To show (3.1), we shall mainly proceed in two steps :

(i) We show that the finite-dimensional distributions of $\tilde{\mu}_\varepsilon^{x,y}$ converge weakly to those of μ_γ as $\varepsilon \downarrow 0$ (Proposition 3.1).
(ii) We show that the family of measures $\{\tilde{\mu}_\varepsilon^{x,y} : \varepsilon > 0\}$ is tight (Proposition 3.2).

From now on, given a measure β on $(\Omega, \mathcal{F})$, $\mathbb{E}_\beta$ denotes the expectation operator associated to β.

Proposition 3.1. *Given any finite sequence* $0 < t_1 < ... < t_k < 1$ *and any* k*-tuple* $(\zeta_1, ..., \zeta_k) \in (\mathbb{R}^n)^k$ *:*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}_\mu\left(\exp\left(i\sum_{j=1}^{k}\langle\zeta_j, z_{t_j}^\varepsilon\rangle\right)\middle| x_1^\varepsilon = y\right) = \mathbb{E}_{\tilde{\nu}}\left(\exp\left(i\sum_{j=1}^{k}\langle\zeta_j, y_{t_j}\rangle\right)\right) \tag{3.2}$$

Proposition 3.2. *We may find positive constants* $\varepsilon_0 > 0$*,* $C > 0$*,* $q > 1$ *and* $\gamma > 1$ *such that*

$$\forall (s,t) \in [0,1]^2,\ t \neq s, \forall \varepsilon \in [0, \varepsilon_0], \quad \mathbb{E}_\mu\left(|z_t^\varepsilon - z_s^\varepsilon|^{2q}\middle| x_1^\varepsilon = y\right) \leq C\,|t-s|^\gamma$$

To prove these above results, we establish a lemma which shall be crucial in our study. The below result deeply relies on the existence of a **positive** density w.r.t. the Lebesgue measure on $\mathbb{R}^n$ for the law of x_1^ε (thanks to the *bracket condition*).

Lemma 3.3. *Let g be a bounded continuous map from $\mathcal{C}([0,1],\mathbb{R}^n)$ to $\mathbb{C}$, then, given any bounded continuous map F from $\mathbb{R}^n$ to $\mathbb{R}$, it holds*

$$\mathbb{E}_\mu\left(g(z^\varepsilon)|x_1^\varepsilon = y\right) = \frac{\displaystyle\int_{\mathbb{R}^n} \mathbb{E}_\mu\left(g(z^\varepsilon)\exp\left(i\langle\xi, \frac{x_1^\varepsilon - y}{\varepsilon}\rangle - \frac{F(x_1^\varepsilon)}{\varepsilon}\right)\right) d\xi}{\displaystyle\int_{\mathbb{R}^n} \mathbb{E}_\mu\left(\exp\left(i\langle\xi, \frac{x_1^\varepsilon - y}{\varepsilon}\rangle - \frac{F(x_1^\varepsilon)}{\varepsilon}\right)\right) d\xi} \tag{3.3}$$

Proof. Denote by $p_1^\varepsilon(x,\bullet)$ the density of the law of x_1^ε w.r.t. the Lebesgue measure on $\mathbb{R}^n$. Write

$$\begin{aligned}&\mathbb{E}_\mu\left(g(z^\varepsilon)\exp\left(i\langle\eta, x_1^\varepsilon\rangle - \frac{F(x_1^\varepsilon)}{\varepsilon}\right)\right)\\ &\qquad = \int_{\mathbb{R}^n} \exp\left(i\langle\eta, y\rangle - \frac{F(y)}{\varepsilon}\right)\mathbb{E}_\mu\left(g(z^\varepsilon)|x_1^\varepsilon = y\right) p_1^\varepsilon(x,y)\,dy\end{aligned} \tag{3.4}$$

and use an inversion Fourier formula to get

$$\begin{aligned}&\mathbb{E}_\mu\left(g(z^\varepsilon)|x_1^\varepsilon = y\right) p_1^\varepsilon(x,y)\, e^{-\frac{F(y)}{\varepsilon}}\\ &\qquad = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathbb{E}_\mu\left(g(z^\varepsilon)\exp\left(i\langle\eta, x_1^\varepsilon - y\rangle - \frac{F(x_1^\varepsilon)}{\varepsilon}\right)\right) d\eta.\end{aligned} \tag{3.5}$$

Observe that, taking $g \equiv 1$, we get

$$p_1^\varepsilon(x,y)\,\exp\left(-\frac{F(y)}{\varepsilon}\right) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathbb{E}_\mu\left(\exp\left(i\langle\eta, x_1^\varepsilon - y\rangle - \frac{F(x_1^\varepsilon)}{\varepsilon}\right)\right) d\eta. \tag{3.6}$$

Next, divide (3.5) by (3.6) and make the change of variables $\xi = \varepsilon\eta$ to get (3.3). □

3.2. **Proof of Proposition 3.1.** The main ingredients of our analysis is Laplace method and Malliavin Calculus. Since our proof is mainly the same as the one of Ben Arous [BA88], we only outline the main steps and give references to the place where the reader may find explanations in [BA88]. On the other hand, since Malliavin calculus is now a classical tool in the analysis of Wiener functionals (see [Nua95] for instance), we do not review the Malliavin calculus used in our proof but only introduce the notation needed.

Fix $0 < t_1 < ... < t_k < 1$ and $(\zeta_1, ..., \zeta_k) \in (\mathbb{R}^n)^k$. In order to lighten our expressions, we let $k = 1$ and we set :

$$\begin{aligned} g(t_1,\zeta_1,\bullet) &= \exp\left(i\langle\zeta_1, \bullet_{t_1}\rangle\right)\\ L_F(\varepsilon,\xi,\bullet) &= \exp\left(i\langle\xi,\bullet\rangle - \frac{F(\bullet)}{\varepsilon}\right)\end{aligned}$$

Moreover, given $h \in \mathbb{H}_0(\mathbb{R}^m)$, we set :

$$G(\varepsilon,h,\omega) = \exp\left(-\frac{1}{\sqrt{\varepsilon}}\sum_{l=1}^m \int_0^1 \dot{h}_s^l\, dB_s^l(\omega) - \frac{1}{2}\sum_{l=1}^m\int_0^1 \left|\dot{h}_s^l\right|^2 ds\right).$$

Thanks to Lemma 3.3, we may write

$$\mathbb{E}_\mu\left(g(t_1,\zeta_1,z^\varepsilon)|x_1^\varepsilon = y\right) \;=\; \frac{\displaystyle\int_{\mathbb{R}^n} \mathbb{E}_\mu\left(g(t_1,\zeta_1,z^\varepsilon)L_F\left(\varepsilon,\xi,z_1^\varepsilon\right)\right)d\xi}{\displaystyle\int_{\mathbb{R}^n} \mathbb{E}_\mu\left(L_F\left(\varepsilon,\xi,z_1^\varepsilon\right)\right)d\xi}$$

The first step is to find the limit as $\varepsilon \downarrow 0$ of the expectations :

$$\mathbb{E}_\mu\left(g(t_1,\zeta_1,z^\varepsilon)\,L_F\left(\varepsilon,\xi,z_1^\varepsilon\right)\right) \text{ and } \mathbb{E}_\mu\left(L_F\left(\varepsilon,\xi,z_1^\varepsilon\right)\right)$$

We only explain our method on the first expectation. To deduce the limit of the second one, the reader may set $g \equiv 1$ in all our expressions below. To study the limit of the first expectation, we write a Cameron-Martin formula corresponding to the transformation on Wiener space $\omega \mapsto \omega + \frac{h}{\sqrt{\varepsilon}}$ (where h is given by Theorem 2.9) :

$$\mathbb{E}_\mu\left(g(t_1,\zeta_1,z^\varepsilon)\,L_F\left(\varepsilon,\xi,z_1^\varepsilon\right)\right) \;=\; \mathbb{E}_\mu\left(g(t_1,\zeta_1,\bar{z}^\varepsilon)\,L_F\left(\varepsilon,\xi,\bar{z}_1^\varepsilon\right)\,G(\varepsilon,h,\omega)\right)$$

where

$$\forall t\in[0,1],\quad \bar{z}_t^\varepsilon \;=\; \frac{\bar{x}_t^\varepsilon - \phi_t(x,h)}{\sqrt{\varepsilon}}$$

and $(\bar{x}_t^\varepsilon : 0 \le t \le 1)$ evolves according to

$$d\bar{x}_t^\varepsilon = \sqrt{\varepsilon}\tilde{X}_l(\bar{x}_t^\varepsilon)\circ dB_t^l + \varepsilon\tilde{X}_0(\bar{x}_t^\varepsilon)dt + \tilde{X}_l(\bar{x}_t^\varepsilon)\,\dot{h}_t^l\,dt, \quad \bar{x}_0^\varepsilon = x, \tag{3.7}$$

The next step is to write a stochastic Taylor expansion of $\bar{x}^\varepsilon$ w.r.t. $\sqrt{\varepsilon}$. In order to indicate the meaning of our expansion, we introduce the Sobolev space $\mathbb{D}^{p,k}$ ($p > 1$, $k \in \mathbb{N}$) of Wiener functionals in $\mathcal{L}^p(\Omega,\mathcal{F},\mu)$ which have a kth-derivative in the sense of Malliavin and we denote by $\|\bullet\|_{p,k}$ the associated Sobolev seminorm. It is well known that, since the coefficients $\tilde{X}_l$ are $\mathcal{C}_b^\infty$, the unique solution of (3.7) is regular in the sense of Malliavin, namely belongs to $\mathbb{D}^{p,k}$ for any $p > 1$ and $k \in \mathbb{N}$. Moreover, $\bar{x}^\varepsilon$ satisfies, for any $p > 1$ and $k \in \mathbb{N}$

$$\forall t\in[0,1],\quad \varlimsup_{\varepsilon\downarrow 0}\|\bar{x}_t^\varepsilon\|_{p,k} \;<\; \infty \tag{3.8}$$

Ben Arous [BA88, formula (3.12) and Lemma (3.10)] shows that, $\bar{x}_t^\varepsilon$, $t \in [0,1]$, may be written (with the notation of Theorem 2.9) as

$$\begin{aligned}\bar{x}_t^\varepsilon = \phi_t(x,h) \;+\; \sqrt{\varepsilon}\,y_t(\omega) \;+\; \frac{\varepsilon}{2}\frac{\partial^2}{\partial h^2}\phi_t(x,h)\,(B(\omega),B(\omega))\\ +\,\frac{\varepsilon}{2}\phi_t^\star\int_0^t \phi_s^{\star-1}\tilde{X}_0(\phi_s(h,x))\,ds \;+\; \varepsilon^{3/2}\,R_3(\varepsilon,\omega)\end{aligned} \tag{3.9}$$

where, for any $p > 1$ and $k \in \mathbb{N}$

$$\|y_t(\omega)\|_{p,k} + \left\|\frac{\partial^2}{\partial h^2}\phi_t(x,h)\,(B(\omega),B(\omega))\right\|_{p,k} + \varlimsup_{\varepsilon\downarrow 0}\|R_3(\varepsilon,\omega)\|_{p,k} \;<\; \infty. \tag{3.10}$$

In the above expressions, $\phi_t^\star$ denote the Jacobian matrix of $x \mapsto \phi_t(x,h)$ and $\phi_t^{\star-1}$ the inverse of $\phi_t^\star$. Assume now that F is $\mathcal{C}_b^\infty$. We now expand $F(\bar{x}_1^\varepsilon)$ using (3.9) [BA88, formula (3.35)] :

$$F\left(\bar{x}_1^\varepsilon\right) \;=\; F(\underbrace{\phi_1(x,h)}_{y}) \;+\; \sqrt{\varepsilon}F^{(1)}(y)(y_1(\omega)) \;+\; +\frac{\varepsilon}{2}\,U(h,\omega) \;+\; \varepsilon^{3/2}V(h,\varepsilon,\omega) \tag{3.11}$$

where

$$\begin{aligned} U(h,\omega) \;=\; & F^{(1)}(y)\left(\frac{\partial^2}{\partial h^2}\phi_t(x,h)\left(B(\omega),B(\omega)\right)\right) + F^{(2)}(y)(y_1(\omega),y_1(\omega)) \\ & +F^{(1)}(y)\left(\phi_t^\star\int_0^t \phi_s^{\star-1}\tilde{X}_0(\phi_s(x,h))\,ds\right) \end{aligned} \tag{3.12}$$

and such that, for any $p>1$ and $k\in\mathbb{N}$, [BA88, Lemma (3.36)]

$$\left\|U(h,\omega)\right\|_{p,k} + \lim_{\varepsilon\downarrow 0}\left\|V(h,\varepsilon,\omega)\right\|_{p,k} \;<\; \infty. \tag{3.13}$$

Ben Arous shows then that, given $(x,y)\notin \mathrm{Cut}(a)$, we may find a $\mathcal{C}_b^\infty$ function $F:\mathbb{R}^n\to\mathbb{R}$ such that [BA88, Lemma (3.8) and Lemma (3.18)]

$$\begin{aligned} F(y) &= -\frac{1}{2}\sum_{l=1}^m\int_0^1\left|\dot{h}_s^l\right|^2\,ds \\ F^{(1)}(y)(\bullet) &= \langle\lambda,\bullet\rangle \\ F^{(1)}(y)(y_1(\omega)) &= -\sum_{l=1}^m\int_0^1\dot{h}_s^l\,dB_s^l(\omega) \end{aligned} \tag{3.14}$$

and there exists $p>1$ such that [BA88, Lemma (3.25)]

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}_\mu\left(\exp\left(-p\left(\frac{1}{2}U(h,\omega)\;+\;\sqrt{\varepsilon}V(h,\varepsilon,\omega)\right)\right)\right) < +\infty. \tag{3.15}$$

Equalities (3.14) implies that

$$\begin{aligned} &\mathbb{E}_\mu\left(g(t_1,\zeta_1,\bar{z}^\varepsilon)\,L_F\left(\varepsilon,\xi,\bar{z}_1^\varepsilon\right)\,G(\varepsilon,h,\omega)\right) \\ &\qquad= \mathbb{E}_\mu\left(g(t_1,\zeta_1,\bar{z}^\varepsilon)\,\exp\left(i\langle\xi,\bar{z}_1^\varepsilon\rangle-\frac{1}{2}U(h,\omega)-\sqrt{\varepsilon}V(h,\varepsilon,\omega)\right)\right). \end{aligned} \tag{3.16}$$

Note now that

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0}E_\mu\left(g(t_1,\zeta_1,\bar{z}^\varepsilon)\,\exp\left(i\langle\xi,\bar{z}_1^\varepsilon\rangle-\frac{1}{2}U(h,\omega)-\sqrt{\varepsilon}V(h,\varepsilon,\omega)\right)\right) \\ &\qquad= \mathbb{E}_\mu\left(g(t_1,\zeta_1,y_{t_1}(\omega))\,\exp\left(i\langle\xi,y_1(\omega)\rangle-\frac{1}{2}U(h,\omega)\right)\right) \end{aligned} \tag{3.17}$$

We now wish to conclude that

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}E_\mu\left(g(t_1,\zeta_1,\bar{z}^\varepsilon)\,\exp\left(i\langle\xi,\bar{z}_1^\varepsilon\rangle-\frac{1}{2}U(h,\omega)-\sqrt{\varepsilon}V(h,\varepsilon,\omega)\right)\right)d\xi \\ &\qquad= \int_{\mathbb{R}^n}\mathbb{E}_\mu\left(g(t_1,\zeta_1,y_{t_1}(\omega))\,\exp\left(i\langle\xi,y_1(\omega)\rangle-\frac{1}{2}U(h,\omega)\right)\right)d\xi \end{aligned} \tag{3.18}$$

To this end, we use the dominated convergence. So we need to show that the right-hand side of (3.17) is integrable w.r.t. ξ over $\mathbb{R}^n$ and that the left-hand side of (3.17) is dominated uniformly in ε by a function which is integrable w.r.t. ξ over $\mathbb{R}^n$. To this end, we are going to use an integration by parts in the sense of Malliavin. First of all, note that (3.9) and (3.10) imply that, for any $p > 1$ and $k \in \mathbb{N}$,

$$\forall t \in [0,1], \quad \lim_{\varepsilon \downarrow 0} \|\bar{z}_t^\varepsilon\|_{p,k} < +\infty. \tag{3.19}$$

On the other hand, Ben Arous [BA88, Lemma (3.36)] proves that, thanks to (3.13) and (3.15), we may find $p_0 > 1$ such that, for any $k \in \mathbb{N}$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \left\| \exp\left(-\frac{1}{2} U(h,\omega) - \sqrt{\varepsilon} V(h,\varepsilon,\omega) \right) \right\|_{p,k} &< \infty, \\ \left\| \exp\left(-\frac{1}{2} U(h,\omega) \right) \right\|_{p_0,k} &< \infty. \end{aligned} \tag{3.20}$$

To be able to use an integration by parts in the sense of Malliavin, we require the Malliavin covariance matrices of $\bar{z}_1^\varepsilon$ and $y_1(\omega)$ to be invertible and that their inverses belong to $\bigcap_{p>1} \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$. Note first that the Malliavin covariance matrix of $y_1(\omega)$ is is deterministic and invertible whenever $(x,y) \notin \mathrm{Cut}(a)$ (see formula (2.8) p. 10). Furthermore, it has been shown that [Mes96, Proposition (4.1) p. 30] the Malliavin covariance matrix $\bar{\sigma}_1^\varepsilon$ of $\bar{z}_1^\varepsilon$ is uniformly non-degenerate, i.e. is, μ-almost surely, invertible and satisfies, for any $p > 1$,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}_\mu \left((\det \bar{\sigma}_1^\varepsilon)^{-p} \right) < +\infty. \tag{3.21}$$

As Ben Arous [BA88, Lemma (3.48)], using sufficiently enough integrations by parts on Wiener space , we may find $\eta > d+1$ and a positive constant C such that, for all $\xi \in \mathbb{R}^n$,

$$\lim_{\varepsilon \downarrow 0} \left| E_\mu \left(g(t_1, \zeta_1, \bar{z}^\varepsilon) \exp\left(i\langle \xi, \bar{z}_1^\varepsilon \rangle - \frac{1}{2} U(h,\omega) - \sqrt{\varepsilon} V(h,\varepsilon,\omega) \right) \right) \right| \le C \left(|\xi| \vee 1 \right)^{-2\eta} \tag{3.22}$$

and

$$\left| \mathbb{E}_\mu \left(\exp\left(i\langle \xi, y_1(\omega) \rangle - \frac{1}{2} U(h,\omega) \right) \right) \right| \le C \left(|\xi| \vee 1 \right)^{-2\eta} \tag{3.23}$$

which, jointly with (3.17), implies (3.18). Take $g \equiv 1$ in (3.18), we then get

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} E_\mu \left(\exp\left(i\langle \xi, \bar{z}_1^\varepsilon \rangle - \frac{1}{2} U(h,\omega) - \sqrt{\varepsilon} V(h,\varepsilon,\omega) \right) \right) d\xi \\ = \int_{\mathbb{R}^n} \mathbb{E}_\mu \left(\exp\left(i\langle \xi, y_1(\omega) \rangle - \frac{1}{2} U(h,\omega) \right) \right) d\xi > 0 \end{aligned} \tag{3.24}$$

The positivity of the right-hand side of (3.24) is establish in [BA88, formula (3.55)]. At this stage of our proof, we have shown that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\mathbb{E}_\mu&\left(\exp\left(i\sum_{j=1}^{k}\langle\zeta_j, z^\varepsilon_{t_j}\rangle\right)\middle| x^\varepsilon_1 = y\right)\\ &= \frac{\displaystyle\int_{\mathbb{R}^n}\mathbb{E}_\mu\left(g(t_1,\zeta_1,y_{t_1}(\omega))\,\exp\left(i\langle\xi, y_1(\omega)\rangle - \frac{1}{2}U(h,\omega)\right)\right)d\xi}{\displaystyle\int_{\mathbb{R}^n}\mathbb{E}_\mu\left(\exp\left(i\langle\xi, y_1(\omega)\rangle - \frac{1}{2}U(h,\omega)\right)\right)d\xi}\end{aligned} \tag{3.25}$$

Since $y_1(\omega)$ is a centered Gaussian which covariance matrix $C(\omega)$ given by (2.8) is invertible. One can get using an inversion Fourier formula :

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}&\mathbb{E}_\mu\left(\exp\left(i\langle\zeta_1, z^\varepsilon_{t_1}\rangle\right)\middle| x^\varepsilon_1 = y\right)\\ &= \frac{\sqrt{2\pi}^d\,(\det C(\omega))^{-1/2}\,\mathbb{E}_\mu\left(g(t_1,\zeta_1,y_{t_1}(\omega))\,\exp\left(-\frac{1}{2}U(h,\omega)\right)\middle| y_1(\omega)=0\right)}{\sqrt{2\pi}^d\,(\det C(\omega))^{-1/2}\,\mathbb{E}_\mu\left(\exp\left(-\frac{1}{2}U(h,\omega)\right)\middle| y_1(\omega)=0\right)}\end{aligned}$$

Next, observe that on $\{y_1(\omega)=0\}$, it holds

$$U(h,\omega) \;=\; S(\omega) \;+\; \langle\lambda, \phi^\star_t\int_0^t \phi^{\star-1}_s\tilde{X}_0(\phi_s(h,x))ds\rangle \tag{3.26}$$

Since the second term in the right-hand side of (3.26) is deterministic, we finally get,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\mathbb{E}_\mu\left(\exp\left(i\langle\zeta_1, z^\varepsilon_{t_1}\rangle\right)\middle| x^\varepsilon_1 = y\right) &= \frac{\mathbb{E}_\mu\left(g(t_1,\zeta_1,y_{t_1}(\omega))\,\exp\left(-\frac{1}{2}S(\omega)\right)\middle| y_1(\omega)=0\right)}{\mathbb{E}_\mu\left(\exp\left(-\frac{1}{2}S(\omega)\right)\middle| y_1(\omega)=0\right)}\\ &= \frac{\mathbb{E}_\nu\left(g(t_1,\zeta_1,y_{t_1}(\omega))\,\exp\left(-\frac{1}{2}S(\omega)\right)\right)}{\mathbb{E}_\nu\left(\exp\left(-\frac{1}{2}S(\omega)\right)\right)}\\ &= \mathbb{E}_{\tilde{\nu}}\left(g(t_1,\zeta_1,y_{t_1}(\omega))\right)\end{aligned}$$

which is the desired result.

3.3. **Proof of Proposition 3.2.** Fix $(s,t)\in[0,1]^2$ such that $s\neq t$. We may redo the same calculations as the previous subsection until (3.21) replacing $g(t_1,\zeta_1,z^\varepsilon)$ by $|z^\varepsilon_t - z^\varepsilon_s|^{2q}$. Next, classical results about stochastic differential equations with $\mathcal{C}^\infty_b$ coefficients yield that, given any $k\in\mathbb{N}$ and $q>1$, we may find $\varepsilon_0>0$, $C>0$ and $\gamma>0$ such that

$$\forall (s,t)\in[0,1]^2,\ s\neq t,\ \forall\varepsilon\in[0,\varepsilon_0],\quad \|z^\varepsilon_t - z^\varepsilon_s\|_{k,2q}\;\le\; C\,|t-s|^\gamma\,. \tag{3.27}$$

From this result, one can show that, we may find $\varepsilon_0 > 0$, $C > 0$, $q > 1$, $\gamma > 1$ and $\eta > d+1$ such that,

$$\forall (s,t) \in [0,1]^2,\ s \neq t,\ \forall \varepsilon \in [0,\varepsilon_0],$$
$$\left| E_\mu \left(|z_t^\varepsilon - z_s^\varepsilon|^{2q} \exp\left(i\langle \xi, \bar{z}_1^\varepsilon \rangle - \frac{1}{2} U(h,\omega) - \sqrt{\varepsilon} V(h,\varepsilon,\omega) \right) \right) \right| \leq C\ |t-s|^\gamma\ (|\xi| \vee 1)^{-2\eta}.$$

The desired result follows then from the positivity of the right-hand side of (3.24).

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