Least Energy Solitary Waves for a System of Nonlinear Schrödinger Equations in $\mathbb{R}^n$

Boyan Sirakov[(1)]

MODALX, UFR SEGMI, Université Paris 10, 92001 Nanterre Cedex, France, and CAMS, EHESS, 54 bd Raspail, 75270 Paris Cedex 06, France

Abstract. In this paper we consider systems of coupled Schrödinger equations which appear in nonlinear optics. The problem has been considered mostly in the one-dimensional case. Here we make a rigorous study of the existence of least energy nonstandard standing waves (solitons) in higher dimensions. We give conditions on the parameters of the system under which it possesses a nonstandard least energy solution, and conditions under which the associated energy functional cannot be minimized on the natural set where the solutions lie.

1 Introduction

The concept of incoherent solitons in nonlinear optics has attracted considerable attention in the last ten years, both from experimental and theoretical point of view. The two experimental studies [16] and [17] demonstrated the existence of solitons made from both spatially and temporally incoherent light. These papers were followed by a large amount of theoretical work on incoherent solitons. We shall quote here [11], [10], [1], [2], where a comprehensive list of references on this subject can be found.

It is shown for instance in the recent works [5], [2] (see also the references there) that, for photorefractive Kerr media, a good approximation describing the propagation of self-trapped mutually incoherent wave packets is the following system of coupled Schrödinger equations

$$2ik_j\frac{\partial\psi_m^j}{\partial t}+\Delta_x\psi_m^j+\alpha k_j^2 I(x,t)\psi_m^j=0, \tag{1}$$

where

$$I(x,t)=\sum_{j=1}^{N_f}\sum_{m=1}^{N_j}\lambda_m^j|\psi_m^j(x,t)|^2.$$

Here ψ_m^j (the (m,j)-component of the beam) is a complex function defined on $\mathbb{R}^n\times\mathbb{R}^+$, $n\leq 3$, Δ is the Laplace operator, N_f is the number of frequencies,

[1]e-mail : sirakov@ehess.fr

N_j is the number of waves at a particular frequency ω_j, k_j is a constant multiple of the frequency ω_j, and λ^j_m are the so-called time averaged mode-occupancy coefficients (we refer to [2] for more precisions on the meaning of the constants in this equation). We note that for this problem all coefficients in (1) are positive.

We will search for soliton (stationary wave) solutions of (1) in the form

$$\psi^j_m(t,x) = e^{i\kappa^j_m t} u^j_m(x), \tag{2}$$

where $u^j_m : \mathbb{R}^n \to \mathbb{R}$ is the spatial profile of the m-th wave at frequency ω_j, and κ^j_m is the propagation speed of this wave. Substituting (2) into (1) and renaming indices and constants leads us to the following real elliptic system for the vector function $u = (u_1, \dots, u_d) : \mathbb{R}^d \to \mathbb{R}^n$

$$-\Delta u_i + \lambda_i u_i = \left(\sum_{j=1}^{d} \mu_{ij} |u_j|^2 \right) u_i, \qquad i = 1, \dots, d. \tag{3}$$

In this paper we consider the case when u can be scaled, i.e. u_i can be replaced by $s_i u_i$, $s_i > 0$, in such a way that $s_j^2 \mu_{ij} = s_i^2 \mu_{ji}$, for all $i \neq j$. Note this is always possible for systems of two equations ($d = 2$). So we can suppose $\mu_{ij} = \mu_{ji}$, and (3) is the Euler-Lagrange system for the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i=1}^{d} \left\{ |\nabla u_i(x)|^2 + \lambda_i |u_i(x)|^2 \right\} dx - \frac{1}{4} \int_{\mathbb{R}^n} \sum_{i,j=1}^{d} \mu_{ij} u_i^2(x) u_j^2(x)\, dx.$$

This functional is well defined if u_i are in the Sobolev space $H^1(\mathbb{R}^n)$, in virtue of the embeddings $H^1(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$, valid for $n \leq 3$.

We will always consider solitons with *finite energy*. We will be in particular interested in existence of *least energy* solutions of (3). The following essential remark has to be done immediately : for the solution to be of the type we are interested in, *at least two* of its components u_i have to be different from the standard zero wave. So it makes sense to consider solutions with minimal energy on the set of solutions $u = (u_i)_i$ of (3), such that $u_i \not\equiv 0$ for at least two different indices i.

To explain in one phrase the essence of the results we obtain, we will show that, somewhat surprisingly, *there always exist ranges of positive parameters in (3), for which this system has a least energy solution, and ranges of positive parameters for which the functional cannot be minimized on the natural set where the eventual solutions lie.* We will see that E, which looks quite "scalar" with respect to the vector u, actually differs in its behaviour from its scalar counterpart, when nonstandard solutions are searched for.

In all the papers that we quoted above the authors considered the one-dimensional case, i.e. $N = 1$. The motivation for our work comes from a recent paper by Lin and Wei [14], which seems to be the first attempt to make a rigorous study of the higher dimensional case. Studying that paper, we could not understand an important point in the proofs (namely, the infimum in Lemma 3 on page 636 and the infimum c' on page 642 in [14] seem to be infinite, due to the invariance of the sets where they are taken with respect to the transformation $u \to tu$, $t \geq 1$). Our first goal was to overcome this problem, but later we discovered that some statements in [14] should also be modified, and that problem (3) has richer structure than expected.

In order to simplify the presentation, from now on we shall work with the system of two equations

$$\left\{ \begin{array}{rcl} \Delta u_1 - u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 & = & 0 \\ u_1, u_2 & \in & H^1(\mathbb{R}^n) \\ \Delta u_2 - \lambda u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 & = & 0 \end{array} \right. \tag{4}$$

(we have adjusted the notations to those of [14]), here $\lambda, \mu_1, \mu_2 > 0$. We stress however that all our results extend straightforwardly to systems with an arbitrary number of equations. Note that the constant 1 in the u_1 term does not introduce a restriction, since this can always be obtained by using renumbering of u_1, u_2 and a scaling of x. This also permits us to suppose $\lambda \geq 1$, without restricting the generality.

A solution $u = (u_1, u_2)$ of (4) which has a zero component ($u_1 \equiv 0$ or $u_2 \equiv 0$) will be called a *standard* solution. The solution $(0, 0)$ will be referred to as the *trivial* solution. We shall search for nonstandard solutions of (4), or, equivalently, for nonstandard critical points of the functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u_1|^2 + u_1^2 + |\nabla u_2|^2 + \lambda u_2^2 \right) - \frac{1}{4} \int_{\mathbb{R}^n} \mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4$$

on the energy space $H := H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. We denote with H_r the set of couples in H who are radially symmetric with respect to a fixed point in $\mathbb{R}^n$.

As in [14] we consider the set

$$\mathcal{N} = \Big\{ u \in H, u_1 \not\equiv 0, u_2 \not\equiv 0, \quad \textstyle\int_{\mathbb{R}^n} (|\nabla u_1|^2 + u_1^2) = \int_{\mathbb{R}^n} \mu_1 u_1^4 + \beta u_1^2 u_2^2,$$

$$\textstyle\int_{\mathbb{R}^n} (|\nabla u_2|^2 + \lambda u_2^2) = \int_{\mathbb{R}^n} \beta u_1^2 u_2^2 + \mu_2 u_2^4 \Big\} .$$

Note that any nonstandard solution of (4) has to belong to $\mathcal{N}$ (multiply the equations in (4) by u_1, u_2, and integrate over $\mathbb{R}^n$). We set

$$A = \inf_{u \in \mathcal{N}} E(u), \qquad A_r = \inf_{u \in \mathcal{N} \cap H_r} E(u). \tag{5}$$

The following proposition shows the role of A and A_r.

Proposition 1.1 *If A (or A_r) is attained by a couple $u \in \mathcal{N}$ then this couple is a solution of (4), provided* $\beta^2 < \mu_1\mu_2$.

We now state our main results. We show that there exist (explicitly given) intervals $I_1, I_2, I_3 \subset \mathbb{R}$ such that I_1 contains zero, I_3 is a neighbourhood of infinity, I_2 is between I_1 and I_3, and A is attained for $\beta \in I_1 \cup I_3$, while it is not attained for $\beta \in I_2$.

Let $w_1(x) = w_1(|x|)$ be the unique positive solution of the scalar equation

$$-\Delta w + w = w^3 \quad \text{in } \mathbb{R}^n. \tag{6}$$

The function w_1 is well studied (see the next section) and will play an important role in our analysis.

Our first result concerns the case $\lambda = 1$. Going back to (1)-(3), we see that this is the case when the propagation speeds are adjusted to the frequencies.

Theorem 1 *Suppose $\lambda = 1$ in system (4).*

(i) If $0 \le \beta < \min\{\mu_1, \mu_2\}$ then $A = A_r$ is attained by the couple

$$(\sqrt{k}w_1, \sqrt{l}w_1), \qquad \textit{where} \quad \left\{ \begin{array}{rcl} \mu_1 k + \beta l &=& 1 \\ \beta k + \mu_2 l &=& 1. \end{array} \right. \tag{7}$$

(ii) If $\min\{\mu_1, \mu_2\} \le \beta \le \max\{\mu_1, \mu_2\}$ and $\mu_1 \ne \mu_2$ then system (4) does not have a nonstandard solution with nonnegative components.

(iii) If $\min\{\mu_1, \mu_2\} \le \beta < \sqrt{\mu_1\mu_2}$ then A and A_r are not attained.

(iv) If $\beta > \max\{\mu_1, \mu_2\}$ then $A = A_r$ is attained by the same couple as in (i), which is of course a solution of (4).

Remark 1. It was stated in [14] that A is attained if $0 < \beta < \sqrt{\mu_1\mu_2}$, independently of λ.
Remark 2. The couple considered in (i) and (iv) is obviously a solution of (4) with $\lambda = 1$, whenever the solution of the linear system in (7) is such that $k > 0, l > 0$. The result here states that this couple is actually *a least energy solution* – this was an open problem, see for example [15], Remark 1.4. We conjecture that under the hypotheses of (i) or (iv) the couple $(\sqrt{k}w_1, \sqrt{l}w_1)$ is the unique positive solution to (4). Note that when $\lambda = \mu_1 = \mu_2 = \beta = 1$ system (4) has an infinity of positive solutions $(\cos\theta\, w_1, \sin\theta\, w_2)$, $\theta \in (0, \pi/2)$.
Remark 3. Note that the minimality statements in Theorem 1 concern all possible solutions, not just the radial ones.

Remark 4. If $\beta \geq 0$ then any nonstandard nonnegative solution of (4) is strictly positive and radial, by the strong maximum principle and the results in [3], see Section 3.5. On the other hand, if A is attained by a couple $(u_1, u_2) \in H$ then it is attained by $(|u_1|, |u_2|) \in H$, so whenever minimizers for A exist and are solutions of (4) with $\beta \geq 0$ we have $A = A_r$.
Remark 5. In the literature on variational problems it is common to speak of ground states as the minimizers of the functional on some set, where all possible solutions have to lie. Also, ground states are often required to be positive. If we comply with this terminology, Theorem 1 (ii)-(iii) give ranges of nonexistence of a nonstandard ground state. However, we do not know if in this case there exists a (changing sign) solution which minimizes the energy on the set of solutions of (4).

The next theorem deals with the general case $\lambda \geq 1$.

Theorem 2 *Suppose $\lambda \geq 1$ in (4) and set $\nu_1 = \mu_1 \lambda^{1-\frac{n}{4}}$, $\nu_2 = \mu_2 \lambda^{\frac{n}{4}-1}$.*

(i) Let ν_0 be the smaller root of the equation

$$\lambda^{-n/4} x^2 - (\nu_1 + \nu_2)x + \nu_1 \nu_2 = 0.$$

If $-\sqrt{\mu_1 \mu_2} < \beta < \nu_0$ then A_r is attained by a solution of (4).

(ii) If $\mu_2 \leq \beta \leq \mu_1$ and $\mu_2 < \mu_1$ then system (4) does not have a nonstandard solution with nonnegative components.

(iii) If $\mu_2 \leq \beta < \sqrt{\mu_1 \mu_2}$ then A and A_r are not attained.

(iv) If $\beta > \lambda^{\frac{n}{4}} \max\{\nu_1, \nu_2\}$ then $A = A_r$ is attained by a solution of (4).

Remark 6. The conditions in (i) and (iv) in Theorem 2 reduce to those from Theorem 1, when $\lambda = 1$. The ranges in Theorem 2 (i) and (iv) are not the best we can get, we have given them in this form to avoid introducing heavy notations at this stage. We will see in the course of the proof how (i) and (iv) can be improved (with the help of the function $h(\lambda)$, defined in Section 3), we refer to Section 3.3, Proposition 3.4 and Section 3.4, Proposition 3.7 for precise statements. When $\lambda \neq 1$ it is open, and quite interesting, to find out what the optimal ranges for existence are. Note also that we do not know if A is attained in the range in Theorem 2 (i).
Remark 7. The statement (i) above includes cases when $\beta < 0$. In many applications this is known as "repulsive interaction".
Remark 8. It will be shown that in cases (iv) we obtain a minimizer even with respect to the standard solutions.

The next section contains further comments on this problem, and some preliminary results. The proofs of the results can be found in Section 3.

2 Preliminaries and Further Comments

In this section we comment our problem more extensively, and recall some known results in the theory of elliptic equations and systems.

Existence and properties of standard solutions of (4) are very well studied. Let us recall some facts. For each $u \in H^1(\mathbb{R}^n)$ we denote

$$\|u\|_\lambda^2 := \int_{\mathbb{R}^n} |\nabla u|^2 + \lambda u^2.$$

Proposition 2.1 *Consider the minimization problems*

$$S_{\lambda,\mu} = \inf_{u \in H^1(\mathbb{R}^n)\setminus\{0\}} \frac{\|u\|_\lambda^2}{\left(\int_{\mathbb{R}^n} \mu u^4\right)^{1/2}}, \qquad T_{\lambda,\mu} = \inf_{u \in \mathcal{M}_0} \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{4}\int_{\mathbb{R}^n} \mu u^4,$$

where $\mathcal{M}_0 = \left\{u \in H^1(\mathbb{R}^n), u \not\equiv 0 \ : \ \|u\|_\lambda^2 = \int_{\mathbb{R}^n} \mu u^4\right\}$. *Then the function*

$$w_{\lambda,\mu}(x) = \mu^{-\frac{1}{2}}\sqrt{\lambda} w_1(\sqrt{\lambda}x)$$

is a minimizer for $T_{\lambda,\mu}$ *and the unique positive solution of the equation*

$$-\Delta w + \lambda w = \mu w^3 \qquad \textit{in } \ \mathbb{R}^n.$$

In addition, we have

$$T_{\lambda,\mu} = \frac{1}{4} S_{\lambda,\mu}^2, \qquad S_{\lambda,\mu} = \mu^{-\frac{1}{2}} \lambda^{1-\frac{n}{4}} S_{1,1}.$$

This proposition is easily proved by scaling and by using known results for (6) (see for example [19], we will give a brief proof in Section 3.4). By [8] any positive solution of (6) is radially symmetric and strictly decreasing in the radial variable. The uniqueness of radial solutions of (6) goes back to Coffman [6], see also Kwong [12].

By Proposition 2.1 system (4) has exactly two nonnegative standard solutions : $(\overline{u}_1, 0)$ and $(0, \overline{u}_2)$, where

$$\overline{u}_1(x) = w_{1,\mu_1}(x), \qquad \overline{u}_2(x) = w_{\lambda,\mu_2}(x). \tag{8}$$

Further, it is known that (6) has an infinity of radial and nonradial solutions, which give an infinity of standard solutions of (4).

We go back to the case of a system. Let us immediately note that the functional E has a sort of "scalar" geometry on H, in the following sense : it can be written as

$$E(u_1, u_2) = \frac{1}{2}\|u\|_H^2 - \frac{1}{4}\int_{\mathbb{R}^n} (Mu^2, u^2),$$

where $u := (u_1, u_2)$, $u^2 := (u_1^2, u_2^2)$, $\|u\|_H^2 := \|u_1\|_1^2 + \|u_2\|_\lambda^2$ is a norm on H, $(Mu^2, u^2) = \mu_1 u_1^2 + 2\beta u_1 u_2 + \mu_2 u_2^2$, and $M = \begin{pmatrix} \mu_1 & \beta \\ \beta & \mu_2 \end{pmatrix}$ is such that

$$c_0(u_1^4 + u_2^4) \leq (Mu^2, u^2) \leq C_0(u_1^4 + u_2^4),$$

for some positive constants c_0, C_0, as long as $-\sqrt{\mu_1\mu_2} < \beta$.

This basically means that all Critical Point Theory (see for example [18], [19]) for scalar functionals can be applied to $E(u_1, u_2)$. For instance, E satisfies the hypotheses of the Symmetric Mountain Pass lemma [18] (or the Fountain Theorem, [19]), which immediately yields the existence of an infinity of solutions of (4), such that $(u_1, u_2) \neq (0, 0)$. *However, a priori nothing prevents these from being standard.*

So, in general, it is unavoidable to distinguish between restricting the solutions (u_1, u_2) to being different from the couple $(0, 0)$ or to being such that $u_1 \not\equiv 0, u_2 \not\equiv 0$. If only the former is done, we will need extra information in order to conclude that we have a nonstandard solution.

Borrowing from the scalar theory, one may envision several ways to prove existence of nonstandard solutions of (4). First, one may try to directly search for critical points of E on H, through use of the Mountain Pass lemma, for example. The drawback of this otherwise very powerful method is that it does not always give enough information on the solutions, nor on their energy level.

Second, one may try to use Constrained Minimization, for example, minimize $\int_{\mathbb{R}^n} (|\nabla u_1|^2 + u_1^2 + |\nabla u_2|^2 + \lambda u_2^2)$ on the set

$$\left\{ u \in H, u_1 \not\equiv 0, u_2 \not\equiv 0, \int_{\mathbb{R}^n} \mu_1 u_1^4 + \beta u_1^2 u_2^2 = 1, \int_{\mathbb{R}^n} \mu_2 u_2^4 + \beta u_1^2 u_2^2 = 1 \right\}.$$

However, one easily sees that, contrary to the scalar case, this approach fails, since even if a minimizer exists, it gives rise to two (as opposed to one) Lagrange multipliers, which cannot be scaled out of the system.

The third approach consists in determining, with the help of the equations we aim to solve, some subset of the energy space where all eventual solutions should belong, and then minimize the functional on this subset (note that E is easily seen not to be bounded below on the whole H). The so-called Nehari manifold is defined by

$$\mathcal{N}_0 := \left\{ u \in H, (u_1, u_2) \not\equiv (0, 0) \ : \ \|u\|_H^2 = \int_{\mathbb{R}^n} (Mu^2, u^2) \right\}.$$

This set has the same properties as the set $\mathcal{M}_0$ in Proposition 2.1, in particular, $\mathcal{N}_0$ is homeomorphic to the unit sphere in H. So, proving that the minimization problem

$$A_0 := \inf_{u\in\mathcal{N}_0} E(u_1, u_2) \tag{9}$$

has a solution (which is a solution of (4)) is analogous to doing the same for $T_{\lambda,\mu}$ in Proposition 2.1, see Section 3.4.

However, except in particular cases (these will be the cases from statements (iv) in our theorems), the minimizer for A_0 can be standard, that is, $\mathcal{N}_0$ is too large, and minimization on it does not give anything interesting. This is where appears the idea to minimize on $\mathcal{N}$ - note that this set no longer has the properties that a Nehari manifold has in the scalar case.

Finally, we make several remarks with respect to the general theory of elliptic systems, developed in recent years (see for example the survey paper [7], and the references there). System (4) is of the so-called *gradient* type, that is, it can be written in the vector form

$$-\Delta u = \nabla f(u),$$

here $f(u) = \frac{1}{4}\left(\mu_1 u_1^4 + 2\beta u_1^2 u_2^2 + \mu_2 u_2^4 - 2u_1^2 - 2\lambda u_2^2\right)$. It is generally thought that gradient systems are not much different from scalar equations. We see here that we have an important example for which it would be wrong to think in this way, if we are interested in finding nonstandard solutions. The reason for this is the fact that system (4) *is not fully coupled*. A general notion of full coupling for nonlinear systems was given and analyzed in [4] ; for a system $-\Delta u = \nabla f(u)$ full coupling would be implied for example by $f_{u_1}(0,s) > 0, f_{u_2}(s,0) > 0$ for $s > 0$. It is the semi-coupled nature of system (4) which causes the phenomena described in Theorems 1 and 2 - if the system were fully coupled (for instance, if there were a term u_1u_2 in $f(u)$), then it would have nonstandard positive ground states for any positive values of its parameters.

3 Proofs of Theorems 1 and 2

The first point in the proofs is to use the functions $w_{\lambda,\mu}$ from Proposition 2.1 in order to obtain an upper bound for A. Then we are going to use this bound in order to study the behaviour of the minimizing sequences for A and A_r.

The proofs of Theorems 1 and 2 will be carried out jointly, to some extent.

3.1 An upper bound on A

Set $w_\lambda(x) = w_{\lambda,1}(x) = \sqrt{\lambda}w_1(\sqrt{\lambda}x)$, respectively $T_\lambda = T_{\lambda,1}$, $S_\lambda = S_{\lambda,1}$ (see Proposition 2.1 for the notations). We introduce the function $h : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$h(\lambda) := \frac{\int_{\mathbb{R}^n} w_1^2(x) w_\lambda^2(x)\, dx}{\int_{\mathbb{R}^n} w_1^4(x)\, dx}.$$

Note that h depends only on λ and n. The following proposition gives some bounds on h.

Proposition 3.1 *For any $\lambda \geq 1$ we have*

$$h(\lambda) \leq \lambda^{1-\frac{n}{4}} \tag{10}$$

and

$$\lambda^{1-\frac{n}{2}} \leq h(\lambda) \leq \sigma \lambda^{1-\frac{n}{2}}, \tag{11}$$

where $\sigma = \sigma(n)$ is the universal constant

$$\sigma = \frac{w_1^2(0) \int_{\mathbb{R}^n} w_1^2(x)\, dx}{\int_{\mathbb{R}^n} w_1^4(x)\, dx}.$$

Proof. We know that w_1 is radial and strictly decreasing in $|x|$. This implies that for $\lambda \geq 1$, $x \in \mathbb{R}^n$,

$$w_1(x) \geq w_1(\sqrt{\lambda} x).$$

Using this, the change of variables $x \to \sqrt{\lambda} x$ and the Hölder inequality we obtain

$$\int_{\mathbb{R}^n} w_1^2(x) w_\lambda^2(x)\, dx \geq \lambda^{1-\frac{n}{2}} \int_{\mathbb{R}^n} w_1^4(\sqrt{\lambda} x) d(\sqrt{\lambda} x) = \lambda^{1-\frac{n}{2}} \int_{\mathbb{R}^n} w_1^4(x)\, dx,$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} w_1^2(x) w_\lambda^2(x)\, dx &\leq \lambda \left(\int_{\mathbb{R}^n} w_1^4(x)\, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} w_1^4(\sqrt{\lambda} x)\, dx \right)^{\frac{1}{2}} \\ &= \lambda^{1-\frac{n}{4}} \int_{\mathbb{R}^n} w_1^4(x)\, dx. \end{aligned}$$

Finally, by the change of variables $x \to x/\sqrt{\lambda}$ we have

$$h(\lambda) = \lambda^{1-\frac{n}{2}} \frac{\int_{\mathbb{R}^n} w_1^2(x/\sqrt{\lambda}) w_1^2(x)\, dx}{\int_{\mathbb{R}^n} w_1^4(x)\, dx} =: \lambda^{1-\frac{n}{2}} h_1(\lambda).$$

By the monotonicity properties of w_1 the function $h_1(\lambda)$ is increasing, and, by Lebesgue monotone convergence, $h_1(\lambda) \to \sigma$ as $\lambda \to \infty$. $\square$

Next, consider the following linear system in $k, l \in \mathbb{R}$:

$$\left\{ \begin{array}{rcrcl} \mu_1 k & + & \beta h(\lambda) l & = & 1 \\ \\ \beta h(\lambda) k & + & \mu_2 \lambda^{2-\frac{n}{2}} l & = & \lambda^{2-\frac{n}{2}}. \end{array} \right. \tag{12}$$

Note that k and l are determined solely by the parameters in system (4). The use of system (12) is seen from the following simple lemma.

Lemma 3.1 *Suppose the parameters $\lambda, \mu_1, \mu_2, \beta$ in (4) are such that the linear system (12) has a solution $k > 0, l > 0$. Then the couple $(\sqrt{k} w_1, \sqrt{l} w_\lambda)$ belongs to $\mathcal{N}$.*

Proof. Recall (Proposition 2.1) that

$$\|w_\lambda\|_\lambda^2 = \int_{\mathbb{R}^n} w_\lambda^4 = 4T_\lambda = S_\lambda^2 = \lambda^{2-\frac{n}{2}} S_1^2, \tag{13}$$

for all $\lambda \geq 1$. Hence (12) and the definition of $h(\lambda)$ imply

$$\left\{ \begin{array}{rcrcl} \left(\int_{\mathbb{R}^n} \mu_1 w_1^4\right) k^2 + \left(\int_{\mathbb{R}^n} \beta w_1^2 w_\lambda^2\right) kl & = & k S_1^2 & = & k\|w_1\|_1^2 \\ \\ \left(\int_{\mathbb{R}^n} \beta w_1^2 w_\lambda^2\right) kl + \left(\int_{\mathbb{R}^n} \mu_2 w_\lambda^4\right) l^2 & = & l S_\lambda^2 & = & l\|w_\lambda\|_\lambda^2, \end{array} \right.$$

and the lemma follows. □

By using this lemma we will obtain an upper bound for the infima we are working with. Recall A_0 is defined in (9), A and A_r are defined in (5).

We have the following estimate on A.

Proposition 3.2 *Suppose the parameters $\lambda, \mu_1, \mu_2, \beta$ in system (4) are such that the linear system (12) has a solution $k > 0, l > 0$. Then*

$$0 < A_0 \leq A \leq A_r \leq \frac{1}{4} \left(k + \lambda^{2-\frac{n}{2}} l\right) S_1^2.$$

Proof. We only have to prove the first and the last inequality in Proposition 3.2. We use the fact that

$$E(u) = \frac{1}{4} \left(\|u_1\|_1^2 + \|u_2\|_\lambda^2\right) = \frac{1}{4} \int_{\mathbb{R}^n} (Mu^2, u^2), \quad \text{for all } u \in \mathcal{N}_0 \supset \mathcal{N}. \tag{14}$$

Then (13) and Lemma 3.1 imply

$$A_r \leq E(\sqrt{k} w_1, \sqrt{l} w_\lambda) = \frac{1}{4} \left(k + \lambda^{2-\frac{n}{2}} l\right) S_1^2.$$

Note that for each $u \in \mathcal{N}_0$, by Hölder and Sobolev inequalities,

$$\|u_1\|_1^2 + \|u_2\|_\lambda^2 = \int_{\mathbb{R}^n} (Mu^2, u^2) \le C_0(\|u_1\|_{L^4}^4 + \|u_2\|_{L^4}^4) \le C_1(\|u_1\|_1^4 + \|u_2\|_\lambda^4)$$

so E is bounded uniformly away from zero on $\mathcal{N}_0$, and $A_0 > 0$. $\square$

Here, and in the sequel, c_0, C_0, C_1 denote positive constants which depend only on the parameters in system (4) and on the dimension n.

Finally, let us list for further reference the conditions under which the solutions of (12) are positive : $k > 0$ and $l > 0$ if either

$$D_\lambda > 0 \qquad \text{and} \qquad \beta h(\lambda) < \min\{\mu_2, \mu_1 \lambda^{2-\frac{n}{2}}\} = \lambda^{1-\frac{n}{4}} \min\{\nu_1, \nu_2\}, \quad (15)$$

or

$$\beta h(\lambda) > \max\{\mu_2, \mu_1 \lambda^{2-\frac{n}{2}}\} = \lambda^{1-\frac{n}{4}} \max\{\nu_1, \nu_2\}, \quad (16)$$

where we have set

$$\nu_1 = \lambda^{1-\frac{n}{4}} \mu_1, \quad \nu_2 = \lambda^{\frac{n}{4}-1} \mu_2, \quad D_\lambda = \mu_1 \mu_2 \lambda^{2-\frac{n}{2}} - \beta^2 h^2(\lambda).$$

In view of the bounds on h we proved in Proposition 3.1, we see that the conditions (15) and (16) are implied by either $-\sqrt{\mu_1 \mu_2} < \beta < \min\{\nu_1, \nu_2\}$ or $\beta > \lambda^{\frac{n}{4}} \max\{\nu_1, \nu_2\}$.

3.2 Behaviour of the minimizing sequences for A. Proof of Theorem 1 (i) and (iv)

The main goal of this section is to find conditions under which each minimizing sequence for A is such that the L^4-norms of both components of the members of the sequence are bounded uniformly away from zero. Careful study of the bounds on the minimizing sequences that we obtain will permit us to prove Theorem 1, parts (i) and (iv).

For each $\lambda \ge 1$, set

$$g(\lambda) = \lambda^{n/4-1} h(\lambda) \quad (17)$$

($g(\lambda) \le 1$ by Proposition 3.1). We have the following result.

Proposition 3.3 *Let $\{u_m\} \subset \mathcal{N}$ be a sequence such that $E(u_m) \to A$ as $m \to \infty$. Then there exists a constant $c_0 > 0$ such that $\|u_{m,1}\|_{L^4(\mathbb{R}^n)} \ge c_0$ and $\|u_{m,2}\|_{L^4(\mathbb{R}^n)} \ge c_0$ for all m, provided*

$$-\infty < \beta < \overline{\nu}_0, \quad (18)$$

where $\overline{\nu}_0$ is the smaller root of the equation

$$g(\lambda)(2 - g(\lambda))x^2 - (\nu_1 + \nu_2)x + \nu_1 \nu_2 = 0.$$

Remark 1. We will show that the hypothesis on β in Theorem 2 (i) can be replaced by $\beta \in (-\sqrt{\mu_1\mu_2}, \overline{\nu}_0)$. It is easy to see that the upper bound in the statement of Theorem 2 (i) implies $\beta < \overline{\nu}_0$. Indeed,

$$1 \geq g(\lambda)(2 - g(\lambda)) \geq \lambda^{-\frac{n}{4}},$$

since we have, by Proposition 3.1,

$$2h(\lambda)\lambda^{1-\frac{n}{4}} - h^2(\lambda) \geq \lambda^{1-\frac{n}{4}} h(\lambda) \geq \lambda^{2-\frac{3n}{4}}. \tag{19}$$

Note that in order to show (19) one uses two inverse inequalities from Proposition 3.1, so $\beta < \overline{\nu}_0$ is a considerably better upper bound than the one in Theorem 2 (i).

Remark 2. An elementary computation shows that for all $\lambda \geq 1$

$$\overline{\nu}_0 \in \left(\frac{\nu_1\nu_2}{\nu_1 + \nu_2}\,,\, \min\{\nu_1, \nu_2\} \right].$$

Proof of Proposition 3.3. Let $\{u_m\} \subset \mathcal{N}$ be a minimizing sequence for A, that is, $u_m \in \mathcal{N}$, and, by (14),

$$E(u_m) = \frac{1}{4}\left(\|u_{m,1}\|_1^2 + \|u_{m,2}\|_\lambda^2\right) = \frac{1}{4}\int_{\mathbb{R}^n} (Mu_m^2, u_m^2) \longrightarrow A,$$

as $m \to \infty$. It follows that $\{u_m\}$ is bounded in H. We recall that $u_{m,i} \not\equiv 0$ for each m, i.

Set

$$y_{m,1} = \left(\int_{\mathbb{R}^n} u_{m,1}^4\right)^{1/2}, \qquad y_{m,2} = \left(\int_{\mathbb{R}^n} u_{m,2}^4\right)^{1/2}.$$

By the Sobolev and Holder inequalities, it follows from the definition of S_λ, Proposition 2.1 and $u_m \in \mathcal{N}$ that

$$S_1 y_{m,1} \leq \|u_{m,1}\|_1^2 = \int_{\mathbb{R}^n} \mu_1 u_{m,1}^4 + \beta u_{m,1}^2 u_{m,2}^2 \leq (\mu_1 y_{m,1}^2 + \beta^+ y_{m,1} y_{m,2}) \tag{20}$$

$$\lambda^{1-\frac{n}{4}} S_1 y_{m,2} \leq \|u_{m,2}\|_\lambda^2 = \int_{\mathbb{R}^n} \beta u_{m,1}^2 u_{m,2}^2 + \mu_2 u_{m,2}^4 \leq (\mu_2 y_{m,2}^2 + \beta^+ y_{m,1} y_{m,2}), \tag{21}$$

where $\beta^+ = \max\{\beta, 0\}$. Proposition 3.3 immediately follows for $\beta \leq 0$.

So, from now on we shall suppose $\beta > 0$. Adding up (20) and (21) results in

$$S_1(y_{m,1} + \lambda^{1-\frac{n}{4}} y_{m,2}) \leq \int_{\mathbb{R}^n} (Mu_m^2, u_m^2) = 4A + o(1), \tag{22}$$

where $o(1) \to 0$ as $m \to \infty$.

Set $z_{m,i} = \dfrac{1}{S_1} y_{m,i}$. Thanks to Proposition 3.2 from (20)–(22) we obtain the following inequalities (k and l denote the positive solutions of (12))

$$\begin{cases} z_{m,1} + \lambda^{1-\frac{n}{4}} z_{m,2} & \leq & k + \lambda^{2-\frac{n}{2}} l + o(1) \\ \mu_1 z_{m,1} + \beta z_{m,2} & \geq & 1 \\ \beta z_{m,1} + \mu_2 z_{m,2} & \geq & \lambda^{1-\frac{n}{4}}. \end{cases} \tag{23}$$

We would like to infer from (23) that the two sequences $\{z_{m,1}\}, \{z_{m,2}\}$ stay uniformly away from zero. For this it is enough to show that the lines

$$l_1 = \{z = (z_1, z_2) \in \mathbb{R}^2 \,:\, z_1 + \lambda^{1-\frac{n}{4}} z_2 = k + \lambda^{2-\frac{n}{2}} l\},$$

$$l_2 = \{z \in \mathbb{R}^2 \,:\, \mu_1 z_1 + \beta z_2 = 1\}, \qquad l_3 = \{z \in \mathbb{R}^2 \,:\, \beta z_1 + \mu_2 z_2 = \lambda^{1-\frac{n}{4}}\},$$

meet, and their crossing points have strictly positive coordinates (these lines are determined by the parameters in system (4)). Indeed, for large m the point $(z_{m,1}, z_{m,2})$ is arbitrarily close to the triangle (or segment, or point) between these crossing points. Since

$$\beta < \overline{\nu}_0 \leq \min\{\nu_1, \nu_2\} \leq \nu_1 \nu_2 = \mu_1 \mu_2, \tag{24}$$

we see that we have to verify the following inequalities

$$\beta \lambda^{1-\frac{n}{4}} < \mu_2, \qquad \beta < \mu_1 \lambda^{1-\frac{n}{4}}, \tag{25}$$

$$\mu_1 (k + \lambda^{2-\frac{n}{2}} l) > 1, \tag{26}$$

$$\mu_2 (k + \lambda^{2-\frac{n}{2}} l) > \lambda^{2-\frac{n}{2}}, \tag{27}$$

$$\beta (k + \lambda^{2-\frac{n}{2}} l) < \lambda^{1-\frac{n}{4}}. \tag{28}$$

Inequalities (25) can be recast as $\beta < \min\{\nu_1, \nu_2\}$, which is true by (24). Since

$$k + \lambda^{1-\frac{n}{4}} l = \frac{\lambda^{2-\frac{n}{2}} \left(\mu_2 + \mu_1 \lambda^{2-\frac{n}{2}} - 2\beta h(\lambda)\right)}{\mu_1 \mu_2 \lambda^{2-\frac{n}{2}} - \beta^2 h^2(\lambda)},$$

and the denominator of this fraction is positive (by (24) and Proposition 3.1), elementary computations show that (26) is equivalent to

$$(\mu_1 \lambda^{2-\frac{n}{2}} - \beta h(\lambda))^2 > 0,$$

while (27) is equivalent to $(\mu_2 - \beta h(\lambda))^2 > 0$, so (26) and (27) hold, thanks to (25) and Proposition 3.1.

Finally, by developing (28) we see that it is equivalent to

$$\left[\frac{2h(\lambda)\lambda^{1-\frac{n}{4}} - h^2(\lambda)}{\lambda^{2-\frac{n}{2}}}\right]\beta^2 - (\nu_1+\nu_2)\beta + \nu_1\nu_2 > 0, \tag{29}$$

which is implied by (18). This finishes the proof of Proposition 3.3. $\square$

Next, we are going to show how inequalities (23) lead to the statement of Theorem 1 (i) and (iv).

Proof of Theorem 1 (i) and (iv). Set $t_{m,1} = z_{m,1} - k$, $t_{m,2} = z_{m,2} - l$. By using system (12) with $\lambda = 1$ we have from inequalities (23), which are valid for $\beta > 0$,

$$\left\{\begin{array}{rcl} t_{m,1} + t_{m,2} & \le & o(1) \\ \mu_1 t_{m,1} + \beta t_{m,2} & \ge & 0 \\ \beta t_{m,1} + \mu_2 t_{m,2} & \ge & 0. \end{array}\right. \tag{30}$$

Now, whenever

$$\beta < \min\{\mu_1, \mu_2\} \qquad \text{or} \qquad \beta > \max\{\mu_1, \mu_2\},$$

the three half-spaces $\{t : t_1+t_2 \le 0\}$, $\{t : \mu_1 t_1+\beta t_2 \ge 0\}$, $\{t : \beta t_1+\mu_2 t_2 \ge 0\}$ meet at most in a triangle in the (t_1, t_2)-plane, and this triangle shrinks to $t_1 = t_2 = 0$ at the limit $m \to \infty$, so we have $z_{m,1} \to k, z_{m,2} \to l$ as $m \to \infty$. Then, by passing to the limit in (22) with $\lambda = 1$, and by using $A \le \frac{1}{4}(k+l)S_1^2$ (Proposition 3.2), we obtain

$$A = \frac{1}{4}(k+l)S_1^2 = E(\sqrt{k}w_1, \sqrt{l}w_1).$$

Parts (i) and (iv) of Theorem 1 are proved. $\square$

3.3 Proof of Proposition 1.1 and Theorem 2 (i)

Proof of Proposition 1.1. Our goal is to show that any minimizer of E restricted to $\mathcal{N}$ is such that $dE(u) = E'(u) = 0$. We write $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$, where $\mathcal{N}_i$ is the set of *nonstandard* $u \in H$ such that $G_i(u) = 0$, with

$$G_1(u) = \int_{\mathbb{R}^n} \left(|\nabla u_1|^2 + u_1^2\right) - \int_{\mathbb{R}^n} \mu_1 u_1^4 + \beta u_1^2 u_2^2,$$

$$G_2(u) = \int_{\mathbb{R}^n} \left(|\nabla u_2|^2 + \lambda u_2^2\right) - \int_{\mathbb{R}^n} \beta u_1^2 u_2^2 + \mu_2 u_2^4.$$

We have, for each $\psi = (\psi_1, \psi_2) \in H$ (setting $\lambda_1 = 1, \lambda_2 = \lambda$),

$$< E'(u), \psi >= \sum_{i=1}^{2} \int \left(\nabla u_i \nabla \psi_i + \lambda_i u_i \psi_i - \mu_i u_i^3 \psi_i - \beta u_i u_j^2 \psi_i\right),\ j \ne i,$$

$$<G_i'(u), \frac{\psi}{2}>= \int \left(\nabla u_i \nabla \psi_i + \lambda_i u_i \psi_i - 2\mu_i u_i^3 \psi_i - \beta u_i u_j (u_i \psi_j + u_j \psi_i)\right), j \neq i.$$

By computing $<G_i'(u), u>$ for $u \in \mathcal{N}_i$ we see that $G_i'(u) \neq 0$ for $i = 1, 2$ and $u \in \mathcal{N}$ (since $u_i \not\equiv 0$ on $\mathcal{N}_i$). Hence, supposing that $u = (u_1, u_2) \in \mathcal{N}$ is a minimizer for E restricted to $\mathcal{N}$, standard minimization theory implies the existence of two Lagrange multipliers $L_1, L_2 \in \mathbb{R}$ such that

$$E'(u) + L_1 G_1'(u) + L_2 G_2'(u) = 0.$$

Setting $G_1(u) = 0$ in the expression $<E'(u)+L_1 G_1'(u)+L_2 G_2'(u), (u_1, 0)>= 0$ we are led to

$$L_1 \int_{\mathbb{R}^n} \mu_1 u_1^4 + L_2 \int_{\mathbb{R}^n} \beta u_1^2 u_2^2 = 0. \tag{31}$$

Similarly, setting $G_2(u) = 0$ in $<E'(u) + L_1 G_1'(u) + L_2 G_2'(u), (0, u_2)>= 0$ we obtain

$$L_1 \int_{\mathbb{R}^n} \beta u_1^2 u_2^2 + L_2 \int_{\mathbb{R}^n} \mu_2 u_2^4 = 0. \tag{32}$$

The system (31)-(32) has the unique solution $L_1 = L_2 = 0$, by the Hölder inequality and the hypothesis of Proposition 1.1. □

Proof of Theorem 2 (i). Suppose we have a minimizing sequence of radial couples $\{u_m\} \subset \mathcal{N}$ for A_r. Then, by standard functional analysis and the compact embedding $H_r^1(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$ the sequences $\{u_{m,i}\}$ converge (up to a subsequence) weakly in $H^1(\mathbb{R}^n)$ and strongly in $L^4(\mathbb{R}^n)$ to a function $u_i \in H^1(\mathbb{R}^n)$. We have, by (14) and standard results on weak convergence,

$$\|u_1\|_1^2 + \|u_2\|_\lambda^2 \leq \liminf \left(\|u_{m,1}\|_1^2 + \|u_{m,2}\|_\lambda^2\right) = 4A_r. \tag{33}$$

In the previous subsection we proved that the L^4-norms of *both* $\{u_{m,1}\}, \{u_{m,2}\}$ are bounded away from zero, so the strong limit $u = (u_1, u_2)$ is *nonstandard*.

In addition, we have

$$\int_{\mathbb{R}^n} (Mu^2, u^2) = \lim_{m\to\infty} \int_{\mathbb{R}^n} (Mu_m^2, u_m^2) = 4 \lim_{m\to\infty} E(u_m) = 4A_r. \tag{34}$$

Remark. It is the last equality which forces us to work only with A_r, that is, to suppose that the minimizing sequence is composed of radial functions, even when $\beta > 0$. If one starts with a minimizing sequence for A and then replaces it by the sequence of the symmetric rearrangements, one is lead to $\int_{\mathbb{R}^n} (Mu^2, u^2) \geq 4A$, while below we shall need the inverse inequality.

Next, let s_1, s_2 be the solutions of the linear system

$$\left\{ \begin{array}{lcl} \left(\int_{\mathbb{R}^n} \mu_1 u_1^4\right) s_1 + \left(\int_{\mathbb{R}^n} \beta u_1^2 u_2^2\right) s_2 & = & \|u_1\|_1^2 \\ \left(\int_{\mathbb{R}^n} \beta u_1^2 u_2^2\right) s_1 + \left(\int_{\mathbb{R}^n} \mu_2 u_2^4\right) s_2 & = & \|u_2\|_\lambda^2. \end{array} \right. \tag{35}$$

This system has a unique solution, by the hypotheses on β and the Hölder inequality.

If $s_1 = s_2 = 1$ we are done, since then $u \in \mathcal{N}$ and by (33) and (34) u is a minimizer, so Proposition 1.1 finishes the proof of Theorem 2 (i).

Lemma 3.2 *Under the hypotheses of Proposition 3.3 the solution of system (35) satisfies* $s_1 > 0, s_2 > 0$.

Before proving Lemma 3.2, let us show how Theorem 2 (i) follows from it. Recall the range given in Theorem 2 (i) is included in (18).

Suppose $(s_1, s_2) \neq (1, 1)$ and set

$$B = \begin{pmatrix} \int_{\mathbb{R}^n} \mu_1 u_1^4 & \int_{\mathbb{R}^n} \beta u_1^2 u_2^2 \\ \int_{\mathbb{R}^n} \beta u_1^2 u_2^2 & \int_{\mathbb{R}^n} \mu_2 u_2^4 \end{pmatrix}.$$

Since $G_1(u_m) = G_2(u_m) = 0$, $u_m \rightharpoonup u$ in H and $u_m \to u$ in $L^4 \times L^4$ we have

$$\begin{aligned} \|u_1\|_1^2 \leq \liminf_{m\to\infty} \|u_{m,1}\|_1^2 &= \liminf_{m\to\infty} \int_{\mathbb{R}^n} \mu_1 u_{m,1}^4 + \beta u_{m,1}^2 u_{m,2}^2 \\ &= \int_{\mathbb{R}^n} \mu_1 u_1^4 + \beta u_1^2 u_{m,2}^2, \end{aligned}$$

and, similarly,

$$\|u_2\|_\lambda^2 \leq \int_{\mathbb{R}^n} \beta u_1^2 u_2^2 + \mu_2 u_2^4. \tag{36}$$

Hence

$$B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \|u_1\|_1^2 \\ \|u_2\|_\lambda^2 \end{pmatrix} \lneqq B \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Set $v_1 = \sqrt{s_1} u_1, v_2 = \sqrt{s_2} u_2$. Then by the definition of s_1, s_2 the couple (v_1, v_2) is on $\mathcal{N}$ but by (14) and (34)

$$\begin{aligned} E(v_1, v_2) &= \frac{1}{4} <B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}> \lneqq \frac{1}{4} <B \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}> \\ &= <\begin{pmatrix} 1 \\ 1 \end{pmatrix}, B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}> \lneqq <B \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}>= 4A_r, \end{aligned}$$

which is a contradiction with the minimality of A_r. Hence $s_1 = s_2 = 1$, so Theorem 2 (i) is proved. □

Remark. Note that we could not use the fact that $E(u) = \frac{1}{4}(\|u_1\|_1^2 + \|u_2\|_\lambda^2)$ on $\mathcal{N}$ to get a contradiction, since we cannot[2] infer from (35) that

$$s_1\|u_1\|_1^2 + s_2\|u_2\|_\lambda^2 < \|u_1\|_1^2 + \|u_2\|_\lambda^2.$$

This is very much in contrast with the situation which one has when minimizing a scalar functional (see the proof of Proposition 3.5 in the next section).

Proof of Lemma 3.2. The lemma is obvious if $\beta \le 0$. So we can suppose $\beta > 0$. For example, let us prove that $s_1 > 0$. We need to show that

$$\|u_1\|_1^2 \int_{\mathbb{R}^n} \mu_2 u_2^4 > \|u_2\|_\lambda^2 \int_{\mathbb{R}^n} \beta u_1^2 u_2^2.$$

By Sobolev and Hölder inequalities this is implied by

$$\begin{aligned} \mu_2\|u_1\|_1^2 \left(\int_{\mathbb{R}^n} u_2^4\right)^{1/2} &> \beta\|u_2\|_\lambda^2 \left(\int_{\mathbb{R}^n} u_1^4\right)^{1/2} \\ \Longleftarrow \qquad \mu_2 \left(\int_{\mathbb{R}^n} u_2^4\right)^{1/2} &> \frac{\beta}{S_1}\|u_2\|_\lambda^2. \end{aligned}$$

By using (36) we see that the last inequality is implied by

$$\begin{aligned} \mu_2 \left(\int_{\mathbb{R}^n} u_2^4\right)^{1/2} &> \frac{\beta}{S_1}\left(\int_{\mathbb{R}^n} \beta u_1^2 u_2^2 + \int_{\mathbb{R}^n} \mu_2 u_2^4\right) \\ \Longleftarrow \qquad 1 &> \frac{\beta}{S_1}\left(\frac{\beta}{\mu_2}\left(\int_{\mathbb{R}^n} u_1^4\right)^{1/2} + \left(\int_{\mathbb{R}^n} u_2^4\right)^{1/2}\right) \end{aligned}$$

Since u is the limit of a minimizing sequence for A, we can use what we proved in the previous section (inequalities (23)-(28)). With the notations used in (23)-(28), the last inequality above can be recast as

$$\lim_{m\to\infty} \beta \left(\frac{\beta}{\mu_2} z_{m,1} + z_{m,2}\right) < 1. \tag{37}$$

By using consecutively the first inequality in (25) and the first inequality in (23), we see that (37) is implied by (28), which we have already shown to hold under the hypothesis of Proposition 3.3.

[2]Indeed, there exist linear systems $a_{i1}x_1 + a_{i2}x_2 = b_i$, $i = 1, 2$, with positive coefficients and positive solutions, such that $a_{i1} + a_{i2} > b_i$, $i = 1, 2$, but $b_1x_1 + b_2x_2 > b_1 + b_2$ – for example $8x_1 + 4x_2 = 11, 2x_1 + 2x_2 = 3$.

To show that $s_2 > 0$ the argument is analogous, by using $S_\lambda = \lambda^{1-n/4} S_1$ and the second inequality in (25). □

Theorem 2 (i) is proved.

Finally, going through the proof of Theorem 2 (i) we see that we only needed the hypotheses of Proposition 3.3 and Proposition 1.1 (see also the remark following Proposition 3.3), so we can state the following result.

Proposition 3.4 *The value A_r is attained by a nonstandard solution of (4), provided*

$$-\sqrt{\mu_1 \mu_2} < \beta < \overline{\nu_0},$$

where $\overline{\nu_0}$ is the smaller root of the equation (see (17))

$$g(\lambda)(2 - g(\lambda))x^2 - (\nu_1 + \nu_2)x + \nu_1 \nu_2 = 0.$$

3.4 Proof of Theorem 2 (iv) and extensions

The idea of the proof of statements (iv) in Theorems 1 and 2 is rather simple : should it turn out that

$$A_0 < \min\{E(\overline{u}_1, 0), E(0, \overline{u}_2)\} \tag{38}$$

($\overline{u}_1, \overline{u}_2$ are defined in (8), A_0 is defined in (9)), then the minimizer for A_0 cannot be standard and is a least energy solution (of course in this case $A_0 = A$). Recall that $(\overline{u}_1, 0), (0, \overline{u}_2)$ have least energy among the standard nontrivial solutions.

We have the following (basically known) fact.

Proposition 3.5 *The minimal value $A_0 > 0$ is attained by a nontrivial (possibly standard) radial solution of (4), provided $\beta > 0$.*

The fact that A_0 is attained by a solution of (4) can be proven for example through the same argument as in Chapter 4 of [19], where the scalar case is considered. We will give here, for the reader's convenience and to permit comparison with the proofs in the previous sections, a direct argument leading to Proposition 3.5.

Before proceeding, we recall some facts about spherical rearrangement (Schwartz symmetrization), see for example [13].

Proposition 3.6 *Suppose $v_1, v_2 \in H^1(\mathbb{R}^n)$ and let v_1^*, v_2^* be the radial functions obtained by Schwarz symmetrization from v_1, v_2. Then for any $p \in [2, 6]$ if $N = 3$, $p \geq 2$ if $N \leq 2$,*

$$\|v_i^*\|_{H^1} \leq \|v_i\|_{H^1}, \qquad \|v_i^*\|_{L^p} = \|v_i\|_{L^p}, \qquad \int_{\mathbb{R}^n} (v_1^*)^2 (v_2^*)^2 \geq \int_{\mathbb{R}^n} v_1^2 v_2^2.$$

Proof of Proposition 3.5. Take a minimizing sequence $\{u_m\} \subset \mathcal{N}_0$ for A_0. Then $\{\|u_m\|_H^2\}$ tends to $4A_0$ (by (14)) so $\{u_m\}$ is bounded in H. By Proposition 3.6 the sequence of rearrangements $u_m^* = (u_{m,1}^*, u_{m,2}^*)$ is bounded in H, and hence converges weakly in H and strongly in $L^4 \times L^4$ to a couple u^*. Hence, by $u_m \in \mathcal{N}_0$ and Proposition 3.6,

$$
\begin{aligned}
\|u^*\|_H^2 &\leq \liminf_{m\to\infty} \|u_m^*\|_H^2 \leq \liminf_{m\to\infty} \|u_m\|_H^2 = \liminf_{m\to\infty} \int (Mu_m^2, u_m^2) \\
&\leq \lim_{m\to\infty} \int (M(u_m^*)^2, (u_m^*)^2) = \int (M(u^*)^2, (u^*)^2),
\end{aligned}
$$

$$
E(u^*) \leq \liminf_{m\to\infty} E(u_m^*) \leq \liminf_{m\to\infty} E(u_m) = A_0.
$$

By the Sobolev inequality, Proposition 3.6 and $u_m \in \mathcal{N}_0$ we have

$$
\|u_{m,1}^*\|_{L^4}^2 + \|u_{m,2}^*\|_{L^4}^2 \leq C_0 \|u_m^*\|_H^2 \leq C_0 \int_{\mathbb{R}^n} M((u_m^*)^2, (u_m^*)^2) \leq C_1 \|u_m^*\|_{L^4 \times L^4}^4,
$$

so $u^* \neq (0,0)$. If $\|u^*\|_H^2 = \int_{\mathbb{R}^n} (M(u^*)^2, (u^*)^2)$, A_0 is attained by u^*. If not, that is $\|u^*\|_H^2 < \int_{\mathbb{R}^n} (M(u^*)^2, (u^*)^2)$, take $s \in (0,1)$ such that $v = su^* \in \mathcal{N}_0$. Then by (14) and Proposition 3.6

$$
E(v) = \frac{1}{4}\|v\|_H^2 < \frac{1}{4}\|u^*\|_H^2 \leq \frac{1}{4} \liminf_{m\to\infty} \|u_m^*\|_H^2 \leq \frac{1}{4} \liminf_{m\to\infty} \|u_m\|_H^2 = A_0,
$$

a contradiction.

Hence u^* is a minimizer and there exists a Lagrange multiplier $L \in \mathbb{R}$ such that $dE(u)\big|_{u=u^*} + L\, d\left(\|u\|_H^2 - \int_{\mathbb{R}^n} (Mu^2, u^2)\right)\big|_{u=u^*} = 0$. Evaluating this differential against u^* gives $L\|u^*\|_H^2 = 0$, i.e. $L = 0$. □

Next, set

$$
J(u) = J(u_1, u_2) = \frac{1}{4} \frac{\left(\|u_1\|_1^2 + \|u_2\|_\lambda^2\right)^2}{\int_{\mathbb{R}^n} (Mu^2, u^2)}.
$$

Lemma 3.3 *Suppose $\beta > 0$. We have*

$$
A_0 = \inf_{u \in H\setminus\{(0,0)\}} J(u) = \inf_{u \in H_r\setminus\{(0,0)\}} J(u). \tag{39}
$$

Proof. It is easy to see, by the Sobolev inequality and Proposition 3.6, that the two infima in (39) are positive and equal. Let B_0 be their value. If u^* is a minimizer for A_0 then $J(u^*) = A_0$ by $u^* \in \mathcal{N}_0$, hence $B_0 \leq A_0$. If $B_0 < A_0$ take $v \neq (0,0)$ such that $J(v) < A_0$. Let $s > 0$ be such that $sv \in \mathcal{N}_0$. Then

$$
\frac{1}{4} s^2 \|v\|_H^2 = E(sv) \geq A_0 > J(v)
$$

implies $\|v\|_H^2 < s^2 \int_{\mathbb{R}^n}(Mv^2, v^2)$, a contradiction with $sv \in \mathcal{N}_0$. $\square$

Further, define the function

$$f(k_1,k_2) := J(\sqrt{k_1}w_1, \sqrt{k_2}w_\lambda) = \frac{1}{4}\frac{(k_1 S_1^2 + k_2\lambda^{2-n/2}S_1^2)^2}{S_1^2(\mu_1 k_1^2 + 2\beta h(\lambda)k_1k_2 + \mu_2\lambda^{2-n/2}k_2^2)},$$

on the set $\mathcal{K} = \{(k_1,k_2) \ : \ k_1 \geq 0, k_2 \geq 0, (k_1,k_2) \neq (0,0)\}$ (recall the definition of $h(\lambda)$ and (13)). Since

$$f(k_1,0) = \frac{1}{4\mu_1}S_1^2 = E(\overline{u}_1, 0), \qquad f(0,k_2) = \frac{1}{4\mu_2}\lambda^{2-\frac{n}{2}}S_1^2 = E(0,\overline{u}_2),$$

we see that for (38) to hold it is sufficient that f does not attain its minimum over $\mathcal{K}$ on the lines $k_1 = 0$ or $k_2 = 0$.

The function f is a fraction of two quadratic forms in (k_1,k_2), and elementary analysis shows that the quantity

$$\frac{(ak_1 + bk_2)^2}{ck_1^2 + 2\gamma k_1k_2 + dk_2^2} \qquad (a,b,c,d,\gamma > 0)$$

does not attain its minimum in $\mathcal{K}$ on the axes if and only if

$$a\gamma - bc > 0, \qquad ad - b\gamma < 0, \tag{40}$$

and then the minimum is attained for $k_1 = b\gamma - ad,\ k_2 = a\gamma - bc$.

Applying this to $f(k_1,k_2)$ we see that (40) becomes

$$\beta h(\lambda) - \mu_1\lambda^{2-\frac{n}{2}} > 0, \qquad \mu_2\lambda^{2-\frac{n}{2}} - \beta h(\lambda)\lambda^{2-\frac{n}{2}} < 0,$$

or, equivalently,

$$\beta g(\lambda) = \beta\frac{h(\lambda)}{\lambda^{1-\frac{n}{4}}} > \max\{\nu_1,\nu_2\}. \tag{41}$$

Inequality (41) is implied by the hypothesis of Theorem 2 (iv) (by Proposition 3.1), so Theorem 2 (iv) is proved. $\square$

Remark. Note that, in the case $\lambda = 1$, the fact that the couple $(\sqrt{k}w_1, \sqrt{l}w_1)$ (defined in Theorem 1) is a minimizer for A was already proved in Section 3.2. Since (38) (which follows from (41)) implies that $A_0 = A$ for $\beta > \max\{\mu_1,\mu_2\}, \lambda = 1$, the couple $(\sqrt{k}w_1, \sqrt{l}w_1)$ is a minimizer for A_0 as well.

It is possible to give other conditions under which (38) holds. For instance, we can compute

$$\min_{(k_1,k_2)\in\mathcal{K}} J(\sqrt{k_1}w_1, \sqrt{k_2}w_1) \qquad \text{and} \qquad \min_{(k_1,k_2)\in\mathcal{K}} J(\sqrt{k_1}w_\lambda, \sqrt{k_2}w_\lambda).$$

We have, by (13),

$$J(\sqrt{k_1}w_1,\sqrt{k_2}w_1)=\frac{1}{4}\frac{\left(k_1S_1^2+k_2(S_1^2+(\lambda-1)\int_{\mathbb{R}^n}w_1^2)\right)^2}{S_1^2(\mu_1k_1^2+2\beta k_1k_2+\mu_2k_2^2)}.$$

We introduce the following universal constant

$$\sigma_0=\frac{\|w_1\|^2_{L^2(\mathbb{R}^n)}}{\|w_1\|^4_{L^4(\mathbb{R}^n)}}=\frac{1}{S_1^2}\int_{\mathbb{R}^n}w_1^2.$$

Since $\|w_1\|^2_{H^1}=\int_{\mathbb{R}^n}w_1^4$ we have $\sigma_0\in(0,1)$. Then

$$J(\sqrt{k_1}w_1,\sqrt{k_2}w_1)=\frac{S_1^2}{4}\frac{\left[k_1+k_2(1+\sigma_0(\lambda-1))\right]^2}{\mu_1k_1^2+2\beta k_1k_2+\mu_2k_2^2},\tag{42}$$

from which it follows that sufficient conditions for (38) are

$$\begin{cases}\beta>\max\{\mu_1b_\lambda,\mu_2b_\lambda^{-1}\}, \quad \text{with} \quad b_\lambda:=1+\sigma_0(\lambda-1)\in[1,\lambda),\\ \\ \dfrac{\lambda^{-2+n/2}\left[2\beta b_\lambda-(\mu_1b_\lambda^2+\mu_2)\right]^2\mu_2}{\mu_1(\beta b_\lambda-\mu_2)^2+2\beta(\beta b_\lambda-\mu_2)(\beta-\mu_1b_\lambda)+\mu_2(\beta-\mu_1b_\lambda)^2}<1.\end{cases}\tag{43}$$

We have obtained (43) by using (40) applied to $J(\sqrt{k_1}w_1,\sqrt{k_2}w_1)$, and by comparing the minimal value given by (40) with $E(0,\overline{u}_2)$. Note that in the fraction in (43) we are dividing a polynomial of degree 2 in β by a polynomial of degree 3 in β.

In order to get simpler to state sufficient conditions for (38) , one could minimize the fraction in (42), with σ_0 replaced by 1 (since $\sigma_0<1$). Then one obtains the following conditions for the corresponding minimum to be attained away from the axes and to be smaller than $\min\{E(\overline{u}_1,0),E(0,\overline{u}_2)\}$: setting $\xi_1=\mu_1\lambda,\ \xi_2=\mu_2/\lambda,\ \gamma_1=\beta-\xi_1,\ \gamma_2=\beta-\xi_2$,

$$\begin{cases}\gamma_1>0, \quad \gamma_2>0, \quad \text{and}\\ \\ \dfrac{(\gamma_1+\gamma_2)^2\,\max\{\xi_1,\lambda^{n/2}\xi_2\}}{\xi_1\gamma_2^2+2\beta\gamma_1\gamma_2+\xi_2\gamma_1^2}<1.\end{cases}\tag{44}$$

For instance, when $\xi_1=\xi_2=\xi$ this condition reads $\beta>(2\lambda^{\frac{n}{2}}-1)\xi$.

Similarly, carrying out the above argument for

$$\begin{aligned}J(\sqrt{k_1}w_\lambda,\sqrt{k_2}w_\lambda) &= \frac{1}{4}\frac{\left(k_1(S_\lambda^2-(\lambda-1)\int_{\mathbb{R}^n}w_\lambda^2)+k_2S_\lambda^2\right)^2}{S_\lambda^2(\mu_1k_1^2+2\beta k_1k_2+\mu_2k_2^2)}\\ &= \frac{S_\lambda^2}{4}\frac{\left(k_1(1-(1-1/\lambda)\sigma_0)+k_2\right)^2}{(\mu_1k_1^2+2\beta k_1k_2+\mu_2k_2^2)}.\end{aligned}$$

(we have again used (13), together with $\|w_\lambda\|^2_{L^2(\mathbb{R}^n)}\|w_\lambda\|^{-4}_{L^4(\mathbb{R}^n)} = \sigma_0/\lambda$), we are led to the following sufficient conditions for (38) :

$$\begin{cases} \beta > \max\{\mu_1 c_\lambda^{-1}, \mu_2 c_\lambda\}, \quad \text{with} \quad c_\lambda := 1-\sigma_0(1-1/\lambda) \in (1/\lambda, 1], \\ \\ \dfrac{\lambda^{2-n/2}\,[2\beta c_\lambda - (\mu_1+\mu_2 c_\lambda^2)]^2\,\mu_1}{\mu_1(\beta-\mu_2 c_\lambda)^2 + 2\beta(\beta-\mu_2 c_\lambda)(\beta c_\lambda-\mu_1)+\mu_2(\beta c_\lambda-\mu_1)^2} < 1. \end{cases} \tag{45}$$

Likewise, minimizing the fraction obtained by replacing c_λ by 1 in the expression of $J(\sqrt{k_1}w_\lambda, \sqrt{k_2}w_\lambda)$ and comparing to $\min\{E(\overline{u}_1,0), E(0,\overline{u}_2)\}$ gives the following sufficient condition : setting $\delta_1 = \beta-\mu_1, \delta_2 = \beta-\mu_2$,

$$\begin{cases} \delta_1 > 0, \quad \delta_2 > 0, \quad \text{and} \\ \\ \dfrac{(\delta_1+\delta_2)^2\ \max\{\lambda^{2-n/2}\mu_1, \mu_2\}}{\mu_1\delta_2^2 + 2\beta\delta_1\delta_2 + \mu_2\delta_1^2} < 1. \end{cases} \tag{46}$$

In particular, if $\mu_1 = \mu_2 = \mu$, this condition reduces to $\beta > (2\lambda^{1-\frac{n}{2}} - 1)\mu$.

To summarize, we state the following proposition.

Proposition 3.7 *The infimum A_0 is attained by a nonstandard radial solution of system (4) provided one of the conditions (41), (43), (44), (45), (46) holds (then of course $A_0 = A = A_r$).*

3.5 Proofs of statements (ii) and (iii) in Theorems 1 and 2

Suppose we have a nonstandard solution $u = (u_1, u_2)$ of system (4), such that $u_1 \geq 0, u_2 \geq 0$ in $\mathbb{R}^n$. Note that each of the functions u_i satisfies a *linear* equation

$$-\Delta u_i + c_i(x)u_i = 0$$

in $\mathbb{R}^n$, where $c_1(x) = 1-\mu_1 u_1^2(x) - \beta u_2^2(x)$, $c_2(x) = \lambda - \beta u_1^2(x) - \mu_2 u_2^2(x)$. So by the Strong Maximum Principle (see for example [9]) each of the functions u_1, u_2 is strictly positive in $\mathbb{R}^n$. By the results in [3] u_1 and u_2 are radial with respect to some point in $\mathbb{R}^n$. Note that solutions of (4) which are in $H^1(\mathbb{R}^n)$ are also in $C^2(\mathbb{R}^n)$ and tend to zero as $x \to \infty$ – this can be proved with the help of a classical "bootstrap" argument.

Next, we multiply the first equation in (4) by u_2, the second equation by u_1, and integrate the resulting equations over $\mathbb{R}^n$. This yields

$$\begin{aligned} \int_{\mathbb{R}^n} (\nabla u_1.\nabla u_2 + u_1 u_2) &= \textstyle\int_{\mathbb{R}^n} u_1 u_2(\mu_1 u_1^2 + \beta u_2^2) \\ \int_{\mathbb{R}^n} (\nabla u_1.\nabla u_2 + \lambda u_1 u_2) &= \textstyle\int_{\mathbb{R}^n} u_1 u_2(\beta u_1^2 + \mu_2 u_2^2), \end{aligned}$$

from which it follows that

$$\int_{\mathbb{R}^n} u_1 u_2 \left[(\lambda - 1) + (\mu_1 - \beta)u_1^2 + (\beta - \mu_2)u_2^2\right] = 0.$$

This equality is in a contradiction with the positivity of u_1 and u_2, as long as the three constants $(\lambda - 1), (\mu_1 - \beta), (\beta - \mu_2)$ are of the same sign or zero, and one of them is not zero. These are statements (ii) in Theorems 1 and 2.

By Proposition 1.1 if a minimizer for A (or A_r) exists and $\beta^2 < \mu_1\mu_2$ then there is a positive solution of system (4) (see also Remark 4 after Theorem 1). So the existence of a minimizer for A (or A_r) gives a contradiction whenever the hypotheses of (ii) are satisfied, and we obtain statements (iii). □

References

[1] Akhmediev N., Ankiewicz A.: Partially coherent solitons on a finite background. Phys. Rev. Lett. 82, 2661-2664 (1999)

[2] Buljan H., Schwartz T., Segev M., Soljacic M., Christoudoulides D.: Polychromatic partially spatially incoherent solitons in a noninstantaneous Kerr nonlinear medium. J. Opt. Soc. Am. B. 21 (2), 397-404 (2004)

[3] Busca J., Sirakov B.: Symmetry results for semilinear elliptic systems in the whole space. J. Diff. Eq. 163 (1), 41-56 (2000)

[4] Busca J., Sirakov B.: Harnack type estimates for nonlinear elliptic systems and applications. Ann. Inst. H. Poincaré Anal. Non Lin. 21 (5), 543-590 (2004)

[5] Christodoulides D., Eugenieva E., Coskun T., Mitchell M., Segev M.: Equivalence of three approaches describing partially incoherent wave propagation in inertial nonlinear media. Phys. Rev. E 63, 035601 (2001)

[6] Coffman C.V.: Uniqueness of the ground state solution for $-\Delta u + u = u^3$ and a variational characterization of other solutions. Arch. Rat. Mech. Anal. 46, 81-95 (1972)

[7] de Figueiredo D. G.: Nonlinear elliptic systems. Anais da Academia Brasileira de Ciências 72 (4), 453-469 (2000)

[8] Gidas B., Ni W.M., Nirenberg L.: Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. Adv. Math. Studies, 7A, 209-243 (1979)

[9] Gilbarg D., Trudinger N.: Elliptic Partial Differential Equations of Second Order, 2nd edition. Springer-Verlag.

[10] Hioe F.T.: Solitary waves for N coupled nonlinear Schrödinger equations. Phys. Rev. Lett. 82, 1152-1155 (1999)

[11] Kutuzov V., Petnikova V.M., Shuvalov V.V., Vysloukh V.A.: Cross-modulation coupling of incoherent soliton modes in photorefractive crystals. Phys. Rev. E 57, 6056-6065 (1998)

[12] Kwong M.K.: Uniqueness of positive solutions of $-\Delta u + u = u^p$ in $\mathbb{R}^n$. Arch. Rat. Mech. Anal. 105, 243-266 (1989)

[13] Lieb E., Loss M.: Analysis. Providence, RI, American Mathematical Scoiety, 1996

[14] Lin T.C., Wei J.: Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $N \leq 3$. Comm. Math. Phys. 255, 629-653 (2005)

[15] Lin T.C., Wei J.: Spikes in two coupled nonlinear Schrödinger equations. Ann. Inst. H. Poincaré, Anal. Non-Lin. 22 (4), 403-439 (2005)

[16] Mitchell M., Chen Z., Shih M., Segev M.: Self-trapping of partially spatially incoherent Light. Phys. Rev. Lett. 77, 490-493 (1996)

[17] Mitchell M., Segev M.: Self-trapping of incoherent white light. Nature (London) 387, 880-882 (1997)

[18] Rabinowitz P.: Minimax methods in critical point theory with applications to differential equations. CBMS Regional Conference Series in Mathematics 65, Providence, RI, 1986

[19] Willem M.: Minimax methods, Springer, 1996