Stochastic Particle Approximations for Two-dimensional Navier-Stokes Equations

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ABSTRACT: We present a probabilistic interpretation of some Navier-Stokes equations which describe the behaviour of the velocity field in a viscous incompressible fluid. We deduce from this approach stochastic particle approximations, which justify the vortex numerical schemes introduced by Chorin to simulate the solutions of the Navier-Stokes equations.

After some recalls on the McKean-Vlasov model, we firstly study a Navier-Stokes equation defined on the whole plane. The probabilistic approach is based on the vortex equation, satisfied by the curl of the velocity field. The equation is then related to a nonlinear stochastic differential equation, and this allows us to construct stochastic interacting particle systems with a "propagation of chaos" property: the laws of their empirical measures converge, as the number of particles tends to infinity, to a deterministic law of which the time-marginals solve the vortex equation. Our approach is inspired by Marchioro and Pulvirenti [26] and we improve their results in a pathwise sense.

Next we study the case of a Navier-Stokes equation defined on a bounded domain, with a no-slip condition at the boundary. In this case, the vortex equation satisfies a Neumann condition at the boundary, which badly depends on the solution. We simplify the model by studying in details the case of a fixed Neumann condition and we finally explain how the results should be adapted in the Navier-Stokes case.

KEYWORDS: 2d Navier-Stokes equation ; vortex method ; interacting particle systems; propagation of chaos.
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1 Introduction

We present in this course a probabilistic interpretation of some Navier-Stokes equations, from which we will deduce stochastic particle approximations and numerical schemes for the solutions of the equations.

The Navier-Stokes equations we consider describe the evolution of the velocity of a viscous and incompressible fluid in dimension two. About twenty years ago, Chorin [9]

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proposed a vortex method to simulate the solutions of these equations, based on the equation satisfied by the vorticity and involving cutoff kernels. His approach was not mathematically proved and many authors tried to give a proof of the convergence of the algorithm.

The main fact for explaining this approach is that in dimension two, the Navier-Stokes equation can be expressed as a simpler equation for the curl of the velocity, called the vortex equation. In the stochastic framework, this equation appears as a McKean-Vlasov equation, in which the coefficient of the drift term can explode. This remark is the basis of the probabilistic interpretation.

In 1982, Marchioro and Pulvirenti [26] have given a probabilistic interpretation of the Navier-Stokes equation thanks to a nonlinear diffusion, for bounded integrable initial data. They have rigorously introduced a cut-off model and some particle systems, and proved for each fixed time the convergence of the expectations of the empirical measures of the particle systems to the solution of the N.S. equation.

Then an open question was the pathwise convergence of these empirical measures to the law of the nonlinear diffusion, or equivalently, the propagation of chaos for the interacting particle systems.

In 1987, Osada [32] proved a propagation of chaos result for an interacting particle system without cut-off by an analytical method based on generators of generalized divergence form, but only for large viscosities and bounded density initial data. The tightness of the laws of the particle systems was always satisfied and the constraint on the viscosity appeared in the identification of the limit laws. This result is not satisfying, since the numerical stochastic particle methods are most efficient in the case of small viscosities, case which was not considered by the author. (See for example the comparison between finite volume deterministic methods and stochastic particle methods in [4]).

In this course, our aim is to obtain some pathwise particle approximations of the solution of the Navier-Stokes equation, by easily simulable systems. We consider two situations. In the first one, the equation is considered in the whole plane with an integrable and bounded initial condition (cf. [29]). We will interpret the vortex equation in a probabilistic point of view and will deduce some pathwise approximations with a precise rate of convergence. The second case will be devoted to equations in a bounded domain of the space, with a Dirichlet condition at the boundary (a no-slip condition). This will correspond to a vortex equation with a Neumann condition at the boundary, as it has been heuristically proven in [10]. We will study the simplified case in which the Neumann condition is fixed, as it has been developed in [21]. We will explain how we should modify the approach to approximate the solution of the Navier-Stokes equation with no-slip condition on the boundary of the domain.

We will essentially consider the framework introduced by Marchioro and Pulvirenti and will define particle systems with cut-off drift coefficients. Then we will study the convergence of these particle systems when the number of particles tends to infinity and the cut-off parameter tends to zero. In both cases above described, one defines a nonlinear diffusion process associated with the vortex (nonlinear) equation. In the bounded domain case, it is a reflected process with space-time random births at the boundary. At the level of processes, the nonlinearity means that the drift coefficient depends on the law of the diffusion process. We define a coupling between independent copies of this nonlinear process and some interacting particle systems with cut-off drift kernels. We work in the path space, and we consider initial data which are not necessarily probability densities. So, we associate with any sample path a signed weight depending on the initial condition. We prove that, when the size of the systems tends to infinity and the size of the cut-off tends to 0 in correlated asymptotics, the weighted empirical measures converge, as probability measures on the path space, to a deterministic probability measure whose time marginals are measure solutions of the vortex equation. This proof is obtained by showing a propagation of chaos result for the particle systems. We deduce from this result an algorithm to simulate the solution of the Navier-Stokes equation.

We will in the second section recall the main results concerning the probabilistic interpretation of the McKean-Vlasov equation and the associated interacting particle systems. We will explain what "propagation of chaos" means. Next, we will describe the two-dimensional Navier-Stokes equation in the whole plane and show the relation with the vortex equation. We will then introduce the probabilistic framework and develop the interacting particle approximations. In the fourth section, we will explain what happens in a bounded domain. We will see that things are really more complicated, especially if we wish to approximate a solution of a Navier-Stokes equation with no-slip condition at the boundary. We will rigorously study a simple case and explain what we should do to obtain the realistic case.

Notation

- For any integer $1 \leq p \leq +\infty$, we denote by L^p the space $L^p(\mathbb{R}^2)$. We will denote by $|.|$ the euclidian norm in $\mathbb{R}^2$, by $\|.\|_\infty$ the L^∞-norm and by $\|.\|_1$ the L^1-norm in $\mathbb{R}^2$.
- For any polish space E, the space $\mathcal{P}(E)$ will be the space of probability measures on E.
- For $p \in \mathcal{P}(E)$ and for any bounded measurable function f defined on E, we denote by $\langle p, f \rangle$ the integral $\int_E f(x)p(dx)$.

- The letter C will denote a positive real constant which can change from line to line.

2 Recalls on the McKean-Vlasov Model

Let us now recall the classical McKean-Vlasov model. The first one to study these equations, following ideas of Kac, was McKean in [27], and Gärtner in [14] introduced the terminology, relying the McKean works concerning stochastic systems in weak interaction, with the problem of the Vlasov equation as limit of particle systems evolving following the laws of the Newtonian mechanics.

2.1 The McKean-Vlasov Equation and the Associated Nonlinear Martingale Problem

The nonlinear partial differential equation, called McKean-Vlasov equation, is a nonlinear Fokker-Planck equation given in dimension d by

$$\frac{\partial p_t}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ij}[x,p_t]p_t\right) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(b_i[x,p_t]p_t\right), \quad p_0 \in \mathcal{P}(I\!\!R^d), \tag{2.1}$$

where p_t is for any time t a probability measure on $I\!\!R^d$, and for $m \in \mathcal{P}(I\!\!R^d)$,

$$\begin{aligned} b[x,m] &= \int_{I\!\!R^d} b(x,y)m(dy), \quad b(x,y) \text{ being a vector of } I\!\!R^d \\ a[x,m] &= \sigma[x,m]\sigma[x,m]^\star, \text{ and} \\ \sigma[x,m] &= \int_{I\!\!R^d} \sigma(x,y)m(dy), \quad \sigma(x,y) \text{ being a matrix of size } (\mathrm{d},\mathrm{k}). \end{aligned}$$

The equation is understood in a weak sense. For nice test functions φ, we have

$$\begin{aligned} \frac{\partial}{\partial_t} < p_t, \varphi > &= < p_t, \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}[.,p_t]\frac{\partial^2\varphi}{\partial x_i\partial x_j}(.) + \sum_{i=1}^{d} b_i[.,p_t]\frac{\partial\varphi}{\partial x_i}(.) > \\ &= < p_t, \mathcal{L}(p_t)\varphi > \end{aligned} \tag{2.2}$$

where the second order differential generator $\mathcal{L}(m)\varphi$ is defined for φ in $C_b^2(I\!\!R^d)$ by

$$\mathcal{L}(m)\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}[x,m]\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x) + \sum_{i=1}^{d} b_i[x,m]\frac{\partial\varphi}{\partial x_i}(x). \tag{2.3}$$

This equation has been studied from a probabilistic point of view by several authors, in particular McKean [27], Tanaka [38], Léonard [24], Gärtner [14], Sznitman [37] and [28]. The probabilistic approach for the study of these nonlinear Fokker-Planck equations consists in looking for underlying processes whose time marginals of the distribution are solutions of the equation. More precisely, one assumes some Markovian behaviour and one tries to define these processes as solutions of nonlinear martingale problems. The martingale problem will be nonlinear in a sense that we now define.

Definition 2.1 *Let $\{X_t,\ t \in [0,T]\}$ be the canonical process on $C([0,T],\ \mathbb{R}^d)$ and let us consider P_0 belonging to $\mathcal{P}(\mathbb{R}^d)$. The probability measure P on $C([0,T],\ \mathbb{R}^d)$ is a solution of the nonlinear martingale problem $(\mathcal{M}_{MV})$ issued from P_0 if for every $\varphi \in C_b^2(\mathbb{R}^d)$,*

$$\varphi(X_t) - \varphi(X_0) - \int_0^t \mathcal{L}(P_s)\varphi(X_s)ds \tag{2.4}$$

is a P-$(\mathcal{F}_t)$ martingale where $P_s = P \circ X_s^{-1}$, $P_0 = P \circ X_0^{-1}$ and $\mathcal{F}_t = \sigma(X_s, s \leq t)$.

Remark 2.2 *1) If we take expectations in (2.4), the family $(P_t)_{t\geq 0}$ is a solution of the evolution equation (2.1). The martingale problem gives more information than the evolution equation. It enables to consider multidimensional time-marginals as $P[X_s \in A, X_t \in B]$ or functionals depending on the whole process, as for example hitting times. So we consider the whole Markov process corresponding to the underlying physical model.*

2) This martingale problem defines a class of generalized Markov processes. Given an initial condition $\mu \in \mathcal{P}(\mathbb{R}^d)$, we are looking for a law P^μ on the canonical space satisfying $P^\mu(X(0) \in A) = \mu(A)$, but we do not demand that $P^\mu(B) = \int P^x(B)\mu(dx)$ (with $P^x = P^{\delta_x}$).

If we denote $p_t(A) = P^\mu(X_t \in A)$, the process $(X_., P^\mu)$ is Markovian in the sense that for any t, the quantity $P_q(X_{t+.} \in B|X_s, s \leq t)$ is a function of X_t and p_t and $\forall x \in \mathbb{R}^d$,

$$P^\mu(X_{t+.} \in B|X_t = x) = P^{p_t}(X_. \in B|X(0) = x).$$

Theorem 2.3 *If the coefficients σ and b are Lipschitz continuous on $\mathbb{R}^{2d}$, and if P_0 has a second order moment, the nonlinear martingale problem $(\mathcal{M}_{MV})$ has a unique solution.*

Proof. In fact, we prove here a stronger result, namely the existence and uniqueness, pathwise (given X_0 and B) and in law, of the solution of the following nonlinear stochastic differential equation :

$$X_t \quad = \quad X_0 + \int_0^t \sigma[X_s, P_s]dB_s + \int_0^t b[X_s, P_s]ds, \tag{2.5}$$

where B is a k-dimensional Brownian motion and X_0 independent of B is distributed according to P_0. The nonlinearity appears through P_s, which is the marginal at time s of the law P of X, This proof is completely detailed in Sznitman [37] Theorem 1.1 and is based on a fixed point theorem. Let $\mathcal{P}(C_T)$ be the space of probability measures on $C_T = C([0,T], \mathbb{R}^d)$ endowed with the weak convergence. This topology is metrisable with the Vaserstein complete metric ρ_T (cf. Rachev [34]), defined for m_1, m_2 by

$$\rho_T(m_1, m_2) = \inf\{\int_{C^T \times C^T} d_T(x,y) \wedge 1 \; m(dx,dy) : m \in \mathcal{P}(C_T \times C_T) \;\text{ with marginals } m_1 \;\text{ and } m_2\}$$

Here, d_T denotes the uniform metric on C_T. One considers the mapping $\psi : \mathcal{P}(C_T) \to \mathcal{P}(C_T)$ which associates with every $m \in \mathcal{P}(C_T)$ the law of X^m defined by :

$$X_t^m \;=\; X_0 + \int_0^t \sigma[X_s^m, m_s]dB_s + \int_0^t b[X_s^m, m_s]ds, \quad t \le T.$$

Observe that if (X) is a solution of (2.5) then the law of (X) is a fixed point of the function ψ and conversely.

By pathwise considerations one proves that for $t \le T$,

$$\rho_t^2(\psi(m^1), \psi(m^2)) \le K \int_0^t \rho_u^2(m^1, m^2)du,$$

for $m^1, m^2 \in \mathcal{P}(C_T)$. Then one deduces from the fixed point theorem the existence and uniqueness of the solution P of the martingale problem (2.4) defined on $[0,T]$. Pathwise uniqueness follows immediately due to the Lipschitz continuity of the coefficients. □

2.2 The Stochastic Interacting Particle System

Let us now describe a way to approximate the solution P of $(\mathcal{M}_{MV})$ previously obtained by the empirical measures of interacting particle systems. It is then natural to "replace" the nonlinearity in $(\mathcal{M}_{MV})$ by the empirical measure of the particles, and that leads to the following definition of the (triangular) systems.

Definition 2.4 *Let us consider independent $\mathbb{R}^k$-valued Brownian motions $(B^i)_{i \in \mathbb{N}^\star}$ and independent random variables $(X_0^i)_{i \in \mathbb{N}^\star}$ with law P_0 and independent of (B^i). For each $n \in \mathbb{N}^\star$, we define the n-particle system $(X^{1n}, X^{2n}, ..., X^{nn})$ as solution of the stochastic differential system*

$$\forall i \in \{1, ..., n\} \;, \quad X_t^{in} = X_0^i + \int_0^t \sigma[X_s^{in}, \mu_s^n]dB_s^i + \int_0^t b[X_s^{in}, \mu_s^n]ds \tag{2.6}$$

where μ^n is the empirical measure of the system, i.e. the (random) probability measure on the path space C_T defined by :

$$\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{in}}, \quad (\delta \text{ denoting the Dirac measure}).$$

The stochastic differential system (2.6) means that for every $1 \leq j \leq d$, $1 \leq i \leq n$,

$$X^{in}_{j,t} = X^i_{j,0} + \int_0^t \sum_{h=1}^k \sigma_{jh}[X^{in}_s, \mu^n_s] dB^i_{h,s} + \int_0^t b_j[X^{in}_s, \mu^n_s] ds.$$

It is easy to prove that under the assumptions of Theorem 2.3, (2.6) has for each n a unique pathwise solution. To compare the behaviour of these particles with the nonlinear problem, we use coupling techniques, consisting in comparing the n-particle system with n independent copies of the limiting equation (2.5) constructed on the same probability space. More precisely, we define for $(B^i)_i$ and $(X^i_0)_i$ previously given, the system $(X^i)_{i \in I\!\!N^\star}$ of independent processes with distribution P by

$$\forall i \in I\!\!N^\star \ , \quad X^i_t = X^i_0 + \int_0^t \sigma[X^i_s, P_s] dB^i_s + \int_0^t b[X^i_s, P_s] ds.$$

Let us prove pathwise estimates comparing $X^{i,n}$ and X^i.

Theorem 2.5 *Let us assume that the functions σ and b are bounded by a real constant M and Lipschitz continuous with Lipschitz constant L.*

Then for all $i \in \{1, ..., n\}$ and for any $T > 0$, one gets

$$\sup_n E(\sup_{t \leq T} |X^{i,n}_t|^2) < +\infty \ ; \ E(\sup_{t \leq T} |X^i_t|^2) < +\infty$$
$$E(\sup_{t \leq T} |X^{i,n}_t - X^i_t|^2) \leq \frac{C_T M^2}{n L^2} \exp(C L^2 T). \tag{2.7}$$

(2.7) obviously implies that the law of every subsystem of size k (k is a fixed integer), issued from the system $(X^{i,n})$, converges when n tends to infinity to the law $P^{\otimes k}$. This property is called propagation of chaos, which means a propagation of independence. Although the particles interact, the initial independence assumption propagates in time when the size of the system tends to infinity. That is mainly due to the exchangeability of the systems and to the fact that the interaction is a function of the empirical measure. Such an interaction is called weak interaction or mean field interaction.

Proof. The two first assertions are standard. Let us give a proof of the third one. The form $\frac{K}{n}$ of the convergence rate has been proved in Sznitman [37], but for all what follows in the next sections, we need to know how K depends on M and L. So we will detail the computation.

We denote by $\mu_.$ the empirical measure of the system $(X^1, ..., X^n)$. Using Burkholder-Davis-Gundy's and Holder's inequalities and the exchangeability of the system

$((X^{1,n},X^1),...,(X^{n,n},X^n))$, we get for any $t \leq T$,

$$
\begin{aligned}
&E(\sup_{s\leq t}|X_s^{in}-X_s^i|^2) \\
&\leq C_1\Big(\sum_{j=1}^d\sum_{h=1}^k\int_0^t E((\sigma_{jh}[X_s^{in},\mu_s^n]-\sigma_{jh}[X_s^i,P_s])^2)ds+\sum_{j=1}^d\int_0^t E((b_j[X_s^{in},\mu_s^n]-b_j[X_s^i,P_s])^2)ds\Big) \\
&\leq C_2\Big(L^2\int_0^t E(\sup_{u\leq s}|X_u^{in}-X_u^i|^2)ds \\
&\quad+\sum_{j=1}^d\sum_{h=1}^k\int_0^t E((\sigma_{jh}[X_s^i,\mu_s]-\sigma_{jh}[X_s^i,P_s])^2)ds+\sum_{j=1}^d\int_0^t E((b_j[X_s^i,\mu_s]-b_j[X_s^i,P_s])^2)ds\Big) \\
&\leq C_3\Big(L^2\int_0^t E(\sup_{u\leq s}|X_u^{in}-X_u^i|^2)ds+\sum_{j=1}^d\sum_{h=1}^k\int_0^t E((\frac{1}{n}\sum_{J=1}^n\sigma_{jh}(X_s^i,X_s^J)-\sigma_{jh}[X_s^i,P_s])^2)ds \\
&\quad+\sum_{j=1}^d\int_0^t E((\frac{1}{n}\sum_{J=1}^n b_j(X_s^i,X_s^J)-b_j[X_s^i,P_s])^2)ds\Big)
\end{aligned}
$$

A convexity inequality is not sufficient to obtain good estimates in the second and the third terms of the last expression. One remarks that since the variables $(X^i)_{1\leq i\leq n}$ are independent with law P, then

$$E(\sigma_{jh}(X_s^i,X_s^J)|X_s^r,\text{for all } r\neq J)=\sigma_{jh}[X_s^i,P_s],\quad \text{for } i\neq J,$$

and then, for $J\neq k$ and $i\neq J$,

$$E((\sigma_{jh}(X_s^i,X_s^J)-\sigma_{jh}[X_s^i,P_s])\cdot(\sigma_{jh}(X_s^i,X_s^k))-\sigma_{jh}[X_s^i,P_s])=0$$

Thus, it suffices to consider the n terms of the form $(\sigma_{jh}(X_s^i,X_s^J)-\sigma_{jh}[X_s^i,P_s])^2$. Hence,

$$E(\sup_{s\leq t}|X_s^{i,n}-X_s^i|^2)\leq CL^2\int_0^t E(|X_r^{i,n}-X_r^i|^2)dr+\frac{C'M^2t}{n}. \tag{2.8}$$

If we take $\phi(t)=E(\sup_{s\leq t}|X_s^{i,n}-X_s^i|^2)+\frac{C'M^2}{nCL^2}$, we have

$$\forall t\leq T\ ,\quad \phi(t)\leq\frac{C'M^2}{nCL^2}+CL^2\int_0^t\phi(r)dr \tag{2.9}$$

By Gronwall's lemma, we conclude that

$$\phi(t)\leq\frac{C'M^2}{nCL^2}\exp(CL^2T) \tag{2.10}$$

where C and C' are two real constants depending on T. □

2.3 Some Words about the Numerical Approach

Let us describe the numerical algorithm we deduce from this study to simulate the solution of the McKean-Vlasov equation. We refer to Bossy [2] and Bossy-Talay [5].

We only consider the case of dimension one, all what follows being easily generalized to every finite dimension. We follow [5] and explain how to simulate the cumulative distribution function of the solution P_t of the McKean-Vlasov equation at time t. This algorithm can easily be adapted to simulate other functionals of P_t, as for example moments of different orders. Let us define $V(t,x) = \int_{I\!R} H(x-y)P_t(dy)$, the cumulative distribution function of P_t where H is the Heavyside function defined by $H(x) = \mathbf{1}_{x\geq 0}$.

From now on, the number n of particles is fixed and the initial law P_0 is assumed to satisfy one of both following assumptions:

(1) P_0 is a Dirac measure at x_0.

(2) P_0 has a continuous density function u_0 such that there exist constants $M > 0$, $\eta \geq 0$ and $\alpha > 0$ with $u_0(x) \leq \eta \exp(-\alpha \frac{x^2}{2})$ for $|x| > M$. (If $\eta = 0$, the measure P_0 has a compact support).

The algorithm starts with an approximation of the initial condition $V(0,.) = V_0(.)$ which is the cumulative distribution function of P_0. One chooses n points $(y_0^1, ..., y_0^n)$ in $I\!R$ such that the piecewise constant function

$$\bar{V}_0(x) = \frac{1}{n}\sum_{i=1}^{n} H(x - y_0^i)$$

approximates V_0 in $L^1(I\!R)$. In the case (1), one takes $y_0^i = x_0$ and in the case (2), a possible choice is to set $y_0^i = \inf\{y; V_0(y) = \frac{i}{n}\}$ if $i = 1, ..., n-1$, and $y_0^n = \inf\{y; V_0(y) = 1 - \frac{1}{2n}\}$.

Then, we discretize in time the interacting particle system following an Euler scheme. We take $\Delta t > 0$ and K is chosen such that $T = K\Delta t$. The discrete times are denoted $t_k = k\Delta t$ with $1 \leq k \leq [K]$. The Euler scheme leads to the following discrete-time system:

$$\begin{aligned} Y_{t_k+1}^i &= Y_{t_k}^i + \frac{1}{n}\sum_{j=1}^{n} b(Y_{t_k}^i, Y_{t_k}^j)\Delta t + \frac{1}{n}\sum_{j=1}^{n} \sigma(Y_{t_k}^i, Y_{t_k}^j)(B_{t_k+1}^i - B_{t_k}^i) \\ Y_0^i &= y_0^i \quad , \quad i = 1, ..., n. \end{aligned} \tag{2.11}$$

Thus we approximate the empirical measure μ_{t_k} by the measure $\bar{\mu}_{t_k} = \frac{1}{n}\sum_{i=1}^{n} \delta_{Y_{t_k}^i}$.

In a similar way, we approximate $V(t_k, .)$ by the cumulative distribution function of $\bar{\mu}_{t_k}$

$$\bar{V}_{t_k}(x) = \frac{1}{n}\sum_{i=1}^{n} H(x - Y_{t_k}^i). \tag{2.12}$$

Then Bossy-Talay in [5] prove that

Theorem 2.6 *Assume that b is a bounded Lipschitz continuous function on $I\!R^2$, that $\sigma \in C_b^1(I\!R^2)$ and that there exists a constant $s > 0$ such that $\sigma(x,y) \geq s$ for every x, y. Assume moreover that P_0 satisfies (1) or (2). Then there exist strictly positive constants C_1 and C_2 depending on σ, b, V_0, T, and C depending on M, η and α, such that for all $k \in \{1, ..., K\}$, one gets*

$$\begin{aligned} \|V_0 - \bar{V}_0\|_{L^1} &\leq \frac{C\sqrt{\log n}}{n} \\ E(\|V(t_k,.) - \bar{V}_{t_k}(.)\|_{L^1}) &\leq C_1\Big(\|V_0 - \bar{V}_0\|_{L^1} + \frac{1}{\sqrt{n}} + \sqrt{\Delta t}\Big). \end{aligned} \tag{2.13}$$

3 The Vortex Equation in the whole plane

We will now adapt these results to the specific case of the two-dimensional Navier-Stokes equation.

3.1 The deterministic framework

Let us consider the velocity flow $u(t,x) \in I\!R^2, t \in I\!R_+, x \in I\!R^2$ of a viscous and incompressible fluid in the whole plane. The equation governing this motion is the Navier-Stokes equation given by

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + (u.\nabla)u(t,x) = \nu\Delta u(t,x) - \nabla p \ ; \\ &\nabla.u(t,x) = 0 \ ; \ u(x,t) \to 0 \text{ as } |x| \to +\infty, \qquad \text{for } 0 \leq t < +\infty, \end{aligned} \tag{3.1}$$

where p is the pressure function and $\nu > 0$ the viscosity (assumed to be constant).

The first step in the probabilistic approach consists in considering the equation satisfied by the vorticity flow $w(t,x) = \text{curl } u(t,x)$. Heuristically, since the divergence of u is zero, w is solution of the nonlinear partial differential equation, called **vortex equation**

$$\begin{aligned} &\frac{\partial w}{\partial t}(t,x) + (u \cdot \nabla)w(t,x) = \nu\Delta w(t,x) \ ; \\ &w(t,x) \to 0 \text{ as } |x| \to +\infty, \qquad \text{for } 0 \leq t < +\infty \ ; \ w_0(x) = w(0,x). \end{aligned} \tag{3.2}$$

This second equation is a one-dimensional equation in which the unknown pressure term has disappeared. It is not closed, and one has to write u in function of w, which is possible since $\nabla.u = 0$. Indeed, the function u can then be formally written as the orthogonal gradient of a courant function ψ, and then $w(t,x) = \text{curl } u(t,x)$ writes $\Delta\psi = w$. The function ψ is then equal to $G \star v$, where $\star$ denotes the convolution and G is the fundamental solution of the Poisson equation in dimension two. For each $r > 0$, $G(r) = -\frac{1}{2\pi}\ln r$.

Then, we deduce that for every $t \geq 0$,

$$u(t,x) = \int_{\mathbb{R}^2} \nabla^{\perp} G(|x-y|) w(t,y) dy = \int_{\mathbb{R}^2} K(x-y) w(t,y) dy \tag{3.3}$$

where $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})$. The equation (3.2) is then closed and we are interested in weak solutions of this equation.

The Biot-Savart kernel $K(x)$, equal to the orthogonal gradient of G, is given for all $x = (x_1, x_2) \in \mathbb{R}^2$ by

$$K(x) = \frac{1}{2\pi} \frac{1}{(x_1^2 + x_2^2)} (-x_2, x_1). \tag{3.4}$$

Note that $\nabla \cdot K = 0$. The main difficulty with the kernel K is its explosion at 0.

However, if $w \in L^{\infty} \cap L^1(\mathbb{R}^2)$, (3.3) makes sense, as it can be seen as follows.

Lemma 3.1 *The function K is bounded at infinity and integrable near zero. Let us introduce $K_1 = \int_{B(0,1)} |K(y)| dy$ and $K_{\infty} = \sup_{y \in B(0,1)^c} |K(y)|$.*

Then for every function $g \in L^1 \cap L^{\infty}$ and $x \in \mathbb{R}^2$,

$$|K * g(x)| \leq K_1 \|g\|_{L^{\infty}} + K_{\infty} \|g\|_{L^1}. \tag{3.5}$$

The proof is very simple, and (3.5) leads to work in the space $L^1 \cap L^{\infty}$, which is well adapted to a probabilistic point of view. In all the following, we will assume that

$$w_0 \in L^1 \cap L^{\infty}. \tag{3.6}$$

At this level, there are two probabilistic approaches in the literature. The first one consists in interpreting classical solutions of the vortex equation, using a Feynman-Kac approach, as the expectations of some stochastic processes. This approach has been developed by Busnello in [8] and allows to prove an existence and uniqueness theorem, but not to obtain a constructive method to simulate the solutions of the equation. So in the following, we will present the second approach which consists in looking for weak solutions of the vortex equation, considered as a Fokker-Planck equation. This approach follows the precursive ideas of Chorin and has been firstly mathematically developed by Marchioro and Pulvirenti in [26] and then in [29].

We have seen that the drift kernel K explodes at 0, so the first step is to approximate it by bounded kernels.

3.2 A cut-off equation

Let us consider as in [26] the cut-off kernel K_ε defined in the following way. We denote by $G(r) = -\frac{1}{2\pi}\ln r$ the fundamental solution of the Poisson equation. One knows that for $x \in I\!R^2$, $K(x) = \nabla^\perp G(|x|)$. For each $\varepsilon > 0$, we consider G_ε defined as $G_\varepsilon(r) = G(r)$ if $|r| \geq \varepsilon$ and arbitrarily extended to an even $C^2(I\!R)$ function such that $|G'_\varepsilon(r)| \leq |G'(r)|$ and $|G''_\varepsilon(r)| \leq |G''(r)|$. Then we define

$$K_\varepsilon(x) = \nabla^\perp G_\varepsilon(|x|).$$

The function K_ε is then bounded by a real M_ε and Lipschitz continuous with a Lipschitz constant L_ε .

Let us also remark that by construction, $\nabla.K_\varepsilon = 0$, and that K_ε satisfies a similar inequality as (3.5): for every $\varepsilon > 0$ and every function $g \in L^1 \cap L^\infty$

$$\|K_\varepsilon * g\|_\infty \leq K_1\|g\|_{L^\infty} + K_\infty\|g\|_{L^1}. \tag{3.7}$$

For each fixed $\varepsilon > 0$, we consider the regularized equation

$$\partial_t w^\varepsilon = \nu\Delta w^\varepsilon - (K_\varepsilon \star w^\varepsilon.\nabla)w^\varepsilon \ ; \ w_0 \in L^1 \cap L^\infty. \tag{3.8}$$

We are in the McKean-Vlasov context, except that the initial condition is not a probability measure. To overpass this problem, we use a trick due to Jourdain [20] to pass from a density initial function to any $w_0 \in L^1 \cap L^\infty$.

We define the bounded function h by

$$\forall x \in I\!R^2 \ , \quad h(x) = \frac{w_0(x)\|w_0\|_1}{|w_0(x)|}, \text{ with the convention } \frac{0}{0} = 0. \tag{3.9}$$

Let us remark that for each $x \in I\!R^2$,

$$-\|w_0\|_1 \leq h(x) \leq \|w_0\|_1, \tag{3.10}$$

and that $w_0(x)dx = h(x)\frac{|w_0(x)|}{\|w_0\|_1}dx$, where $\frac{|w_0|}{\|w_0\|_1}$ is thus a probability density.

Now, for Q a probability measure on $C([0,+\infty), I\!R^2)$, we define the family $(\tilde{Q}_t)_{t\geq 0}$ of weighted signed measures on $I\!R^2$ by

$$\forall B \text{ Borel subset of } I\!R^2 \ , \quad \tilde{Q}_t(B) = E^Q(1_B(X_t)h(X_0)), \tag{3.11}$$

where X denotes the canonical process on $C([0,+\infty), I\!R^2)$. (One associates with each sample-path a signed weight depending on the initial position).

The following lemma will be usefull in the following.

Lemma 3.2 *1) For each $t \geq 0$, the signed measure $\tilde{Q}_t$ is bounded, and its total mass is less than $\|w_0\|_1$.*

2) If Q_t is absolutely continuous with respect to the Lebesgue measure, then the same holds for $\tilde{Q}_t$.

Proof. Since the function h is bounded by $\|w_0\|_1$ and using (3.11), the lemma is obvious. □

The equation (3.8) understood in its weak form leads naturally to the following non-linear stochastic differential equation.

Definition 3.3 *Let us consider a random $I\!R^2$-valued variable X_0 with distribution $\frac{|w_0(x)|}{\|w_0\|_1}dx$ and B be a 2-dimensional Brownian motion independent of X_0. A solution $Z^\varepsilon \in C(I\!R_+, I\!R^2)$ of the nonlinear stochastic differential equation satisfies $\forall t \in I\!R_+$*

$$Z_t^\varepsilon = X_0 + \sqrt{2\nu}B_t + \int_0^t K_\varepsilon \star \tilde{P^\varepsilon}_s(Z_s^\varepsilon)ds,$$
$$P_s^\varepsilon = \mathcal{L}(Z_s^\varepsilon) \text{ and } \tilde{P}_s^\varepsilon \text{ is related to } P_s^\varepsilon \text{ by } (3.11). \tag{3.12}$$

Since K_ε is Lipschitz continuous and bounded, we can adapt the previous section to show the existence and pathwise uniqueness of the solution Z^ε of this equation. By the Girsanov theorem, one knows moreover that for each time $t > 0$, its law P_t^ε has a density p_t^ε with respect to the Lebesgue measure. That also implies the existence of a density $\tilde{p}_t^\varepsilon$ for the weighted measure $\tilde{P}_t^\varepsilon$.

By Itô's formula, one proves that for any $T > 0$, the probability measure P^ε on $\mathcal{P}(C([0,T], I\!R^2))$ is solution of the nonlinear martingale problem $(\mathcal{M}^\varepsilon)$: for any $\phi \in C_b^2(I\!R^2)$ and $t \leq T$,

$$\phi(X_t) - \phi(X_0) - \int_0^t K^\varepsilon * \tilde{P}_s(X_s).\nabla\phi(X_s)ds - \nu\int_0^t \Delta\phi(x_s)ds \tag{3.13}$$

is a P^ε-martingale, where X is the canonical process on $C([0,T], I\!R^2)$, P_0 is equal to $\frac{|w_0(x)|}{\|w_0\|_1}dx$, and $\tilde{P}_s^\varepsilon$ is related to $P_s^\varepsilon = P^\varepsilon \circ X_s^{-1}$ by (3.11).

Multiplying all terms of (3.13) by $h(X_0)$, taking expectations and using that $\nabla.K_\varepsilon = 0$, we show that $(\tilde{p}_t^\varepsilon)_t$ is a weak solution of the equation (3.8), and further for each $t > 0$,

$$\|\tilde{p}_t^\varepsilon\|_{L^1} \leq E^{P^\varepsilon}(|h(X_0)|) \leq \|w_0\|_{L^1}. \tag{3.14}$$

That implies in particular that the drift coefficient $K_\varepsilon \star \tilde{p}^\varepsilon$ is bounded, uniformly in time, by $\|w_0\|_{L^1}$. Then one can apply analysis results due to Friedman [12] to deduce that the solution $\tilde{p}^\varepsilon$ of (3.8) is continuous on $[0,T] \times I\!R^d$ and belongs to $C^{1,2}((0,T] \times I\!R^d)$. But since $\nabla.K_\varepsilon = 0$, one proves that $\tilde{p}^\varepsilon$ is also a strong solution of the equation

$$\partial_t v = \nu \Delta v - K_\varepsilon \star v.\nabla v \ ; \ v_0 = w_0$$

Then one knows (cf. [13] or [22]) that $\tilde{p}^\varepsilon$ satisfies a Feynman-Kac formula and can be written for any $t \leq T$ as $E(w_0(Y_t^{x,\varepsilon}))$, where the process $(Y_t^{x,\varepsilon})$ is defined by

$$Y_t^{x,\varepsilon} = x + \sqrt{2\nu} W_t + \int_0^t K_\varepsilon \star \tilde{p}_s^\varepsilon (Y_s^{x,\varepsilon}) ds$$

from which we deduce that

$$\sup_{t \leq T} \|\tilde{p}_t^\varepsilon\|_{L^\infty} \leq \|w_0\|_{L^\infty}. \tag{3.15}$$

Hence the good space to define the solutions is the space

$$\mathcal{H} = \{q \in L^\infty([0,T], L^1 \cap L^\infty); \ \sup_{t \leq T} \|q_t\|_{L^1} \leq \|w_0\|_{L^1} \text{ and } \sup_{t \leq T} \|q_t\|_{L^\infty} \leq \|w_0\|_{L^\infty}\}$$

and we define, for $q \in \mathcal{V}$ and $t \leq T$, the norm

$$|||q_t||| = \|q_t\|_{L^1} + \|q_t\|_{L^\infty}.$$

Let us now prove the uniqueness of the solution of (3.8) in this space. We use to this aim the mild form of the equation and firstly remark that the heat kernel G_t^ν on $I\!R^2$ defined by $G_t^\nu(x) = \frac{1}{4\pi t\nu} e^{-\frac{|x|^2}{4t\nu}}$ satisfies

Lemma 3.4

$$\|\nabla_x G_t^\nu\|_{L^1} \leq \frac{C}{\sqrt{\nu t}} \tag{3.16}$$

where C is a real constant, and then $\int_0^t \|\nabla_x G_{t-s}^\nu\|_{L^1} ds < +\infty$.

Proposition 3.5 *1) Each weak solution $w^\varepsilon \in \mathcal{H}$ of (3.8) is a.s. solution of the evolution equation*

$$w^\varepsilon(t,x) = G_t^\nu \star w_0(x) + \int_0^t \int_{I\!R^2} \nabla_x G_{t-s}^\nu (x-y).K_\varepsilon \star w_s^\varepsilon(y) w_s^\varepsilon(y) dy ds. \tag{3.17}$$

2) There exists a unique weak solution of (3.8) in $\mathcal{H}$.

Proof. 1) Using Fubini's theorem (allowed by Lemma 3.16 and (3.7)), we easily prove that for every function $\psi(t,x) \in C_b^{1,2}(I\!R_+ \times I\!R^2)$,

$$\int_{I\!R^2} \psi(t,x) w_t^\varepsilon(x) dx = \int_{I\!R^2} \psi(0,x) w_0(x) dx + \nu \int_0^t \int_{I\!R^2} \Delta\psi(s,x) w_s^\varepsilon(x) dx ds$$
$$+ \int_0^t \int_{I\!R^2} \psi'_s(s,x) w_s^\varepsilon(x) dx ds + \int_0^t \int_{I\!R^2} (K_\varepsilon \star w_s^\varepsilon(x).\nabla_x \psi(s,x)) w_s^\varepsilon(x) dx ds.$$

Then by choosing for a fixed time t, $\psi(s,x) = \int_{\mathbb{R}^2} G^\nu_{t-s}(x-y)\phi(y)dy$ for $\phi \in C^2_b(\mathbb{R}^2)$ and thanks again to Fubini's theorem, we obtain the mild equation (3.17).

2) Let q and q' be two solutions of (3.8) belonging to $\mathcal{H}$. Then

$$q_t(x)-q'_t(x) = \int_0^t \int \nabla_x G^\nu_{t-s}(x-y)(K_\varepsilon \star q_s(y)(q_s(y)-q'_s(y)) + K_\varepsilon \star (q_s(y)-q'_s(y))q'_s(y))dyds.$$

Thus by (3.16) and (3.7), a simple computation gives (the constant C may change from line to line),

$$\begin{aligned} & |||q_t(.) - q'_t(.)||| \\ \leq & \int_0^t ||| \int \nabla_x G^\nu_{t-s}(x-y)(K_\varepsilon \star q_s(y)(q_s(y) - q'_s(y)) + K_\varepsilon \star (q_s(y) - q'_s(y))q'_s(y))dy|||ds \\ \leq & \ C(K_1 + K_\infty)(\|w_0\|_{L^1} + \|w_0\|_{L^\infty}) \int_0^t \|\nabla_x G^\nu_{t-s}\|_{L^1} |||q_s(.) - q'_s(.)|||ds \\ \leq & \ \frac{C}{\sqrt{\nu}} \int_0^t \frac{|||q_s(.) - q'_s(.)|||}{\sqrt{t-s}} ds. \end{aligned}$$

By an iteration, we obtain

$$\begin{aligned} |||q_t(.) - q'_t(.)||| & \leq C \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \frac{|||q_u(.) - q'_u(.)|||}{\sqrt{s-u}} duds \\ & \leq C \int_0^t |||q_u(.) - q'_u(.)||| \int_u^t \frac{1}{\sqrt{t-s}\sqrt{s-u}} dsdu \\ & \leq C \int_0^t |||q_u(.) - q'_u(.)|||du. \end{aligned}$$

Therefore, by Gronwall's lemma,

$$\sup_{t\in[0,T]} |||q_t(.) - q'_t(.)||| = 0$$

and the uniqueness in (3.8) is proved. □

3.3 Existence and uniqueness of the vortex equation

We are now able to study the existence and uniqueness of a solution of the vortex equation in $\mathcal{H}$.

Theorem 3.6 *Assume $w_0 \in L^1 \cap L^\infty$. There exists a unique weak solution $w \in \mathcal{H}$ to the vortex equation (3.2). This solution is moreover solution of the evolution equation*

$$w_t(x) = G^\nu_t \star w_0(x) + \int_0^t \int_{\mathbb{R}^2} \nabla_x G^\nu_{t-s}(x-y).K \star w_s(y)w_s(y)dyds. \tag{3.18}$$

Proof. The uniqueness is proved following a similar proof as the one in Proposition 3.2, using (3.5) instead of (3.7).

Let us now show the existence. Let us first remark that the space $L^1 \cap L^\infty$ endowed with the norm $|||.|||$ is a complete space.

For each $\varepsilon > 0$, we have constructed a solution $(\tilde{p}_t^\varepsilon)_t$ of (3.8). Our aim is now to prove that this family is Cauchy (in ε) and that the limit point satisfies the vortex equation. Let us fix $\varepsilon > 0$ and $\varepsilon' > 0$ and consider the two families $(\tilde{p}_t^\varepsilon)_t$ and $(\tilde{p}_t^{\varepsilon'})_t$ previously defined. They are solution of the corresponding mild equations (3.17) and we write

$$\begin{aligned}\tilde{p}_t^\varepsilon(x) - \tilde{p}_t^{\varepsilon'}(x) &= \int_0^t \int \nabla_x G_{t-s}^\nu(x-y)(K_\varepsilon \star \tilde{p}_s^\varepsilon(y)\tilde{p}_s^\varepsilon(y) - K_{\varepsilon'} \star \tilde{p}_s^{\varepsilon'}(y)\tilde{p}_s^{\varepsilon'}(y))dyds \\ &= \int_0^t \int \nabla_x G_{t-s}^\nu(x-y)(K_\varepsilon \star (\tilde{p}_s^\varepsilon(y) - \tilde{p}_s^{\varepsilon'}(y))\tilde{p}_s^\varepsilon(y) \\ &\quad + (K_\varepsilon \star \tilde{p}_s^{\varepsilon'}(y) - K_{\varepsilon'} \star \tilde{p}_s^{\varepsilon'}(y))\tilde{p}_s^\varepsilon(y) + K_{\varepsilon'} \star \tilde{p}_s^{\varepsilon'}(y)(\tilde{p}_s^\varepsilon(y) - \tilde{p}_s^{\varepsilon'}(y))dyds.\end{aligned}$$

Since the functions $(\tilde{p}_t^\varepsilon)_t$ and $(\tilde{p}_t^{\varepsilon'})_t$ belong to $\mathcal{H}$ and by (3.16) and (3.7), we will do similar calculations as before to estimate the first and third terms of the right hand side. Now, to control the second term, we need the following lemma.

Lemma 3.7 *For each $t \leq T$, for each $x \in \mathbb{R}^2$,*

$$\left|\int_{\mathbb{R}^2} (K_{\varepsilon'}(x-y) - K_\varepsilon(x-y))\tilde{p}_t^{\varepsilon'}(y)dy\right| \leq 2\varepsilon' \|w_0\|_\infty. \tag{3.19}$$

Proof. Let us assume that $\varepsilon' > \varepsilon$. Since $K_{\varepsilon'}$ and K_ε coincide for $|x| \geq \varepsilon'$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} (K_{\varepsilon'}(x-y) - K_\varepsilon(x-y))\tilde{p}_t^{\varepsilon'}(y)dy\right| \\ \leq\; & \int_{|x-y|\leq\varepsilon'} |K_{\varepsilon'}(x-y) - K_\varepsilon(x-y)||\tilde{p}_t^{\varepsilon'}(y)|dy \\ \leq\; & \int_{|x-y|\leq\varepsilon'} (|K_{\varepsilon'}(x-y)| + |K_\varepsilon(x-y)|)|\tilde{p}_t^{\varepsilon'}(y)|dy \\ \leq\; & 2\int_{|x-y|\leq\varepsilon'} |K(x-y)||\tilde{p}_t^{\varepsilon'}(y)|dy \leq 2\|w_0\|_\infty \int_{|z|\leq\varepsilon'} |K(z)|dz \quad \text{by (3.15)} \\ \leq\; & 2\varepsilon'\|w_0\|_\infty \quad \text{by an easy computation.}\end{aligned}$$

We have used that by definition and for every $\varepsilon > 0$, $|K_\varepsilon(x-y)| \leq |K(x-y)|$. □

Let us now come back to the proof of Theorem 3.6. We apply Lemma 3.7 to show that

$$|||\tilde{p}_t^\varepsilon - \tilde{p}_t^{\varepsilon'}||| \;\leq\; C_1\varepsilon' + C_2 \int_0^t \frac{1}{\sqrt{t-s}} |||\tilde{p}_s^\varepsilon - \tilde{p}_s^{\varepsilon'}|||ds \tag{3.20}$$

and by iteration of this inequality and Gronwall's lemma, we finally obtain that for $\varepsilon' > \varepsilon$,

$$\sup_{t\leq T} |||\tilde{p}_t^\varepsilon - \tilde{p}_t^{\varepsilon'}||| \;\leq\; C\varepsilon' \exp^{C'T}. \tag{3.21}$$

The family $\varepsilon \to \tilde{p}^\varepsilon$ is then a Cauchy family in $L^\infty([0,T], L^1 \cap L^\infty)$.

Hence, there exists a function $w \in L^\infty([0,T], L^1 \cap L^\infty)$, such that for each $t \leq T$

$$\lim_{\varepsilon \to 0} \sup_{t \leq T} |||\tilde{p}_t^\varepsilon - w_t||| = 0$$

That implies that

$$\sup_{t \leq T} \|w_t\|_{L^1} \leq \|w_0\|_{L^1} \; ; \; \sup_{t \leq T} \|w_t\|_{L^\infty} \leq \|w_0\|_{L^\infty}. \tag{3.22}$$

Next, with similar arguments as before, one can pass to the limit in ε in the mild equation (3.17) and show that w is solution of the linear equation

$$q(t,x) = G_t^\nu * w_0(x) + \int_0^t \int \nabla_x G_{t-s}^\nu (x-y) K * w_s(y) q(s,y) dy. \tag{3.23}$$

It is not hard to deduce from it that w is a weak solution of the vortex equation. The theorem is proved. □

3.4 The Nonlinear Process Associated with the Vortex Equation

As previously, the equation (3.2) leads naturally to a nonlinear martingale problem. Let us fix $T > 0$.

Definition 3.8 *The probability measure $P \in \mathcal{P}(C([0,T], \mathbb{R}^2))$ is solution of the nonlinear martingale problem $(\mathcal{M})$ if for each $\phi \in C_b^2(\mathbb{R}^2)$ and $t \leq T$,*

$$\phi(X_t) - \phi(X_0) - \int_0^t K * \tilde{P}_s(X_s).\nabla\phi(X_s) ds - \nu \int_0^t \Delta\phi(x_s) ds$$

is a P-martingale, where X is the canonical process on $C([0,T], \mathbb{R}^2)$, P_0 is equal to $\frac{|w_0(x)|}{||w_0||_1} dx$, and $\tilde{P}_s$ is related by (3.11) to $P_s = P \circ X_s^{-1}$.

This nonlinear martingale problem is related to the following nonlinear stochastic differential equation.

Definition 3.9 *Let us consider a $\mathbb{R}^2$-valued random variable X_0 with distribution $\frac{|w_0(x)|}{||w_0||_1} dx$ and B be a 2-dimensional Brownian motion independent of X_0. A solution $X \in C(\mathbb{R}_+, \mathbb{R}^2)$ of the nonlinear stochastic differential equation satisfies $\forall t \in \mathbb{R}_+$*

$$\begin{aligned} &X_t = X_0 + \sqrt{2\nu} B_t + \int_0^t K * \tilde{P}_s(X_s) ds, \\ &P_s \text{ is the marginal at time } s \text{ of the law of } X_s. \end{aligned} \tag{3.24}$$

Notations: We denote by $\hat{\mathcal{P}}_\infty(C([0,T],\mathbb{R}^2))$ the space of probability measures on $C([0,T],\mathbb{R}^2)$ such that for each $s \leq T$, the time-marginal P_s (and then $\tilde{P}_s$) has a density with respect to the Lebesgue measure, belonging to $L^\infty(\mathbb{R}^2)$. Then there exists (cf. Meyer [31] p.194) a measurable version $(s,x) \to \tilde{p}(s,x)$ in L^∞ such that for $s \in [0,T]$, $\tilde{P}_s(dx) = \tilde{p}(s,x)dx$.

We will prove the following theorem.

Theorem 3.10 *Let us consider $w_0 \in L^1 \cap L^\infty$. Then there exists a unique solution $P \in \hat{\mathcal{P}}_\infty(C([0,T],\mathbb{R}^2))$ to the martingale problem $(\mathcal{M})$ such that $P_0(dx) = \frac{|w_0(x)|}{\|w_0\|_1}dx$.*

Moreover, for each $t \in [0,T]$, the "weighted" density $\tilde{p}(t,x)$ is a.s. equal to w_t defined in Theorem 3.6.

Proof. 1) If $P \in \hat{\mathcal{P}}_\infty(C([0,T],\mathbb{R}^2))$ is a solution of $(\mathcal{M})$ and $\tilde{p}_t$ a measurable version of the densities of $\tilde{P}_t$, then by multiplying by $h(X_0)$ and by taking the expectation in $(\mathcal{M})$, we get that $\tilde{p}$ is a weak solution of (3.2) with initial condition w_0. Then by the uniqueness given in Theorem 3.6, for each $t \in [0,T]$, $\tilde{p}_t(x) = w_t(x)$ *a.s.*.

2) let us consider this unique weak solution w of (3.2) issued from w_0. Then w_t is for each t a bounded integrable function. We say that $P^w \in \mathcal{P}(C([0,T],\mathbb{R}^2))$ is solution of the classical martingale problem $(\mathcal{M}^w)$ if for each $\phi \in C_b^2(\mathbb{R}^2)$,

$$\phi(X_t) - \phi(X_0) - \nu \int_0^t \Delta\phi(X_s)ds - \int_0^t K * w_s(X_s).\nabla\phi(X_s)ds$$

is a P^w-martingale and $P_0^w(dx) = \frac{|w_0(x)|}{\|w_0\|_1}dx$. This martingale problem is well-posed. Indeed, by Lemma 3.1 and (3.22), for each $s \geq 0$,

$$\|K * w_s\|_\infty \leq K_1\|w_s\|_\infty + K_\infty\|w_s\|_1 \leq K_1\|w_0\|_\infty + K_\infty\|w_0\|_1 \tag{3.25}$$

so the drift coefficient is bounded, and by Girsanov's Theorem, we get the existence and uniqueness of the solution of $(\mathcal{M}^w)$. Moreover, every time marginal of P^w admits a density p_s^w, and multiplying by $h(X_0)$ all the terms of the martingale problem and taking expectations, we obtain immediately that $(\tilde{p}_s^w)_{s\in[0,T]}$ is solution of the weak equation: for each $\phi \in C_b^2(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2} \phi(x)\tilde{p}_t^w(x)dx \quad &= \quad \int_{\mathbb{R}^2} \phi(x)w_0(x)dx + \nu\int_0^t\int_{\mathbb{R}^2} \Delta\phi(x)\tilde{p}_s^w(x)dxds \\ &\quad + \int_0^t\int_{\mathbb{R}^2}(K * w_s(x).\nabla\phi(x)\tilde{p}_s^w(x)dxds. \end{aligned}\tag{3.26}$$

Then the flow $\tilde{p}_t$ is solution of

$$\frac{\partial\tilde{p}^w}{\partial t} + (K * w.\nabla)\tilde{p}^w = \nu\Delta\tilde{p}^w \ ; \quad \tilde{p}_0^w = w_0.$$

Adapting what we have done in Section 3.2, since the divergence of K is zero and w_0 belongs to $L^\infty \cap L^1$, we are able to prove that for each $t \in [0,T]$,

$$\|\tilde{p}_t^w\|_\infty \leq \|w_0\|_\infty. \tag{3.27}$$

3) Let us now prove the existence and uniqueness of a solution of (3.26) in $L^\infty([0,T], L^1 \cap L^\infty)$. We obtain as before by Fubini's theorem and thanks to Lemma 3.1 that $\tilde{p}^w$ is solution of the following evolution equation

$$\tilde{p}^w(t,x) = G_t^\nu * w_0(x) + \int_0^t \int \nabla_x G_{t-s}^\nu (x-y) K * w_s(y) \tilde{p}^w(s,y) dy. \tag{3.28}$$

We can easily prove the uniqueness of the solution of (3.28) in $L^\infty([0,T], L^\infty)$. Since $\tilde{p}^w$ and w are solutions of (3.28),

$$\sup_{t\in[0,T]} \|\tilde{p}^w(t,.) - w(t,.)\|_\infty = 0.$$

So the probability measure P^w is solution of the nonlinear martingale problem $(\mathcal{M})$.

4) Let us now prove the uniqueness of a solution of this martingale problem. Let P and Q be two solutions. By the same reasoning as above, it is easy to prove that $(\tilde{P}_t)_{t\geq 0}$ and $(\tilde{Q}_t)_{t\geq 0}$ are equal to $(w(t,x)dx)_{t\geq 0}$. Hence P and Q are solutions of the classical well-posed martingale problem $(\mathcal{M}^w)$ and are then equal, and Theorem 4.7 is proved. $\square$

3.5 The Particle Approximations

3.5.1 The Cut-off Case

Let us firstly adapt to the cut-off case the results of Section 2.

Definition 3.11 *Consider a sequence $(B^i)_{i\in\mathbb{N}}$ of independent Brownian motions on $\mathbb{R}^2$ and a $\mathbb{R}^2$-valued sequence of independent variables $(Z_0^i)_{i\in\mathbb{N}}$ distributed according $\frac{|w_0(x)|}{\|w_0\|_1}dx$, and independent of $(B^i)_{i\in\mathbb{N}}$. For a fixed ε, for each $n \in \mathbb{N}^*$, and $1 \leq i \leq n$, let us consider the interacting processes defined by*

$$Z_t^{in,\varepsilon} = Z_0^i + \sqrt{2\nu} B_t^i + \int_0^t K_\varepsilon * \tilde{\mu}_s^{n,\varepsilon}(Z_s^{in,\varepsilon}) ds \tag{3.29}$$

where $\tilde{\mu}^{n,\varepsilon} = \frac{1}{n}\sum_{j=1}^n h(Z_0^j)\delta_{Z^{jn,\varepsilon}}$ is the weighted empirical measure of the system. (That is a random finite measure on $C(\mathbb{R}_+, \mathbb{R}^2)$).

We also define the limiting independent processes by

$$\bar{Z}_t^{i,\varepsilon} = Z_0^i + \sqrt{2\nu} B_t^i + \int_0^t K_\varepsilon * \tilde{P}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon}) ds, \quad P_s^\varepsilon \text{ is the law of } \bar{Z}_s^{i,\varepsilon} \tag{3.30}$$

Proposition 3.12 *1) For each $T > 0$ and for each n, there exits a unique (pathwise) solution to the interacting particle system (3.29) in $C([0,T], \mathbb{R}^{2n})$ and a unique (pathwise) solution to the nonlinear equation (3.30) in $C([0,T], \mathbb{R}^2)$.*

2) For each $T > 0$,

$$E(\sup_{t\leq T} |Z_t^{in,\varepsilon} - \bar{Z}_t^{i,\varepsilon}|) \leq \frac{M_\varepsilon}{\sqrt{n} L_\varepsilon} \exp(\|w_0\|_1 T L_\varepsilon). \tag{3.31}$$

Proof. By the boundedness of h, the proof of the first assertion is obvious. The second assertion is obtained by an adaptation of the proof of Theorem 2.5. Since the diffusion coefficient is a constant, one can lead the computation in L^1. □

3.5.2 The Approximating Interacting Particle System

We now define the interacting particle system we are interested in.

We now consider $T > 0$ and a sequence (ε_n) tending to 0 such that

$$\lim_n \frac{M_{\varepsilon_n}}{\sqrt{n} L_{\varepsilon_n}} \exp(\|w_0\|_1 T L_{\varepsilon_n}) = 0. \tag{3.32}$$

For each n and given independent Brownian motions $(B^i)_{1\leq i\leq n}$, we consider a coupling between the n particle system $(Z^{in} = Z^{in,\varepsilon_n})$ defined with the drift K_{ε_n} as in (3.29), and the corresponding n independent limiting processes $\bar{Y}^{in} = \bar{Z}^{i,\varepsilon_n}$ defined for each $t \leq T$ and n by

$$\bar{Y}_t^{in} = Z_0^i + \sqrt{2\nu} B_t^i + \int_0^t K_{\varepsilon_n} * \tilde{P}_s^n(\bar{Y}_s^{in}) ds, \tag{3.33}$$

where P_s^n is the law of the $\bar{Y}_s^{in}$.

By similar arguments as before, P_s^n admits a density function p_s^n and then $\tilde{P}_s^n$ admits a density function $\tilde{p}_s^n$ belonging to $\mathcal{H}$, weak solution of the equation

$$\frac{\partial \tilde{p}^n}{\partial t} = \nu \Delta \tilde{p}^n - (K_{\varepsilon_n} * \tilde{p}^n \cdot \nabla) \tilde{p}^n \ ; \ \tilde{p}_0^n = w_0. \tag{3.34}$$

and solution of the mild equation

$$\tilde{p}_t^n(x) = G_t^\nu * w_0(x) + \int_0^t \nabla_x G_{t-s}^\nu * (\tilde{p}_s^n . K_{\varepsilon_n} * \tilde{p}_s^n)(x) ds. \tag{3.35}$$

Let us now introduce for each n the coupling of processes $(Z^{in}, \bar{Y}^{in}, \bar{X}^i)_{1\leq i\leq n}$, where $(\bar{X}^i)$ are independent copies of X defined as in (3.24) on a certain probability space and $Z^{in}, \bar{Y}^{in}$ are driven, for each i respectively, following the same Brownian motion as $\bar{X}^i$.

We will now compare the two processes $\bar{Y}^{in}$ and $\bar{X}^i$. So we need to estimate $w - \tilde{p}^n$. Using (3.23) and (3.35), we obtain

$$\tilde{p}^n_t(x) - w_t(x) = \int_0^t \int_{I\!R^2} \nabla_x G^\nu_{t-s}(x-y).\Big(K_{\varepsilon^n} * \tilde{p}^n_s(y)\tilde{p}^n_s(y) - K * w_s(y) w_s(y)\Big) dyds \tag{3.36}$$

We will prove the

Theorem 3.13 *There exist positive real constants C_1 and C_2 such that*

$$\sup_{t\leq T} |||\tilde{p}^n_t - w_t||| \leq \frac{C_1}{\sqrt{\nu}} \varepsilon_n |||w_0|||^2 \sqrt{T} \exp(C_2 |||w_0||| T), \tag{3.37}$$

The proof begins with the lemma.

Lemma 3.14 *For each $t \leq T$, for each $x \in I\!R^2$,*

$$|\int_{I\!R^2} (K_{\varepsilon_n}(x-y) - K(x-y))\tilde{p}^n_t(y) dy| \leq 2\varepsilon_n \|w_0\|_\infty \tag{3.38}$$

$$\|K_{\varepsilon_n} * \tilde{p}^n_t - K * w_t\|_\infty \leq 2\varepsilon_n \|w_0\|_\infty + (K_1 + K_\infty)|||\tilde{p}^n_t - w_t|||. \tag{3.39}$$

Proof. 1) (3.38) is proved as (3.19).

2) For $x \in I\!R^2$,

$$\begin{aligned} |K_{\varepsilon_n} * \tilde{p}^n_t(x) - K * w_t(x)| &\leq 2\varepsilon_n \|w_0\|_\infty + \int_{I\!R^2} |K(x-y)||\tilde{p}^n_t(y) - w_t(y)| dy \\ &\leq 2\varepsilon_n \|w_0\|_\infty + K_\infty \|\tilde{p}^n_t - w_t\|_1 + K_1 \|\tilde{p}^n_t - w_t\|_\infty. \end{aligned}$$

□

Let us now prove Theorem 3.13.

Proof. We consider (3.36). Then,

$$\begin{aligned} & |\tilde{p}^n_t(x) - w_t(x)| \\ \leq\ & |\int_0^t \int_{I\!R^2} \nabla_x G^\nu_{t-s}(x-y).\Big(\tilde{p}^n_s(y)(K_{\varepsilon_n} * \tilde{p}^n_s(y) - K * w_s(y)) + K * w_s(y)(\tilde{p}^n_s(y) - w_s(y))\Big) dyds| \\ \leq\ & \int_0^t \int_{I\!R^2} |\nabla_x G^\nu_{t-s}(x-y)| \Big(|\tilde{p}^n_s(y)| (2\varepsilon_n \|w_0\|_\infty + 2(K_\infty + K_1)|||w_0||| |||\tilde{p}^n_s - w_s|||) \Big) dyds \\ & \quad \text{by (3.39), (3.25) and (3.5)} \\ \leq\ & \frac{C}{\sqrt{\nu}} \int_0^t \frac{1}{\sqrt{t-s}} \Big(\|w_0\|_\infty (2\varepsilon_n \|w_0\|_\infty + 2(K_\infty + K_1)|||w_0||| |||\tilde{p}^n_s - w_s|||) \Big) ds \\ \leq\ & \frac{C}{\sqrt{\nu}} \Big(4\varepsilon_n \|w_0\|^2_\infty \sqrt{T} + 2(K_\infty + K_1)|||w_0||| \int_0^t \frac{1}{\sqrt{t-s}} |||\tilde{p}^n_s - w_s||| ds \Big). \end{aligned}$$

Consider now the L^1-norm of $\tilde{p}^n_t - w_t$ and by similar computation,

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\tilde{p}^n_t(x) - w_t(x)|dx \\
\leq\ & \int_0^t \frac{1}{\sqrt{t-s}} \|\tilde{p}^n_s\|_1 (2\varepsilon_n \|w_0\|_\infty + (K_\infty + K_1)|||\tilde{p}^n_s - w_s|||)ds + \int_0^t \frac{1}{\sqrt{t-s}} \|K * w_s\|_\infty \|\tilde{p}^n_s - w_s\|_1 ds \\
\leq\ & \frac{A}{\sqrt{\nu}} \Big(4\varepsilon_n |||w_0|||^2 \sqrt{T} + 2(K_\infty + K_1)|||w_0||| \int_0^t \frac{1}{\sqrt{t-s}} |||\tilde{p}^n_s - w_s|||ds \Big).
\end{aligned}$$

By associating the two previous results, we obtain

$$|||\tilde{p}^n_t - w_t||| \leq \frac{C}{\sqrt{\nu}} \Big(8\varepsilon_n |||w_0|||^2 \sqrt{T} + 2|||w_0||| \int_0^t \frac{1}{\sqrt{t-s}} |||\tilde{p}^n_s - w_s|||ds \Big).$$

We iterate twice this inequality and obtain finally by Gronwall's lemma that

$$\sup_{t \leq T} |||\tilde{p}^n_t - w_t||| \leq \frac{C_1}{\sqrt{\nu}} \varepsilon_n |||w_0|||^2 \sqrt{T} \exp(C_2 |||w_0||| T).$$

□

Adding now (3.39) and (3.37), we deduce the

Corollary 3.15 *For each $t \leq T$*

$$\|K_{\varepsilon_n} * \tilde{p}^n_t - K * w_t\|_\infty \leq A_T \varepsilon_n, \tag{3.40}$$

where $A_T = 2\|w_0\|_\infty + (K_\infty + K_1)\frac{C_1}{\sqrt{\nu}} \varepsilon_n |||w_0|||^2 \sqrt{T} \exp(C_2 |||w_0||| T)$.

We are now able to obtain our main theorem.

Theorem 3.16 *Let us consider independent processes, solutions of the stochastic differential equations defined on $[0,T]$, $T > 0$ by*

$$\bar{X}^i_t = Z^i_0 + \sqrt{2\nu} B^i_t + \int_0^t K * \tilde{p}_s(\bar{X}^i_s) ds, \tag{3.41}$$

where $(B^i)_{i \in \mathbb{N}}$ are independent Brownian motions on $\mathbb{R}^2$ and $(Z^i_0)_{i \in \mathbb{N}}$ are $\mathbb{R}^2$-valued iid random variables independent of $(B^i)_{i \in \mathbb{N}}$ with law $P_0(dx) = \frac{|w_0(x)|}{\|w_0\|_1} dx$, $w_0 \in L^1 \cap L^\infty$. The function $\tilde{p}_s$ is the density of the signed measure $\tilde{P}_s$ associated with the law P_s of $\bar{X}^i_s$ by (3.11)

Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers which tends to 0 and such that

$$\lim_n \frac{M_{\varepsilon_n}}{\sqrt{n} L_{\varepsilon_n}} \exp(\|w_0\|_1 T L_{\varepsilon_n}) = 0.$$

We consider the coupled n-particle system $(Z^{in})_{1 \leq i \leq n}$ defined by

$$Z^{in}_t = Z^i_0 + \sqrt{2\nu} B^i_t + \int_0^t \frac{1}{n} \sum_{j=1}^n h(Z^j_0) K_{\varepsilon_n}(Z^{in}_s - Z^{jn}_s) ds. \tag{3.42}$$

Then, for each $1 \leq i \leq n$,

$$\lim_{n\to+\infty} E\Big(\sup_{t\leq T}|Z_t^{in} - \bar{X}_t^i|\Big) = 0, \tag{3.43}$$

(in the precise asymptotics given by (3.47)).

That implies the propagation of chaos and the convergence in law (uniformly in time), of the weighted empirical measures $\tilde{\mu}_s^n = \frac{1}{n}\sum_{i=1}^n h(Z_0^{in})\delta_{Z_s^{in}}$ to $\tilde{P}_s$ and $\tilde{P}_s = w_s(x)dx$ where w is solution of the vortex equation with initial datum w_0.

Proof.

$$\begin{aligned} E\Big(\sup_{t\leq T}|Z_t^{in} - \bar{X}_t^i|\Big) &\leq E\Big(\sup_{t\leq T}|Z_t^{in} - \bar{Y}_t^{i,n}|\Big) + E\Big(\sup_{t\leq T}|\bar{Y}_t^{i,n} - \bar{X}_t^i|\Big) \\ &\leq \frac{M_{\varepsilon_n}}{nL_{\varepsilon_n}}\exp(\|w_0\|_1 T L_{\varepsilon_n}) + E\Big(\sup_{t\leq T}|\bar{Y}_t^{i,n} - \bar{X}_t^i|\Big). \end{aligned} \tag{3.44}$$

But

$$|\bar{Y}_t^{i,n} - \bar{X}_t^i| \leq \int_0^t |K_{\varepsilon_n} * \tilde{p}_s^n(\bar{Y}_s^{i,n}) - K_{\varepsilon_n} * \tilde{p}_s^n(\bar{X}_s^i)|ds + \int_0^t |K_{\varepsilon_n} * \tilde{p}_s^n(\bar{X}_s^i) - K * w(\bar{X}_s^i)|ds.$$

The second right hand-side term is controled thanks to Corollary 3.15. It remains to study the first right hand-side term, what we do following [26] Lemma 3.1 and Theorem 3.1. It is proved that for x and z in $I\!R^2$,

$$|K_{\varepsilon_n} * \tilde{p}_s^n(x) - K_{\varepsilon_n} * \tilde{p}_s^n(z)| \leq C_0(\|w_0\|_1 + \|w_0\|_\infty)\phi(x,z),$$

where $\phi(x,z) = \tilde{\phi}(|x-z|)$, and $\tilde{\phi}(r) = r(1-\ln r)$ if $0 < r < 1$ and $\tilde{\phi}(r) = 1$ if $r \geq 1$. Let us remark that the the function $\tilde{\phi}$ is non-decreasing and concave.

Then, by noting $C = C_0(\|w_0\|_1 + \|w_0\|_\infty)$, one deduces that

$$|\bar{Y}_t^{i,n} - \bar{X}_t^i| \leq A_T\varepsilon_n + C\int_0^t \phi(\bar{Y}_s^{i,n}, \bar{X}_s^i)ds.$$

Thus,

$$\begin{aligned} E\Big(\sup_{u\leq t}|\bar{Y}_u^{i,n} - \bar{X}_u^i|\Big) &\leq A_T\varepsilon_n + C\int_0^t E\Big(\sup_{u\leq s}\phi(\bar{Y}_u^{i,n}, \bar{X}_u^i)\Big)ds \\ &\leq A_T\varepsilon_n + C\int_0^t E\Big(\sup_{u\leq s}\phi(\bar{Y}_u^{i,n}, \bar{X}_u^i)\Big)ds \\ &\leq A_T\varepsilon_n + C\int_0^t E\Big(\tilde{\phi}(\sup_{u\leq s}|\bar{Y}_u^{i,n} - \bar{X}_u^i|)\Big)ds \\ &\qquad \text{since } \tilde{\phi} \text{ is non decreasing} \\ &\leq A_T\varepsilon_n + C\int_0^t \tilde{\phi}(E\Big(\sup_{u\leq s}|\bar{Y}_u^{i,n} - \bar{X}_u^i|\Big))ds \quad \text{by concavity of } \tilde{\phi} \end{aligned}$$

Let us denote by $H(t) = E\Big(\sup_{u\le t}|\bar{Y}^{i,n}_u - \bar{X}^i_u|\Big)$. Then by the previous computations,

$$H(t) \le A_T\varepsilon_n + C\int_0^t \tilde{\phi}(H(s))ds. \tag{3.45}$$

As in [26], we introduce the solution $h(x_0,t)$ of the equation

$$z'(t) = C\tilde{\phi}(z(t)) \quad ; \quad z(0) = x_0 > 0.$$

Then, if $x_0 < 1$, and if $t_0 = \inf\{t, h(x_0,t) > 1\}$,

$$\begin{aligned} h(x_0,t) &= x_0^{\exp(-Ct)}\exp(1-e^{-Ct}) \ \text{ if } h(x_0,t) < 1\ ,\ t < t_0 \\ &= 1 + C(t-t_0) \ \text{ if } h(x_0,t) \ge 1,\ t \ge t_0 \end{aligned}$$

and if $x_0 \ge 1$, $h(x_0,t) = x_0 + Ct$. Hence, by (3.45), we get $H(t) \le h(A_T\varepsilon_n, t)$. But since ε_n tends to 0 as n tends to infinity, for n sufficiently large, we deduce that

$$H(t) \le (A_T\varepsilon_n)^{\exp(-Ct)}\exp(1-e^{-Ct})$$

and finally

$$E\Big(\sup_{u\le T}|\bar{Y}^{i,n}_u - \bar{X}^i_u|\Big) \le (A_T\varepsilon_n)^{\exp(-CT)}\exp(1-e^{-CT}). \tag{3.46}$$

Now, by (3.44), and (3.46), we finally conclude that

$$E\Big(\sup_{t\le T}|Z^{in}_t - \bar{X}^i_t|\Big) \le \frac{M_{\varepsilon_n}}{\sqrt{n}L_{\varepsilon_n}}\exp(\|w_0\|_1 T L_{\varepsilon_n}) + (A_T\varepsilon_n)^{\exp(-CT)}\exp(1-e^{-CT}), \tag{3.47}$$

where $C = C_0(\|w_0\|_1 + \|w_0\|_\infty)$ and then tends to 0 when n tends to infinity. □

3.5.3 Numerical Results

We finally deduce from this study an algorithm for the simulation of the solution of the vortex equation. To approximate numerically this solution, it is necessary to discretize in time the particle system with an Euler scheme, as it has been described in Section 2, Theorem 2.6. Mimicking the result obtained by Bossy-Talay [3] for the Burgers equation, and if $\bar{\mu}^n_{l\Delta t}$ denotes the weighted empirical measure of the discretized system, one hopes that $K_{\varepsilon_n} \star \bar{\mu}^n_{l\Delta t}$ converges to $K \star w_{l\Delta t}$ in L^1 with rate $\mathcal{O}(\sqrt{\Delta t} + \frac{1}{\sqrt{n}})$, where Δt denotes the time-step. The simulations realized with specific kernels K_ε confirm this behaviour.

3.6 Generalization to a Finite Measure Initial Condition

It is possible to generalize what we have done to the case of a finite measure initial condition, instead of $w_0 \in L^1 \cap L^\infty$. We refer to [30] for details. In that case, the study of the vortex equation exposed by Giga-Miyakawa-Osada in [16] involves analytical results, in particular on generators of generalized divergence form. The solutions of the vortex equation live in L^q-spaces, with $1 < q < 2$. This induces new difficulties, even if the particle approximation method is essentially similar to the previous one. Due to the explosions of the estimates (3.46), we obtain a convergence in law for the particle systems, instead of a L^1-convergence as in the bounded case.

4 The case of a bounded domain

We now consider a Navier-Stokes equation in a bounded domain Θ of $I\!R^2$ satisfying the no-slip boundary condition:

$$\partial_t u(t,x) + (u.\nabla)u(t,x) = \nu\Delta u(t,x) - \nabla p \quad \text{in } \Theta;$$
$$\nabla.u(t,x) = 0 \quad \text{in } \Theta \ ; \ u(0,x) = u_0(x) \text{ for } x \in \Theta \ ; \ u(x,t) = (0,0) \text{ for } x \in \partial\Theta,$$

where p is the pressure and $\nu > 0$ the viscosity coefficient. A probabilistic approach of this equation, based on branching processes, has already been developed by Benachour, Roynette, Vallois [1] and generalized in dimension 3 by Giet [15]. But even if the authors propose some particle approximations, the convergence of the method is not shown and the particle systems they describe are not for use in practice. Our purpose is to construct some easily simulable particle systems, and to rigorously show the convergence of some associated weighted empirical measures to a deterministic finite measure associated with the solution of the Navier-Stokes equation.

As in Section 3, we try to associate a vortex equation with this Navier-Stokes equation. This approach would consist in replacing the Biot and Savart kernel by the orthogonal gradient K of the Green function of the Dirichlet problem in the domain. But one then only obtains the nullity of the normal component of the velocity on the boundary. To obtain in addition the nullity of the tangential component, we are inspired by Cottet [10], who proves that by adding a Neumann condition to the vortex equation, one obtains an admissible vorticity field in the sense that the associated velocity satisfies *a posteriori* the no-slip condition.

This Neumann condition badly depends on the vorticity and is really hard to take into account. So we will mainly deal in the following with a vortex equation in a bounded domain of R^2 with a given Neumann condition at the boundary. We obtain the existence and uniqueness of the solution of this vortex equation in an appropriate space. We are interested in proving the convergence of Monte-Carlo approximations to this solution. To our knowledge, there was so far no proof of convergence of deterministic or stochastic particle methods in this simplified case.

We associate with the vortex equation a nonlinear diffusive and reflected process, with random birth at the boundary governed by the Neumann condition. We construct interacting normally reflected particle systems with space-time random births at the boundary and prove the propagation of chaos to the law of the nonlinear process associated with the vortex equation. We are inspired by the paper of Sznitman [36], which concerns interacting and reflected McKean-Vlasov particle systems living in a bounded domain. Some additional difficulties appear here, due to the singular interacting kernel K and to the space-time random births. Moreover, since w_0 and g are not probability densities, we introduce as in Section 3 some weights associated with the initial position.

We will be then able to describe the simulation algorithm, and we will finally explain how to adapt this numerical approach if we assume the Cottet condition (cf. [10]) to come back to the Navier-Stokes case.

Notation: If Θ is a bounded domain of $\mathbb{R}^2$, the Sobolev space $H^1(\Theta)$ consists in functions which belong together with their first order distribution derivatives to $L^2(\Theta)$.

4.1 The model

Let $T > 0$. Let us consider a function g defined on $\partial\Theta$. We are interested in the equation

$$\begin{aligned} &\partial_t w(t,x) + \nabla.(wKw)(t,x) = \nu\Delta w(t,x) \quad \text{in }]0,T] \times \Theta; \\ &w(0,x) = w_0(x) \text{ in } \Theta \ ; \ \partial_n w = \nabla w.n = g \text{ on }]0,T] \times \partial\Theta \end{aligned} \tag{4.1}$$

where $n(x)$ denotes the outward normal to $\partial\Theta$ at the point x and $Kw(t,x) = \int_\Theta K(x,y)w(t,y)dy$. The kernel $K(x,y)$ is equal to $\nabla_x^\perp G(x,y) = (-\partial_{x_2}G(x,y), \partial_{x_1}G(x,y))$ where $G(x,y)$ is the fundamental solution of the Poisson equation

$$\Delta_x G(x,y) = \delta_y(x), \ x \in \Theta \ ; \ G(x,y) = 0, \ x \in \partial\Theta \tag{4.2}$$

Let us remark the important properties of the kernel K:

$$\forall x \neq y \in \bar{\Theta} \ , \ \nabla_x.K(x,y) = 0 \ ; \quad \forall x \in \partial\Theta, \forall y \in \bar{\Theta}, \ K(x,y).n(x) = 0 \tag{4.3}$$

In all the following, we will moreover assume

Hypotheses (H):

The domain Θ of $I\!R^2$ is bounded, simply connected and of class $\mathcal{C}^4$.

$$w_0 \in L^2(\Theta) \quad ; \quad g(t,x) \in L^2_t([0,T], L^2_x(\partial\Theta, d\sigma)), \tag{4.4}$$

where $d\sigma(x)$ denotes the surface measure on the boundary.

Thanks to the assumptions made on Θ, the following properties hold for the Green function G and the kernel $K = (K_1, K_2)$:

Lemma 4.1 *$\exists C_0 > 0$, $\forall x \neq y \in \bar{\Theta}$,*

$$|G(x,y)| \leq C_0(1 + |\ln|x-y||); \quad |K(x,y)| \leq \frac{C_0}{|x-y|}$$
$$|\nabla_x K_i(x,y)| + |\nabla_y K_i(x,y)| \leq \frac{C_0}{|x-y|^2} \quad \textit{for} \ \ i = 1,2.$$

Proof. For $y = (y_1, y_2) \in I\!R^2$, let $y^\perp = (-y_2, y_1)$ and $y^* = y/|y|^2$ if $y \neq (0,0)$.
In case Θ is the unit disk $B(0,1)$ of $I\!R^2$, one has the following explicit expression for the Green function (see [17] p.19)

$$G_0(x,y) = \frac{1}{2\pi} \ln\left(\frac{|x-y|}{|y||x-y^*|}\right). \tag{4.5}$$

We remark that

$$\forall x,y \in \bar{B}(0,1),\ |x-y^*| \geq |y||x-y^*| = \sqrt{|x-y|^2 + (|x|^2-1)(|y|^2-1))} \geq |x-y|. \tag{4.6}$$

As a consequence,

$$|2\pi G_0(x,y)| \leq -\ln|x-y| 1_{\{|x-y|\leq 1\}} + \ln(|y||x-y^*|) 1_{\{|y||x-y^*|\geq 1\}}.$$

As $|y||x-y^*| = |x|y| - y/|y|| \leq 2$, we conclude that $|2\pi G_0(x,y)| \leq |\ln|x-y|| + \ln(2)$.
We also deduce from (4.6) the bound on the corresponding kernel

$$K(x,y) = \frac{1}{2\pi}\left(\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-y^*)^\perp}{|x-y^*|^2}\right) = \frac{1}{2\pi}\left(((x-y)^*)^\perp - ((x-y^*)^*)^\perp\right).$$

To estimate ∇K_i, we combine (4.6) and the fact that each term of the Jacobian matrix of $z \to z^*$ is bounded by $1/|z|^2$.
When Θ is a general bounded and simply connected domain of class $\mathcal{C}^3$, according to [33], there is a conformal mapping from $B(0,1)$ onto Θ which extends to a one-to-one $\mathcal{C}^2$ mapping from $\bar{B}(0,1)$ to $\bar{\Theta}$ denoted by f and such that Df, $(Df)^{-1}$ and $D^2 f$ are bounded on $\bar{B}(0,1)$. Since the Green function for Θ is given by

$$G(x,y) = G_0(f^{-1}(x), f^{-1}(y)),$$

the estimations on G, K and ∇K_i follow from those obtained for the unit disk and the just mentionned properties of f. $\square$

We are interested in weak solutions of (4.1) defined in the following sense

Definition 4.2 *We say that* $w : [0,T]\times\Theta \to \mathbb{R}$ *is a weak solution of (4.1) if* $w(0,.) = w_0$ *and*

(i) $w \in L^\infty_t(L^2_x)\cap L^2_t(H^1_x)$ *where* $L^\infty_t(L^2_x)$ *and* $L^2_t(H^1_x)$ *stand respectively for* $L^\infty([0,T],L^2(\Theta))$ *and* $L^2([0,T],H^1(\Theta))$

(ii) for any $v\in H^1(\Theta)$, $\frac{d}{dt}\int_\Theta w_t v + \nu\int_\Theta \nabla w_t.\nabla v = \int_\Theta w_t K w_t.\nabla v + \nu\int_{\partial\Theta} g_t v d\sigma$.

Before stating the existence of a unique weak solution to (4.1), we are going to check the following Lemma which prepares the study of the nonlinear term in (4.1).

Lemma 4.3

$$\forall 2 < p \leq +\infty,\ \exists C>0,\ \forall w\in L^p(\Theta),\ Kw\in\mathcal{C}(\bar\Theta) \text{ and } \|Kw\|_{L^\infty}\leq C\|w\|_{L^p} \quad (4.7)$$

$$\exists C>0,\ \forall w\in L^2(\Theta),\ \|Kw\|_{L^2}\leq C\|w\|_{L^2} \quad (4.8)$$

Proof. For $\alpha>0$, let $K_\alpha(x,y) = \mathbf{1}_{\{|x-y|>\alpha\}}K(x,y)$. By Lebesgue's theorem and using the continuity of K away from the diagonal, we obtain the continuity of $x\in\bar\Theta\mapsto K_\alpha(x,.)\in L^q_y$, for each $q\geq 1$. When in addition $q<2$, according to Lemma 3.3, $K_\alpha(x,.)$ converges to $K(x,.)$ in L^q_y uniformly on $\bar\Theta$, when α tends to 0. We deduce that $K(x,.)$ is continuous in L^q_y and obtain (4.7) by Hölder inequality.
Let $w\in L^2(\Theta)$. Using Lemma 3.3 and Cauchy-Schwarz inequality, we get

$$\|Kw\|^2_{L^2}\leq\int_\Theta\left(\int_\Theta\frac{C_0}{|x-y|}dy\right)\left(\int_\Theta\frac{C_0 w^2(y)}{|x-y|}dy\right)dx\leq\left(\sup_{x\in\bar\Theta}\int_\Theta\frac{C_0}{|x-y|}dy\right)^2\|w\|^2_{L^2}.$$

$\square$

We can now state the following existence and uniqueness theorem, and refer to [21] for the proof. It consists in obtaining energy estimates and in adapting the Galerkin approximation method.

Theorem 4.4 *Under hypotheses* **(H)**, *equation (4.1) has a unique solution* w *in the sense of Definition 4.2. In addition,* $w\in C([0,T],L^2_x)\cap L^4_t(L^4_x)$.

In order to give a probabilistic interpretation to the obtained weak solution of (4.1), we introduce the semi-group $P^\nu_t(x,y)$ associated with $\sqrt{2\nu}$ times the Brownian motion normally reflected on the boundary and prove the following mild representation

Proposition 4.5 *Let w denote the weak solution of (4.1) given by Theorem 4.4. Then $\forall t \in [0,T]$, dx a.e. in Θ,*

$$w_t(x) = P_t^\nu w_0(y) + \int_0^t \nabla P_{t-s}^\nu.(w_s K w_s)(x) ds + \nu \int_0^t \int_{\partial\Theta} P_{t-s}^\nu(y,x) g(s,y) d\sigma(y) ds \quad (4.9)$$

where $\nabla P_{t-s}^\nu.(w_s K w_s)(x) = \int_\Theta \nabla_y P_{t-s}^\nu(y,x).w_s(y) K w_s(y) dy$.

Proof. Let $t \in]0,T]$ and φ be a smooth function on $\bar{\Theta}$ with a vanishing normal derivative at the boundary : $\partial_n \varphi(x) = 0$ for $x \in \partial\Theta$. According to [23] Theorem 5.3 p.320, the boundary value problem

$$\partial_s \psi + \nu \Delta \psi = 0 \text{ on } [0,t] \times \Theta \; ; \quad \partial_n \psi = 0 \text{ on } [0,t] \times \partial\Theta \; ; \quad \psi(t,.) = \varphi(.) \text{ on } \Theta$$

admits a classical solution $\psi(s,x)$ which is $\mathcal{C}^{1,2}$ on $[0,t] \times \bar{\Theta}$. By the Feynman-Kac approach, this solution has the following representation : $\psi(s,x) = P_{t-s}^\nu \varphi(x)$. Clearly $\psi \in L^\infty([0,t], H^1(\Theta))$ and $\partial_s \psi \in L^2([0,t], (H^1)'_x(\Theta))$. By [39], Lemma 1.2 p. 261, we deduce that in $\mathcal{D}'(]0,t[)$,

$$\frac{d}{ds} \int_\Theta w_s \psi(s,.) = \int_\Theta w_s \partial_s \psi(s,.) - \nu \int_\Theta \nabla w_s . \nabla \psi(s,.) + \int_\Theta w_s K w_s . \nabla \psi(s,.) + \nu \int_{\partial\Theta} g_s \psi(s,.) d\sigma.$$

By the equation satisfied by ψ, the sum of the two first terms of the r.h.s. is nil. Hence

$$\begin{aligned} \int_\Theta w_t(x) \varphi(x) dx &= \int_\Theta w_0(x) \psi(0,x) dx + \int_0^t \int_\Theta w_s K w_s(x) . \nabla \psi(s,x) dx ds \\ &+ \nu \int_0^t \int_{\partial\Theta} \psi(s,x) g(s,x) d\sigma(x) ds. \end{aligned}$$

By the symmetry of P^ν and hypotheses **(H)**, $\int_0^t \int_{\partial\Theta} \int_\Theta P_{t-s}^\nu(x,y) |\varphi(y)| dy |g(s,x)| d\sigma(x) ds \leq \sup |\varphi| \|g\|_{L_t^1(L_x^1(\partial\Theta))} < +\infty$. Hence, by Fubini's theorem the last term of the r.h.s. is equal to $\nu \int_\Theta \varphi(x) \int_0^t \int_{\partial\Theta} P_{t-s}^\nu(y,x) g(s,y) d\sigma(y) ds dx$. We conclude the proof by applying similarly Fubini's theorem to the other terms of the r.h.s. and remarking that the derived equality holds for any smooth function φ with vanishing normal derivative.
To justify the use of Fubini's theorem in the second term, we need the following estimations given by [35] (a.13) and (a.14) p.600 :

$$\forall x \in \bar{\Theta},\ \forall y \in \bar{\Theta},\ |\nabla_x P_t^\nu(x,y)| \leq C_1 / t^{3/2} \text{ and } \|\nabla_x P_t^\nu(x,y)\|_{L_y^1(\Theta)} \leq C_1/\sqrt{t}. \quad (4.10)$$

Indeed the first one ensures that $\nabla \psi(s,x) = \int_\Theta \nabla_x P_{t-s}^\nu(x,y) \varphi(y) dy$. By the second one and (4.8),

$$\int_0^t \int_\Theta |w_s K w_s|(x) \int_\Theta |\nabla_x P_{t-s}^\nu(x,y)| |\varphi(y)| dy dx ds \leq C \sup |\varphi| \|w\|^2_{L_t^\infty(L_x^2)} \int_0^t (t-s)^{-1/2} ds.$$

□

4.2 The Probabilistic Interpretation of the Vortex Equation with a Neumann Boundary Condition

We are again in a McKean-Vlasov context and we will associate a nonlinear martingale problem.

To bypass the difficulty due to the Neumann condition involving g, we essentially follow Fernandez-Méléard [11], proving that this term is related to space-time random births located at the boundary. We have also to take into account to the bounded domain instead of the whole space and the diffusion processes we consider will be reflected on the boundary. There are also births inside the domain at time 0 and the functions w_0 and g are not probability densities. As in Section 3, we follow Jourdain [20] to overpass this problem.

Let $\|w_0\|_1 = \int_\Theta |w_0|$ and $\|g\|_1 = \int_{[0,T]\times\partial\Theta} |g| d\sigma dt$. To govern the times and positions of births we introduce on $[0,T] \times \bar{\Theta}$ the probability measure

$$P_0(dt, dx) = \mathbf{1}_{\{x\in\Theta\}}\delta_{\{0\}}(dt)\frac{|w_0(x)|}{\|w_0\|_1 + \nu\|g\|_1}dx + \mathbf{1}_{\{x\in\partial\Theta\}}\frac{\nu|g(t,x)|}{\|w_0\|_1 + \nu\|g\|_1}dtd\sigma(x). \quad (4.11)$$

We also consider for $t \in [0,T]$ and $x \in \bar{\Theta}$ the measurable function

$$h(t,x) = \mathbf{1}_{\{t=0, x\in\Theta\}}\frac{w_0(x)}{|w_0(x)|}(\|w_0\|_1 + \nu\|g\|_1) + \mathbf{1}_{\{x\in\partial\Theta\}}\frac{g(t,x)}{|g(t,x)|}(\|w_0\|_1 + \nu\|g\|_1) \quad (4.12)$$

with values in $\{-(\|w_0\|_1 + \nu\|g\|_1), 0, \|w_0\|_1 + \nu\|g\|_1\}$. Let us remark that if φ a bounded measurable function on $[0,T] \times \bar{\Theta}$, then

$$\int_{[0,T]\times\bar{\Theta}} \varphi(t,x)h(t,x)P_0(dt,dx) = \int_\Theta \varphi(0,x)w_0(x)dx + \nu \int_{[0,T]\times\partial\Theta} \varphi(t,x)g(t,x)dtd\sigma(x) \quad (4.13)$$

Let $(\tau, (X_t)_{t\leq T}, (k_t)_{t\leq T})$ denote the canonical process on $[0,T]\times\mathcal{C}([0,T],\bar{\Theta})\times\mathcal{C}([0,T],\mathbb{R}^2)$. For a probability measure Q on this space, we define the family $(\tilde{Q}_t)_{t\in[0,T]}$ of signed measures on $\bar{\Theta}$ by

$$\forall B \in \mathcal{B}(\bar{\Theta}), \quad \tilde{Q}_t(B) = E^Q(h(\tau, X_0)\mathbf{1}_{\{\tau\leq t\}}\mathbf{1}_B(X_t)), \quad (4.14)$$

It is easy to check that for each $t \in [0,T]$, the signed measure $\tilde{Q}_t$ is bounded with a total mass less than $\|w_0\|_1 + \nu\|g\|_1$.

To give a probabilistic interpretation to the equation, we are inspired by Sznitman [36] and Bossy-Jourdain [6] for the reflected contribution and by Fernandez-Méléard [11] for the space-time random births.

Definition 4.6 *Let $T > 0$. We denote by $\mathcal{P}_T$ the space of probability measures Q on $[0,T] \times \mathcal{C}([0,T], \bar{\Theta}) \times \mathcal{C}([0,T], I\!R^2)$ such that for each $t \in [0,T]$, the signed measure $\tilde{Q}_t$ has a density $\tilde{q}_t$ with respect to the Lebesgue measure on Θ and the measurable version $\tilde{q}$ belongs to $L^\infty_t(L^2_x) \cap L^2_t(H^1_x)$.*

Definition 4.7 *A probability measure $P \in \mathcal{P}_T$ is solution of the nonlinear martingale problem $(\mathcal{M}_B)$ if*

1) $P \circ (\tau, X_0, k_0)^{-1} = P_0 \otimes \delta_{(0,0)}$

2) for each $\phi \in \mathcal{C}^2_b(I\!R^2)$,

$$M^\phi_t = \phi(X_t + k_t) - \phi(X_0) - \int_0^t \mathbf{1}_{\{\tau \leq s\}} \Big(K\tilde{p}_s(X_s).\nabla\phi(X_s + k_s) + \nu\Delta\phi(X_s + k_s) \Big) ds$$

is a P-martingale, for the filtration $\mathcal{F}_t = \sigma(\tau, (X_s, k_s), s \leq t)$ ($\tilde{p}(s,x)$ denotes a measurable version of the densities of $\tilde{P}_s$).

3) P a.s., $\forall t \in [0,T]$,

$$\int_0^t d|k|_s < +\infty \ , \ |k|_t = \int_0^t \mathbf{1}_{\{X_s \in \partial\Theta\}} \mathbf{1}_{\{\tau \leq s\}} d|k|_s \ , \ k_t = \int_0^t n(X_s) d|k|_s.$$

The following lemma states the link between $(\mathcal{M}_B)$ and the vortex equation (4.1).

Lemma 4.8 *If $P \in \mathcal{P}_T$ solves $\mathcal{M}_B$ then $\tilde{p}$ is a weak solution of (4.1).*

Proof. By Definition 4.7 1), (4.14) and (4.13), $\tilde{p}_0 = w_0$.

According to Definition 4.7 2), $\beta_t = X_t - X_0 - \int_0^t \mathbf{1}_{\{\tau \leq s\}} K\tilde{p}_s(X_s)ds + k_t$ is a P-continuous martingale with bracket $< \beta >_t = 2\nu(t-\tau)^+ I_2$ where I_2 denotes the 2×2 identity matrix, which implies that $\beta_t = 0$ for $t \in [0,\tau]$. Using moreover Definition 4.7 3), we deduce that $X_t = X_0$ for $t \in [0,\tau]$. Hence for $\psi \in \mathcal{C}^{1,2}([0,T] \times \bar{\Theta})$,

$$\int_0^T \partial_s\psi(s, X_s)ds + \psi(0, X_0) = \psi(\tau, X_0) + \int_0^T \mathbf{1}_{\{\tau \leq s\}} \partial_s\psi(s, X_s)ds.$$

If moreover $\forall (s,x) \in [0,T] \times \partial\Theta$, $\partial_n\psi(s,x) = 0$, by Itô's formula, we deduce that

$$\begin{aligned} \psi(T, X_t) &= \psi(\tau, X_0) + \int_0^T \nabla\psi(s, X_s).d\beta_s \\ &\quad + \int_0^T \mathbf{1}_{\{\tau \leq s\}} (\partial_s\psi(s, X_s) + K\tilde{p}_s(X_s).\nabla\psi(s, X_s) + \nu\Delta\psi(s, X_s))ds \end{aligned}$$

Multiplying by the $\mathcal{F}_0$-measurable variable $h(\tau, X_0)$, taking expectations and using the definition of $\tilde{p}$ and (4.13), we deduce that

$$\begin{aligned} \int_{\bar{\Theta}} \psi(T,x)\tilde{p}(T,x)dx &= \int_{\bar{\Theta}} \psi(0,x)w_0(x)dx + \nu \int_0^T \int_{\partial\Theta} \psi(s,x)g(s,x)d\sigma(x)ds \\ &\quad + \int_0^T \int_{\bar{\Theta}} (\partial_s\psi(s,x) + K\tilde{p}_s(x).\nabla\psi(s,x) + \nu\Delta\psi(s,x))\tilde{p}(s,x)dxds, \end{aligned}$$

For the choice $\psi(s,x) = \varphi(s)v(x)$ where v is a $\mathcal{C}^2$ function on $\bar\Theta$ such that $\partial_n v = 0$ on $\partial\Theta$ and $\varphi \in \mathcal{D}(]0,T[)$, we obtain

$$\int_0^T \left(\varphi'(s)\int_{\bar\Theta}\tilde p_s v + \varphi(s)\left(\int_\Theta \tilde p_s K\tilde p_s.\nabla v + \nu\int_\Theta \tilde p_s\Delta v + \nu\int_{\partial\Theta} g_s v d\sigma\right)\right) ds = 0.$$

As $P \in \mathcal{P}_T$, $\tilde p \in L^2_t(H^1_x)$. By Green's formula for functions in $H^1(\Theta)$ ([7] p.197) and since $\partial_n v$ vanishes on the boundary, ds a.e. in $[0,T]$, $\int_\Theta \tilde p_s \Delta v = -\int_\Theta \nabla\tilde p_s.\nabla v$.
Since Θ is $\mathcal{C}^4$, the $\mathcal{C}^2(\bar\Theta)$-functions with a vanishing normal derivative are dense in $H^1(\Theta)$ and we conclude that $\tilde p$ satisfies Definition 4.2 (ii). $\square$

Theorem 4.9 *Under Hypotheses* **(H)**, *the martingale problem* $(\mathcal{M}_B)$ *has a unique solution* P. *In addition, the corresponding* $\tilde p$ *is a weak solution of (4.1) and satisfies (4.9).*

Proof. 1) Uniqueness

Let P^1 and P^2 be two solutions of $(\mathcal{M}_B)$. Then according to Lemma 4.8, $\tilde p^1$ and $\tilde p^2$ are weak solutions of (4.1). According to Theorem 4.4, $\tilde p_1 = \tilde p_2 = w$. Hence P^1 and P^2 both solve the martingale problem defined like $(\mathcal{M}_B)$ but with known drift coefficient Kw_s replacing $K\tilde p_s$ in Definition 4.7 2). Since $w \in L^4_t(L^4_x)$, by (4.7), $\|Kw_s\|_{L^\infty_x} \in L^4_t$.
Let Γ denote the first marginal of the probability measure P_0 on $[0,T]\times\bar\Theta$ and for $i = 1,2$ and $u \in [0,T]$, $p^i(u,.)$ be a regular conditional probability on $[0,T]\times\mathcal{C}([0,T],\bar\Theta)\times\mathcal{C}([0,T],I\!R^2)$ endowed with P^i given $\tau = u$.

Then $d\Gamma(u)$ a.e., $p^i(u,.)$ a.s., $\tau = u$, Definition 4.7 3) is satisfied and $p^i(u,.)\circ(X_0,k_0)^{-1}$ is equal to

$$\mathbf{1}_{\{u=0\}}\frac{|w_0(x)|dx}{\|w_0\|_1}\otimes\delta_{(0,0)} + \mathbf{1}_{\{u>0\}}\frac{|g(u,x)|d\sigma(x)}{\int_{\partial\Theta}|g(u,y)|d\sigma(y)}\otimes\delta_{(0,0)} \tag{4.15}$$

and $\forall\phi \in C^2_b(I\!R^2)$,

$$\phi(X_t+k_t) - \phi(X_0) - \int_0^t \mathbf{1}_{\{u\le s\}}\Big(Kw_s(X_s).\nabla\phi(X_s+k_s) + \nu\Delta\phi(X_s+k_s)\Big)ds$$

is a $p^i(u,.)$-martingale.
Reasoning like in the proof of Lemma 4.8, we obtain that $d\Gamma(u)$ a.e., $p^i(u,.)$ a.s., $X_t = X_0$ and $k_t = (0,0)$ for $t \in [0,u]$. With (4.15), we deduce that $d\Gamma(u)$ a.e., $p^1(u,.)\circ(X_u,k_u)^{-1} = p^2(u,.)\circ(X_u,k_u)^{-1}$ and that for $i = 1,2$, $p^i(u,.)$ is equal to the image of $p^i(u,.)\circ((X_{t+u},k_{t+u})_{t\in[0,T-u]})^{-1}$ by the mapping

$$(X_t,k_t)_{t\ge 0} \in \mathcal{C}([0,T-u],\bar\Theta\times I\!R^2) \to (X_{(t-u)^+},k_{(t-u)^+})_{t\in[0,T]} \in \mathcal{C}([0,T],\bar\Theta\times I\!R^2).$$

Moreover $d\Gamma(u)$ a.e. , $W_t = \frac{1}{\sqrt{2\nu}}\left(X_{t+u} - X_u - \int_u^{t+u} Kw_s(X_s)ds + k_{t+u}\right)$ is a $p^i(u,.)$ Brownian motion. Since $s \to \|Kw_s\|_{L^\infty}$ is square integrable, combining trajectorial uniqueness for the Brownian motion normally reflected at the boundary of Θ (see [25]), Girsanov's theorem and the equality $p^1(u,.) \circ (X_u, k_u)^{-1} = p^2(u,.) \circ (X_u, k_u)^{-1}$ which holds $d\Gamma(u)$ a.e., we deduce that $d\Gamma(u)$ a.e.,

$$p^1(u,.) \circ ((X_{t+u}, k_{t+u})_{t\in[0,T-u]})^{-1} = p^2(u,.) \circ ((X_{t+u}, k_{t+u})_{t\in[0,T-u]})^{-1}.$$

Hence $d\Gamma(u)$ p.p. $p^1(u,.) = p^2(u,.)$ and $P^1 = P^2$.

2) Existence.

Let w be the solution of the vortex equation given by Theorem 4.4. We recall that $\|Kw_s\|_{L^\infty} \in L^4_t$. We construct a solution to the linear martingale problem defined like $(\mathcal{M}_B)$ but with known drift coefficient $Kw_s(.)$ replacing $K\tilde{p}_s$ in Definition 4.7 2) and we check that this probability measure solves $(\mathcal{M}_B)$.

Let (τ, X_0) be a random variable with law P_0 independent from $(W_t)_{t\in[0,T]}$ a two-dimensional Brownian motion. Existence and trajectorial uniqueness hold for the stochastic differential equation with normal reflection

$$X_t = X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{\tau\le s\}} dW_s - k_t \; ; \quad |k|_t = \int_0^t \mathbf{1}_{\{X_s\in\partial\Theta\}}\mathbf{1}_{\{\tau\le s\}} d|k|_s \; ; \; k_t = \int_0^t n(X_s)d|k|_s.$$

Moreover $\forall t \in [0,T]$, X_t admits

$$x \to \frac{1}{\|w_0\|_1 + \nu\|g\|_{L^1([0,t]\times\partial\Theta)}}\left(|w_0|P_t^\nu(x) + \nu\int_0^t\int_{\partial\Theta} |g|(s,y)P_{t-s}^\nu(y,x)\sigma(dy)ds\right)$$

as a density w.r.t. the Lebesgue measure on $\bar{\Theta}$. Since $\|Kw_s\|_{L^\infty}$ is square integrable, by Girsanov's theorem we deduce that the martingale problem defined like $(\mathcal{M}_B)$, but with Kw_s replacing $K\tilde{p}_s$, admits a solution P such that $\forall t \in [0,T]$, the measure $\tilde{P}_t$ has a density. Let $\tilde{p}$ denote a measurable version of the densities.

We set $t \in [0,T]$. Reasoning like in the proof of Lemma 4.8, we obtain that for $\psi \in \mathcal{C}^{1,2}([0,t]\times\bar{\Theta})$ such that $\forall (s,x) \in [0,t]\times\partial\Theta$, $\partial_n\psi(s,x) = 0$,

$$\begin{aligned}\int_{\bar{\Theta}} \psi(t,x)\tilde{p}(t,x)dx &= \int_{\bar{\Theta}} \psi(0,x)w_0(x)dx + \nu\int_0^t\int_{\partial\Theta} \psi(s,x)g(s,x)d\sigma(x)ds \\ &+ \int_0^t\int_{\bar{\Theta}} (\partial_s\psi(s,x) + Kw_s(x).\nabla\psi(s,x) + \nu\Delta\psi(s,x))\tilde{p}(s,x)dxds.\end{aligned}$$

Choosing $\psi(s,x) = P_{t-s}^\nu\varphi(x)$ like in the proof of Proposition 4.5 and remarking that because of (4.10) and the uniform in time bound $\|\tilde{p}_t\|_{L^1} \le \|w_0\|_1 + \nu\|g\|_1$,

$$\int_0^t\int_{\Theta^2} |\nabla_x P_{t-s}^\nu(x,y)||\varphi(y)||\tilde{p}_s(x)||Kw_s(x)|dxdyds \le C\int_0^t \frac{\|Kw_s\|_{L^\infty}ds}{\sqrt{t-s}} < +\infty,$$

we deduce by Fubini's theorem that

$$dx\ a.e.,\ \tilde{p}_t(x) = P_t^\nu w_0(x) + \int_0^t \nabla P_{t-s}^\nu.(\tilde{p}_s K w_s)(x) ds + \nu \int_{(0,t]\times\partial\Theta} P_{t-s}^\nu(x,y) g(s,y) d\sigma(y) ds.$$

Now, using the mild equation (4.9) satisfied by w and (4.10), we obtain

$$\exists C>0,\ \forall t\in[0,T],\ \|\tilde{p}_t - w_t\|_{L^1} \le C\int_0^t \|\tilde{p}_s - w_s\|_{L^1} \frac{\|Kw_s\|_{L^\infty}}{\sqrt{t-s}} ds. \tag{4.16}$$

By iterating this bound, then using Hölder's inequality, we obtain

$$\begin{aligned}
\|\tilde{p}_t - w_t\|_{L^1} &\le C\int_0^t \|\tilde{p}_s - w_s\|_{L^1}\|Kw_s\|_{L^\infty} \int_s^t \frac{\|Kw_u\|_{L^\infty}\, du}{\sqrt{t-u}\sqrt{u-s}} ds \\
&\le C\int_0^t \|\tilde{p}_s - w_s\|_{L^1}\|Kw_s\|_{L^\infty}\|Kw\|_{L_t^4(L_x^\infty)} \left(\int_s^t ((t-u)(u-s))^{-2/3} du\right)^{3/4} ds.
\end{aligned}$$

Hence (4.16) holds with $(t-s)^{-1/2}$ replaced by $(t-s)^{-1/4}$ in the r.h.s. After the next iteration we obtain that (4.16) holds with $(t-s)^{-1/2}$ replaced by 1 and conclude by Gronwall's lemma that $\forall t\in[0,T],\ \tilde{p}_t = w_t$. $\square$

4.3 Stochastic Approximations of the Solution of the Vortex Equation

4.3.1 The Case of a Cut-off Kernel

As in Section 3, we introduce a cut-off kernel K_ε preserving the properties (4.3). More precisely we consider an increasing $\mathcal{C}^2$-function η from $I\!R_+$ to $I\!R_+$, such that $\eta(x)=x$ for $x\le\frac{1}{2}$ and $\eta(x)=1$ for $x\ge 1$. For $\varepsilon\le 1$, we set

$$\begin{aligned}
G_\varepsilon(x,y) &= \eta\left(\frac{|x-y|^3}{\varepsilon^3}\right) G(x,y) : ; &(4.17)\\
K_\varepsilon(x,y) &= \nabla_x^\perp G_\varepsilon(x,y) \\
&= \eta\left(\frac{|x-y|^3}{\varepsilon^3}\right) K(x,y) + \eta'\left(\frac{|x-y|^3}{\varepsilon^3}\right)\frac{3(x-y)^\perp|x-y|}{\varepsilon^3} G(x,y) &(4.18)
\end{aligned}$$

The following Lemma states usefull properties of this cutoff kernel :

Lemma 4.10 *1) There exists a constant C independent of ε, such that*

$$\begin{aligned}
\nabla_x.K_\varepsilon(x,y) = 0 \quad &; \quad K_\varepsilon(x,y)\cdot n(x) = 0 \text{ for } x\in\partial\Theta, \\
K_\varepsilon(x,y) &= K(x,y) \quad \text{if } |x-y|\ge\varepsilon \\
\forall x,y\in\bar\Theta\ ,\ |K_\varepsilon(x,y)| &\le \frac{C(1+|\ln|x-y||)}{|x-y|} &(4.19)
\end{aligned}$$

2) $\sup_{x\in\bar\Theta}\|K(x,.)-K_\varepsilon(x,.)\|_{L_y^p}$ tends to 0 as ε tends to 0 as soon as $p<2$.

3) For ε sufficiently small, the kernel K_ε is bounded by $M_\varepsilon \le \frac{C|\ln\varepsilon|}{\varepsilon}$ and Lipschitz continuous in both variables with constant $L_\varepsilon \le \frac{C|\ln\varepsilon|}{\varepsilon^2}$ where C does not depend on ε.

Proof. The two first properties in 1) are obvious and 2) is an easy consequence of (4.19).

By Lemma 4.1 and the above definition of η, the norm of first term of the r.h.s. of (4.18) is smaller than $C_0(\frac{1}{|x-y|} \wedge \sup_{r\in[0,\varepsilon 2^{-1/3}]} \frac{r^2}{\varepsilon^3}) \leq C_0(\frac{1}{|x-y|} \wedge \frac{1}{\epsilon})$. By the estimate of G in Lemma 3.3 and since $\eta'(x) = 0$ for $x > 1$, the second term of the r.h.s. of (4.18) is smaller than $3C_0\|\eta'\|_\infty$ times

$$\left(\frac{1+|\ln|x-y||}{|x-y|} \wedge \sup_{r\in[0,\varepsilon]} \frac{r^2(1+|\ln(r)|)}{\varepsilon^3}\right) \leq \left(\frac{1+\ln|x-y|}{|x-y|} \wedge \frac{1+|\ln(\varepsilon)|}{\varepsilon}\right)$$

as $\varepsilon \leq 1$. We deduce both (4.19) and the upper-bound in $C|\ln(\varepsilon)|/\varepsilon$. To prove that K_ε is Lipschitz continuous, we use in a similar way Lemma 3.3 combined with the definition of η to check that the gradient of each coordinate of K_ε w.r.t. either x or y is bounded by $C|\ln(\varepsilon)|/\varepsilon^2$ (the contribution of the first term of the r.h.s. of (4.18) is C/ε^2 whereas the one of the second term is $C|\ln(\varepsilon)|/\varepsilon^2$). □

With a slight adaptation of Sznitman [36] to take into account the random births on the boundary, we obtain the existence and pathwise uniqueness of the following interacting particle systems.

Definition 4.11 *Consider a sequence $(B^i)_{i\in\mathbb{N}}$ of independent Brownian motions on $\mathbb{R}^2$ and a sequence of independent variables $(\tau^i, Z_0^i)_{i\in\mathbb{N}}$ with values in $[0,T]\times\bar{\Theta}$ distributed according to P_0, and independent of the Brownian motions. For a fixed ε, for each $n\in\mathbb{N}^*$, and $1\leq i\leq n$, let us consider the interacting processes defined by*

$$\begin{aligned}
&Z_t^{in,\varepsilon} \in \bar{\Theta}, \forall t\in[0,T]\\
&Z_t^{in,\varepsilon} = Z_0^i + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{\tau^i\leq s\}} dB_s^i + \int_0^t \mathbf{1}_{\{\tau^i\leq s\}} K_\varepsilon \tilde{\mu}_s^{n,\varepsilon}(Z_s^{in,\varepsilon})ds - k_t^{in,\varepsilon}\ ;\\
&|k^{in,\varepsilon}|_t = \int_0^t \mathbf{1}_{\{Z_s^{in,\varepsilon}\in\partial\Theta\}} \mathbf{1}_{\{\tau^i\leq s\}} d|Z^{in,\varepsilon}|_s\ ;\ k_t^{in,\varepsilon} = \int_0^t n(Z_s^{in,\varepsilon}) d|k^{in,\varepsilon}|_s
\end{aligned} \qquad (4.20)$$

where $\tilde{\mu}_s^{n,\varepsilon} = \frac{1}{n}\sum_{j=1}^n h(\tau^j, Z_0^j)\mathbf{1}_{\{\tau^j\leq s\}}\delta_{Z_s^{jn,\varepsilon}}$ is the weighted empirical measure of the system.

Let us remark that the particles either have birth at time 0 inside the domain and evolve as diffusive particles with normal reflecting boundary conditions, or have birth at a random time on the boundary of the domain, and evolve after birth as the other ones. Moreover, all particles, as soon as they are born, interact together following a mean field depending on the parameter ε.

Again according to [36], we also get the existence and pathwise uniqueness of the limiting processes (when n tends to infinity and ε is fixed), coupled with the interacting processes, as follows.

Definition 4.12 *We define* $\bar{Z}^{i,\varepsilon}$ *by*

$$\begin{aligned}
&\bar{Z}_t^{i,\varepsilon} \in \bar{\Theta}, \forall t \in [0,T] \\
&\bar{Z}_t^{i,\varepsilon} = Z_0^i + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{\tau^i \le s\}} dB_s^i + \int_0^t \mathbf{1}_{\{\tau^i \le s\}} K_\varepsilon \tilde{Q}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon}) ds - \bar{k}_t^{i,\varepsilon} \ ; \\
&|\bar{k}^{i,\varepsilon}|_t = \int_0^t \mathbf{1}_{\{\bar{Z}_s^{i,\varepsilon} \in \partial\Theta\}} \mathbf{1}_{\{\tau^i \le s\}} d|\bar{k}^{i,\varepsilon}|_s \ ; \ \bar{k}_t^{i,\varepsilon} = \int_0^t n(\bar{Z}_s^{i,\varepsilon}) d|\bar{k}^{i,\varepsilon}|_s
\end{aligned} \tag{4.21}$$

where Q^ε *is the common law of* $(\tau^i, \bar{Z}^{i,\varepsilon}, \bar{k}^{i,\varepsilon})$*, and* $\tilde{Q}_s^\varepsilon$ *is defined from* Q^ε *by (4.14).*

Sznitman also proves a propagation of chaos result, but without precise estimates on the rate of convergence. In order to get such estimates, we denote by H a $\mathcal{C}_b^2(\bar{\Theta})$-extension of the distance-function $d(.,\partial\Theta)$ (defined on a restriction to Θ of a neighbourhood of $\partial\Theta$). The function H satisfies (see [17])

$$\nabla H = -n \text{ on } \partial\Theta. \tag{4.22}$$

We also recall that the domain Θ (since $\mathcal{C}^4$) satisfies the uniform "exterior sphere" condition:

$$\exists C_{sp} \ge 0 \ , \forall x \in \partial\Theta \ , \ \forall x' \in \bar{\Theta} \ , \ C_{sp}|x - x'|^2 + n(x).(x - x') \ge 0. \tag{4.23}$$

Proposition 4.13 *For* $t \le T$*, for each* $i \in \{1, ..., n\}$*,*

$$\begin{aligned}
E(\sup_{s \le t} |Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2) &\le 2d(\Theta)\sqrt{\frac{A_\varepsilon}{n}} \exp(K_H(1 + (\|w_0\|_1 + \nu\|g\|_1)(M_\varepsilon/2 + L_\varepsilon)t)) \\
E(\sup_{s \le t} |k_s^{in,\varepsilon} - \bar{k}_s^{i,\varepsilon}|) &\le E(\sup_{s \le t} |Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|) + 2t(\|w_0\|_1 + \nu\|g\|_1)\left(L_\varepsilon E(\sup_{s \le t} |Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|) + \frac{M_\varepsilon}{\sqrt{n}}\right)
\end{aligned}$$

where K_H *is a constant only depending only on the upper-bounds of the function* H *and its derivatives,* $d(\Theta)$ *is the diameter of* Θ *and* $A_\varepsilon = \frac{4(\|w_0\|_1 + \nu\|g\|_1)^2 M_\varepsilon^2}{2 + (\|w_0\|_1 + \nu\|g\|_1)(M_\varepsilon + 2L_\varepsilon)}$.

Remark 4.14 *The convergence rate given above is not optimal in* n*. Indeed one can check that* $E(\sup_{s \le t} |Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^4)$ *is smaller than* $\frac{C_1 M_\varepsilon^4 t}{n^2 C_2} \exp(C(M_\varepsilon^2 + 4L_\varepsilon^2)t^2)$*, but in the next section, we will choose* $\varepsilon = \varepsilon_n$ *depending on* n *and converging to* 0 *in such a way that* $E(\sup_{s \le t} |Z_s^{in,\varepsilon_n} - \bar{Z}_s^{i,\varepsilon_n}|^2)$ *tends to* 0*. The estimation given in the proposition allows a quicker (but still very slow) convergence of* ε_n *to* 0 *than the previous one.*

Proof. We compare the two processes $Z^{in,\varepsilon}$ and $\bar{Z}^{i,\varepsilon}$. We denote for simplicity Z, k, $\bar{Z}$ and $\bar{k}$ instead of $Z^{in,\varepsilon}$, $k^{in,\varepsilon}$, $\bar{Z}^{i,\varepsilon}$ and $\bar{k}^{i,\varepsilon}$, $h_t = H(Z_t)$, $\bar{h}_t = H(\bar{Z}_t)$, $h'_t = \nabla H(Z_t)$, $\bar{h}'_t =$

$\nabla H(\bar{Z}_t)$, $h''_t = \Delta H(Z_t)$, $\bar{h}''_t = \Delta H(\bar{Z}_t)$, $b_t = K_\varepsilon \tilde{\mu}^{n,\varepsilon}_s(Z_t)$ and $\bar{b}_t = K_\varepsilon \tilde{Q}^\varepsilon_t(\bar{Z}_t)$. Computing $d\exp(-2C_{sp}(h_t + \bar{h}_t))|Z_t - \bar{Z}_t|^2$ by Itô's formula, we get

$$\begin{aligned} & 1_{\{\tau_i \le t\}} \exp(-2C_{sp}(h_t + \bar{h}_t)) \times \Big[2(Z_t - \bar{Z}_t).(d\bar{k}_t - dk_t) - 2C_{sp}|Z_t - \bar{Z}_t|^2(d|k|_t + d|\bar{k}|_t) \\ - \quad & 2C_{sp}|Z_t - \bar{Z}_t|^2 \Big(\sqrt{2\nu}(h'_t + \bar{h}'_t)dB^i_t + \Big\{ h'_t b_t + \bar{h}'_t \bar{b}_t + \nu(-2C_{sp}|h'_t + \bar{h}'_t|^2 + h''_t + \bar{h}''_t) \Big\} dt \Big) \\ + \quad & 2(Z_t - \bar{Z}_t).(b_t - \bar{b}_t)dt \Big] \end{aligned} \tag{4.24}$$

Because of the "exterior sphere" condition, the local time terms of the first line have a non-positive contribution after integration over time. We deduce that for K_H a constant which can be computed and depends only on upper-bounds of the function H and its derivatives,

$$\begin{aligned} E(|Z^{in,\varepsilon}_t - \bar{Z}^{i,\varepsilon}_t|^2) \quad \le \quad & K_H \Big((1 + M_\varepsilon(\|w_0\|_1 + \nu\|g\|_1)) \int_0^t E(|Z^{in,\varepsilon}_s - \bar{Z}^{i,\varepsilon}_s|^2) ds \\ & + \int_0^t E(|Z^{in,\varepsilon}_s - \bar{Z}^{i,\varepsilon}_s||K_\varepsilon \tilde{\mu}^{n,\varepsilon}_s(Z^{in,\varepsilon}_s) - K_\varepsilon \tilde{Q}^\varepsilon_s(\bar{Z}^{i,\varepsilon}_s)|) ds \Big) \end{aligned} \tag{4.25}$$

Using the Lipschitz continuity of K_ε, the boundedness of h and the exchangeability of the processes $(Z^{in,\varepsilon}, \bar{Z}^{i,\varepsilon})$, $1 \le i \le n$, we obtain

$$\begin{aligned} E \quad & (\quad |Z^{in,\varepsilon}_s - \bar{Z}^{i,\varepsilon}_s||K_\varepsilon \tilde{\mu}^{n,\varepsilon}_s(Z^{in,\varepsilon}_s) - K_\varepsilon \tilde{Q}^\varepsilon_s(\bar{Z}^{i,\varepsilon}_s)|) \\ \le \quad & (\|w_0\|_1 + \nu\|g\|_1) L_\varepsilon E(|Z^{in,\varepsilon}_s - \bar{Z}^{i,\varepsilon}_s|(|Z^{in,\varepsilon}_s - \bar{Z}^{i,\varepsilon}_s| + \frac{1}{n}\sum_{j=1}^n |Z^{jn,\varepsilon}_s - \bar{Z}^{j,\varepsilon}_s|)) \\ & + E(|Z^{in,\varepsilon}_s - \bar{Z}^{i,\varepsilon}_s||\frac{1}{n}\sum_{j=1}^n h(\tau_j, Z^j_0) \mathbf{1}_{\{\tau^j \le s\}} K_\varepsilon(\bar{Z}^{i,\varepsilon}_s, \bar{Z}^{j,\varepsilon}_s) - K_\varepsilon \tilde{Q}^\varepsilon_s(\bar{Z}^{i,\varepsilon}_s)|) \\ \le \quad & (1 + 2(\|w_0\|_1 + \nu\|g\|_1)L_\varepsilon) E(|Z^{in,\varepsilon}_s - \bar{Z}^{i,\varepsilon}_s|^2) \\ & + E(|\frac{1}{n}\sum_{j=1}^n h(\tau_j, Z^j_0) \mathbf{1}_{\{\tau^j \le s\}} K_\varepsilon(\bar{Z}^{i,\varepsilon}_s, \bar{Z}^{j,\varepsilon}_s) - K_\varepsilon \tilde{Q}^\varepsilon_s(\bar{Z}^{i,\varepsilon}_s)|^2) \end{aligned}$$

After expansion of $E(|\frac{1}{n}\sum_{j=1}^n h(\tau_j, Z^j_0)\mathbf{1}_{\{\tau^j \le s\}} K_\varepsilon(\bar{Z}^{i,\varepsilon}_s, \bar{Z}^{j,\varepsilon}_s) - K_\varepsilon \tilde{Q}^\varepsilon_s(\bar{Z}^{i,\varepsilon}_s)|^2)$, many terms disappear by independence of the variables which are centered conditionnally to $\bar{Z}^{i,\varepsilon}$ and it only remains n bounded terms. We deduce that

$$\begin{aligned} E(|Z^{in,\varepsilon}_t - \bar{Z}^{i,\varepsilon}_t|^2) \le K_H \quad \Big(\quad & (2 + (\|w_0\|_1 + \nu\|g\|_1)(M_\varepsilon + 2L_\varepsilon)) \int_0^t E(|Z^{in,\varepsilon}_s - \bar{Z}^{i,\varepsilon}_s|^2) ds \\ & + \frac{4(\|w_0\|_1 + \nu\|g\|_1)^2 M_\varepsilon^2 t}{n} \Big) \end{aligned} \tag{4.26}$$

Using Gronwall's Lemma, we obtain that both sides of (4.25) and (4.26) are smaller than

$$f(t) = \frac{4(\|w_0\|_1 + \|\nu g\|_1)^2 M_\varepsilon^2}{n(2 + (\|w_0\|_1 + \nu\|g\|_1)(M_\varepsilon + 2L_\varepsilon))} \exp(K_H(2 + (\|w_0\|_1 + \nu\|g\|_1)(M_\varepsilon + 2L_\varepsilon))t).$$

Integrating (4.24) w.r.t. time, dealing with the stochastic integral thanks to Doob's inequality and using that the r.h.s. of (4.25) is smaller than $f(t)$, we get

$$\begin{aligned}
E(\sup_{s\le t}|Z_s^{in,\varepsilon}-\bar Z_s^{i,\varepsilon}|^2) &\le \Big(K_H\int_0^t E(|Z_s^{in,\varepsilon}-\bar Z_s^{i,\varepsilon}|^4)ds\Big)^{1/2}+f(t)\\
&\le d(\Theta)\Big(K_H\int_0^t E(|Z_s^{in,\varepsilon}-\bar Z_s^{i,\varepsilon}|^2)ds\Big)^{1/2}+f(t)\\
&\le d(\Theta)\sqrt{f(t)}+f(t)\ \text{ since the r.h.s. of (4.26) is smaller than } f(t)
\end{aligned}$$

The l.h.s. being smaller than $d(\Theta)^2$, it is smaller than $2d(\Theta)\sqrt{f(t)}$ when $f(t)\ge d(\Theta)^2$ and the r.h.s. is smaller than $2d(\Theta)\sqrt{f(t)}$ otherwise. We deduce the desired estimate for $E(\sup_{s\le t}|Z_s^{in,\varepsilon}-\bar Z_s^{i,\varepsilon}|^2)$.
Now remarking that

$$\sup_{s\le t}|k_s^{in,\varepsilon}-\bar k_s^{i,\varepsilon}|\le\int_0^t|K_\varepsilon\tilde\mu_s^{n,\varepsilon}(Z_s^{in,\varepsilon})-K_\varepsilon\tilde Q_s^\varepsilon(\bar Z_s^{i,\varepsilon})|ds+\sup_{s\le t}|Z_s^{in,\varepsilon}-\bar Z_s^{i,\varepsilon}|$$

and using arguments developed above we obtain the other estimate. □

Remark 4.15 *Let us remark that if $\bar\Theta$ is a convex region then the rate of convergence is easier to obtain. Indeed the constant C_{sp} defined in (4.23) can be chosen equal to 0 :*

$$\forall x\in\partial\Theta\ ,\ \forall x'\in\bar\Theta\ ,\ n(x).(x-x')\ge 0. \tag{4.27}$$

In the expression of $|Z_t^{in,\varepsilon}-\bar Z_t^{i,\varepsilon}|^2$ given by Itô's formula, the local times terms are nonpositive and therefore

$$\begin{aligned}
E(\sup_{s\le t}|Z_s^{in,\varepsilon}-\bar Z_s^{i,\varepsilon}|^2) &\le (1+2(\|w_0\|_1+\nu\|g\|_1)L_\varepsilon))\int_0^t E(\sup_{u\le s}|Z_u^{in,\varepsilon}-\bar Z_u^{i,\varepsilon}|^2)ds\\
&\quad+\frac{4(\|w_0\|_1+\nu\|g\|_1)^2M_\varepsilon^2 t}{n}
\end{aligned}$$

and we conclude by Gronwall's Lemma.

4.3.2 Convergence of the Limiting Laws

We will prove that the laws Q^ε of $(\tau^1,\bar Z^{1,\varepsilon},\bar k^{1,\varepsilon})$ converge to the unique solution P of $(\mathcal{M}_B)$ as ε tends to 0. We are first going to check that the drift coefficient $K_\varepsilon\tilde Q_s^\varepsilon$ converges to $K\tilde p_s$.
By Girsanov's theorem, it turns out that $\forall s>0$, the measure $\tilde Q_s^\varepsilon$ admits a density function q_s^ε. Moreover, reasoning like in the proof of Theorem 4.9 and using the boundedness of K_ε, we show that q^ε is the unique solution in $L_T^1=\{p_t,\sup_{t\le T}\|p_t\|_{L^1}<+\infty\}$ of the equation

$$q_t^\varepsilon(x)=P_t^\nu w_0(x)+\int_0^t\nabla_x P_{t-s}^\nu.(q_s^\varepsilon K_\varepsilon q_s^\varepsilon)(x)ds+\nu\int_0^t\int_{\partial\Theta}P_{t-s}^\nu g(s,y)d\sigma(y)ds. \tag{4.28}$$

On the other hand, thanks to Lemma 4.19 1), we can apply to the equation

$$\begin{aligned} \partial_t w(t,x) + \nabla.(wK_\varepsilon w)(t,x) &= \nu\Delta w(t,x) \ \text{ in } \Theta; \\ w(x,0) = w_0 \text{ in } \Theta \ ; \ \partial_n w &= \nabla w.n = g \text{ on } \partial\Theta \end{aligned} \tag{4.29}$$

all what we have done for the equation (4.1). Then there exists a unique weak solution w^ε belonging to $L^\infty_t(L^2_x) \cap L^2_t(H^1_x)$. Now, like in Proposition 4.5, we obtain that w^ε is also solution of (4.28). Since it belongs to L^1_T (Θ is bounded), we conclude that $w^\varepsilon = q^\varepsilon$. Thanks to (4.19), one can check that

$$\sup_{\varepsilon\in(0,1]} \Big(\|q^\varepsilon\|_{L^\infty_t(L^2_x)} + \|q^\varepsilon\|_{L^2_t(H^1_x)} + \|\partial_t q^\varepsilon\|_{L^2_t(H^{1'}_x)} + \|q^\varepsilon\|_{L^4_t(L^4_x)} \Big) < +\infty. \tag{4.30}$$

Remark 4.16 *Similarly the non-negative measures $B \in \mathcal{B}(\bar\Theta) \to E^{Q^\varepsilon}(\mathbf{1}_{\{\tau\le t\}} 1_B(X_t))$ have densities p^ε_t w.r.t. the Lebesgue measure which are the unique solution in L^1_T of the mild equation obtained by replacing respectively w_0 and g by $|w_0|/(\|w_0\|_1 + \nu\|g\|_1)$ and $|g|/(\|w_0\|_1 + \nu\|g\|_1)$ in (4.28).*
Identifying p^ε with the unique weak solution of the problem obtained from (4.29) by replacing w_0 and g in the same way, we check that (4.30) holds for p^ε.

We can now prove the convergence of q^ε to w.

Proposition 4.17

$$\lim_{\varepsilon\to 0} \|q^\varepsilon - w\|_{L^2_t(L^2_x)} = 0 \ ; \ \lim_{\varepsilon\to 0} \|K_\varepsilon q^\varepsilon - Kw\|_{L^2_t(L^2_x)} = 0.$$

Proof. Thanks to (4.30), one can extract from each sequence q^{ε_n} with ε_n tending to 0, a sub-sequence (still denoted q^{ε_n} for simplicity), which converges strongly in $L^2_t(L^2_x)$ and in $L^2_t(H^1_x)$ and weakly* in $L^\infty_t(L^2_x)$ to $\tilde w$. We can show that $\tilde w$ is a weak solution of (4.1) and conclude that $\tilde w = w$ by uniqueness for this equation.

Let $1 < p < 2$. Combining the Sobolev inequality $\|q^{\varepsilon_n}_s\|_{L^{\frac{p}{p-1}}} \le C\|q^{\varepsilon_n}_s\|_{H^1}$, Lemma 4.10 2) and (4.30), we obtain that the term $(K_{\varepsilon_n} - K)q^{\varepsilon_n}_s$ converges to 0 in $L^2_t(L^\infty_x)$. Now, writing

$$\|K_\varepsilon q^\varepsilon - Kw\|_{L^2_t(L^2_x)} \le \|K(q^\varepsilon - w)\|_{L^2_t(L^2_x)} + \|(K_\varepsilon - K)q^\varepsilon\|_{L^2_t(L^2_x)},$$

and using (4.8), one easily deduces the second assertion. □

Theorem 4.18 *The probability measures Q^ε on $[0,T]\times\mathcal{C}([0,T],\bar\Theta)\times\mathcal{C}([0,T],\mathbb{R}^2)$ converge weakly to the unique solution P of $(\mathcal{M}_B)$, as ε tends to 0.*

Proof. The weak convergence topology being metrizable, we check that $(Q^n = Q^{\varepsilon_n})_{n\in I\!N}$ converges weakly to P when the sequence ε_n tends to 0 as n tends to $+\infty$. Let us firstly prove the uniform tightness of the sequence $(Q^n)_n$, next identify the limiting points.

1) By (4.19) and (4.30), we easily obtain that

$$\sup_n \|K_{\varepsilon_n} q_s^{\varepsilon_n}\|_{L_t^4(L_x^\infty)} < +\infty. \tag{4.31}$$

Then the Kolmogorov tightness criterion is satisfied for the laws of

$$\bar{Y}_t^{1,\varepsilon_n} = Z_0^1 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{\tau^1\le s\}} dB_s^1 + \int_0^t \mathbf{1}_{\{\tau^1\le s\}} K_{\varepsilon_n} q_s^n(\bar{Z}_s^{1,\varepsilon_n})ds.$$

Now the uniform tightness of the laws Q^n of the processes $(\tau^1, \bar{Z}^{1,\varepsilon_n}, \bar{k}^{1,\varepsilon_n})$ is a simple consequence of the fact that the application sending $y \in \mathcal{C}([0,T], I\!R^2)$ on the solution $(x,k) \in \mathcal{C}([0,T],\bar{\Theta}) \times \mathcal{C}([0,T], I\!R^2)$ of the Skorohod problem is continuous (See [25]).

2) Let us now denote by Q^∞ a limit value of a convergent subsequence still denoted by (Q^n) for simplicity and prove by arguments inspired from Sznitman ([36]) that $Q^\infty = P$.

If as usual (τ, X, k) denotes the canonical process on $[0,T]\times\mathcal{C}([0,T],\bar{\Theta})\times\mathcal{C}([0,T], I\!R^2)$, let us define, for $p\in I\!N^*$, $0\le s_1\le ...\le s_p\le s< t\le T$, $\phi\in\mathcal{C}_b^2(I\!R^2)$, $g\in\mathcal{C}_b([0,T],(\bar{\Theta}\times I\!R^2)^p)$ the function

$$\begin{aligned} G_n(\tau,X,k) \quad &= \quad g(\tau, X_{s_1}, k_{s_1}, ..., X_{s_p}, k_{s_p})\Big(\phi(X_t+k_t) - \phi(X_s+k_s) \\ &\quad - \int_s^t \mathbf{1}_{\{\tau\le u\}}\Big(\nu\Delta\phi(X_u+k_u) + K_{\varepsilon_n} q_u^{\varepsilon_n}(X_u).\nabla\phi(X_u+k_u)\Big)du\Big) \end{aligned}$$

Then $E^{Q^n}(G_n(\tau,X,k)) = 0$. Now if we define the function G by replacing $K_{\varepsilon_n} q_s^{\varepsilon_n}$ by Kw_s in (4.32), we want to prove that $E^{Q^\infty}(G(\tau,X,k)) = 0$.

$$E^{Q^\infty}(G(\tau,X,k)) = E^{Q^\infty}(G(\tau,X,k)) - E^{Q^n}(G(\tau,X,k)) + E^{Q^n}(G(\tau,X,k) - G^n(\tau,X,k)).$$

Since $w\in L_t^4(L_x^4)$, by (4.7), ds a.e. in $[0,T]$ $x\in\bar{\Theta}\to Kw_s(x)$ is continuous and $Kw_s \in L_t^4(L_x^\infty)$. We deduce that $G(\tau,X,k)$ is a continuous function on the path space, and the first term of the r.h.s. tends to 0 as n tends to infinity. On the other hand, using Remark 4.16 and Proposition 4.17, we obtain

$$\begin{aligned} E^{Q^n}|G^n(\tau,X,k) - G(\tau,X,k)| \quad &\le \quad CE\Big(\int_0^t \mathbf{1}_{\{\tau^1\le s\}}|K_{\varepsilon_n} q_s^{\varepsilon_n}(\bar{Z}_s^{1,\varepsilon_n}) - Kw_s(\bar{Z}_s^{1,\varepsilon_n})|ds\Big) \\ &\le \quad C\|p^{\varepsilon_n}\|_{L_t^2(L_x^2)}\|K_{\varepsilon_n} q^{\varepsilon_n} - Kw\|_{L_t^2(L_x^2)} \to 0 \text{ as } n\to+\infty. \end{aligned}$$

Hence $E^{Q^\infty}(G(\tau,X,k)) = 0$. Since $\forall n,\ Q^n\circ(\tau,X_0,k_0)^{-1} = P_0\otimes\delta_{(0,0)}$, $Q^\infty\circ(\tau,X_0,k_0)^{-1} = P_0\otimes\delta_{(0,0)}$. We are now going to prove that Q^∞-almost surely,

$$|k|_T<\infty \text{ and } \forall t\in[0,T],\ |k|_t = \int_0^t \mathbf{1}_{\{X_s\in\partial\Theta\}}\mathbf{1}_{\{\tau\le s\}}d|k|_s\ ;\ k_t = \int_0^t n(X_s)d|k|_s.$$

As according to the proof of Theorem 4.9, P is the unique solution of the linear martingale problem defined like $\mathcal{M}_T$ but with known drift coefficient Kw_s, we will conclude that $Q^\infty = P$. According to the following Lemma the proof of which is postponed,

Lemma 4.19 *For any $A \geq 0$, the following subset of* $[0,T] \times \mathcal{C}([0,T], \bar{\Theta}) \times \mathcal{C}([0,T], I\!R^2)$

$$F_A = \left\{ (u,x,k) : \ |k|_T = \int_0^T \mathbf{1}_{\{u \leq s\}} \mathbf{1}_{\{x_s \in \partial\Theta\}} d|k|_s \leq A \text{ and } \forall t \in [0,T], \ k_t = \int_0^t n(x_s) d|k|_s \right\}$$

is closed.

we have

$$Q^\infty \left(\bigcup_{A>0} F_A \right) \geq 1 - \lim_{A \to +\infty} \liminf_{n \to +\infty} Q^n(F_A^c) \geq 1 - \lim_{A \to +\infty} \frac{\sup_{n \in I\!N} E^{Q^n} |k|_T}{A}.$$

Therefore it is enough to check that $\sup_{n \in I\!N} E|\bar{k}^{1,\varepsilon_n}|_T < +\infty$ to conclude the proof. Since $\nabla H = -n$ on $\partial\Theta$, applying Itô's formula to compute $H(\bar{Z}_T^{1,\varepsilon_n})$, we get that $|\bar{k}^{1,\varepsilon_n}|_T$ is equal to

$$H(\bar{Z}_T^{1,\varepsilon_n}) - H(Z_0^1) - \int_0^T \mathbf{1}_{\{\tau^1 \leq s\}} \left((\nu \Delta H + K_{\varepsilon_n} q_s^{\varepsilon_n} . \nabla H)(\bar{Z}_s^{1,\varepsilon_n}) ds + \sqrt{2\nu} \nabla H(\bar{Z}_s^{1,\varepsilon_n}) . dB_s^1 \right).$$

Taking expectations and using (4.31), we obtain the desired result. □

Proof.of Lemma 4.19 Let $(u^n, x^n, k^n) \in F_A$ converge to (u,x,k) as $n \to +\infty$. Since $\sup_n |k^n|_T \leq A$, by extraction of a subsequence, we can suppose that the measure $d|k^n|$ (resp. dk^n) converges weakly to a positive measure da with mass smaller than A (resp. to db_s). Of course $db_s = \lambda(s) da_s$ for some measurable function $\lambda : [0,T] \to I\!R^2$ and since k^n converges uniformly on $[0,T]$ to k, $db_s = dk_s$. Since $d(x_s^n, \partial\Theta)$, where $d(., \partial\Theta)$ denotes the (continuous) distance from the boundary function, converges uniformly on $[0,T]$ to $d(x_s, \partial\Theta)$,

$$\int_0^T d(x_s, \partial\Theta) da_s = \lim_n \int_0^T d(x_s^n, \partial\Theta) d|k|_s^n = 0.$$

We deduce that da_s a.e. and therefore $d|k|_s$ a.e., $x_s \in \partial\Theta$. Since the functions k^n which are equal to $(0,0)$ on $[0, u^n]$ converge uniformly to k, this function is equal to $(0,0)$ on $[0,u]$ and $|k|_u = 0$. To check the only lacking property : $dk_s = n(x_s) d|k|_s$, we remark that

$$\forall f \in \mathcal{C}([0,T], I\!R_+), \ \forall g \in \mathcal{C}([0,T], \bar{\Theta}), \ \int_0^T f(s) \left((x_s - g(s)).dk_s + C_{sp} |x_s - g(s)|^2 da_s \right) \geq 0$$

by taking the limit $n \to +\infty$ in the similar inequalities satisfied with (x, dk, da) replaced by $(x^n, dk^n, d|k^n|)$ according to the uniform "exterior sphere" condition (4.23). We deduce that $dk_s = |\lambda(s)| n(x_s) da_s$ which implies the desired property. □

4.4 The Convergence Theorem

We now consider a sequence (ε_n) tending to 0 as n tends to infinity, in such a way that

$$\lim_{n\to+\infty} L_{\varepsilon_n}^2\sqrt{\frac{A_{\varepsilon_n}}{n}}\exp(K_H(1+(\|w_0\|_1+\nu\|g\|_1)(M_{\varepsilon_n}/2+L_{\varepsilon_n})T))+\frac{M_{\varepsilon_n}}{\sqrt{n}}=0. \quad (4.32)$$

Of course, the convergence of ε_n to 0 is very slow and this choice is certainely not optimal.

Let us now consider for each n the system of processes (τ^i, Z^{in}, k^{in}) where $Z^{in} = Z^{in,\varepsilon_n}$ and $k^{in} = k^{in,\varepsilon_n}$ are defined as in (4.35) but with K_{ε_n} replacing K_ε. We are now able to obtain our main theorem.

Theorem 4.20 *1) The laws of the n-particle system* $(\tau^i, Z^{in}, k^{in})_{1\le i\le n}$*, are* P*-chaotic (where* P *is the solution of the problem* $(\mathcal{M}_B)$*):*

$$\forall p\in I\!N^*\ ,\ \mathcal{L}((\tau^1,Z^{1n},k^{1n}),...,(\tau^p,Z^{pn},k^{pn}))\overset{weakly}{\Longrightarrow} P^{\otimes p}\ \text{as}\ n\to+\infty. \quad (4.33)$$

2) The approximate velocity field converges to Kw*:*

$$\lim_{n\to+\infty} E(\|K_{\varepsilon_n}\tilde{\mu}_t^{n,\varepsilon_n}(x)-Kw_t(x)\|^2_{L^2_t(L^2_x)})=0. \quad (4.34)$$

Proof.

1) Since the processes $(\tau^i, \bar{Z}^{i,\varepsilon_n}, \bar{k}^{i,\varepsilon_n})_i$ are independent, Theorem 4.18 implies that for every fixed $p\in I\!N^*$, the law of $((\tau^1,\bar{Z}^{1,\varepsilon_n},\bar{k}^{1,\varepsilon_n}),...,(\tau^p,\bar{Z}^{p,\varepsilon_n},\bar{k}^{p,\varepsilon_n}))$ converges weakly to $P^{\otimes p}$. Let $\mathcal{C}_T=[0,T]\times\mathcal{C}([0,T],\bar{\Theta})\times\mathcal{C}([0,T],I\!R^2)$. We endow $\mathcal{C}_T^p$ with the uniform metric and $\mathcal{P}(\mathcal{C}_T^p)$ with the Vaserstein metric $\rho(\mu,\nu)=\inf\left\{\int_{\mathcal{C}_T^p\times\mathcal{C}_T^p} d(x,y)\wedge 1 R(dx,dy); R \text{ has marginals } \mu \text{ and } \nu\right\}$ compatible with the topology of the weak convergence. Hence

$$\rho(\mathcal{L}((\tau^1,\bar{Z}^{1,\varepsilon_n},\bar{k}^{1,\varepsilon_n}),...,(\tau_p,\bar{Z}^{p,\varepsilon_n},\bar{k}^{p,\varepsilon_n})),P^{\otimes p})\to 0 \text{ as } n\to+\infty.$$

By Proposition 4.13, and (4.32)

$$\lim_{n\to+\infty} E\left(d\left(((\tau^1,Z^{1n},k^{1n}),...,(\tau^p,Z^{pn},k^{pn})),((\tau^1,\bar{Z}^{1,\varepsilon_n},\bar{k}^{1,\varepsilon_n}),...,(\tau_p,\bar{Z}^{p,\varepsilon_n},\bar{k}^{p,\varepsilon_n}))\right)\right)=0$$

which ensures that

$$\lim_{n\to+\infty}\rho\left(\mathcal{L}\left((\tau^1,Z^{1n},k^{1n}),...,(\tau^p,Z^{pn},k^{pn})\right),\mathcal{L}\left((\tau^1,\bar{Z}^{1,\varepsilon_n},\bar{k}^{1,\varepsilon_n}),...,(\tau^p,\bar{Z}^{p,\varepsilon_n},\bar{k}^{p,\varepsilon_n})\right)\right)=0.$$

We conclude that $\rho(\mathcal{L}((\tau^1,Z^{1n},k^{1n}),...,(\tau^p,Z^{pn},k^{pn})),P^{\otimes p})$ converges to 0.

2) On the other hand,

$$E(|K_{\varepsilon_n}\tilde{\mu}_t^{n,\varepsilon_n}(x) - Kw_t(x)|^2) \le 3E\Bigg(\Big|K_{\varepsilon_n}\tilde{\mu}_t^{n,\varepsilon_n}(x) - \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\tau^i\le t\}} h(\tau^i, Z_0^i) K_{\varepsilon_n}(x, \bar{Z}_t^{i,\varepsilon_n})\Big|^2$$

$$+\Big|\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\tau^i\le t\}} h(\tau^i, Z_0^i) K_{\varepsilon_n}(x, \bar{Z}_t^{i,\varepsilon_n}) - K_{\varepsilon_n}\tilde{Q}_t^{\varepsilon_n}(x)\Big|^2 + |K_{\varepsilon_n}\tilde{Q}_t^{\varepsilon_n}(x) - Kw_t(x)|^2\Bigg)$$

$$\le 3\Bigg((\|w_0\|_1 + \nu\|g\|_1)^2 (L_{\varepsilon_n}^2 E(\sup_{s\le t}|Z_s^{in} - \bar{Z}_s^{i,\varepsilon_n}|^2) + \frac{4M_{\varepsilon_n}^2}{n}) + |K_{\varepsilon_n} q_t^{\varepsilon_n}(x) - Kw_t(x)|^2\Bigg).$$

We conclude using (4.32), Proposition 4.13 and Proposition 4.17. □

Remark 4.21 *Since the laws $\mathcal{L}((\tau^1, Z^{1n}, k^{1n}), ..., (\tau^n, Z^{nn}, k^{nn}))$ are exchangeable, the propagation of chaos is equivalent to the convergence in probability of the empirical measures to P, as probability measures on the path space (cf. [37]). As a consequence, if the space of finite measures on $\bar{\Theta}$ is endowed with the weak convergence topology, the random finite measures $\tilde{\mu}_t^{n,\varepsilon_n} = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\tau^i\le t\}} h(\tau^i, Z_0^i)\delta_{Z_t^{in}}$ converge in measure to $w_t(x)dx$ for any $t \in [0,T]$, w being the unique solution of the vortex equation.*

4.4.1 Numerical Comments

We finally deduce from this study a simulation algorithm for the solution of the equation (4.1). To approximate numerically this solution, it is necessary to discretize in time the particle system. This can be achieved thanks to the Euler scheme for reflected diffusions proposed by Gobet [18]. Adapting these results to our case with identity diffusion matrix and normal reflection, and as in Bossy-Jourdain [6], one could hope to prove that if $\bar{\mu}_{l\Delta t}^n$ denotes the weighted empirical measure of the discretized system, $K_{\varepsilon_n}\bar{\mu}_{l\Delta t}^n$ converges to $Kw_{l\Delta t}$ in $L^\infty(\Theta)$, with the rate $\mathcal{O}(\Delta t + \frac{1}{\sqrt{n}})$, where Δt denotes the time step.

4.5 Some Comments on the Generalization to the Navier-Stokes Case

In [10], the Neumann condition obtained by Cottet has the form

$$\partial_n w = \int_{\partial\Theta} Bw(x)d\sigma(x)$$

on $\partial\Theta$, with a specific kernel $B(x,y)$ which is a sophisticated derivation operator with a bad behaviour when $x = y$. The dependence of the Neumann condition on w makes things more complicated, since this condition can not be interpreted as the law of some births on the boundary. So after having cutoff the kernel B by bounded kernels B_ε, the particle system we consider is a system who creates new particles on the boundary with a rate depending on the empirical measure of the alive particles. So the number of particles do not stay constant, and we will consider at each time the empirical measure of the particles alive at time t, which is a finite (point) measure on $\bar{\Theta}$.

This work is in progress. So, let us summary describe this sytem, without proof of convergence, to give an idea of the numerical algorithm.

Let $n \in I\!N^*$ and $(Z_0^{in,\varepsilon})_{1\le i\le n}$ denote independent initial random variables with law $\frac{|w_0|(x)}{\|w_0\|_1}dx$ independent from a sequence $(B^i)_{i\ge 1}$ of two-dimensional Brownian motions. For $1 \le i \le n$ we assign the weight $s_i = \frac{w_0}{|w_0|}(Z_0^{in,\varepsilon})$ to the i-th particle. Let also $(\tau_k, Z_k, U_k)_{k\ge 1}$ be a sequence of independent random variables with law

$$C_\varepsilon|\partial\Theta| 1_{\{t\ge 0\}} e^{-C_\varepsilon|\partial\Theta|t} dt \otimes \frac{d\sigma(z)}{|\partial\Theta|} \otimes 1_{[0,1]}(u)du,$$

where C_ε is an upper-bound of the kernel B_ε and $|\partial\Theta| = \int_{\partial\Theta} d\sigma(x)$.

We set $T_0 = 0$. The system with $N_0 = n$ initial particles is constructed inductively for $k \ge 1$ as follows :

- $T_k = T_{k-1} + \frac{\tau_k}{N_{k-1}}$.
- On the time-interval $[T_{k-1}, T_k]$, the number of particles remains equal to N_{k-1} and their positions $Z_t^{in,\varepsilon}$, $1 \le i \le N_{k-1}$ evolve according to the following stochastic differential equation with normal reflection :

$$\begin{aligned} &Z_t^{in,\varepsilon} \in \bar{\Theta}, \forall t \in [T_{k-1}, T_k]; \\ &Z_t^{in,\varepsilon} = Z_{T_{k-1}}^{in,\varepsilon} + \sqrt{2\nu}B_t^i + \frac{\|w_0\|_1}{n}\int_0^t \sum_{j=1}^{N_{k-1}} s_j K_\varepsilon(Z_s^{in,\varepsilon}, Z_s^{jn,\varepsilon})ds - k_t^{in,\varepsilon} \ ; \\ &|k^{in,\varepsilon}|_t = \int_0^t \mathbf{1}_{\{Z_s^{in,\varepsilon}\in\partial\Theta\}} d|Z^{in,\varepsilon}|_s \ ; \ k_t^{in,\varepsilon} = \int_0^t n(Z_s^{in,\varepsilon}) d|k^{in,\varepsilon}|_s \end{aligned} \tag{4.35}$$

- At time T_k,
  - either $U_k \le \frac{|\sum_{i=1}^{N_{k-1}} s_i B_\varepsilon(Z_k, Z_{T_k}^{in,\varepsilon})|}{N_{k-1}C_\varepsilon}$ and we create a new particle : $N_k = N_{k-1}+1$, $Z_{T_k}^{N_k n,\varepsilon} = Z_k$, s_{N_k} equal to the sign of $\sum_{i=1}^{N_{k-1}} s_i B_\varepsilon(Z_k, Z_{T_k}^{in,\varepsilon})$.
  - or the converse inequality holds and no particle is created : $N_k = N_{k-1}$.

As $\forall k \in I\!N$, $N_k \le n+k$, $\lim_{k\to+\infty} T_k \ge \sum_{k\in I\!N} \frac{\tau_{k+1}}{n+k}$ and

$$E(e^{-\lim_{k\to+\infty} T_k}) \le E\left(\prod_{k\in I\!N} e^{-\frac{\tau_{k+1}}{n+k}}\right) = \prod_{k\in I\!N}\left(1 - \frac{1}{1+C_\varepsilon|\partial\Theta|(n+k)}\right) = 0.$$

We deduce that a.s. $\lim_{k\to+\infty} T_k = +\infty$ and the particle system is defined on the time interval $[0, +\infty)$.

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