A COMMUTATIVITY CRITERION FOR CERTAIN ALGEBRAS OF INVARIANT DIFFERENTIAL OPERATORS ON NILPOTENT HOMOGENEOUS SPACES

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ABSTRACT. Let G be a connected, simply connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$, H a connected closed subgroup of G with Lie algebra $\mathfrak{h}$ and f a linear form on $\mathfrak{g}$ satisfying $f([\mathfrak{h},\mathfrak{h}]) = \{0\}\cdot$ Let χ_f be the unitary character of H with differential $\sqrt{-1}f$ at the origin. Let τ_f be the unitary representation of G induced from the character χ_f of H. We consider the algebra $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ of differential operators invariant under the action of G on the bundle with basis G/H associated to these data. We show that $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ is commutative if and only if τ_f is of finite multiplicities. This proves a conjecture of Corwin-Greenleaf and Duflo.

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1. Introduction

Let G be a connected, simply connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$ and H a connected closed subgroup of G with Lie algebra $\mathfrak{h}$. Every linear form f on $\mathfrak{g}$ satisfying $f([\mathfrak{h},\mathfrak{h}]) = \{0\}$ defines a unitary character χ_f of H given by $\chi_f(\exp X) = e^{\sqrt{-1}f(X)}$ for all X in $\mathfrak{h}$. We form the unitary representation τ_f of G induced from χ_f in a Hilbert space $\mathcal{H}_{\tau_f}$. More precisely, let $C^\infty(G,H,f)$ be the vector space of C^∞ complex functions ϕ on G satisfying the following covariance relation:

$$\phi(gh) = \chi_f^{-1}(h)\,\phi(g), \quad \forall h \in H,\ \forall g \in G.$$

Consider the vector subspace of $C^\infty(G,H,f)$ of elements with compact support modulo H equipped with the norm

$$\|\phi\|^2 = \int_{G/H} |\phi(g)|^2\,\mathrm{d}\dot{g}$$

where $\mathrm{d}\dot{g}$ denotes a left G-invariant measure on G/H. The Hilbert space $\mathcal{H}_{\tau_f}$ is the completion of this space relatively to this norm. The representation τ_f of G is defined as the left translations on $\mathcal{H}_{\tau_f}$:

$$\tau_f(g)(\phi)(g') = \phi(g^{-1}g'), \quad \forall (g,g') \in G \times G, \quad \forall \phi \in \mathcal{H}_{\tau_f}. \tag{1.1}$$

The unitary representation τ_f of G decomposes into a continuous sum of unitary irreducible representations of G:

$$\tau_f \simeq \int_{\widehat{G}}^{\oplus} m(\pi)\pi \, \mathrm{d}\mu(\pi) \tag{1.2}$$

where $m(\pi)$ denotes the multiplicity of π and μ a Plancherel measure of τ_f on the unitary dual $\widehat{G}$ of G. It is well known that for μ-almost all π in $\widehat{G}$, either the multiplicities $m(\pi)$ appearing in (1.2) are finite and admit a uniform bound or they are infinite (see [3] Section 1, and [10] Theorem 1.1). In the first (resp. second) case, we shall say that τ_f is of finite (resp. infinite) multiplicities.

Also, the multiplicities of τ_f have a nice geometric interpretation in terms of a certain affine subspace of the vector dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Let $\Gamma_{\mathfrak{g},\mathfrak{h},f}$ be the set of all forms on $\mathfrak{g}$ identical to f on $\mathfrak{h}$:

$$\Gamma_{\mathfrak{g},\mathfrak{h},f} = \{\ell \in \mathfrak{g}^* \mid \ell(Y) = f(Y), \ \ \forall Y \ \in \mathfrak{h}\}. \tag{1.3}$$

For a linear form ℓ on $\mathfrak{g}$, let us denote by $\Omega_\ell = G \cdot \ell$ the coadjoint orbit of G through ℓ. The celebrated orbit method associates with ℓ a unitary irreducible representation π_ℓ of G in a suitable Hilbert space $\mathcal{H}_\ell$ which only depends on Ω_ℓ, up to equivalence. Moreover, the correspondence $\Omega_\ell \mapsto \pi_\ell$ is a bijection. Corwin, Greenleaf and Grélaud proved in [3] Section 1 and [4] Theorem 1.2, that for irreducible unitary representations $\pi = \pi_\ell$ of G:

(*i*) $m(\pi)$ is the number of coadjoint orbits of H in $\Omega_\ell \cap \Gamma_{\mathfrak{g},\mathfrak{h},f}$, $\mathrm{d}\mu(\pi)$-a.e. In particular, $m(\pi) \neq 0 \iff \Omega_\ell \cap \Gamma_{\mathfrak{g},\mathfrak{h},f} \neq \emptyset$.

(*ii*) τ_f is of finite multiplicities $\iff \dim H \cdot \ell = \frac{1}{2} \dim G \cdot \ell$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h},f}$.

In the sequel, *for generic* ℓ *in* $\Gamma_{\mathfrak{g},\mathfrak{h},f}$ means that the property holds for ℓ belonging to some non-empty Zariski-open subset of $\Gamma_{\mathfrak{g},\mathfrak{h},f}$.

Finally, let $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ be the algebra of linear differential operators leaving the space $C^\infty(G,H,f)$ invariant and commuting with the left translation L of G:

$$\begin{aligned} &D \in \mathcal{D}(\mathfrak{g},\mathfrak{h},f) \iff \\ &\qquad D(L(g)\phi) = L(g)(D\phi), \ \ \forall g \in G, \ \ \forall \phi \in C^\infty(G,H,f). \end{aligned} \tag{1.4}$$

The question now is: Is there any relation between the commutativity of the algebra $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ and the multiplicities of the unitary representation τ_f of G? Corwin and Greenleaf proved (Theorem 1.1 of [5]) that if τ_f is of finite multiplicities then $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ is commutative. Also (Question 5, p. 747 of [5]), they stated the following

Conjecture: $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ *is commutative if and only if* τ_f *is of finite multiplicities.*

This conjecture is also related to the Question 6 asked by Duflo in [6], when applied to the case of connected, simply connected, real nilpotent Lie groups. Till now, it had been proved only in special cases: when $\dim \mathfrak{h} = 2$ by

Baklouti and Fujiwara ([1], théorème 4.8); when $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ by Baklouti and Ludwig ([2], Theorem 1.4); when $\mathfrak{h}$ has an $\mathrm{ad}\,\mathfrak{h}$-invariant supplementary subspace in $\mathfrak{g}$ (in particular when $\mathfrak{h}$ is 1-dimensional) by Fujiwara, Lion and Mehdi ([8], Corollary 1). In the present paper, we prove that it is true in full generality as it was announced in [9].

The following important points of our proof of should be emphasized:

1) We just have to prove the implication

$$\tau_f \text{ is of infinite multiplicities } \Rightarrow \mathcal{D}(\mathfrak{g},\mathfrak{h},f) \text{ is non commutative,}$$

as the converse has already been established by Corwin and Greenleaf.

2) Assume $\dim \mathfrak{g}/\mathfrak{h} \geq 1$ and let $\mathfrak{g}'$ be an ideal of codimension one in $\mathfrak{g}$ that contains $\mathfrak{h}$. As explained at the end of Section 4 of [5], it is easily seen that $\mathcal{D}(\mathfrak{g}',\mathfrak{h},f') \subset \mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ where $f' \overset{\text{def.}}{=} f|_{\mathfrak{g}'}$. Therefore, if $\mathcal{D}(\mathfrak{g}',\mathfrak{h},f')$ is non-commutative so is $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$. So, we can assume $\mathcal{D}(\mathfrak{g}',\mathfrak{h},f')$ commutative. By an induction argument on the dimension of $\mathfrak{g}$, we have at the level of $\mathfrak{g}'$:

$$\mathcal{D}(\mathfrak{g}',\mathfrak{h},f') \text{ commutative } \Rightarrow \tau_{f'} \overset{\text{def.}}{=} \mathrm{Ind}\underset{H}{\overset{G'}{\uparrow}} \chi_{f'} \text{ is of finite multiplicities.}$$

Therefore, we can now assume $\tau_{f'}$ of finite multiplicities.

3) Recall, from the assertion (2) above that τ_f is of finite multiplicities if and only if $\dim H\cdot\ell = \frac{1}{2}\dim G\cdot\ell$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h},f}$. The facts that τ_f is of infinite and $\tau_{f'}$ of finite multiplicities then imply that $\dim H\cdot\ell = \dim H\cdot\ell'$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h},f}$.

4) We shall prove that $\mathcal{D}(\mathfrak{g}',\mathfrak{h},f')$ is properly contained in $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ if and only if $\dim H\cdot\ell = \dim H\cdot\ell'$ or equivalently that $\mathcal{D}(\mathfrak{g}',\mathfrak{h},f') = \mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ if and only if $\dim H\cdot\ell = \dim H\cdot\ell' + 1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h},f}$ (Theorem 5.2). This is the most important and difficult result of the article.

5) Now, Theorem 1 of [8] asserts that if $\tau_{f'}$ (resp. τ_f) is of finite (resp. infinite) multiplicities and $\mathcal{D}(\mathfrak{g}',\mathfrak{h},f')$ properly contained in $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$, then $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ is not commutative. This proves the conjecture.

It should be stressed that our proofs are of entirely algebraic nature.

This article is organized as follows. Our main notations are introduced in Section 2. In Section 3, we recall several important properties of the algebra $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ which we shall refer to later on. Section 4 is devoted to the proof of two lemmas that will be crucial in the sequel. The proofs of our main results are contained in Section 5. Finally, we completely work out an example in Section 6.

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2. Notations

We fix once for all a linear form f on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ satisfying $f([\mathfrak{h},\mathfrak{h}]) = \{0\}$. To simplify our notations, given a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, we will drop $f|_{\mathfrak{k}}$ if used as an index, whenever there is no ambiguity. For instance, taking $\mathfrak{k} = \mathfrak{g}$ and $f|_{\mathfrak{k}} = f$, we shall write $C^\infty(G,H)$, $\mathcal{D}(\mathfrak{g},\mathfrak{h})$ and $\Gamma_{\mathfrak{g},\mathfrak{h}}$ instead of $C^\infty(G,H,f)$, $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ and $\Gamma_{\mathfrak{g},\mathfrak{h},f}$ respectively.

Suppose that $\mathfrak{g}$ is of dimension n. We fix a flag $\mathcal{S}$ of ideals of $\mathfrak{g}$

$$\mathfrak{g}_0 = \{0\} \subsetneq \mathfrak{g}_1 \subsetneq \cdots \subsetneq \mathfrak{g}_{n-1} \subsetneq \mathfrak{g}_n = \mathfrak{g}. \tag{2.1}$$

Denote by $\mathcal{I}$ the (possibly empty) ordered set of indices

$$\mathcal{I} = \{i_1 < i_2 < \cdots < i_d\} \tag{2.2}$$

such that $\mathfrak{h} \cap \mathfrak{g}_{i_s-1} \neq \mathfrak{h} \cap \mathfrak{g}_{i_s}$. We have $1 \leq i_1$ and $i_d \leq n$. For s in $\{1,2,\ldots,d\}$, we let $\mathfrak{h}_s = \mathfrak{h} \cap \mathfrak{g}_{i_s}$ and obtain the following flag of ideals of $\mathfrak{h}$:

$$\mathfrak{h}_0 = \{0\} \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_d = \mathfrak{h}. \tag{2.3}$$

Then we define the (complementary) ordered set $\mathcal{J}$ of indices

$$\mathcal{J} = \{j_1 < j_2 < \cdots < j_p\} = \{1,2,\ldots,n\} \setminus \mathcal{I}. \tag{2.4}$$

Let $\mathfrak{k}_r = \mathfrak{h} + \mathfrak{g}_{j_r}$ for all r in $\{1,2,\ldots,p\}$ and put $\mathfrak{k}_0 = \mathfrak{h}$. One obtains a sequence of subalgebras of $\mathfrak{g}$

$$\mathfrak{h} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_p = \mathfrak{g} \tag{2.5}$$

such that $\dim \mathfrak{k}_r/\mathfrak{k}_{r-1} = 1$.

Observe that $\operatorname{card}\mathcal{I} = \dim \mathfrak{h} = d$ and that $\operatorname{card}\mathcal{J} = \dim \mathfrak{g}/\mathfrak{h} = p$.

We pick an element $Y_s \in \mathfrak{h}_s$ such that $Y_s \notin \mathfrak{h}_{s-1}$ for s in $\{1,2,\ldots,d\}$, and an element $X_r \in \mathfrak{k}_r$ such that $X_r \notin \mathfrak{k}_{r-1}$, for r in $\{1,2,\ldots,p\}$. In this way, we obtain a set of n elements

$$\{Y_1, Y_2, \ldots, Y_d, X_1, X_2, \ldots, X_p\}$$

forming a Malcev basis of $\mathfrak{g}$. It is clear that this Malcev basis is strong if we choose the order defined by the flag $\mathcal{S}$. However, this basis is weak if we choose the order defined by $\mathfrak{h}_s$ and $\mathfrak{k}_r$.

Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{g}$ and ℓ in $\mathfrak{g}^*$. It will be convenient to let $\mathfrak{l}^{B_\ell}$ be the subspace orthogonal of $\mathfrak{l}$ in $\mathfrak{g}$ relatively to the antisymmetric bilinear form B_ℓ on $\mathfrak{g} \times \mathfrak{g}$ defined by $B_\ell(X,Y) = \ell([X,Y])$, so that

$$\mathfrak{l}^{B_\ell} = \{X \in \mathfrak{g} \mid \ell([X,Y]) = 0,\ \forall Y \in \mathfrak{l}\}.$$

When $\mathfrak{l}$ is an ideal, $\mathfrak{l}^{B_\ell}$ is a subalgebra of $\mathfrak{g}$. In particular, the Lie subalgebra $\mathfrak{g}^{B_\ell}$ will be denoted by $\mathfrak{g}(\ell)$. It is the Lie algebra of the stabilizer group of ℓ under the coadjoint action of G, but this fact will not be used here.

Let $\mathfrak{m}$ be another subalgebra of $\mathfrak{g}$. We shall also set $\ell' = \ell|_{\mathfrak{g}'}$, $f' = f|_{\mathfrak{g}'}$ and denote by $\mathfrak{m}(\ell)$ and $\mathfrak{m}(\ell')$ the subalgebras $\mathfrak{m} \cap \mathfrak{g}(\ell)$ and $\mathfrak{m} \cap \mathfrak{g}'^{B_\ell}$ of $\mathfrak{m}$.

If $\dim \mathfrak{g}/\mathfrak{h} \geq 1$, $\mathfrak{g}'$ will always denote an ideal of codimension 1 of $\mathfrak{g}$ which contains $\mathfrak{h}$ in the sequel. Also, we shall choose the flag $\mathcal{S}$ so that $\mathfrak{g}_{n-1} = \mathfrak{g}'$. Similarly, if $\dim \mathfrak{h} \geq 1$, $\mathfrak{h}'$ will always denote a subalgebra of codimension 1 in $\mathfrak{h}$. The flag (2.3) will be such that $\mathfrak{h}_{d-1} = \mathfrak{h}'$. If $\mathfrak{g}'$ and $\mathfrak{h}'$ both exists, then $\dim \mathfrak{g} \geq 2$ and $\mathfrak{g} \supsetneq \mathfrak{g}' \supseteq \mathfrak{h} \supsetneq \mathfrak{h}'$.

In what follows, we shall use the following wide-spread convention: if we consider the elements $\{X_r \in \mathfrak{k}_r \mid 1 \leq r \leq p\}$ and $\{Y_s \in \mathfrak{h}_s \mid 1 \leq s \leq d\}$ defined above, then for each p-uple $J = \{j_1, j_2, \dots, j_p\} \in \mathbb{N}^p$, d-uple $K = \{k_1, k_2, \dots, k_d\} \in \mathbb{N}^d$ and $(d-1)$-uple $L = \{l_1, l_2, \dots, l_{d-1}\} \in \mathbb{N}^{d-1}$, we denote respectively by X^J, Y^K and Y'^L the elements $X^J = X_p^{j_p} X_{p-1}^{j_{p-1}} \dots X_1^{j_1}$, $Y^K = Y_d^{k_d} Y_{d-1}^{k_{d-1}} \dots Y_1^{k_1}$ and $Y'^L = Y_{d-1}^{l_{d-1}} \dots Y_1^{l_1}$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. As is customary, we will denote by $|I|$ (resp. $|K|$, resp. $|L|$) the sum $j_1 + j_2 + \dots + j_p$ (resp. $k_1 + k_2 + \dots + k_d$, resp. $l_1 + l_2 + \dots + l_{d-1}$). We shall also consider the elements $\{\widehat{Y}_s \mid \widehat{Y}_s \stackrel{\text{def.}}{=} Y_s + if(Y_s),\ 1 \leq s \leq d\}$ of $\mathcal{U}(\mathfrak{g})$, so that $\widehat{Y}^K = \widehat{Y}_d^{k_d} \widehat{Y}_{d-1}^{k_{d-1}} \dots \widehat{Y}_1^{k_1}$ and $\widehat{Y}'^L = \widehat{Y}_{d-1}^{l_{d-1}} \dots \widehat{Y}_1^{l_1}$.

In this section, we denote by $\mathcal{O}$ the subset of $\Gamma_{\mathfrak{g},\mathfrak{h}}$ formed by the elements ℓ such that $\dim \mathfrak{g}_i(\ell)$ is minimal for $1 \leq i \leq n$, where the $\mathfrak{g}_i$'s are the components of the flag $\mathcal{S}$ of $\mathfrak{g}$ in (2.1). It can easily be verified (see e.g. [7], section 3.3) that $\mathcal{O}$ is a Zariski-open subset of $\Gamma_{\mathfrak{g},\mathfrak{h}}$. Now, it can also be seen ([7], proposition 3.2.2) that if $\mathfrak{g}_i = \mathfrak{g}_{i-1} + \mathfrak{g}_i(\ell)$ for *one* element ℓ of $\mathcal{O}$, the same property holds for *all* elements of this subset. So we define the finite subset of positive integers

$$\begin{aligned} T(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) &\stackrel{\text{def.}}{=} \{1 \leq i \leq n \mid \mathfrak{g}_i = \mathfrak{g}_{i-1} + \mathfrak{g}_i(\ell),\ \forall \ell \in \mathcal{O}\} \\ &= \{m_1 < \dots < m_t\}. \end{aligned}$$

More generally, we define the subset $T_i(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) = T(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) \cap \{0, \dots, i\}$ of $T(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S})$, for any integer $0 \leq i \leq n$.

In a similar way, we can also consider the non-empty Zariski-open subset $\mathcal{O}_H$ of $\Gamma_{\mathfrak{g},\mathfrak{h}}$ formed by the elements ℓ such that $\dim \mathfrak{h}_i(\ell)$ is minimal for $1 \leq i \leq d$, where the $\mathfrak{h}_i$'s are components of the flag (2.3) of $\mathfrak{h}$. As before, it can be verified that if $\mathfrak{h}_i = \mathfrak{h}_{i-1} + \mathfrak{h}_i(\ell)$ for *one* element ℓ of $\mathcal{O}_H$, the same property holds for *all* elements of this subset. So we define the following subsets of $T(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S})$:

$$T^H(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) = \{1 \leq j \leq n \mid j = i_s,\ \mathfrak{h}_s = \mathfrak{h}_{s-1} + \mathfrak{h}_s(\ell),\ \forall \ell \in \mathcal{O}_H\},$$

and $\quad T_i^H(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) = T^H(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) \cap \{0, \dots, i\} \qquad$ for $0 \leq i \leq n$.

Finally, we let $U(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) = T(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) \setminus T^H(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S})$ and for any integer $1 \leq i \leq n$, define the subset $U_i(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) = U(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}) \cap \{1, \dots, i\}$ of $U(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S})$.

In the sequel, we shall respectively write

$T(\mathfrak{g}, \mathfrak{h}, \mathcal{S})$, $T_i(\mathfrak{g}, \mathfrak{h}, \mathcal{S})$, $T^H(\mathfrak{g}, \mathfrak{h}, \mathcal{S})$, $T_i^H(\mathfrak{g}, \mathfrak{h}, \mathcal{S})$, $U(\mathfrak{g}, \mathfrak{h}, \mathcal{S})$ and $U_i(\mathfrak{g}, \mathfrak{h}, \mathcal{S})$

or even simply $T(\mathcal{S})$, $T_i(\mathcal{S})$, $T^H(\mathcal{S})$, $T_i^H(\mathcal{S})$, $U(\mathcal{S})$ and $U_i(\mathcal{S})$

in place of $T(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}), T_i(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}), T^H(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S}), T_i^H(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S})$, $U(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S})$ and $U_i(\mathfrak{g}, \mathfrak{h}, f, \mathcal{S})$.

The algebraic description of $\mathcal{D}(\mathfrak{g},\mathfrak{h})$ given in [5] by means of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ will also be useful. More precisely, let $\mathfrak{a}_\mathfrak{h}$ be the vector subspace of $\mathcal{U}(\mathfrak{g})$ generated by the elements $\widehat{Y} = Y + \sqrt{-1}f(Y)$, $Y \in \mathfrak{h}$, and $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$ the left-ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{a}_\mathfrak{h}$. Denote by $\mathcal{U}(\mathfrak{g},\mathfrak{h})$ the subalgebra of $\mathcal{U}(\mathfrak{g})$ defined by

$$\mathcal{U}(\mathfrak{g},\mathfrak{h}) = \{A \in \mathcal{U}(\mathfrak{g}) \mid [A,Y] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h},\ \forall\, Y \in \mathfrak{h}\}. \tag{2.6}$$

We note L and R the natural extension to the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the left and right actions of $\mathfrak{g}$, defined respectively by

$$L(Y)(\phi)\,(g) = \frac{d}{dt}\phi(e^{-tY}g)|_{t=0}$$

and

$$R(Y)(\phi)\,(g) = \frac{d}{dt}\phi(ge^{tY})|_{t=0},$$

for all Y in $\mathfrak{g}$ and ϕ in $C^\infty(G)$.

It is well known (see e.g. Theorem 4.1 of [5]) that the map $\mathcal{U}(\mathfrak{g},\mathfrak{h}) \to \mathcal{D}(\mathfrak{g},\mathfrak{h})$, $A \mapsto R(A)$ is onto and that its kernel is $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$. In particular, it induces the isomorphism of algebras:

$$\mathcal{U}(\mathfrak{g},\mathfrak{h})/\mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h} \simeq \mathcal{D}(\mathfrak{g},\mathfrak{h}),$$

Since $\mathcal{U}(\mathfrak{g},\mathfrak{h}) \cap \mathcal{U}(\mathfrak{g}') = \mathcal{U}(\mathfrak{g}',\mathfrak{h})$, we see that

$$\begin{aligned} &1)\quad \mathcal{D}(\mathfrak{g}',\mathfrak{h}) = \mathcal{D}(\mathfrak{g},\mathfrak{h}) \iff \mathcal{U}(\mathfrak{g},\mathfrak{h}) \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h} \\ &2)\quad \mathcal{D}(\mathfrak{g}',\mathfrak{h}) \subsetneq \mathcal{D}(\mathfrak{g},\mathfrak{h}) \iff \mathcal{U}(\mathfrak{g},\mathfrak{h}) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}, \end{aligned} \tag{2.7}$$

so that either 1) or 2) holds.

We shall denote by $C\mathcal{D}(\mathfrak{g},\mathfrak{h})$ the center of $\mathcal{D}(\mathfrak{g},\mathfrak{h})$ and set

$$\mathcal{U}_C(\mathfrak{g},\mathfrak{h}) = R^{-1}(C\mathcal{D}(\mathfrak{g},\mathfrak{h})) = \{A \in \mathcal{U}(\mathfrak{g},\mathfrak{h}) \mid [A,\mathcal{U}(\mathfrak{g},\mathfrak{h})] \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}\}.$$

We shall associate to $\mathfrak{g}', \mathfrak{k}_r$ and $\mathfrak{h}'$ the same objects as we did with $\mathfrak{g}$ and $\mathfrak{h}$. For instance, we shall consider the vector subspace $\mathfrak{a}_{\mathfrak{h}'}$ of $\mathcal{U}(\mathfrak{g})$ generated by the elements $Y + \sqrt{-1}f(Y)$, where $Y \in \mathfrak{h}'$. Accordingly, as in (2.6), we shall use the following notations:

$$\begin{aligned} \mathcal{U}(\mathfrak{g},\mathfrak{h}') = \mathcal{U}(\mathfrak{g},\mathfrak{h}',f) \quad &= \{A \in \mathcal{U}(\mathfrak{g}) \mid [A,Y] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'},\ \forall\, Y \in \mathfrak{h}'\}, \\ \mathcal{U}(\mathfrak{k}_r,\mathfrak{h}) = \mathcal{U}(\mathfrak{k}_r,\mathfrak{h},f|_{\mathfrak{k}_r}) &= \{A \in \mathcal{U}(\mathfrak{k}_r) \mid [A,Y] \in \mathcal{U}(\mathfrak{k}_r)\mathfrak{a}_\mathfrak{h},\ \forall\, Y \in \mathfrak{h}\}. \end{aligned}$$

Let $\ell \in \mathfrak{g}^*$. Recall that the H-orbit $H \cdot \ell$ of ℓ in $\mathfrak{g}^*$ is said to be *saturated relatively to* $\mathfrak{g}'$ if and only if $H \cdot \ell + \mathfrak{g}'^{\perp} = H \cdot \ell$. The following simple equivalences hold for any $\ell \in \mathfrak{g}^*$ (see e.g. [7], proposition 6.4.1):

$$(2.8)\qquad \begin{aligned} 1)\quad \mathfrak{h}(\ell)\neq\mathfrak{h}(\ell') &\iff \dim\mathfrak{h}(\ell)=\dim\mathfrak{h}(\ell')-1\\ &\iff H\cdot\ell \text{ is saturated relatively to } \mathfrak{g}'^{\perp}\\ &\iff \dim H\cdot\ell\neq\dim H\cdot\ell'\\ &\iff \dim H\cdot\ell=\dim H\cdot\ell'+1\\ &\iff \mathfrak{h}^{B_\ell}\subset\mathfrak{g}'.\\ 2)\quad \mathfrak{h}(\ell)=\mathfrak{h}(\ell') &\iff H\cdot\ell \text{ is not saturated relatively to } \mathfrak{g}'^{\perp}\\ &\iff \dim H\cdot\ell=\dim H\cdot\ell'\\ &\iff \mathfrak{g}=\mathfrak{g}'+\mathfrak{h}^{B_\ell}\end{aligned}$$

so that either the equivalent properties of 1) or those of 2) are true. We shall prove in our main result Theorem 5.2, that the first (resp. the second) properties of (2.7) are equivalent to the first (resp. the second) properties of (2.8). However, the notion of saturation of H-orbits which is useful for the construction of H-invariant polynomial functions and, through the operation of symmetrization, for the construction of invariant differential operators, will only be used here in the last section for the study of an example and will not intervene in the proof of our main results.

3. Some properties of $\mathcal{D}(\mathfrak{g},\mathfrak{h})$

A simple consequence of the Poincaré-Birkhoff-Witt Theorem is that the families

$$\{X^J\widehat{Y}^K \mid (J,K)\in\mathbb{N}^p\times\mathbb{N}^d\}$$

and

$$\{X^J\widehat{Y}^K \mid (J,K)\in\mathbb{N}^p\times\mathbb{N}^d,\ |K|>0\}$$

respectively form a basis of $\mathcal{U}(\mathfrak{g})$ and of $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ (see Lemma 4.2 of [5]). Observe that the elements $\{X^J \mid J\in\mathbb{N}^p\}$ form a basis of a supplementary subspace S of $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ in $\mathcal{U}(\mathfrak{g})$.

Till the end of the section, we assume $n\geq d\geq 1$ so that $\mathfrak{h}'$ and Y_d exist. Let $Y^K=\widehat{Y}_d^k\widehat{Y}'^L$. Keeping the notations of Section 2, we see that the families

$$\{X^J\widehat{Y}_d^k\widehat{Y}'^L \mid (J,k,L)\in\mathbb{N}^p\times\mathbb{N}\times\mathbb{N}^{d-1},\ |L|>0\}$$

and

$$\{X^J\widehat{Y}_d^k \mid (J,k)\in\mathbb{N}^p\times\mathbb{N}\}$$

respectively form a basis of $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$ and of a supplementary subspace of $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$ in $\mathcal{U}(\mathfrak{g})$.

We now recall several properties of the algebra $\mathcal{D}(\mathfrak{g},\mathfrak{h})$ which are due to Baklouti and Fujiwara (see [1]). For the convenience of the reader, and in order to make this paper as self-contained as possible, we shall also give their proofs.

Lemma 3.1. *(i)* $\mathfrak{g}_{i_d-1}S\subset S\oplus\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$.
(ii) $[\mathfrak{h},S]\subset S\oplus\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$.

Proof. *(i)* The idea of the proof is very simple, though complicated to write down. For every $0\leq q\leq p$ and $k\in\mathbb{N}$, let us denote by $S_{q,k}$ the subspace of S generated by the $X^J = X_q^{i_q} \dots X_1^{i_1}$, $J \in \mathbb{N}^q$ and $|J| \le k$. In particular, $S_{0,k} = S_{q,0} = \mathbb{C}$. The index q is useful in the proof to check that certain monomials are well ordered. It is enough to prove by induction on k that

($\star$) $Y'X^J \in S_{q,k+1} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$, $\forall Y' \in \mathfrak{h}'$, $\forall X^J \in S_{q,k}$,

($\star\star$) $X_r X^J \in S_{\sup(r,q),k+1} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$, $\forall r$ such that $j_r \le i_d - 1$, $\forall X^J \in S_{q,k}$.

This is clear for $k = 0$. Let $k > 0$. Suppose the result has been proved up to rank $k-1$. If we take X_r as above, with $r \ge q$, we have $X_r X^J \in S_{r,k+1}$ so that the result is obvious. Now, if we take $T \in \mathfrak{g}_{i_d - 1}$ such that either $T = Y' \in \mathfrak{h}'$ or $T = X_r$ with $r < q$, then we can write X^J as $X^J = X_q X^{J'}$ with $X^{J'} = X_q^{j_q - 1} \dots X_1^{j_1}$ and use the identity:

$$TX^J = [T, X_q]X^{J'} + X_q T X^{J'}.$$

We have $[T, X_q] \in \mathfrak{g}_{j_q - 1} \cap \mathfrak{g}_{i_d - 1}$ and $X^{J'} \in S_{q,k-1}$. Hence, by induction

$$[T, X_q]X^{J'} \in S_{q,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}.$$

Similarly, $TX^{J'} \in S_{q,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. Thus $X_q T X^{J'} \in S_{q,k+1} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$.

(*ii*) We show by induction on k that

$$[Y, X^J] \in S_{q,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}, \ \forall Y \in \mathfrak{h}, \ \forall X^J \in S_{q,k}.$$

This is clear for $k = 0$. Assume this true at rank $k-1$, $k \ne 0$. Then we have

$$[Y, X^J] = [Y, X_q]X^{J'} + X_q[Y, X^{J'}].$$

Since $[Y, X_q] \in \mathfrak{g}_{j_q - 1} \cap \mathfrak{g}_{i_d - 1}$ and $X^{J'} \in S_{q,k-1}$, we know from the assertion (*i*) that $[Y, X_q]X^{J'} \in S_{q,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. Finally, by induction we have $[Y, X^{J'}] \in S_{q,k-1} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. Hence, $X_q[Y, X^{J'}] \in S_{q,k} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. □

Lemma 3.2. *Let W be an element of $\mathcal{U}(\mathfrak{g})$ written as*

$$W \equiv \sum_{k \le k'} A_k \widehat{Y}_d^k \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}, \quad \text{where the } A_k\text{'s belong to } S.$$

Let $Y \in \mathfrak{h}$ and, making use of the previous lemma, let $(B_k)_{k \le k'}$ be the elements of S such that $[Y, A_k] \equiv B_k \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. Then, we have

$$\begin{aligned} [Y, W] &\equiv [Y, \sum_{k \le k'} A_k \widehat{Y}_d^k] && \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'} \\ &\equiv \sum_{k \le k'} [Y, A_k]\widehat{Y}_d^k && \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'} \\ &\equiv \sum_{k \le k'} B_k \widehat{Y}_d^k && \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}. \end{aligned}$$

Proof. The first and third identities follow from the inclusion $\mathfrak{h} \subset \mathcal{U}(\mathfrak{g}, \mathfrak{h}')$ and the second one, from the relations $[Y, A_k \widehat{Y}_d^k] = [Y, A_k]\widehat{Y}_d^k + A_k[Y, \widehat{Y}_d^k]$ and $[Y, \widehat{Y}_d] \subset \mathfrak{h}' \cap \ker f \in \mathfrak{a}_{\mathfrak{h}'}$. □

Proposition 3.3. *(i) We have* $[\mathfrak{h}, \mathcal{U}(\mathfrak{g},\mathfrak{h}) \cap S] \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. *In particular*

$$\mathcal{U}(\mathfrak{g},\mathfrak{h}) \cap S \subset \mathcal{U}(\mathfrak{g},\mathfrak{h}').$$

(ii) We have the decomposition $\mathcal{U}(\mathfrak{g},\mathfrak{h}) = (\mathcal{U}(\mathfrak{g},\mathfrak{h}) \cap S) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, *so that the restriction to* $\mathcal{U}(\mathfrak{g},\mathfrak{h}) \cap S$ *of the projection of* $\mathcal{U}(\mathfrak{g},\mathfrak{h})$ *on* $\mathcal{D}(\mathfrak{g},\mathfrak{h})$ *is a bijection and every element of*

$$\mathcal{D}(\mathfrak{g},\mathfrak{h}) \simeq \mathcal{U}(\mathfrak{g},\mathfrak{h})/\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$$

admits a unique representative in S. *This representative belongs to* $\mathcal{U}(\mathfrak{g},\mathfrak{h}')$.

Proof. *(i)* The assertion *(ii)* of Lemma 3.1 yields

$$[\mathfrak{h}, \mathcal{U}(\mathfrak{g},\mathfrak{h}) \cap S] \subset (S \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}) \cap \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}} \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}.$$

(ii) The result follows from the decomposition $\mathcal{U}(\mathfrak{g}) = S \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ and from the inclusion $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}} \subset \mathcal{U}(\mathfrak{g},\mathfrak{h})$. □

Proposition 3.4. *Let* $(A_k)_{0\le k\le k'}$ *be a family of elements of* S, *we have*

$$\sum_k A_k \widehat{Y}_d^k \in \mathcal{U}(\mathfrak{g},\mathfrak{h}') \iff A_k \in \mathcal{U}(\mathfrak{g},\mathfrak{h}'),\ \forall k.$$

Proof. let $Y' \in \mathfrak{h}'$. For all k, we define the element B_k of S such that $[Y', A_k] \equiv B_k \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$ and obtain from Lemma 3.2

$$[Y', \sum_k A_k \widehat{Y}_d^k] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'} \iff B_k = 0, \quad \forall k.$$

□

Proposition 3.5. *Suppose that* $\dim \mathfrak{g} \ge 3$ *and* $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{h}$ *and* $\mathfrak{h}'$ *are like in Section 2 then*:

(i) We have the equivalence

$$\mathcal{U}(\mathfrak{g},\mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'} \iff \mathcal{U}(\mathfrak{g},\mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}.$$

(ii) Assuming moreover that $\mathcal{U}(\mathfrak{g},\mathfrak{h}') \subset \mathcal{U}(\mathfrak{g},\mathfrak{h})$, *then we have*

$$\mathcal{U}(\mathfrak{g},\mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'} \iff \mathcal{U}(\mathfrak{g},\mathfrak{h}) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}.$$

Proof. *(i)* $\Leftarrow$ is obvious. For $\Rightarrow$, let W' be an element of $\mathcal{U}(\mathfrak{g},\mathfrak{h}') \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'})$. We may write W' as $W' \equiv \sum_{k\le k'} A_k \widehat{Y}_d^k \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$, where the A_k's belong to S. As W' is not in $\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$, one of the A_k's, say A_{k_0}, is not in $\mathcal{U}(\mathfrak{g}')$. In other words, X_p does occur in A_{k_0}. Then, Proposition 3.4 implies that $A_{k_0} \in \mathcal{U}(\mathfrak{g},\mathfrak{h}')$. As $(\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}) \cap S = \mathcal{U}(\mathfrak{g}') \cap S$ and $A_{k_0} \in S \setminus \mathcal{U}(\mathfrak{g}')$, A_{k_0} does not belong to $\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. This proves $\Rightarrow$.

(ii) The implication $\Rightarrow$ is a direct consequence of *(i)* and of the assumption. For $\Leftarrow$, the decompositions

$$\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}} = (\mathcal{U}(\mathfrak{g}') \cap S) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$$

and

$$\mathcal{U}(\mathfrak{g},\mathfrak{h}) = (\mathcal{U}(\mathfrak{g},\mathfrak{h}) \cap S) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$$

along with the right-hand side property imply that $\mathcal{U}(\mathfrak{g},\mathfrak{h})\cap S \not\subset \mathcal{U}(\mathfrak{g}')\cap S = (\mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}})\cap S$. So the result follows from the inclusion $\mathcal{U}(\mathfrak{g},\mathfrak{h})\cap S \subset \mathcal{U}(\mathfrak{g},\mathfrak{h}')$ of Proposition 3.3. □

Proposition 3.6. *Assume* $\mathcal{U}(\mathfrak{g},\mathfrak{h}')\cap S \subset \mathcal{U}(\mathfrak{g},\mathfrak{h})\cap S$. *Then*

$$[\mathfrak{h},\mathcal{U}(\mathfrak{g},\mathfrak{h}')] \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}.$$

(*In other words,* $\mathfrak{h}\subset \mathcal{U}_C(\mathfrak{g},\mathfrak{h}')$.)

Proof. Let us write an element W of $\mathcal{U}(\mathfrak{g},\mathfrak{h}')$ under the form

$$W \equiv \sum_{k\le k'} A_k \widehat{Y}_d^k \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'},$$

where the A_k's belong to S. Now, Proposition 3.4 implies that $A_k \in \mathcal{U}(\mathfrak{g},\mathfrak{h}')\cap S$ so that, using the assumption, $A_k\in\mathcal{U}(\mathfrak{g},\mathfrak{h})\cap S$. Finally, using the first assertion of Proposition 3.3, we have

$$[\mathfrak{h},W] \subset \sum_{k\le k'} \left([\mathfrak{h},A_k]\,\widehat{Y}_d^k + A_k[\mathfrak{h},\widehat{Y}_d^k]\right) \subset \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}.$$

□

The next result ([1], lemme 4.1) is independent of the previous ones and of a different nature:

Proposition 3.7. *Suppose that* $\dim\mathfrak{g}\ge 2$ *and* $\mathfrak{g}$, $\mathfrak{g}'$ *and* $\mathfrak{h}$ *are like in Section 2, so that* X_p *exists.*

(*i*) *Let* $m\ge 1$ *and* $W=\sum_{k=0}^{m} X_p^k A_k$, *where each* A_k *belongs to* $\mathcal{U}(\mathfrak{g}')$, *be an element of* $\mathcal{U}(\mathfrak{g},\mathfrak{h})$. *Then* $A_m\in\mathcal{U}(\mathfrak{g}',\mathfrak{h})$ *and* $mX_pA_m + A_{m-1}\in \mathcal{U}(\mathfrak{g},\mathfrak{h})$.

(*ii*) *Let* $W = X_pU+V$ *with* $U,V\in\mathcal{U}(\mathfrak{g}')$. *If* $U\notin \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$, *then* $W\notin \mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

(*iii*) *Suppose* $\mathcal{U}(\mathfrak{g},\mathfrak{h})\not\subset \mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, *then there exists an element* $W = X_pU+V$ *of* $\mathcal{U}(\mathfrak{g},\mathfrak{h})$ *such that* $U\in\mathcal{U}(\mathfrak{g}',\mathfrak{h})\setminus\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$ *and* $V\in\mathcal{U}(\mathfrak{g}')$. *In particular,* $W\notin \mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

Proof. (*i*) The Poincaré-Birkhoff-Witt Theorem clearly implies that

$$\mathcal{U}(\mathfrak{g}) = \bigoplus_j X_p^j\,\mathcal{U}(\mathfrak{g}') \quad\text{and}\quad \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}} = \bigoplus_j X_p^j\,\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}. \tag{3.1}$$

We set $\mathfrak{i}_{m-2} = \bigoplus_{j=0}^{m-2} X_p^j\,\mathcal{U}(\mathfrak{g}')$. Then we have for all $Y\in\mathfrak{h}$:

$$\begin{aligned}[W,Y] &\equiv X_p^m[A_m,Y] + \sum_{j=1}^{m} X_p^{j-1}[X_p,Y]X_p^{m-j}A_m + X_p^{m-1}[A_{m-1},Y] \\ &\qquad\qquad \mod \mathfrak{i}_{m-2} \\ &\equiv X_p^m[A_m,Y] + X_p^{m-1}(m[X_p,Y]A_m + [A_{m-1},Y]) \\ &\qquad\qquad \mod \mathfrak{i}_{m-2}\in\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}\end{aligned}$$

so that $A_m\in\mathcal{U}(\mathfrak{g}',\mathfrak{h})$ and $mX_pA_m + A_{m-1}\in\mathcal{U}(\mathfrak{g},\mathfrak{h})$.

(ii) Using (3.1), we see that if $W \in \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ then $U \in \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$ because in this case,

$$W \in (X_p\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')) \cap (\bigoplus_j X_p^j \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}} \oplus \mathcal{U}(\mathfrak{g}')) = X_p\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}} \oplus \mathcal{U}(\mathfrak{g}').$$

(iii) Let $W' = \sum_{k=0}^{m} X_p^k A_k \in \mathcal{U}(\mathfrak{g},\mathfrak{h}) \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}})$. The assertion (i) implies that $A_m \in \mathcal{U}(\mathfrak{g}',\mathfrak{h})$ and that $W = mX_pA_m + A_{m-1} \in \mathcal{U}(\mathfrak{g},\mathfrak{h})$. Without loss of generality, we can assume that $A_m \notin \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$ so that using (ii), we see that W has the required property. □

The following important statement will be needed to prove our main result Theorem 5.2, (see [1] théorème 4.4 for its original proof).

Theorem 3.8. *Under the notations and the assumptions of Proposition 3.7, suppose that*

$$\mathcal{U}(\mathfrak{g},\mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'} \text{ and } \mathcal{U}(\mathfrak{g}',\mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}',\mathfrak{h}).$$

Then we have $\mathcal{U}(\mathfrak{g},\mathfrak{h}) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

Proof. First, let us prove that there exists an element $W = X_pU + V \in \mathcal{U}(\mathfrak{g},\mathfrak{h}')$ with $U, V \in \mathcal{U}(\mathfrak{g}')$ such that

(a) $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ or equivalently such that $U \notin \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$.
(b) $(\operatorname{ad} Y_d)W \in \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

It will be used below in various situations to construct an element of $\mathcal{U}(\mathfrak{g},\mathfrak{h})$ that does not belong to $\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

We know from the assertion (i) of Proposition 3.5 that actually $\mathcal{U}(\mathfrak{g},\mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Therefore, we can find $U_s \in \mathcal{U}(\mathfrak{g}')$, $0 \le s \le r$ with $r > 0$ and $U_r \notin \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$ so that $\sum_{0\le s\le r} X_p^s U_s \in \mathcal{U}(\mathfrak{g},\mathfrak{h}')$. Now, the first assertion of Proposition 3.7, applied at the level of $\mathfrak{g}$ and $\mathfrak{h}'$, says that the element $rX_pU_r + U_{r-1}$ belongs to $\mathcal{U}(\mathfrak{g},\mathfrak{h}')$ whereas the second assertion of the same proposition says that it satisfies (a) above. Next, let $m' \in \mathbb{N}$ be the greatest integer such that

$$W = (\operatorname{ad} Y_d)^{m'}(rX_pU_r + U_{r-1}) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}.$$

We see that W does satisfy (a) and (b).

Now we introduce a few notations that will only be useful in the present proof. For $S, T \in \mathcal{U}(\mathfrak{g})$, we set $\{S, T\} = ST + TS$. Also, for $s \in \mathbb{N}$, we write $S_s = (\operatorname{ad} Y_d)^s S$ so that $S = S_0$. Then for $r \in \mathbb{N} \setminus \{0\}$, we define

$$\mathcal{T}_r(S) = \{S_0, S_{2r}\} - \{S_1, S_{2r-1}\} + \cdots + (-1)^{r-1}\{S_{r-1}, S_{r+1}\} + (-1)^r S_r^2.$$

Since $Y_d \in \mathcal{U}(\mathfrak{g},\mathfrak{h}')$, we see that $\mathcal{T}_r(S)$ belongs to $\mathcal{U}(\mathfrak{g},\mathfrak{h}')$ whenever S does. Moreover, if r is large enough to satisfy $S_{2r+1} \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ then $(\operatorname{ad} Y_d)\mathcal{T}_r(S) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ so that $\mathcal{T}_r(S) \in \mathcal{U}(\mathfrak{g},\mathfrak{h})$.

We let m be the smallest integer such that $(\operatorname{ad} Y_d)^m W \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. We shall now consider different cases depending on the value of m:

- If $m = 1$, the result is obvious since $W \in \mathcal{U}(\mathfrak{g}, \mathfrak{h}) \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}})$.

- If $m = 2q + 1$ with $q \geq 1$, the remarks just above imply that

$$\mathcal{T}_q(W) = \{W_0, W_{2q}\} - \{W_1, W_{2q-1}\} + \cdots + (-1)^{q-1}\{W_{q-1}, W_{q+1}\} + (-1)^q W_q^2$$

belongs to $\mathcal{U}(\mathfrak{g}, \mathfrak{h})$. We want to prove that $\mathcal{T}_q(W) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. We have $\mathcal{T}_q(W) \equiv 2WW_{2q} \equiv 2X_pUW_{2q} \mod \mathcal{U}(\mathfrak{g}')$. So we get our result, using the facts that U and W_{2q} do not belong to $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$ and that the ring $\mathcal{U}(\mathfrak{g}', \mathfrak{h})/\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$ has no zero divisors.

- If $m = 2q$ with $q > 1$, then we see that for any $c \in \mathbb{C}$,

$$(\operatorname{ad} Y_d)^{2q+1}(W(W_{2q-2} + cW_{2q-1})) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$$

so that if we set $\widetilde{W}(c) \stackrel{\text{def.}}{=} W(W_{2q-2} + cW_{2q-1})$, we have $\mathcal{T}_q(\widetilde{W}(c)) \in \mathcal{U}(\mathfrak{g}, \mathfrak{h})$ by the remark above.

We want to prove that $\mathcal{T}_q(\widetilde{W}(c)) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ for some $c \in \mathbb{C} \setminus \{0\}$. For some time, we consider $\widetilde{W}(c)$ and $\mathcal{T}_q(\widetilde{W}(c))$ as elements of $\mathcal{U}(\mathfrak{g})[c]$, the algebra of polynomials of c with coefficients in $\mathcal{U}(\mathfrak{g})$. Noting that $\mathfrak{a}_{\mathfrak{h}} \subset \mathcal{U}_C(\mathfrak{g}, \mathfrak{h}')$, we have modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}[c]$ for any $q \geq 1$:

$$\begin{array}{llcll}
\widetilde{W}(c)_0 & = W(W_{2q-2} + cW_{2q-1}) & & & \\
\widetilde{W}(c)_1 & \equiv W_1(W_{2q-2} + cW_{2q-1}) & + & WW_{2q-1} & \\
\widetilde{W}(c)_2 & \equiv W_2(W_{2q-2} + cW_{2q-1}) & + & 2W_1W_{2q-1} & \in \mathcal{U}(\mathfrak{g}')[c] \\
\vdots & & & & \\
\widetilde{W}(c)_{2q-1} & \equiv W_{2q-1}(W_{2q-2} + cW_{2q-1}) & + & (2q-1)W_{2q-2}W_{2q-1} & \in \mathcal{U}(\mathfrak{g}')[c] \\
\widetilde{W}(c)_{2q} & \equiv & & 2qW_{2q-1}^2 & \in \mathcal{U}(\mathfrak{g}')
\end{array}$$

Now we check that

$$\mathcal{T}_q(\widetilde{W}(c)) \in cX_p\mathcal{U}(\mathfrak{g}') \oplus X_p\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')[c] \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}[c]$$

and work out the component relative to $cX_p\mathcal{U}(\mathfrak{g}')$. Using the fact that $q > 1$, we obtain

$$\begin{aligned}
\mathcal{T}_q(\widetilde{W}(c)) &\equiv \{\widetilde{W}(c)_0, \widetilde{W}(c)_{2q}\} - \{\widetilde{W}(c)_1, \widetilde{W}(c)_{2q-1}\} \\
&\qquad\qquad \mod \mathcal{U}(\mathfrak{g}')[c] \oplus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}[c] \\
&\equiv 2\widetilde{W}(c)_0\widetilde{W}(c)_{2q} - 2\widetilde{W}(c)_1\widetilde{W}(c)_{2q-1} \\
&\qquad\qquad \mod \mathcal{U}(\mathfrak{g}')[c] + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}[c] \\
&\equiv 2c(2q-1)\, WW_{2q-1}^3 \\
&\qquad\qquad \mod X_p\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')[c] + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}[c] \\
&\equiv 2c(2q-1)\, X_pUW_{2q-1}^3 \\
&\qquad\qquad \mod X_p\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}')[c] + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}[c]
\end{aligned}$$

Considering now $\widetilde{W}(c)$ as an element of $\mathcal{U}(\mathfrak{g})$, we infer from this that there exists $\widetilde{U} \in \mathcal{U}(\mathfrak{g}')$ such that $\widetilde{W}(c) \equiv X_p\left(2c(2q-1)\, UW_{2q-1}^3 + \widetilde{U}\right) \mod \mathcal{U}(\mathfrak{g}') +$

$\mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$. Then, noting that

$$X_p\mathcal{U}(\mathfrak{g}') \cap [\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}] = X_p\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\mathfrak{h},$$

that the ring $\mathcal{U}(\mathfrak{g}', \mathfrak{h})/\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\mathfrak{h}$ has no zero divisors and that UW_{2q-1}^3 does not belong to $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\mathfrak{h}$ since U and W_{2q-1} do not, we see that $\widetilde{W}(c) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$ for any c such that

$$2c(2q-1)\, UW_{2q-1}^3 + \widetilde{U} \notin \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\mathfrak{h}.$$

Note that this method yields no results for $m = 2$.

- If $m = 2$ then, for the first time in this proof, we have to use the relation $\mathcal{U}(\mathfrak{g}', \mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}', \mathfrak{h})$. It implies easily that there exists $T \in \mathcal{U}(\mathfrak{g}', \mathfrak{h}') \setminus \mathcal{U}(\mathfrak{g}', \mathfrak{h})$ such that $(\operatorname{ad} Y_d)^2 T \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$. Then, we see that $(\operatorname{ad} Y_d)^3(WT) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$, that $WT \in \mathcal{U}(\mathfrak{g}, \mathfrak{h}')$ and that modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$

$$\begin{aligned} \mathcal{T}_1(WT) &= \{(WT)_0, (WT)_2\} - (WT)_1^2 \\ &\equiv \{WT, 2W_1T_1\} - (W_1T + WT_1)^2 \quad \in \mathcal{U}(\mathfrak{g}, \mathfrak{h}). \end{aligned}$$

We want to prove that $\mathcal{T}_1(WT) \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$. We have

$$\mathcal{T}_1(WT) \equiv -W^2T_1^2 \equiv -X_p^2\,(UT_1)^2 \mod X_p\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}.$$

We note that $X_p^2\mathcal{U}(\mathfrak{g}') \cap [X_p\mathcal{U}(\mathfrak{g}') \oplus \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}] = X_p^2\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\mathfrak{h}$. Then, the fact that U and T_1 do not belong to $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_\mathfrak{h}$ implies the result in this case. □

4. Two fundamental lemmas

Here, we state and prove two lemmas that will be crucial for the proof of Theorem 5.2. We are concerned with the study of certain special elements of $\mathcal{U}(\mathfrak{g})$ that we call $\Gamma_{\mathfrak{g},\mathfrak{h}}$-*central*.

Let ℓ be a linear form on $\mathfrak{g}$. Recall that π_ℓ denote the unitary irreducible representation of G in the Hilbert space $\mathcal{H}_\ell$ associated, via the orbit method, with the coadjoint orbit $G \cdot \ell$ of ℓ. We shall keep the same symbol π_ℓ to denote the associate representation of $\mathcal{U}(\mathfrak{g})$ in the space $\mathcal{H}_\ell^\infty$ of C^∞-vectors of $\mathcal{H}_\ell$. An element $A \in \mathcal{U}(\mathfrak{g})$ is then said to be $\Gamma_{\mathfrak{g},\mathfrak{h}}$-central if $\pi_\ell(A)$ is scalar for all ℓ in a non-empty Zariski-open subset $\mathcal{O}$ of $\Gamma_{\mathfrak{g},\mathfrak{h}}$, namely if there exists a complex function θ_A on $G \cdot \mathcal{O}$ such that (Section 1.1 of [7])

$$\pi_\ell(A) = \theta_A(\ell) Id_{\mathcal{H}_\ell} \qquad \text{for } \ell \in G \cdot \mathcal{O}. \tag{4.1}$$

It turns out that in this case, $\pi_\ell(A)$ is scalar for *all* ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$ and therefore that we can assume that θ_A is defined on the whole of $G \cdot \Gamma_{\mathfrak{g},\mathfrak{h}}$. Moreover, this function is G-invariant and its restriction to $\Gamma_{\mathfrak{g},\mathfrak{h}}$ is polynomial and H-invariant ([7], théorème 2.1.1).

As in [7] Définition 3.4.1, we consider for every integer $1 \leq k \leq t$, a $\Gamma_{\mathfrak{g},\mathfrak{h}}$-central element σ_k satisfying

(1) $\sigma_k = \xi_k X_{m_k} + R_k$ where ξ_k and R_k belong to $\mathcal{U}(\mathfrak{g}_{m_k-1})$.

(2) ξ_k is $\Gamma_{\mathfrak{g},\mathfrak{h}}$-central.

(3) The function $\ell \mapsto \theta_{\xi_k}(\ell)$ does not vanish on a non-empty Zariski-open subset of $\Gamma_{\mathfrak{g},\mathfrak{h}}$.

Using results of Corwin and Greenleaf ([5], Theorem 3.1), it can be seen that there exists such a σ_k. Indeed, they construct elements which verify more properties than ours.

To establish our lemmas, we recall some elementary properties of $\Gamma_{\mathfrak{g},\mathfrak{h}}$-central elements in $\mathcal{U}(\mathfrak{g})$ (see [7] Proposition 1.4.1 and also Lemma 5.1 of [5]). Let $\mathcal{U}(\mathfrak{g},\Gamma_{\mathfrak{g},\mathfrak{h}})$ be the algebra of $\Gamma_{\mathfrak{g},\mathfrak{h}}$-central elements in $\mathcal{U}(\mathfrak{g})$ and $Z(\mathfrak{g},\mathfrak{h}) = \{\theta_A \mid A \in \mathcal{U}(\mathfrak{g},\Gamma_{\mathfrak{g},\mathfrak{h}})\}$ the algebra of complex functions satisfying (4.1). With the general notations of Section 2, one has the following maps:

$^t : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}), \quad A \mapsto {}^tA$, the main linear antiautomorphism of $\mathcal{U}(\mathfrak{g})$. (In particular, we have ${}^tX = -X$ and ${}^t(XY) = YX$ for $X, Y \in \mathfrak{g}^{\mathbb{C}}$).

$\alpha : \mathcal{U}(\mathfrak{g},\Gamma_{\mathfrak{g},\mathfrak{h}}) \to Z(\mathfrak{g},\mathfrak{h}), \quad A \mapsto \theta_A.$

$\varpi : \mathcal{U}_C(\mathfrak{g},\mathfrak{h}) \to C\mathcal{D}(\mathfrak{g},\mathfrak{h}), \quad A \mapsto R(A).$

The antiautomorphism ^{t} of $\mathcal{U}(\mathfrak{g})$ is one-to-one and maps $\mathcal{U}(\mathfrak{g},\Gamma_{\mathfrak{g},\mathfrak{h}})$ into $\mathcal{U}_C(\mathfrak{g},\mathfrak{h})$, while α and ϖ are onto. It can be shown that there exists a map $\delta : Z(\mathfrak{g},\mathfrak{h}) \to C\mathcal{D}(\mathfrak{g},\mathfrak{h})$ such that $\delta(\theta_A) = L(A) = R({}^tA)$, which is an injection of the commutative algebra $Z(\mathfrak{g},\mathfrak{h})$ into $C\mathcal{D}(\mathfrak{g},\mathfrak{h})$, so that the commutative diagram 1 below holds. (For these results, see [7] Proposition 1.4.1, and also Lemma 5.1 of [5]).

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g},\Gamma_{\mathfrak{g},\mathfrak{h}}) & \xrightarrow{\;t\;} & \mathcal{U}_C(\mathfrak{g},\mathfrak{h}) \\ \alpha \downarrow & & \downarrow \varpi \\ Z(\mathfrak{g},\mathfrak{h}) & \xrightarrow{\;\delta\;} & C\mathcal{D}(\mathfrak{g},\mathfrak{h}) \end{array}$$

Diagram 1

With respect to the flag (2.1) of $\mathfrak{g}$, we can also define the algebras

$$Z_i(\mathfrak{g},\mathfrak{h}) \overset{\text{def.}}{=} \{\theta_A \mid A \in \mathcal{U}(\mathfrak{g}_i) \cap \mathcal{U}(\mathfrak{g},\Gamma_{\mathfrak{g},\mathfrak{h}})\}, \quad 0 \le i \le n.$$

So that $\qquad \{0\} = Z_0(\mathfrak{g},\mathfrak{h}) \subseteq Z_1(\mathfrak{g},\mathfrak{h}) \subseteq \cdots \subseteq Z_n(\mathfrak{g},\mathfrak{h}) = Z(\mathfrak{g},\mathfrak{h}).$

The main results of [7] can be stated as follows:

($\star$) For all integer $0 \le i \le n$, the family $\{\theta_{\sigma_j} \mid m_j \in T_i(\mathcal{S})\}$ is a system of rational generators of $Z_i(\mathfrak{g},\mathfrak{h})$ (théorème 4.1.1 of [7]).

($\star\star$) For all integer $0 \le i \le n$, the family $\{\theta_{\sigma_j} \mid m_j \in U_i(\mathcal{S})\}$ is a transcendence basis of the algebra $Z_i(\mathfrak{g},\mathfrak{h})$ (théorème 4.1.2 of [7]).

We are now ready to prove our lemmas.

Lemma 4.1. *Assume that* $\dim \mathfrak{h} \geq 1$. *Let* $i_s \in T^H(\mathcal{S})$. *In particular, we have* $\mathfrak{h}_s = \mathfrak{h}_{s-1} + \mathfrak{h}_s(\ell)$ *for generic* ℓ *in* $\Gamma_{\mathfrak{g},\mathfrak{h}}$ *and there exists* k $(1 \leq k \leq t)$, *such that* $m_k = i_s$. *Then the following assertions hold*:

(*i*) *There exists a polynomial* P *satisfying*

$$P({}^t\sigma_1, \ldots, {}^t\sigma_k) \equiv 0 \mod \mathcal{U}(\mathfrak{g}_{m_k})\mathfrak{a}_{\mathfrak{h}_s} \tag{4.2}$$

such that the coefficient of the dominant power of ${}^t\sigma_k$ *is not zero modulo* $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

(*ii*) *There exists a polynomial* Q *satisfying*

$$Q({}^t\sigma_1, \ldots, {}^t\sigma_k, Y_s) \equiv 0 \mod \mathcal{U}(\mathfrak{g}_{m_k-1})\mathfrak{a}_{\mathfrak{h}_{s-1}}$$

such that the coefficient of the dominant power of Y_s *is not zero modulo* $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

Proof. (*i*) Since $m_k \in T^H(\mathcal{S})$, m_k does not belong to $U(\mathcal{S})$. So from $(\star\star)$ above, the family $\{\theta_{\sigma_j} \mid j \in U_{m_k-1}(\mathcal{S})\}$ is a transcendence basis for $Z_{m_k}(\mathfrak{g}, \mathfrak{h})$. In particular, the element θ_{σ_k} of $Z_{m_k}(\mathfrak{g}, \mathfrak{h})$ is algebraic over the ring generated by this family and, a fortiori, by the family $\{\theta_{\sigma_j} \mid j \in T_{m_k-1}(\mathcal{S})\}$. In other words, there exists a polynomial P of k variables such that

$$P(\theta_{\sigma_1}, \ldots, \theta_{\sigma_k}) = \sum_{j=0}^{m} P_j(\theta_{\sigma_1}, \ldots, \theta_{\sigma_{k-1}})\, \theta_{\sigma_k}^j = 0$$

with $P_m(\theta_{\sigma_1}, \ldots, \theta_{\sigma_{k-1}}) \neq 0$. We deduce, from the commutativity of Diagram 1, that

$$\varpi(P({}^t\sigma_1, \ldots, {}^t\sigma_k)) = \delta(P(\theta_{\sigma_1}, \ldots, \theta_{\sigma_k})) = 0,$$

with $\varpi(P_m({}^t\sigma_1, \ldots, {}^t\sigma_{k-1})) \neq 0$. Therefore $P({}^t\sigma_1, \ldots, {}^t\sigma_k) \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}} \cap \mathcal{U}(\mathfrak{g}_{m_k}) = \mathcal{U}(\mathfrak{g}_{m_k})\mathfrak{a}_{\mathfrak{h}_s}$. Similarly, $P_m({}^t\sigma_1, \ldots, {}^t\sigma_{k-1}) \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

(*ii*) First, observe that $\widehat{Y}_s \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}} \subset \mathcal{U}_C(\mathfrak{g}, \mathfrak{h})$. Therefore, using the assertion (1) of this section which implies that ${}^t\sigma_k = -Y_s{}^t\xi_k + {}^tR_k$, we see that the identity (4.2) can be rewritten as follows:

$$P({}^t\sigma_1, \ldots, {}^t\sigma_k + (Y_s + \sqrt{-1}f(Y_s))^t\xi_k) \equiv 0 \mod \mathcal{U}(\mathfrak{g}_{m_k-1})\mathfrak{a}_{\mathfrak{h}_{s-1}}, \tag{4.3}$$

so that Y_s disappears. Also, we note that in (4.3), the elements ${}^t\sigma_r$ $(1 \leq r \leq k)$, ${}^t\xi_k$ and Y_s belong to $\mathcal{U}_C(\mathfrak{g}, \mathfrak{h})$. Developing P in Y_s, we obtain

$$\varpi\big[P_m({}^t\sigma_1, \ldots, {}^t\sigma_{k-1})({}^t\xi_k Y_s)^m + \sum_{j=0}^{m-1} \widetilde{Q}_j({}^t\sigma_1, \ldots, {}^t\sigma_k, {}^t\xi_k)\, Y_s^j\big] = 0, \tag{4.4}$$

where the $\widetilde{Q}_j$'s are some polynomials of degree less than or equal to m in ${}^t\xi_k$. Now, from the assertion $(\star)$ above, there exist two polynomials S and T of $k-1$ variables such that

$$\varpi(S({}^t\sigma_1, \ldots, {}^t\sigma_{k-1})^t\xi_k) = \varpi(T({}^t\sigma_1, \ldots, {}^t\sigma_{k-1}))$$

with $\varpi(S({}^t\sigma_1, \ldots, {}^t\sigma_{k-1})^t\xi_k) \neq 0$ and $\varpi(T({}^t\sigma_1, \ldots, {}^t\sigma_{k-1})) \neq 0$.

Multiplying (4.4) by $\varpi(S({}^t\sigma_1,\dots,{}^t\sigma_{k-1})^m)$, we obtain

$$\varpi[T({}^t\sigma_1,\dots,{}^t\sigma_{k-1})^m P_m({}^t\sigma_1,\dots,{}^t\sigma_{k-1})Y_s^m + \sum_{j=0}^{m-1} Q_j({}^t\sigma_1,\dots,{}^t\sigma_k)Y_s^j] = 0,$$

for suitable polynomials Q_j. This proves (*ii*). □

Lemma 4.2. *Assume* $\dim\mathfrak{h}\geq 1$ *and* $\mathfrak{h}=\mathfrak{h}'+\mathfrak{h}(\ell)$ *for generic* ℓ *in* $\Gamma_{\mathfrak{g},\mathfrak{h}}$. *Then* $R(Y_d)$ *is algebraic over* $C\mathcal{D}(\mathfrak{g},\mathfrak{h}')$.

In other words, if $d\in T^H(\mathcal{S})$, *there exists a polynomial* Q *of* Y_d *with coefficients in* $\mathcal{U}_C(\mathfrak{g},\mathfrak{h}')$ *satisfying* $Q(Y_d)\in\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$ *and such that the coefficient of the leading power of* Y_d *is not zero modulo* $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$.

Proof. First, keeping the notation of the previous lemma, let us show that

$$T_{i_d}(\mathfrak{g},\mathfrak{h}',\mathcal{S})=T_{i_d}(\mathcal{S}).$$

In this proof, let $\mathrm{p}:\mathfrak{g}^*\to\mathfrak{g}^*_{i_d-1}$ be the natural projections $\ell\mapsto\ell|_{\mathfrak{g}_{i_d-1}}$. We shall use the fact that it is both open and continuous for the Zariski topology. We note that $\mathrm{p}(\Gamma_{\mathfrak{g},\mathfrak{h}})\subset\Gamma_{\mathfrak{g}_{i_{d-1}},\mathfrak{h}'}$ and that $\mathrm{p}^{-1}(\mathrm{p}(\Gamma_{\mathfrak{g},\mathfrak{h}}))\subset\Gamma_{\mathfrak{g},\mathfrak{h}'}$. Let $1\leq r\leq i_d$. Since $[\mathfrak{g},\mathfrak{g}_{i_d}]\subset\mathfrak{g}_{i_d-1}$, the restriction of any $\ell\in\mathfrak{g}^*$ to $[\mathfrak{g},\mathfrak{g}_{i_d}]$ does not change if we replace ℓ by any element of $\mathrm{p}^{-1}(\mathrm{p}(\ell))$ and neither $\mathfrak{g}_r(\ell)$.

Suppose $r\in T_{i_d}(\mathcal{S})$ (resp. $r\notin T_{i_d}(\mathcal{S})$). This amounts to saying that $\mathfrak{g}_r=\mathfrak{g}_{r-1}+\mathfrak{g}_r(\ell)$ (resp. $\mathfrak{g}_r(\ell)\subset\mathfrak{g}_{r-1}$) on a non-empty Zariski-open subset $\mathcal{O}$ of $\Gamma_{\mathfrak{g},\mathfrak{h}}$. The same relation holds on the non-empty Zariski-open subset $\mathrm{p}^{-1}(\mathrm{p}(\mathcal{O}))$ of $\Gamma_{\mathfrak{g},\mathfrak{h}'}$. Therefore, $r\in T(\mathfrak{g},\mathfrak{h}',\mathcal{S})$ (resp. $r\notin T(\mathfrak{g},\mathfrak{h}',\mathcal{S})$). We have then proved that $T_{i_d}(\mathfrak{g},\mathfrak{h}',\mathcal{S})=T_{i_d}(\mathcal{S})$.

So there exists a partial sequence $(\sigma_r)_{1\leq r\leq k}$ of Corwin-Greenleaf $\Gamma_{\mathfrak{g},\mathfrak{h}'}$-central elements that satisfy properties (1), (2) and (3) of the present section, where we replace $\mathfrak{h}$ by $\mathfrak{h}'$. For any $\Gamma_{\mathfrak{g},\mathfrak{h}'}$-central element σ, $\pi_\ell(\sigma)$ is scalar for all ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}'}$ and consequently, for all ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}\subset\Gamma_{\mathfrak{g},\mathfrak{h}'}$. Therefore, $(\sigma_r)_{1\leq r\leq k}$ is also a sequence of Corwin-Greenleaf $\Gamma_{\mathfrak{g},\mathfrak{h}}$-central elements. We can take this sequence, choose $s=d$ and apply (*ii*) of Lemma 4.1. The result follows from the fact that the ${}^t\sigma_r$'s now belong to $\mathcal{U}_C(\mathfrak{g},\mathfrak{h})\cap\mathcal{U}_C(\mathfrak{g},\mathfrak{h}')$. □

5. Proof of the Conjecture

In the proof of the next theorem, we will use several times the following simple lemma:

Lemma 5.1. *Let* $\mathfrak{k}$ *and* $\mathfrak{k}'$ *be two subalgebras of* $\mathfrak{g}$ *such that* $\mathfrak{h}\subset\mathfrak{k}'\subset\mathfrak{k}$. *Let* $\mathfrak{g}''$ *be an ideal of* $\mathfrak{g}$ *such that* $[\mathfrak{k},\mathfrak{h}]\subset\mathfrak{g}''$. *We set* $\mathfrak{h}''\overset{\text{def.}}{=}\mathfrak{h}\cap\mathfrak{g}''$. *Then, the following properties are equivalent*:

(*i*) $\mathfrak{h}''\cap\mathfrak{k}^{B_\ell}=\mathfrak{h}''\cap\mathfrak{k}'^{B_\ell}$ (*resp.* $\dim\mathfrak{h}''\cap\mathfrak{k}^{B_\ell}=\dim\mathfrak{h}''\cap\mathfrak{k}'^{B_\ell}-1$) *for generic* ℓ *in* $\Gamma_{\mathfrak{g},\mathfrak{h}}$.

(*ii*) $\mathfrak{h}''\cap\mathfrak{k}^{B_\ell}=\mathfrak{h}''\cap\mathfrak{k}'^{B_\ell}$ (*resp.* $\dim\mathfrak{h}''\cap\mathfrak{k}^{B_\ell}=\dim\mathfrak{h}''\cap\mathfrak{k}'^{B_\ell}-1$) *for generic* ℓ *in* $\Gamma_{\mathfrak{k},\mathfrak{h}''}$.

Proof. In this proof, let $\mathfrak{k}''\overset{\text{def.}}{=}\mathfrak{k}\cap\mathfrak{g}''$, $\mathrm{p}:\mathfrak{g}^*\to\mathfrak{k}''^*$ and $\mathrm{q}:\mathfrak{k}^*\to\mathfrak{k}''^*$ be the natural projections $\ell\mapsto\ell|_{\mathfrak{k}''}$. We shall use the fact that they are both open and continuous for the Zariski topology. We note that $\mathrm{p}(\Gamma_{\mathfrak{g},\mathfrak{h}})\subset\Gamma_{\mathfrak{k}'',\mathfrak{h}''}$, $\mathrm{q}^{-1}(\mathrm{p}(\Gamma_{\mathfrak{g},\mathfrak{h}}))\subset\Gamma_{\mathfrak{k},\mathfrak{h}''}$ and $\mathrm{p}^{-1}(\mathrm{q}(\Gamma_{\mathfrak{k},\mathfrak{h}''}))\subset\Gamma_{\mathfrak{g},\mathfrak{h}''}$.

Let $\ell\in\mathfrak{g}^*$ and $\lambda\in\mathrm{q}^{-1}(\mathrm{p}(\ell))\subset\mathfrak{k}^*$. Since $[\mathfrak{k},\mathfrak{h}'']\subset\mathfrak{k}''$, we have $\ell|_{[\mathfrak{k},\mathfrak{h}'']}=\lambda|_{[\mathfrak{k},\mathfrak{h}'']}$. Therefore, $\mathfrak{h}''\cap\mathfrak{k}^{B_\ell}=\mathfrak{h}''\cap\mathfrak{k}^{B_\lambda}$ and $\mathfrak{h}''\cap\mathfrak{k}'^{B_\ell}=\mathfrak{h}''\cap\mathfrak{k}'^{B_\lambda}$. So, we see that if one of the assertions (*i*) holds on a non-empty Zariski-open subset $\mathcal{O}$ of $\Gamma_{\mathfrak{g},\mathfrak{h}}$, then it is also verified on the non-empty Zariski-open subset $\mathrm{q}^{-1}(\mathrm{p}(\mathcal{O}))$ of $\Gamma_{\mathfrak{k},\mathfrak{h}''}$. So (*i*) $\Rightarrow$ (*ii*) holds.

Conversely, let $\ell\in\mathfrak{k}^*$ and $\lambda\in\mathrm{p}^{-1}(\mathrm{q}(\ell))\subset\mathfrak{g}^*$. Since $[\mathfrak{k},\mathfrak{h}'']\subset\mathfrak{k}''$, we have $\ell|_{[\mathfrak{k},\mathfrak{h}'']}=\lambda|_{[\mathfrak{k},\mathfrak{h}'']}$. Therefore, $\mathfrak{h}''\cap\mathfrak{k}^{B_\ell}=\mathfrak{h}''\cap\mathfrak{k}^{B_\lambda}$ and $\mathfrak{h}''\cap\mathfrak{k}'^{B_\ell}=\mathfrak{h}''\cap\mathfrak{k}'^{B_\lambda}$. So, we see that if one of the assertion (*ii*) holds on a non-empty Zariski-open subset $\mathcal{O}$ of $\Gamma_{\mathfrak{k},\mathfrak{h}''}$, then it is also verified on the non-empty Zariski-open subset $\mathrm{p}^{-1}(\mathrm{q}(\mathcal{O}))\cap\Gamma_{\mathfrak{g},\mathfrak{h}}$ of $\Gamma_{\mathfrak{g},\mathfrak{h}}$. So (*ii*) $\Rightarrow$ (*i*) also holds. □

The conjecture will be a by-product of the following theorem. Note that (2.7) and (2.8) give a number of equivalent ways to express it.

Theorem 5.2. *Let G be a connected, simply connected, nilpotent real Lie group with non-zero dimensional Lie algebra $\mathfrak{g}$, H a proper, closed, connected subgroup of G with Lie algebra $\mathfrak{h}$. Let $\mathfrak{g}'$ be an ideal of codimension one of $\mathfrak{g}$ containing $\mathfrak{h}$. Let f be a linear form on $\mathfrak{g}$ such that $f([\mathfrak{h},\mathfrak{h}])=\{0\}$. Then the following properties are equivalent*:

(*i*) $\mathcal{U}(\mathfrak{g},\mathfrak{h})\subset\mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

(*ii*) *The H-orbits $H\cdot\ell$ are saturated relatively to $\mathfrak{g}'^{\perp}$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$.*

Proof. Using (2.8), it will be convenient to prove the following equivalent form of the Theorem

$$\mathcal{U}(\mathfrak{g},\mathfrak{h})\subset\mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}\iff\dim\mathfrak{h}(\ell)=\dim\mathfrak{h}(\ell')-1\ \text{ for generic }\ell\text{ in }\Gamma_{\mathfrak{g},\mathfrak{h}}.$$

We shall use an induction both on the dimension of $\mathfrak{g}$ and on the dimension of $\mathfrak{h}$. First, we consider two situations which can be settled directly. They include all cases such that $\dim\mathfrak{g}\le 2$.

When $\mathfrak{h}$ is 0-dimensional, clearly $\mathfrak{h}(\ell)=\mathfrak{h}(\ell')=\{0\}$ for all $\ell\in\mathfrak{g}^*$. Moreover, $\mathcal{U}(\mathfrak{g},\mathfrak{h})$ (resp. $\mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$) is equal to $\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{U}(\mathfrak{g}')$). Therefore, the theorem is obvious in this case.

Now, we turn to the case $\mathfrak{h}=\mathfrak{g}'$. In this situation, $\mathcal{U}(\mathfrak{g}')\subset\mathcal{U}(\mathfrak{g},\mathfrak{h})$.

If ℓ-generically on $\Gamma_{\mathfrak{g},\mathfrak{h}}$, we have $\mathfrak{h}(\ell)=\mathfrak{h}(\ell')$ then $\mathfrak{g}=\mathfrak{h}+\mathfrak{g}(\ell)$. Hence

$$f([\mathfrak{g},\mathfrak{h}])=\ell([\mathfrak{g},\mathfrak{h}])=\ell([\mathfrak{h},\mathfrak{h}])=f([\mathfrak{h},\mathfrak{h}])=\{0\}.$$

Therefore, $[\mathfrak{g},\mathfrak{h}]\subset\mathfrak{a}_{\mathfrak{h}}$ and $\mathcal{U}(\mathfrak{g},\mathfrak{h})=\mathcal{U}(\mathfrak{g})$. Now, since $X_p\notin\mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, we have $\mathcal{U}(\mathfrak{g},\mathfrak{h})\not\subset\mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, as expected.

If we have $\dim\mathfrak{h}(\ell)=\dim\mathfrak{h}(\ell')-1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$, then there exists an element Y in $\mathfrak{h}$ such that $f([X_p,Y])\neq 0$. We are led to a contradiction if we assume $\mathcal{U}(\mathfrak{g},\mathfrak{h}) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. In this case, using the assertion (*iii*) of Proposition 3.7, we can choose an element W of $\mathcal{U}(\mathfrak{g},\mathfrak{h})$ such that $W = X_p U + V$, with $U, V \in \mathcal{U}(\mathfrak{g}') \subset \mathcal{U}(\mathfrak{g},\mathfrak{h})$ and $U \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Then, we have

$$[W,Y] = [X_p,Y]\,U + X_p\,[U,Y] + [V,Y] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}},$$

which implies $[X_p,Y]\,U \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Since the ring $\mathcal{U}(\mathfrak{g},\mathfrak{h})/\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ has no zero divisors, we see that $[X_p,Y] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, contradicting the fact that $[X_p,Y] \in \mathfrak{h} \setminus \ker f$.

We may now assume that $\mathfrak{h}$ has dimension $d \geq 1$ with $\mathfrak{h} \neq \mathfrak{g}'$, so that choosing $\mathfrak{h}'$ as in Section 2, we have $\mathfrak{h}' \subsetneq \mathfrak{h} \subsetneq \mathfrak{g}' \subsetneq \mathfrak{g}$ and $\mathfrak{g}$ is of dimension at least 3. By induction, we shall also assume that the theorem is true for all cases such that the dimension of $\mathfrak{g}$ is strictly less than n and that, when the dimension of $\mathfrak{g}$ is n, for all cases such that the dimension of $\mathfrak{h}$ is strictly less than d. Several situations may occur:

either $\mathfrak{h}(\ell) = \mathfrak{h}(\ell')$ or $\dim \mathfrak{h}(\ell) = \dim \mathfrak{h}(\ell') - 1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$;
either $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ or $\dim \mathfrak{h}'(\ell) = \dim \mathfrak{h}'(\ell') - 1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}'}$.

Using Lemma 5.1, where we replace $\mathfrak{k}$ by $\mathfrak{g}$, $\mathfrak{k}'$ by $\mathfrak{g}'$, $\mathfrak{g}''$ by $\mathfrak{g}'$ and $\mathfrak{h}$ by $\mathfrak{h}'$, we see that saying that $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ (resp. $\dim \mathfrak{h}'(\ell) = \dim \mathfrak{h}'(\ell') - 1$) for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}'}$ is equivalent to saying that the same property holds ℓ-generically on $\Gamma_{\mathfrak{g},\mathfrak{h}}$. Moreover, if $\mathfrak{h}(\ell) = \mathfrak{h}(\ell')$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$, clearly $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$.

These remarks lead us to consider three cases:

1) Case: $\dim \mathfrak{h}(\ell) = \dim \mathfrak{h}(\ell') - 1$ and $\dim \mathfrak{h}'(\ell) = \dim \mathfrak{h}'(\ell') - 1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$

First, we see that $\mathcal{U}(\mathfrak{g},\mathfrak{h}') \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$ by applying the induction hypothesis to $\mathfrak{g}$ and $\mathfrak{h}'$. Now, we claim that $\mathcal{U}(\mathfrak{g},\mathfrak{h}) \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Indeed, assume there exists an element $W \in \mathcal{U}(\mathfrak{g},\mathfrak{h})$ such that $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Then, the second assertion of Proposition 3.3 would imply the existence of $W' \in \mathcal{U}(\mathfrak{g},\mathfrak{h}') \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'})$ such that $W' \equiv W \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. This is absurd, since it contradicts the induction hypothesis.

2) Case: $\mathfrak{h}(\ell) = \mathfrak{h}(\ell')$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$

We observed before, that in this case, $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}'}$. Using the induction hypothesis applied to $\mathfrak{g}$ and $\mathfrak{h}'$, we know that $\mathcal{U}(\mathfrak{g},\mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. We now consider two subcases.

a) $\mathcal{U}(\mathfrak{g}',\mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}',\mathfrak{h})$

The previous remark and Theorem 3.8 give at once, as we expect:

$$\mathcal{U}(\mathfrak{g},\mathfrak{h}) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$$

b) $\mathcal{U}(\mathfrak{g}',\mathfrak{h}') \subset \mathcal{U}(\mathfrak{g}',\mathfrak{h})$

Our first goal will be to prove that the hypothesis implies that $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell)$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$, so that we can make use of Lemma 4.2. (In other words, we shall prove that $d \in T^H(\mathcal{S})$.)

First, using an induction argument, we show that for $0 \le r \le p-1$, $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h} \cap \mathfrak{k}_r^{B_\ell}$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. This is obvious if $r = 0$ since $\mathfrak{h} \cap \mathfrak{k}_0^{B_\ell} = \mathfrak{h}$. Assuming $r > 0$ and the property true up to rank $r-1$, we note the inclusion $\mathcal{U}(\mathfrak{k}_r, \mathfrak{h}') \subset \mathcal{U}(\mathfrak{k}_r, \mathfrak{h})$. Then, replacing $\mathfrak{g}$ by $\mathfrak{k}_r$ in the assertion *(ii)* of Proposition 3.5, we have

either $\mathcal{U}(\mathfrak{k}_r, \mathfrak{h}) \not\subset \mathcal{U}(\mathfrak{k}_{r-1}) + \mathcal{U}(\mathfrak{k}_r)\mathfrak{a}_{\mathfrak{h}}$,
or $\mathcal{U}(\mathfrak{k}_r, \mathfrak{h}') \subset \mathcal{U}(\mathfrak{k}_{r-1}) + \mathcal{U}(\mathfrak{k}_r)\mathfrak{a}_{\mathfrak{h}'}$
and $\mathcal{U}(\mathfrak{k}_r, \mathfrak{h}) \subset \mathcal{U}(\mathfrak{k}_{r-1}) + \mathcal{U}(\mathfrak{k}_r)\mathfrak{a}_{\mathfrak{h}}$.

Therefore, by induction on the dimension of G we have

either $\mathfrak{h} \cap \mathfrak{k}_r^{B_\ell} = \mathfrak{h} \cap \mathfrak{k}_{r-1}^{B_\ell}$ for generic ℓ in $\Gamma_{\mathfrak{k}_r,\mathfrak{h}}$,
or $\dim \mathfrak{h} \cap \mathfrak{k}_r^{B_\ell} = \dim \mathfrak{h} \cap \mathfrak{k}_{r-1}^{B_\ell} - 1$ for generic ℓ in $\Gamma_{\mathfrak{k}_r,\mathfrak{h}}$
and $\dim \mathfrak{h}' \cap \mathfrak{k}_r^{B_\ell} = \dim \mathfrak{h}' \cap \mathfrak{k}_{r-1}^{B_\ell} - 1$ for generic ℓ in $\Gamma_{\mathfrak{k}_r,\mathfrak{h}'}$.

Using Lemma 5.1 where we replace $\mathfrak{k}$ by $\mathfrak{k}_r$, $\mathfrak{k}'$ by $\mathfrak{k}_{r-1}$, $\mathfrak{g}''$ either by $\mathfrak{g}$ or $\mathfrak{g}_{i_d-1}$ so that $\mathfrak{h}''$ is replaced either by $\mathfrak{h}$ or $\mathfrak{h}'$, this can be rewritten as follows:

either $\mathfrak{h} \cap \mathfrak{k}_r^{B_\ell} = \mathfrak{h} \cap \mathfrak{k}_{r-1}^{B_\ell}$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$,
or $\dim \mathfrak{h} \cap \mathfrak{k}_r^{B_\ell} = \dim \mathfrak{h} \cap \mathfrak{k}_{r-1}^{B_\ell} - 1$ *and* $\dim \mathfrak{h}' \cap \mathfrak{k}_r^{B_\ell} = \dim \mathfrak{h}' \cap \mathfrak{k}_{r-1}^{B_\ell} - 1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$.

In the first situation, it is obvious that $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h} \cap \mathfrak{k}_r^{B_\ell}$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$ by induction on $r = \dim \mathfrak{k}_r/\mathfrak{h}$, while in the second one, we would be led to a contradiction if we assumed $\mathfrak{h} \cap \mathfrak{k}_r^{B_\ell} \subset \mathfrak{h}'$. Observing that $\dim \mathfrak{h} \cap \mathfrak{k}_{r-1}^{B_\ell} = \dim \mathfrak{h}' \cap \mathfrak{k}_{r-1}^{B_\ell} + 1$ by induction and that $\mathfrak{h} \cap \mathfrak{k}_r^{B_\ell} = \mathfrak{h}' \cap \mathfrak{k}_r^{B_\ell}$, we would have in this case

$$\dim \mathfrak{h} \cap \mathfrak{k}_r^{B_\ell} = \dim \mathfrak{h}' \cap \mathfrak{k}_r^{B_\ell} = \dim \mathfrak{h}' \cap \mathfrak{k}_{r-1}^{B_\ell} - 1 = \dim \mathfrak{h} \cap \mathfrak{k}_{r-1}^{B_\ell} - 2,$$

which is impossible since $\mathfrak{k}_{r-1}$ is of codimension 1 in $\mathfrak{k}_r$. So in all cases, we have $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h} \cap \mathfrak{k}_r^{B_\ell}$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. Applying this result with $r = p-1$, we have $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell')$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$, so that $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell)$ since $\mathfrak{h}(\ell) = \mathfrak{h}(\ell')$.

Next, we will show that $\mathcal{U}(\mathfrak{g}, \mathfrak{h}) \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Since the second assertion of proposition 3.5 says that $\mathcal{U}(\mathfrak{g}, \mathfrak{h}') \not\subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, let $W = X_pU + V \in \mathcal{U}(\mathfrak{g}, \mathfrak{h}')$ with $U \in \mathcal{U}(\mathfrak{g}', \mathfrak{h}') \setminus \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ and $V \in \mathcal{U}(\mathfrak{g}')$. We want to show that $[W, Y_d] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Actually, we will prove that $[W, Y_d] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$.

We have $Y_d \in \mathcal{U}(\mathfrak{g}, \mathfrak{h}')$. Also, replacing $\mathfrak{g}$ by $\mathfrak{g}'$ in Proposition 3.6, we know that $Y_d \in \mathcal{U}_C(\mathfrak{g}', \mathfrak{h}')$. Hence

$$[W, Y_d] = [X_pU + V, Y_d] \in (\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'} + \mathcal{U}(\mathfrak{g}')) \cap \mathcal{U}(\mathfrak{g}, \mathfrak{h}') = \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'} + \mathcal{U}(\mathfrak{g}', \mathfrak{h}'),$$

so that $[[W, Y_d], Y_d] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$.

Now, we can apply Lemma 4.2 which says that there exist $m > 0$ and $Q_j \in \mathcal{U}_C(\mathfrak{g}, \mathfrak{h}')$, $0 \le j \le m$, with $Q_m \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$, so that

$$\sum_{j=0}^{m} Q_j\, Y_d^j \equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}.$$

We choose m minimal for such an identity to hold. The adjoint action of W reads as follows:

$$\left(\sum_{j=1}^{m} j\, Q_j\, Y_d^{j-1}\right)[W, Y_d] \equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}.$$

We have $\left(\sum_{j=1}^{m} jQ_j\, Y_d^{j-1}\right) \not\equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. If $m > 1$, this is due to the minimality condition and if $m = 1$, to the fact that $Q_1 \not\equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. As the ring $\mathcal{U}_C(\mathfrak{g}, \mathfrak{h}')/\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$ is entire, we have $[W, Y_d] \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}'}$. The proof is complete in this case.

3) Case: $\dim \mathfrak{h}(\ell) = \dim \mathfrak{h}(\ell') - 1$ and $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$

Note that the condition $\dim \mathfrak{h}(\ell) = \dim \mathfrak{h}(\ell') - 1$ for one ℓ, implies that the center $\mathfrak{z}$ of $\mathfrak{g}$ is contained in $\mathfrak{g}'$. The assumption $\mathfrak{h}'(\ell) = \mathfrak{h}'(\ell')$ will be used only in the situation 3)c)α) below.

We shall prove the inclusion $\mathcal{U}(\mathfrak{g}, \mathfrak{h}) \subset \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Using the assertion (*iii*) of Proposition 3.7, it is enough for this to show that if $W = X_pU + V \in \mathcal{U}(\mathfrak{g}, \mathfrak{h})$ with $U \in \mathcal{U}(\mathfrak{g}', \mathfrak{h})$ and $V \in \mathcal{U}(\mathfrak{g}')$ then necessarily $U \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

We shall consider three subcases depending on $\widetilde{\mathfrak{z}} \stackrel{\text{def.}}{=} \mathfrak{z} \cap \mathfrak{h} \cap \ker f$ and on $\mathfrak{z}$.

a) $\widetilde{\mathfrak{z}} \neq \{0\}$

This subcase can be settled easily by applying the induction hypothesis to the quotients $\mathfrak{g}/\widetilde{\mathfrak{z}}$ and $\mathfrak{h}/\widetilde{\mathfrak{z}}$.

b) $\widetilde{\mathfrak{z}} = \{0\}$ and $\dim \mathfrak{z} \geq 2$

In this subcase, either $\dim \mathfrak{z} \cap \mathfrak{h} = 1$ or $\mathfrak{z} \cap \mathfrak{h} = \{0\}$. Both situations can be dealt with using similar methods. We leave the proof in the first one to the reader and assume hereafter that $\mathfrak{z} \cap \mathfrak{h} = \{0\}$. We can then choose two linearly independent elements Z_1 and Z_2 in $\mathfrak{z}$ such that $(\mathbb{R}Z_1 + \mathbb{R}Z_2) \cap \mathfrak{h} = \{0\}$. We set $\widehat{\mathfrak{h}} \stackrel{\text{def.}}{=} \mathfrak{h} \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$.

We recall that the Y_s's, $1 \leq s \leq d$, form a basis of $\mathfrak{h}$. We take a supplementary basis $(T_r)_{3 \leq r \leq p-1}$ of $\mathfrak{h} \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$ in $\mathfrak{g}'$, associated to a supplementary basis of $\mathfrak{h}$ in $\mathfrak{g}'$ given by $\{Z_1, Z_2, (T_r)_{3 \leq r \leq p-1}\}$. We consider the vector subspaces S_1 of $\mathcal{U}(\mathfrak{g}')$ and S_1^* of $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$ generated by the families $(T^J\, \widehat{Y}^K)_{J \in \mathbb{N}^{p-3},\, K \in \mathbb{N}^d}$ and $(T^J\, \widehat{Y}^K)_{J \in \mathbb{N}^{p-3},\, K \in \mathbb{N}^d, |K|>0}$. In this proof, $\{Z_1^*, Z_2^*, (Y_s^*)_{1 \leq s \leq d}, (T_r^*)_{3 \leq r \leq p-1}, X_p^*\}$ denotes the dual basis in $\mathfrak{g}^*$ of the basis $\{Z_1, Z_2, (Y_s)_{1 \leq s \leq d}, (T_r)_{3 \leq r \leq p-1}, X_p\}$ of $\mathfrak{g}$.

We have $\widehat{f}([\widehat{\mathfrak{h}}, \widehat{\mathfrak{h}}]) = \{0\}$ for any $\widehat{f} \in \Gamma_{\mathfrak{g},\mathfrak{h}}$. Also, the family

$$\begin{aligned}\{T^J\, \widehat{Y}^K (Z_1 + \sqrt{-1}\widehat{f}(Z_1))^j\, (Z_2 + \sqrt{-1}\widehat{f}(Z_2))^k \mid \\ J \in \mathbb{N}^{p-3},\ K \in \mathbb{N}^d, \qquad j, k \in \mathbb{N},\ |K| + j + k > 0\}\end{aligned}$$

form a basis for $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\widehat{\mathfrak{h}},\widehat{f}}$ where, as one expects, $\mathfrak{a}_{\widehat{\mathfrak{h}},\widehat{f}}$ is the vector subspace of $\mathcal{U}(\mathfrak{g})$ generated by the elements $Z_1 + \sqrt{-1}\widehat{f}(Z_1)$, $Z_2 + \sqrt{-1}\widehat{f}(Z_2)$ and

$Y + \sqrt{-1}\widehat{f}(Y)$, $Y \in \mathfrak{h}$. In particular, any element U_* of $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\widehat{\mathfrak{h}},\widehat{f}}$ can be written in a unique way as

$$U_* = \sum_{j,k} U_*^{(j,k)} \left(Z_1 + \sqrt{-1}\widehat{f}(Z_1)\right)^j \left(Z_2 + \sqrt{-1}\widehat{f}(Z_2)\right)^k \tag{5.1}$$

with $U_*^{(j,k)} \in S_1$ if $j + k \neq 0$ and $U_*^{(0,0)} \in S_1^*$. The sum being finite.

We note that $W \in \mathcal{U}(\mathfrak{g}, \widehat{\mathfrak{h}}, \widehat{f})$ and $U \in \mathcal{U}(\mathfrak{g}', \widehat{\mathfrak{h}}, \widehat{f})$.

We know that there exists a non-empty Zariski-open subset $\mathcal{O}_0$ of $\Gamma_{\mathfrak{g},\mathfrak{h}}$ whose elements ℓ are such that $\dim \mathfrak{h}(\ell)$ and $\dim \mathfrak{h}'(\ell')$ are minimal. If we take any $\widehat{f}$ in $\mathcal{O}_0$ and replace $\mathfrak{h}$ by $\widehat{\mathfrak{h}}$, f by $\widehat{f}$ and $\Gamma_{\mathfrak{g},\mathfrak{h}}$ by $\Gamma_{\mathfrak{g},\hat{\mathfrak{h}},\hat{f}}$, we see that the conditions of 3)a) above are fulfilled. This implies that $U \in \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\widehat{\mathfrak{h}},\widehat{f}}$.

We now fix $\widehat{f}$ in $\mathcal{O}_0$. We set $\widehat{Z_1} = Z_1 + \sqrt{-1}\widehat{f}(Z_1)$ and $\widehat{Z_2} = Z_2 + \sqrt{-1}\widehat{f}(Z_2)$. Replacing U_* by U in formula (5.1) gives

$$U = \sum_{j,k} U^{(j,k)} \,\widehat{Z_1}^j \,\widehat{Z_2}^k \tag{5.2}$$

with $U^{(j,k)} \in S_1$ if $j + k \neq 0$ and $U^{(0,0)} \in S_1^*$. The sum being finite.

It is elementary to prove that there exists a non-empty Zariski-open subset $\mathcal{O}$ of $\mathbb{R}^2$ whose elements (u, v) are such that $\widehat{f}_{u,v} = (\widehat{f} + uZ_1^* + vZ_2^*) \in \mathcal{O}_0$. Replacing U_* by U and $\widehat{f}$ by $\widehat{f}_{u,v}$ in (5.1), we also see that for such elements

$$U = \sum_{j,k} U_{u,v}^{(j,k)} \,(\widehat{Z_1} + \sqrt{-1}u)^j \,(\widehat{Z_2} + \sqrt{-1}v)^k \tag{5.3}$$

with $U_{u,v}^{(j,k)} \in S_1$ if $j + k \neq 0$ and $U_{u,v}^{(0,0)} \in S_1^*$. The sum being finite.

Using the equality $U = \sum_{j,k} U^{(j,k)} \,(\widehat{Z_1} + \sqrt{-1}u - \sqrt{-1}u)^j \,(\widehat{Z_2} + \sqrt{-1}v - \sqrt{-1}v)^k$, formula (5.2) and formula (5.3) yield

$$U_{u,v}^{(0,0)} = \sum_{j,k} (-\sqrt{-1}u)^j (-\sqrt{-1}v)^k U^{(j,k)} \in S_1^* \subset \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}, \quad \forall (u, v) \in \mathcal{O}.$$

From this relation, we see that for all $(j, k) \in \mathbb{N}^2$ and all $J \in \mathbb{N}^{p-3}$, the component of $U^{(j,k)}$ on T^J vanishes. This implies that $U^{(j,k)} \in S_1^* \subset \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$. In particular, we have $U \in \mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$ as we expected. The proof of the theorem in this subcase is complete.

Before going any further in the study of Case 3), we note that its general assumptions imply that $\mathfrak{h} = \mathfrak{h}' + \mathfrak{h}(\ell')$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$, and that $\mathfrak{h}(\ell) = \mathfrak{h}'(\ell)$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. In other words, we have $i_d \in T^H(\mathfrak{g}', \mathfrak{h}, \mathcal{S})$ and $i_d \notin T^H(\mathcal{S})$. This result can be found in [7], proposition 6.4.3. For the convenience of the reader, let us recall its proof. By subtracting, we have for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$,

$$\dim \mathfrak{h}(\ell) - \dim \mathfrak{h}'(\ell) = \dim \mathfrak{h}(\ell') - \dim \mathfrak{h}'(\ell') - 1.$$

Then $\dim \mathfrak{h}(\ell) - \dim \mathfrak{h}'(\ell)$ and $\dim \mathfrak{h}(\ell') - \dim \mathfrak{h}'(\ell')$ are both either 0 or 1. So necessarily, $\dim \mathfrak{h}(\ell) - \dim \mathfrak{h}'(\ell) = 0$ and $\dim \mathfrak{h}(\ell') - \dim \mathfrak{h}'(\ell') = 1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$, as we expected.

The proposition 6.4.3 of [7] also says that $T^H(\mathfrak{g}', \mathfrak{h}, \mathcal{S})$ and $T^H_{n-1}(\mathcal{S})$ always differ from at most one element. Hence, in the present situation, we have

$$T^H(\mathfrak{g}', \mathfrak{h}, \mathcal{S}) = T^H_{n-1}(\mathcal{S}) \cup \{i_d\}.$$

These results will be used for the proof in c)α) below.

c) $\underline{\dim \mathfrak{z} = 1 \text{ and } \widetilde{\mathfrak{z}} = \{0\}}$

We let $\mathfrak{z}'$ be the center of $\mathfrak{g}'$. Choose $Z \in \mathfrak{z} \setminus \{0\}$ so that $\mathfrak{z} = \mathbb{R}Z$, and $Y \in \mathfrak{g}_2 \setminus \mathfrak{g}_1$. Denote by $\widetilde{\mathfrak{g}}$ the centralizer of $\mathfrak{g}_2$ (or equivalently of Y) in $\mathfrak{g}$. We shall consider four subcases:

α) $\underline{\mathfrak{g}' = \widetilde{\mathfrak{g}}}$

This is equivalent to saying that $\mathfrak{g}_2 \subset \mathfrak{z}'$. In this case, we have $f([X_p, Y]) \neq \{0\}$ and $\mathfrak{h} \subset \mathfrak{g}'$. The assumption of α) implies that $2 \in T(\mathfrak{g}', \mathfrak{h}, \mathcal{S})$ and $2 \notin T(\mathcal{S})$. It is also easily verified (see [7], proposition 6.4.4) that $T(\mathfrak{g}', \mathfrak{h}, \mathcal{S})$ and $T_{n-1}(\mathcal{S})$ always differ from at most one element. So in the present situation, we have

$$T(\mathfrak{g}', \mathfrak{h}, \mathcal{S}) = T_{n-1}(\mathcal{S}) \cup \{2\}. \tag{5.4}$$

Our first goal will be to prove that $R(Y)$ is algebraic over $C\mathcal{D}(\mathfrak{g}, \mathfrak{h})$. In other words, we shall prove that there exists a polynomial P of Y satisfying $P(Y) = \sum_{j=0}^{m} P_j Y^j \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, with the coefficients P_j in $\mathcal{U}_C(\mathfrak{g}, \mathfrak{h})$ and the coefficient P_m of the leading power Y^m of P is different from zero modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$.

This result is easily proved when $2 \in \mathcal{I}$ (or equivalently when $(\mathfrak{h} \cap \mathfrak{g}_2/(\mathfrak{h} \cap \mathfrak{g}_1) \neq \{0\})$. Indeed, in that case, there exists a real number a such that $Y + aZ \in \mathfrak{h}$. Obviously $Y + aZ + if(Y + aZ) \in \mathfrak{a}_{\mathfrak{h}}$ with $Z \in \mathcal{U}_C(\mathfrak{g}, \mathfrak{h})$. This gives the expected polynomial relation.

We now assume that $2 \notin \mathcal{I}$. In this case, $i_d \neq 2$ or equivalently $i_d > 2$. Using the fact mentioned above, that $i_d \in T^H(\mathfrak{g}', \mathfrak{h}, \mathcal{S}) \subset T_{n-1}(\mathfrak{g}, \mathfrak{h}, \mathcal{S})$ along with (5.4), we see that $i_d \in T(\mathcal{S}) \setminus T^H(\mathcal{S}) = U(\mathcal{S})$. This can also quickly and directly be seen by observing that the equality $\dim \mathfrak{h}(\ell') - \dim \mathfrak{h}'(\ell') = 1$ which is verified ℓ-generically on $\Gamma_{\mathfrak{g},\mathfrak{h}}$, implies that there exists $Y(\ell) \in \mathfrak{h}(\ell') \setminus \mathfrak{h}'$, hence $Y(\ell) \in \mathfrak{g}_{i_d}(\ell') \setminus \mathfrak{g}_{i_{d-1}}$. Then, we have $\ell([X_p, Y(\ell)]) \neq 0$ since $i_d \notin T^H(\mathcal{S})$. Therefore

$$\ell\big([X_p, \ell([X_p, Y])\, Y(\ell) - \ell([X_p, Y(\ell)])\, Y]\big) = 0.$$

Hence, we see that $\ell([X_p, Y])\, Y(\ell) - \ell([X_p, Y(\ell)]\, Y \in \mathfrak{g}_{i_d}(\ell) \setminus \mathfrak{g}_{i_{d-1}}$ and that $i_d \in T(\mathcal{S})$.

Taking $m_k = i_d$ and using the results of Section 4, we can choose a partial sequence $(\sigma_r)_{1 \leq r \leq k}$ of Corwin-Greenleaf $\Gamma_{\mathfrak{g},\mathfrak{h}}$-central elements that satisfy the properties (1), (2) and (3) of the current Section 4.

We have seen that the family $\big(\varpi({}^t\sigma_r)\big)_{1\leq r\leq k}$ algebraically generates the subalgebra $\delta(Z_{m_k}(\mathfrak{g},\mathfrak{h}))$ of $C\mathcal{D}(\mathfrak{g},\mathfrak{h})$. It can be shown that the σ_r's are also $\Gamma_{\mathfrak{g}',\mathfrak{h}}$-central elements (see the proof of the proposition 6.2.1 in [7]). Moreover, Y is a $\Gamma_{\mathfrak{g}',\mathfrak{h}}$-central element. So the family $\{Y\}\cup(\sigma_r)_{1\leq r\leq k}$ forms a partial sequence of Corwin-Greenleaf $\Gamma_{\mathfrak{g}',\mathfrak{h}}$-central elements and $\varpi(Y)\cup\big(\varpi({}^t\sigma_r)\big)_{1\leq r\leq k}$ algebraically generates the subalgebra $\delta(Z_{m_k}(\mathfrak{g}',\mathfrak{h}))$ of $C\mathcal{D}(\mathfrak{g}',\mathfrak{h})$.

Since $m_k\in T^H(\mathfrak{g}',\mathfrak{h},\mathcal{S})$, we know from the first assertion of Lemma 4.1 applied at the level of $\mathfrak{g}'$, that ${}^t\sigma_k$ algebraically depends on the family $\{Y\}\cup({}^t\sigma_r)_{1\leq r\leq k-1}$ modulo $\mathcal{U}(\mathfrak{g}')\mathfrak{a}_{\mathfrak{h}}$. Thus, we obtain a polynomial P such that

$$P({}^t\sigma_1,\dots,{}^t\sigma_{k-1},Y,{}^t\sigma_k)\equiv 0 \mod \mathcal{U}(\mathfrak{g}_{i_d})\mathfrak{a}_{\mathfrak{h}} \tag{5.5}$$

where the leading power of ${}^t\sigma_k$ has a non-zero coefficient modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. We rewrite (5.5) as follows:

$$\sum_{j=0}^{m} P_j({}^t\sigma_1,\dots,{}^t\sigma_k)\,Y^j\equiv 0 \mod \mathcal{U}(\mathfrak{g}_{i_d})\mathfrak{a}_{\mathfrak{h}}, \tag{5.6}$$

for some polynomials $P_j({}^t\sigma_1,\dots,{}^t\sigma_k)$, $0\leq j\leq m$ and $m\geq 0$. Let us write $P_j\overset{\text{def.}}{=}P_j({}^t\sigma_1,\dots,{}^t\sigma_k)$. The P_j's are elements of $\mathcal{U}_C(\mathfrak{g},\mathfrak{h})$.

Since $m_k\in T(\mathcal{S})\setminus T^H(\mathcal{S})$, we also know from the assertion $(\star\star)$ of Section 4 that ${}^t\sigma_k$ is algebraically independent of the family $({}^t\sigma_r)_{1\leq r\leq k-1}$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Since, in the formula (5.6), at least one of the P_j's really contains ${}^t\sigma_k$, it does not belong to $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Without loss of generality, we can assume that P_m has this property. In particular, $m\geq 1$ and (5.6) is a non-trivial relation. We have proved that Y depends algebraically on $\mathcal{U}_C(\mathfrak{g},\mathfrak{h})$ modulo $\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. From now on, we choose P in such a way that its degree in Y is minimal.

Next, as $[W,Y]=ZU=UZ$, we apply the adjoint action of W on the formula (5.6) to see that

$$\Big(\sum_{j=1}^{m} jP_j\,Y^{j-1}\Big)\,UZ\equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}.$$

We have $\Big(\sum_{j=1}^{m} jP_j\,Y^{j-1}\Big)\not\equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. If $m>1$ this is due to the minimality condition, while if $m=1$ to the fact that $P_1\not\equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Also, it is clear that $Z\notin\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. The ring $\mathcal{U}(\mathfrak{g},\mathfrak{h})/\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ has no zero divisors, so we see that $U\in\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ in all instances, as we expected. The theorem is proved in this situation.

$\beta)$ $\mathfrak{g}'\neq\widetilde{\mathfrak{g}}$, $\mathfrak{h}\subset\widetilde{\mathfrak{g}}$ and $\dim\mathfrak{h}(\ell)=\dim\mathfrak{h}\cap\widetilde{\mathfrak{g}}^{B_\ell}$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$

First, we can choose X_p in $\widetilde{\mathfrak{g}}$ and find X in $\mathfrak{g}'$ so that

$$\mathfrak{g}'=(\mathfrak{g}'\cap\widetilde{\mathfrak{g}})\oplus\mathbb{R}X \quad\text{and}\quad \widetilde{\mathfrak{g}}=(\mathfrak{g}'\cap\widetilde{\mathfrak{g}})\oplus\mathbb{R}X_p.$$

In the following diagrams, an arrow symbolizes the canonical injection between two subalgebras of $\mathfrak{h}$ orthogonal through B_ℓ to two ideals of $\mathfrak{g}$. The second ideal being of codimension one in the first. The figures beside the arrows give the increase of dimension between the source and the target spaces for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. Clearly, their possible values are 0 or 1.

Here, the only possible values of

$$\dim \mathfrak{h} \cap \left(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}\right)^{B_\ell} - \dim \mathfrak{h} \cap \mathfrak{g}^{B_\ell} = \dim \mathfrak{h} \cap \left(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}\right)^{B_\ell} - \dim \mathfrak{h}(\ell)$$

are 0, 1 or 2 since $\mathfrak{g}' \cap \widetilde{\mathfrak{g}}$ is of codimension 2 in $\mathfrak{g}$. We have by assumption for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$,

$$\dim \mathfrak{h}(\ell) = \dim \mathfrak{h}(\ell') - 1 \quad \text{and} \quad \dim \mathfrak{h}(\ell) = \dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell}.$$

We also have

$$\begin{aligned} \dim \mathfrak{h} \cap & (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - \dim \mathfrak{h}(\ell) \\ &= (\dim \mathfrak{h} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - \dim \mathfrak{h}(\ell')) + (\dim \mathfrak{h}(\ell') - \dim \mathfrak{h}(\ell)) \\ &= (\dim \mathfrak{h} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - \dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell}) + (\dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell} - \dim \mathfrak{h}(\ell)). \end{aligned}$$

Thus, we see that $\dim \mathfrak{h} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - \dim \mathfrak{h}(\ell) = 1$, so that Diagram 2 below holds.

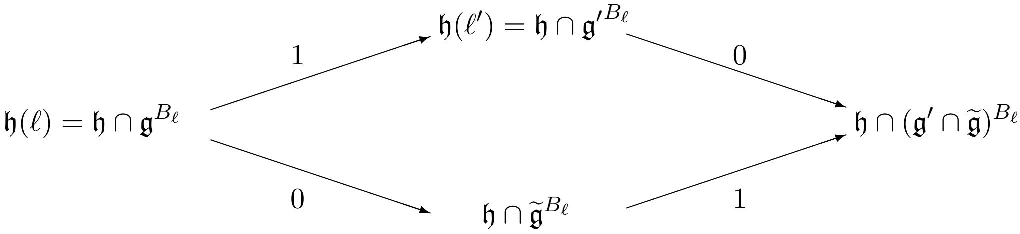

Diagram 2

We take $W = X_pU + V$ as above and show that we are led to a contradiction if we assume that $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Indeed, we shall prove that this implies the existence of an element $\widetilde{W} = X_p\widetilde{U} + \widetilde{V}$ in $\mathcal{U}(\widetilde{\mathfrak{g}}, \mathfrak{h})$ with $\widetilde{V} \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$ and $\widetilde{U} \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}, \mathfrak{h}) \setminus \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})\mathfrak{a}_{\mathfrak{h}}$, leading, by induction on n, to a contradiction since $\dim \mathfrak{h} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - \dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell} = 1$.

For $b \in \mathbb{N}$, we define the subspaces $S_b = \displaystyle\sum_{i=0}^{b-1} X^i \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$ if $b \geq 1$ and $S_0 = \{0\}$ in $\mathcal{U}(\mathfrak{g}')$. It is easy to rewrite W under the form

$$W = \sum_{i=0}^{a} X^i X_p U_i + \sum_{i=0}^{b} X^i V_i = \sum_{i=0}^{a} X^a X_p U_a + X^b V_b + W_b \tag{5.7}$$

for suitable integers a and b. Here, $U_i, V_i \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$ and $W_b \in S_b$. Without loss of generality, we can choose W so that $U_i, V_i \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}) \setminus \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})\mathfrak{a}_{\mathfrak{h}}$ and $b \leq a$.

Indeed if we assume $b > a$, applying the first assertion of Proposition 3.7, where we replace $\mathfrak{g}'$ by $\widetilde{\mathfrak{g}}$ and X_p by X, we know that $V_b \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}, \mathfrak{h})$. Next, as $\dim \mathfrak{h} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} = \dim \mathfrak{h}(\ell')$, we can apply the induction hypothesis to obtain an element $XA + B$ of $\mathcal{U}(\mathfrak{g}', \mathfrak{h})$ with $A \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}, \mathfrak{h}) \setminus (\mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})\mathfrak{a}_\mathfrak{h})$ and $B \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$. We see that

$$W' = WA^b - (XA + B)^b V_b = X^a X_p U_a A^b + X^{b'} V_{b'} + W_{b'}$$

is an element of $\mathcal{U}(\mathfrak{g}, \mathfrak{h}) \setminus (\mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h})$ such that $b' < b$, $V_{b'} \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$ and $W_{b'} \in S_{b'}$.

So, repeating this procedure if necessary, we may assume $b \leq a$ in (5.7). If $b = a$ (resp. $b < a$) then applying again Proposition 3.7, we see that $\widetilde{W} = X_p U_a + V_a \in \mathcal{U}(\widetilde{\mathfrak{g}}, \mathfrak{h})$ (resp. $\widetilde{W} = X_p U_a \in \mathcal{U}(\widetilde{\mathfrak{g}}, \mathfrak{h})$) with $U_a \notin \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})\mathfrak{a}_\mathfrak{h}$. This contradicts the fact that $\dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell} = \dim \mathfrak{h} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - 1$ as shown on Diagram 2 and the induction hypothesis.

γ) <u>$\mathfrak{g}' \neq \widetilde{\mathfrak{g}}$, $\mathfrak{h} \subset \widetilde{\mathfrak{g}}$ and $\dim \mathfrak{h}(\ell) = \dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell} - 1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$</u>

The assumptions imply that $\mathfrak{h}^{B_\ell} \subset \mathfrak{g}'$ and $\mathfrak{h}^{B_\ell} \subset \widetilde{\mathfrak{g}}$ because of the last equivalence of 2) in (2.8). Therefore, we have $\mathfrak{h}^{B_\ell} \subset \mathfrak{g}' \cap \widetilde{\mathfrak{g}}$. For the same reason, this implies in turn that $\dim \mathfrak{h} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - \dim \mathfrak{h}(\ell') = 1$ and $\dim \mathfrak{h} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - \dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell} = 1$. Hence, we obtain Diagram 3 below:

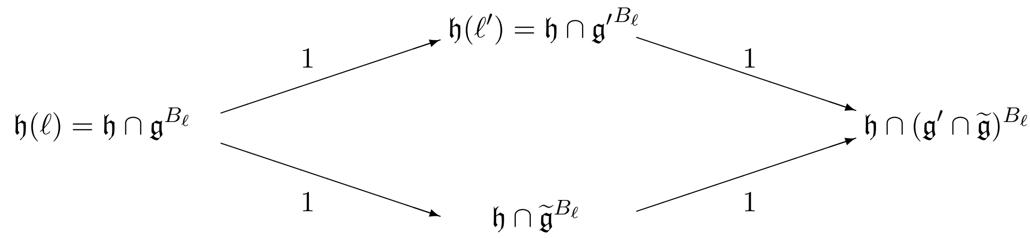

Diagram 3

We take $W = X_p U + V$ as above and show that we are led to a contradiction if we assume $U \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$. We know that $U \in \mathcal{U}(\mathfrak{g}', \mathfrak{h})$. Let us write U under the form $\sum_{i=0}^{m} X^i U_i$ with $U_i \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$. Replacing $\mathfrak{g}$ by $\mathfrak{g}'$ and $\mathfrak{g}'$ by $\mathfrak{g}' \cap \widetilde{\mathfrak{g}}$, and using the fact that $\dim \mathfrak{h} \cap (\widetilde{\mathfrak{g}} \cap \mathfrak{g}')^{B_\ell} - \dim \mathfrak{h}(\ell') = 1$ for generic ℓ in $\Gamma_{\mathfrak{g}',\mathfrak{h}}$, we apply the induction hypothesis on the dimension of $\mathfrak{g}$ to see that $\mathcal{U}(\mathfrak{g}', \mathfrak{h}) \subset \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}) + \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\mathfrak{h}$. Thus, we have $U_i \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$ for $i \neq 0$. Therefore, without loss of generality, we may assume $U = U_0 \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$.

Next, without loss of generality, we may also assume that V can be written under the form $V = \sum_{i=0}^{m} X^i V_i$ with $V_i \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}) \setminus \mathcal{U}(\mathfrak{g})\mathfrak{a}_\mathfrak{h}$. We are led to a contradiction if we suppose $m \geq 1$. Indeed, in this case, the first assertion of Proposition 3.7 says that

$$mXV_m + V_{m-1} \in \mathcal{U}(\mathfrak{g}', \mathfrak{h}) \setminus (\mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}) + \mathcal{U}(\mathfrak{g}')\mathfrak{a}_\mathfrak{h})$$

which as just seen, is impossible.

So, we can choose $W = X_p U_0 + V_0$ with U_0 and V_0 in $\mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$ and $U_0 \notin \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})\mathfrak{a}_{\mathfrak{h}}$. We have $X_p U_0 + V_0 \in \mathcal{U}(\widetilde{\mathfrak{g}}, \mathfrak{h}) \setminus (\mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}) + \mathcal{U}(\widetilde{\mathfrak{g}})\mathfrak{a}_{\mathfrak{h}})$. Finally, we use the equality $\dim \mathfrak{h} \cap (\widetilde{\mathfrak{g}} \cap \mathfrak{g}')^{B_\ell} - \dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell} = 1$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. It implies by induction that $\mathcal{U}(\widetilde{\mathfrak{g}}, \mathfrak{h}) \subset \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}) + \mathcal{U}(\widetilde{\mathfrak{g}})\mathfrak{a}_{\mathfrak{h}}$. This gives a contradiction. The theorem is proved in this situation.

δ) $\underline{\mathfrak{h} \not\subset \widetilde{\mathfrak{g}}}$

In this case, we set $\widetilde{\mathfrak{h}} = \mathfrak{h} \cap \widetilde{\mathfrak{g}}$ and choose $X \in \mathfrak{h}$ so that $\mathfrak{h} = \widetilde{\mathfrak{h}} \oplus \mathbb{R}X$ and $\mathfrak{g} = \widetilde{\mathfrak{g}} \oplus \mathbb{R}X$. Clearly, $\mathfrak{h} \subset X^{B_\ell}$. So, we have

$$\dim \mathfrak{h}(\ell) = \dim \mathfrak{h}(\ell') - 1$$

and $$\dim \mathfrak{h}(\ell) = \dim \mathfrak{h} \cap \widetilde{\mathfrak{g}}^{B_\ell} \quad \text{for } \ell \text{ generic in } \Gamma_{\mathfrak{g},\mathfrak{h}}.$$

Therefore, just as in β) above and for the same reasons, Diagram 2 holds.

Now it is easy to see that for any ideal $\mathfrak{g}_*$ of $\mathfrak{g}$ containing Y, we have $\mathfrak{h} \cap \mathfrak{g}_*^{B_\ell} = \widetilde{\mathfrak{h}} \cap \mathfrak{g}_*^{B_\ell}$ for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. Thus, we can replace $\mathfrak{h}$ by $\widetilde{\mathfrak{h}}$ in Diagram 2 above so that in particular,

$$\dim \widetilde{\mathfrak{h}} \cap (\mathfrak{g}' \cap \widetilde{\mathfrak{g}})^{B_\ell} - \dim \widetilde{\mathfrak{h}} \cap \widetilde{\mathfrak{g}}^{B_\ell} = 1 \quad \text{for generic } \ell \text{ in } \Gamma_{\mathfrak{g},\mathfrak{h}}. \tag{5.8}$$

We show that the assumption $W \notin \mathcal{U}(\mathfrak{g}') + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ leads to a contradiction. Indeed, in the present situation, we can write $W = X_p(\sum_i U_i X^i) + \sum_i V_i X^i$ with $U_i, V_i \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$, so that we have

$$W \equiv X_p \sum_i (-\sqrt{-1}\, f(X))^i\, U_i + \sum_i (-\sqrt{-1} f(X))^i V_i \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}.$$

Hence, without loss of generality, we may assume that $W = X_p U + V$ with $U, V \in \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}})$, so that $W \in \mathcal{U}(\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{h}})$ and $W \notin \mathcal{U}(\mathfrak{g}' \cap \widetilde{\mathfrak{g}}) + \mathcal{U}(\widetilde{\mathfrak{g}})\mathfrak{a}_{\widetilde{\mathfrak{h}}}$. Replacing $\mathfrak{g}$ by $\widetilde{\mathfrak{g}}$, $\mathfrak{g}'$ by $\mathfrak{g}' \cap \widetilde{\mathfrak{g}}$ and $\mathfrak{h}$ by $\widetilde{\mathfrak{h}}$ in the induction hypothesis on the dimension of $\mathfrak{g}$, we see that this is incompatible with (5.8).

This completes the proof of the theorem. □

We can now prove the conjecture of Corwin-Greenleaf and Duflo as stated in the introduction.

Corollary 5.3. *Let G be a connected, simply connected, nilpotent real Lie group with Lie algebra $\mathfrak{g}$ and H be a closed connected subgroup of G with Lie algebra $\mathfrak{h}$. Let f be a linear form on $\mathfrak{g}$ such that $f([\mathfrak{h}, \mathfrak{h}]) = \{0\}$. Let χ_f be the unitary character of H defined by $\chi_f(\exp(X)) = e^{\sqrt{-1}f(X)}$ for all X in $\mathfrak{h}$. Let $\tau_f = \operatorname{Ind}_H^G \chi_f$ be the unitary representation of G, induced from χ_f and defined by (1.1). Let $\mathcal{D}(\mathfrak{g}, \mathfrak{h})$ be the algebra of G-invariant linear differential operators defined by (1.4). Then, the following two assertions are equivalent*

(a) *τ_f is of finite multiplicities.*

(b) *$\mathcal{D}(\mathfrak{g}, \mathfrak{h})$ is commutative.*

Proof. $(a) \Rightarrow (b)$ is a fundamental result of Corwin-Greenleaf (Theorem 1.1 of [5]).

Let us prove that *non*-$(a) \Rightarrow$ *non*-(b) by induction on the dimension of $\mathfrak{g}$. First, recall from the assertion (ii) of Section 1 that for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$

$$\begin{aligned}\tau_f \text{ is of finite multiplicities}&\\ &\Longleftrightarrow \dim H\cdot\ell = \tfrac{1}{2}\dim G\cdot\ell\\ &\Longleftrightarrow 2(\dim\mathfrak{h}-\dim\mathfrak{h}(\ell)) = \dim\mathfrak{g}-\dim\mathfrak{g}(\ell).\end{aligned}$$

Thus, it suffices to prove that $2(\dim\mathfrak{h}-\dim\mathfrak{h}(\ell)) < \dim\mathfrak{g}-\dim\mathfrak{g}(\ell)$ implies that $\mathcal{D}(\mathfrak{g},\mathfrak{h})$ is not commutative. In this case, $\mathfrak{h}\neq\mathfrak{g}$. Let $\mathfrak{g}'$ be a subalgebra of codimension one in $\mathfrak{g}$ that contains $\mathfrak{h}$. If already $\mathcal{D}(\mathfrak{g}',\mathfrak{h})\subset\mathcal{D}(\mathfrak{g},\mathfrak{h})$ is non-commutative then obviously, $\mathcal{D}(\mathfrak{g},\mathfrak{h})$ has the same property. So we may assume that $\mathcal{D}(\mathfrak{g}',\mathfrak{h})$ is commutative or equivalently, using the induction argument, that $2(\dim\mathfrak{h}-\dim\mathfrak{h}(\ell')) = \dim\mathfrak{g}'-\dim\mathfrak{g}'(\ell')$. Subtracting both relations, we have

$$2(\dim\mathfrak{h}(\ell')-\dim\mathfrak{h}(\ell)) < 1+\dim\mathfrak{g}'(\ell')-\dim\mathfrak{g}(\ell)\leq 2$$

which implies $\mathfrak{h}(\ell')=\mathfrak{h}(\ell)$. Then, Theorem 5.2 asserts that there exists an element $W\in\mathcal{U}(\mathfrak{g},\mathfrak{h})$ such that $W\notin\mathcal{U}(\mathfrak{g}')+\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Finally, using Theorem 1 of [8], we obtain an element T in $\mathcal{U}(\mathfrak{g}',\mathfrak{h})$ such that $[W,T]\notin\mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, as expected. This proves that $\mathcal{D}(\mathfrak{g},\mathfrak{h})$ is not commutative. ☐

6. Example

In this section, $\mathfrak{g}$ will denote the real nilpotent Lie algebra of dimension 7 generated by the vectors $\{X_i,\ 1\leq i\leq 7\}$ with the following non-zero brackets:

$$\begin{aligned}&[X_1,X_3]=X_2,\ [X_1,X_4]=X_3,\ \ [X_1,X_5]=X_4,\ [X_1,X_7]=X_6,\\ &[X_4,X_5]=X_6,\ [X_5,X_6]=X_2,\ \ [X_4,X_7]=-X_2.\end{aligned}$$

It is clear that the center of $\mathfrak{g}$ is $\mathfrak{z}=\mathbb{R}X_2$.

We choose the flag $\mathcal{S}$ of $\mathfrak{g}$ defined in (2.1) as follows:

$$\{0\}=\mathfrak{g}_0\subset\mathfrak{g}_1\subset\mathfrak{g}_2\subset\cdots\subset\mathfrak{g}_6\subset\mathfrak{g}_7=\mathfrak{g}$$

where

$$\begin{aligned}&\mathfrak{g}_0=\{0\},\qquad \mathfrak{g}_1=\mathbb{R}X_2=\mathfrak{z},\qquad \mathfrak{g}_2=\mathbb{R}X_2\oplus\mathbb{R}X_3,\\ &\mathfrak{g}_3=\mathbb{R}X_2\oplus\mathbb{R}X_3\oplus\mathbb{R}X_6,\qquad \mathfrak{g}_4=\mathbb{R}X_2\oplus\mathbb{R}X_3\oplus\mathbb{R}X_6\oplus\mathbb{R}X_4,\\ &\mathfrak{g}_5=\mathbb{R}X_2\oplus\mathbb{R}X_3\oplus\mathbb{R}X_6\oplus\mathbb{R}X_4\oplus\mathbb{R}X_5,\\ &\mathfrak{g}_6=\mathbb{R}X_2\oplus\mathbb{R}X_3\oplus\mathbb{R}X_6\oplus\mathbb{R}X_4\oplus\mathbb{R}X_5\oplus\mathbb{R}X_1,\ \mathfrak{g}_7=\mathfrak{g}.\end{aligned}$$

Next, we equip $\mathfrak{g}^*$ with the dual basis $\{X_j^*\mid 1\leq j\leq 7\}$ of the basis $\{X_j\mid 1\leq j\leq 7\}$ of $\mathfrak{g}$. We take $\mathfrak{h}\overset{\text{def.}}{=}\mathbb{R}X_4$ and fix $f\overset{\text{def.}}{=}\lambda X_4^*$ so that the affine space defined by (1.3) is simply

$$\Gamma_{\mathfrak{g},\mathfrak{h}}=\{\sum_{j=1}^{7}\xi_jX_j^*\mid\xi_4=\lambda\}.$$

We now describe the *generic* H-orbits, namely the H-orbits of maximal dimension in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. They are contained in the following Zariski-open subset:

$$\mathcal{O} = \{\sum_{j=1}^{7} \xi_j X_j^* \in \Gamma_{\mathfrak{g},\mathfrak{h}} \mid \xi_6 \neq 0\}.$$

A simple and direct calculation shows that if $\ell = \sum_{j=1}^{7} \xi_j X_j^* \in \Gamma_{\mathfrak{g},\mathfrak{h}}$, then for all real number t, $\mathrm{Ad}^*(\exp(-tX_4))(\ell) = \sum_{j=1}^{7} \xi_j(t) X_j^*$ where

$$\begin{aligned} &\xi_1(t) = \xi_1 - t\xi_3, \quad \xi_2(t) = \xi_2, \quad \xi_3(t) = \xi_3, \quad \xi_4(t) = \lambda, \\ &\xi_5(t) = \xi_5 + t\xi_6, \quad \xi_6(t) = \xi_6, \quad \xi_7(t) = \xi_7 - t\xi_2. \end{aligned} \tag{6.1}$$

The set of indices of $\mathcal{I}$ and $\mathcal{J}$ defined by (2.2) and (2.4) are respectively $\{4\}$ and $\{1, 2, 3, 5, 6, 7\}$, so that the sequence of subalgebras (2.5) becomes:

$\mathfrak{k}_0 = \mathfrak{h} = \mathbb{R}X_4$
$\mathfrak{k}_1 = \mathbb{R}X_2 \oplus \mathbb{R}X_4$
$\mathfrak{k}_2 = \mathbb{R}X_2 \oplus \mathbb{R}X_3 \oplus \mathbb{R}X_4$
$\mathfrak{k}_j = \mathfrak{g}_{j+1}$ for $3 \leq j \leq 6$

Then, we consider the associated sequence of subalgebras:

$$\mathcal{D}(\mathfrak{k}_1, \mathfrak{h}) \subseteq \mathcal{D}(\mathfrak{k}_2, \mathfrak{h}) \subseteq \cdots \subseteq \mathcal{D}(\mathfrak{k}_5, \mathfrak{h}) \subseteq \mathcal{D}(\mathfrak{k}_6, \mathfrak{h}) = \mathcal{D}(\mathfrak{g}, \mathfrak{h}). \tag{6.2}$$

Theorem 5.2 says exactly which of these inclusions is proper or is an equality.

Using the calculation (6.1) and setting here $\ell_j \stackrel{\text{def.}}{=} \ell|_{\mathfrak{k}_j}$ for all $\ell \in \mathfrak{g}^*$, $1 \leq j \leq 6$, we obtain

$$\begin{aligned} &\dim H \cdot \ell_j = 0, \qquad \forall \ell \in \mathcal{O}, \quad 0 \leq j \leq 3, \\ &\dim H \cdot \ell_j = 1, \qquad \forall \ell \in \mathcal{O}, \quad 4 \leq j \leq 6. \end{aligned} \tag{6.3}$$

Following the sequence $\mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_6$, there is only one jump in the dimensions of the generic H-orbits in $\Gamma_{\mathfrak{g},\mathfrak{h}}$ which arises when passing from $\mathfrak{k}_3$ to $\mathfrak{k}_4$. Thus, Theorem 5.2 implies that $\mathcal{D}(\mathfrak{k}_j, \mathfrak{h})$ is properly contained in $\mathcal{D}(\mathfrak{k}_{j+1}, \mathfrak{h})$ for $1 \leq j \leq 6$, except for $j = 3$ where we have the equality $\mathcal{D}(\mathfrak{k}_3, \mathfrak{h}) = \mathcal{D}(\mathfrak{k}_4, \mathfrak{h})$. So (6.2) reads as

$$\mathcal{D}(\mathfrak{k}_1, \mathfrak{h}) \subsetneq \mathcal{D}(\mathfrak{k}_2, \mathfrak{h}) \subsetneq \mathcal{D}(\mathfrak{k}_3, \mathfrak{h}) = \mathcal{D}(\mathfrak{k}_4, \mathfrak{h}) \subsetneq \mathcal{D}(\mathfrak{k}_5, \mathfrak{h}) \subsetneq \mathcal{D}(\mathfrak{g}, \mathfrak{h}). \tag{6.4}$$

In other words, there exists a non zero element of $\mathcal{D}(\mathfrak{k}_{j+1}, \mathfrak{h})$ that does not belong to $\mathcal{D}(\mathfrak{k}_j, \mathfrak{h})$ for all j, except for $j = 3$. To check this, we shall construct explicitly such a new element. We proceed as in [8], by applying the symmetrization map to suitable H-invariant polynomials arising from the Pukanszky parametrization of the generic H-orbits in $\Gamma_{\mathfrak{g},\mathfrak{h}}$ (see [11]).

More precisely, setting $u \stackrel{\text{def.}}{=} \xi_5 + t\xi_6$ in the calculations (6.1), we parametrize the generic H-orbits in $\Gamma_{\mathfrak{g},\mathfrak{h}}$ as follows: if $\ell = \sum_{j=1}^{7} \xi_j X_j^* \in \mathcal{O}$, then we let $\mathrm{Ad}^*(\exp(-tX_4))(\ell) = \sum_{j=1}^{7} r_j(u)X_j^*$ for all real number t, with

$$r_1(u) = \frac{\xi_1\xi_6 + \xi_3\xi_5 - u\xi_3}{\xi_6}, \quad r_2(u) = \xi_2, \quad r_3(u) = \xi_3, \quad r_4(u) = \lambda,$$
$$r_5(u) = u, \quad r_6(u) = \xi_6, \quad r_7(u) = \frac{\xi_7\xi_6 + \xi_2\xi_5 - u\xi_2}{\xi_6}.$$

This gives us the following H-invariant polynomials on $\mathfrak{g}^*$: $\xi_1\xi_6 + \xi_3\xi_5$, ξ_2, ξ_3, ξ_6 and $\xi_7\xi_6 + \xi_2\xi_5$. Applying to these polynomials the symmetrization map $\mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, we obtain the elements $X_1X_6 + X_3X_5$, X_2, X_3, X_6 and $X_7X_6 + X_2X_5$ of $\mathcal{U}(\mathfrak{g}, \mathfrak{h})$. So, we have the following new element of $\mathcal{U}(\mathfrak{g}, \mathfrak{h})$:

X_2, when passing from $\mathbb{C} = \mathcal{U}(\mathfrak{k}_0, \mathfrak{h})$ to $\mathcal{U}(\mathfrak{k}_1, \mathfrak{h})$,
X_3, when passing from $\mathcal{U}(\mathfrak{k}_1, \mathfrak{h})$ to $\mathcal{U}(\mathfrak{k}_2, \mathfrak{h})$,
X_6, when passing from $\mathcal{U}(\mathfrak{k}_2, \mathfrak{h})$ to $\mathcal{U}(\mathfrak{k}_3, \mathfrak{h})$,
$X_1X_6 + X_3X_5$, when passing from $\mathcal{U}(\mathfrak{k}_4, \mathfrak{h})$ to $\mathcal{U}(\mathfrak{k}_5, \mathfrak{h})$,
$X_7X_6 + X_2X_5$, when passing from $\mathcal{U}(\mathfrak{k}_5, \mathfrak{h})$ to $\mathcal{U}(\mathfrak{g}, \mathfrak{h})$.

Let us check that no new element of $\mathcal{D}(\mathfrak{g}, \mathfrak{h})$ arises when we pass from $\mathcal{D}(\mathfrak{k}_3, \mathfrak{h})$ to $\mathcal{D}(\mathfrak{k}_4, \mathfrak{h})$. Assume there exists A in $\mathcal{U}(\mathfrak{k}_4, \mathfrak{h})$ that does not belong to $\mathcal{U}(\mathfrak{k}_3) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$. Using the assertion (*iii*) of Proposition 3.7, we may write A as $A = X_5U + V$ with U and V in $\mathcal{U}(\mathfrak{k}_3)$. Then, observing that $\mathfrak{k}_3$ is commutative, it is easy to see that $[A, X_4] = -X_6U$. Since $X_6 \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, this forces $U \in \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ and then $V \in \mathcal{U}(\mathfrak{k}_3, \mathfrak{h})$. Thus, we have $A \in \mathcal{U}(\mathfrak{k}_3, \mathfrak{h}) + \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$ thereby proving that $\mathcal{D}(\mathfrak{k}_3, \mathfrak{h}) = \mathcal{D}(\mathfrak{k}_4, \mathfrak{h})$.

Now we turn to the question of the commutativity of $\mathcal{D}(\mathfrak{g}, \mathfrak{h})$. After a straightforward calculation, we observe that the generic G-orbits in $\Gamma_{\mathfrak{g},\mathfrak{h}}$ are 6-dimensional while the generic H-orbits are 1-dimensional by (6.3). So, the representation $\mathrm{Ind}\underset{H}{\overset{G}{\uparrow}}\chi_f$ is of infinite multiplicities by the assertion (*iii*) of Section 1. Therefore, we expect from Corollary 5.3 that $\mathcal{D}(\mathfrak{g}, \mathfrak{h})$ is not commutative.

Indeed, for $1 \leq j \leq 3$, the groups $K_j = \exp \mathfrak{k}_j$ are abelian. So, the mutiplicities of the representations $\mathrm{Ind}\underset{H}{\overset{K_j}{\uparrow}}\chi_{f_j}$ are finite while the algebras $\mathcal{D}(\mathfrak{k}_j, \mathfrak{h})$ are commutative. This agrees with Corollary 5.3 as well as with the fundamental result of Corwin-Greenleaf (Theorem 1.1 of [5]).

For $j = 4$, the dimension of the K_4-orbits $K_4 \cdot \ell_4$ are 2-dimensional while, by (6.3), the H-orbits $H \cdot \ell_4$ are 1-dimensional for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. Hence, we deduce that $\mathrm{Ind}\underset{H}{\overset{K_4}{\uparrow}}\chi_{f_4}$ is of finite multiplicities. At the same time, (6.4) implies that $\mathcal{D}(\mathfrak{k}_4, \mathfrak{h})$ is commutative, as expected from Corollary 5.3.

For $j = 5$, the orbits $K_5 \cdot \ell_5$ are 4-dimensional while the orbits $H \cdot \ell_5$ are 1-dimensional by (6.3), for generic ℓ in $\Gamma_{\mathfrak{g},\mathfrak{h}}$. Hence, we deduce that $\mathrm{Ind}\underset{H}{\overset{K_5}{\uparrow}}\chi_{f_5}$ is of infinite multiplicities. At the same time, X_3 and $X_1X_6 + X_3X_5$ are two elements of $\mathcal{U}(\mathfrak{k}_5, \mathfrak{h})$ satisfying $[X_3, X_1X_6 + X_3X_5] = -X_2X_6$. Since $X_2X_6 \notin \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}$, we see that

$$[X_3, X_1X_6 + X_3X_5] \not\equiv 0 \mod \mathcal{U}(\mathfrak{g})\mathfrak{a}_{\mathfrak{h}}.$$

This proves that $\mathcal{D}(\mathfrak{k}_5, \mathfrak{h})$ is not commutative in agreement with Corollary 5.3.

These results show that $\mathrm{Ind}\overset{G}{\underset{H}{\uparrow}} \chi_f$ is of infinite multiplicities and $\mathcal{D}(\mathfrak{g}, \mathfrak{h})$ not commutative.

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