On strongly Petrovskiĭ's parabolic SPDEs in arbitrary dimension

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Abstract

In this paper we show that the Cahn-Hilliard stochastic SPDE has a function valued solution in dimension 4 and 5 when the perturbation is driven by a space-correlated Gaussian noise. This is done proving general results on SPDEs with globally Lipschitz coefficients associated with operators on smooth domains of $\mathbb{R}^d$ which are parabolic in the sense of Petrovskiĭ, and do not necessarily define a semi-group of operators. We study the regularity of the trajectories of the solutions and the absolute continuity of the law at some given time and position.
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1 Introduction - Weak solution

Let Q be a compact subset of $\mathbb{R}^d$, σ and b_i $1 \le i \le N$ be real-valued functions defined on $[0,T] \times Q \times \mathbb{R}$ and $(k_i, 1 \le i \le N)$ be multi-indices. Let F denote a one-dimensional (d+1)-parameter Gaussian noise (either a space-time white noise, or a space correlated noise), $A(t,x,D_x)$ denote a differential operator of order $2n$. Consider the following stochastic partial differential equation

$$\frac{\partial}{\partial t} u(t,x) = A(t,x,D_x)\, u(t,x) + \sigma(t,x,u(t,x))\, \dot{F}(t,x) + \sum_{i=1}^{N} D_x^{k_i}\big(b_i(t,x,u(t,x))\big)\,, \qquad (1.1)$$

with some homogeneous boundary conditions, denoted by **(BC)**, defined for $1 \le q \le n$ by

$$B_q(t,x,D_x)u(t,x) = \sum_{|k|\le r_q} B_{q,k}(t,x)\, D_x^k u(t,x) = 0 \text{ on } [0,T] \times \partial Q,\ r_q \le 2n-1\,,$$

and the initial condition $u(0,.) = u_0$. We suppose that the operator $L = \frac{\partial}{\partial t} - A(t,x,D_x)$ is uniformly parabolic in the sense of Petrovskiĭ and that the boundary conditions are complementary and satisfy the normality assumption (for the complete definition see e.g. S.D. Eidelman and N.V. Zhitarashu [13], p. 2-17). The most important class of uniformly strongly parabolic operator in the sense of Petrovskiĭ is defined by:

$$A(t,x,D_x) = \sum_{|k|\le 2n} a_k(x,t) D_x^k\,,$$

where there exists a positive constant δ_0 such that for any $(x,t) \in Q \times [0,T]$, $\xi \in \mathbb{R}^d$,

$$(-1)^n \Big(\sum_{|k|=2n} a_k(x,t)\, \xi^k \Big) \le -\delta_0 |\xi|^{2n}.$$

In this particular case, the definitions can be found in [18], p.113-121. A simple example is provided by $A(t,x,D_x) = a_1(t,x)D_n(a_2(t,x)D_n)$, where D_n is a differential operator of order n, and $(-1)^n \sup\{(a_1a_2)(t,x) \,:\, (t,x)\in[0,T]\times Q\} < 0$. When $n=2$ and $D_2=\Delta$, the Dirichlet boundary conditions ($u=\Delta u = 0$ on $[0,T]\times\partial Q$) or Neumann boundary conditions ($\frac{\partial}{\partial\nu}u = \frac{\partial}{\partial\nu}\Delta u = 0$ on $[0,T]\times\partial Q$, where ν denotes the outer normal) are normal and complementary.

A similar equation has been studied by Z. Brzezniak and S. Peszat [3] when $Q=\mathbb{R}^d$ for smooth bounded coefficients a_k which do not depend on the time parameter t. Using time-homogeneous semi-group techniques, these authors prove the existence and uniqueness of a mild solution to (1.1) in some weighted L^p spaces (or the set of continuous functions having some decay property at infinity), depending on the hypothesis on the initial condition u_0. Unlike this paper, we allow the case where the differential operator does not yield a time-homogeneous semi-group and work with martingale-measures as in J.B. Walsh [28].

Let $l\geq 0$ be an integer and $\lambda\in]0,1[$. According to S.D. Eidelman and N.V. Ivasisen [12], if ∂Q is of class $\mathcal{C}^{2n+l+\lambda}$, the coefficients $a_k(t,x)$ are of class $\mathcal{C}^{2n(l+\lambda),l+\lambda}([0,T]\times Q)$ for $|k|\leq 2n$, the coefficients $B_{q,k}(t,x)$ are of class $\mathcal{C}^{(2n-r_q+l+\lambda)2n,2n-r_q+l+\lambda}([0,T]\times\partial Q)$, then if G denotes the Green function associated with L and the boundary conditions **(BC)**, for $|a|+2nb\leq 2n+l$, one has for every $t>s\geq 0$ and $x,y\in Q$

$$\left| D_x^a \frac{\partial^b}{\partial t^b} G(t,x;s,y)\right| \leq C(t-s)^{-(\alpha+a\,\delta+b\,\eta)} \exp\left(-c\,\frac{|x-y|^\beta}{(t-s)^\gamma}\right), \tag{1.2}$$

$$\text{with } \alpha=\frac{d}{2n},\ \beta=\frac{2n}{2n-1},\ \gamma=\frac{1}{2n-1},\ \delta=\frac{1}{2n} \text{ and } \eta=1. \tag{1.3}$$

Note that in some cases, it is possible to extend this upper estimate on G to the case the subset Q is not smooth (see e.g. [4] for the case $A=-\Delta^2$ on $Q=[0,\pi]^d$ and homogeneous Neumann's boundary conditions). Therefore, for $d<2n$, the integral $\int_0^t\int_Q G^2(t,x;s,y)\,dyds$ converges, so that the stochastic integrals of $G(t,x;s,y)$ with respect to the space-time white noise $F(ds,dy)$ are well-defined. Usual arguments show that in the particular case of selfadjoint operators $A(t,x,D_x)$, such as $A=a_1(t)D_n(a_2(t,x)D_n)$ with appropriate normal and complementary Dirichlet boundary conditions, the Green function $G(t,.;s,.)$ is symmetrical in x and y, so that $D_y^aG(t,x;s,y) = D_y^{\tilde a}G(t,x;s,y)$ with $|a|=|\tilde a|$; then (1.2) holds for D_y^a instead of D_x^a.

We now generalize the setting of [28], in order to define a "weak" solution to (1.1), which is an alternative to mild solutions. Return time, and consider the adjoint operator $L^* = -\frac{\partial}{\partial t} - A^*(t,x,D_x)$ and the adjoint boundary conditions $B_q^*=0$, $1\leq q\leq n$, on $[0,t]\times\partial Q$; then for fixed $t>0$, exchanging the role of (t,x) and (s,y), $G(t,x;s,y)$ is the fundamental solution to the adjoint problem on the time interval $[0,t]$. Thus, for any smooth function ϕ on Q, the function

$$v(s,y) = \int_Q G(t,x;s,y)\,\phi(x)\,dx \tag{1.4}$$

is the solution to the equation $L^*v=0$ on $[0,t]\times Q$, with adjoint boundary conditions ($B_q^*v = 0, 1\leq q\leq n$) on $[0,t]\times\partial Q$, and such that $v(t,.)=\phi$. Then for Dirichlet's systems ($r_q=q-1$, $1\leq q\leq n$) or in particular cases (see e.g. example 1.1), for v defined by (1.4) and "regular" u, the following Green formula holds (see e.g. [13], p. 231 or [18], p. 133):

$$\int_0^t\int_Q (Lu)(s,y)\,v(s,y)\,dyds + \int_Q u(0,y)\,v(0,y)\,dy = \int_Q u(t,x)\,v(t,x)\,dx\,.$$

Furthermore, if integration by parts yields

$$\int_0^t\int_Q D_y^{k_i}(b_i(s,y,u(s,y))\,v(s,y))\,dyds = (-1)^{|k_i|}\int_0^t\int_Q b_i(s,y,v(s,y))\,D_y^{k_i}v(s,y)\,dyds$$

and if $\int_0^t \int_Q \sigma(s,y,u(s,y))\, v(s,y) F(ds,dy)$ is defined as the stochastic integral with respect to a worthy martingale measure, then even if u is not "regular", we can define the weak solution to (1.1), by requiring that the following form of equation (1) holds for every function v in $\mathcal{C}^{2n,1}([0,T]\times Q)$:

$$\int_0^t \int_Q \Big[\sigma(s,y,u(s,y))\, v(s,y)\, F(ds,dy) + \sum_{i=1}^N (-1)^{|k_i|} b_i(s,y,u(s,y)) D_y^{k_i} v(s,y)\, dyds \Big] + \int_Q u(0,y)\, v(0,y)\, dy = \int_Q u(t,x)\, v(t,x)\, dx \; . \tag{1.5}$$

Using (stochastic) Fubini's theorem we obtain the following evolution equation, which is equivalent to (1.5):

$$\begin{aligned} u(t,x) \;=\; & \int_Q G(t,x;0,y)\, u_0(y)\, dy + \int_0^t \int_Q G(t,x;s,y)\, \sigma(s,y,u(s,y))\, F(ds,dy) \\ & + \sum_{i=1}^N \int_0^t \int_Q H_i(t,x;s,y)\, b_i(s,y,u(s,y))\, dy\, ds \,, \end{aligned} \tag{1.6}$$

where $H_i(t,x;s,y) = (-1)^{|k_i|} D_y^{k_i} G(t,x;s,y)$; if $G(t,x;s,y)$ is symmetric in x and y, the upper estimate (1.2) implies that $|H_i(t,x;s,y)| \le C\,(t-s)^{-(\alpha+|k_i|\delta)} \exp\Big(-c\, \frac{|x-y|^\beta}{(t-s)^\gamma}\Big)$. We give an example where all the requirements (except that on the existence of the stochastic integral) are fulfilled.

Example 1.1 *The boundary of the set Q is of class $\mathcal{C}^{4+l+\lambda}$, the functions $a_1(t) \in \mathcal{C}^{4(l+\lambda)}([0,T])$, $a_2(t,x) \in \mathcal{C}^{4(l+\lambda),l+\lambda}([0,T]\times Q)$, $A(t,x,D_x) = a_1(t)\, \Delta(a_2(t,x)\, \Delta)$ and $\sup_{(t,x)\in[0,T]\times Q} a_1(t)\, a_2(t,x) < 0$.*

Case 1 *Dirichlet's boundary conditions: $u = \Delta u = 0$ on $[0,T]\times \partial Q$, either $0 \le |k_i| \le 1$, or $|k_i| = 2$ and $b_i(s,y,z) = \tilde{b}_i(z)$ for some function $\tilde{b}_i$ of class $\mathcal{C}_2$ such that $\tilde{b}_i(0) = 0$.*

Case 2 *Neumann's boundary conditions: let ν denote the outer normal, $\frac{\partial}{\partial\nu} u = \frac{\partial}{\partial\nu}\Delta u = 0$, $\frac{\partial}{\partial\nu} a_2(t,x) = 0$ on $[0,T]\times\partial Q$, $k_i = 0$ or $|k_i| = 2$, k_i has even components and $b_i(s,y,z) = \tilde{b}_i(z)$ for some function $\tilde{b}_i$ of class $\mathcal{C}_2$.*

For $d \ge 2n$, the function $G^2(s,x;s,y)$ need not be in $L^2([0,T]\times Q, dsdy)$, so that the Gaussian noise F need not be the space-time white noise; we require the noise F to be a Gaussian process which is white in time, but has a space correlation defined in terms of a function f depending on the difference of two vectors of $\mathbb{R}^d$ (or such that when $|x-y| \to 0$, the product $f(x,y)\,|x-y|^a$ remains bounded for some $a > 0$). We just mention a few previous papers on this subject, stressing the type of noise which is used. A particular case of this noise (where the function f only depends of the norm $|x-y|$, such as $f(x-y) = |x-y|^{-a}$ for $0 < a < d$) has been used in C. Mueller [22], R. Dalang and N. Frangos [7], A. Millet and M. Sanz-Solé [21] in the case $Q = \mathbb{R}^2$ for the wave operator. In these papers, the existence and uniqueness of a continuous solution is proved by precise estimates of integrals involving the corresponding Green function. A more general covariance structure (depending on the radon measure μ with Fourier transform f, or more generally a tempered distribution $\Gamma = \hat{\mu}$) has been used in [3], [16], [17], [26],[24] for the wave and heat operators on $\mathbb{R}^d$; in the last references, the existence of a solution is proved in some weighted L^p-space, or in the space of continuous functions with some decay at infinity, and the method uses the Fourier transform of the Green function G (see e.g. S. Peszat and J. Zabczyk [26] for a detailed account of existence and uniqueness results to parabolic SPDEs with a semi-group structure in any dimension). This general covariance was also used by R. Dalang [6], who proves the existence of continuous processes solutions to the heat and wave stochastic

SPDEs by means of an extension of stochastic integrals with respect to martingale measures for distribution-valued integrands. In these references, the coefficients of the differential operator A do not depend on (t, x).

On the other hand, several attempts have been made to find function-valued solutions to "highly non-linear" stochastic SPDEs, namely PDEs with a polynomial forcing term b_i (such as the Burgers PDE ($d = 1$, $A = \Delta$, $N = 1$, $a_1 = 1$ and $b_1(t, x, y) = y^2$), or the Cahn-Hilliard's PDE ($d \leq 3$, $A = -\Delta^2$, and $\sum_i D_x^{k_i} b_i(t, x, u(t, x)) = \Delta R(u(t, x))$, where R is a polynomial of odd degree with positive dominant coefficient) and with a stochastic perturbation driven by the space-time white noise. Thus, G. Da Prato A. Debussche and Temam [9] and then I. Gyöngy [15] have proved the existence of a function-valued solution to the stochastic Burgers equation in dimension 1. G. Da Prato and A. Debussche [8] have proved the existence of a function-valued solution to the stochastic Cahn-Hilliard equation in dimension 1 (up to 3) when the perturbation is driven by a space-time white noise (a Gaussian noise with some spatial correlation). C. Cardon-Weber [4] and [5] has proved the existence of a function-valued solution to the stochastic Cahn-Hilliard equation in dimension $d \leq 3$ when R is a polynomial of degree 3 and when the stochastic perturbation is driven by the space-time white noise. The method used in these papers is the following: using a truncation procedure and the existence and uniqueness results proved in the case of globally Lipschitz coefficients, one proves the existence and uniqueness of a solution to the SPDE where the polynomial coefficients have been changed. Then the uniqueness property of the solution allows to use concatenation to obtain the existence of a solution up to some stopping time. Finally, a priori estimates for a deterministic PDE obtained by isolating the stochastic integral (whose behavior is controlled by means of the Garsia lemma), prove that this stopping time is the terminal time T. These last estimates use methods of analysis which heavily depend on the specific form of the PDE, and no general scheme can be given. Let us finally mention that, using semi-group techniques, Z. Brzezniak and S. Peszat [3] have proved the existence of solutions to some SPDE with a polynomial drift term (when $N = 1$, $a_1 = 0$, and when the operator A is of order 2 and yields a semi-group of operators). Also in [20], the existence of the solution to a stochastic wave equation in dimension 2 with a non-uniformly Lipschitz drift has been proved, while the existence to the stochastic KDV equation has been shown by A. Debussche and A. de Bouard [11].

The aim of this paper is two-fold. On one hand, we prove that in this general context with time and space dependent coefficients, the upper estimates (1.2) of the Green function G and its time and space derivatives are sufficient to ensure the existence and uniqueness of the solution u to (1.6), provided that some integrability condition of the covariance function f on a neighborhood of 0 is required. We prove that, when the Green function G has a lower estimate by $t^{-\alpha}$ on the diagonal (which can be the case when it admits an explicit eigenvectors-eigenvalues expansion), this condition is necessary to be able to consider stochastic integrals of G. As in [28] and [6], we use stochastic integrals with respect to martingale measures. We give sufficient conditions on the covariance function f for the trajectories of u to be Hölder-continuous. We then use these results to extend the existence and uniqueness of a function-valued solution to the stochastic Cahn-Hilliard equation when $Q = [0, \pi]^d$ or a bounded "smooth" subset of $\mathbb{R}^d$, $d = 4, 5$. We give necessary and sufficient conditions on the covariance function f to ensure that the stochastic integral of the corresponding Green function G is well-defined. We study regularity properties of the trajectories of u and prove that, if $Q = [0, \pi]^d$ and the diffusion coefficient σ is strictly elliptic, the law of $u(t, x)$ has a density for $t > 0$ and $x \in Q$. This extends the results proved in [8] and [4] to higher dimensions. For the sake of simplicity, we mostly restrict ourselves to the case F is a space-correlated noise; this could be avoided in "small" dimension for arbitrary Petrovskiĭ's parabolic SPDEs. Also note that the proof of the existence of a solution to the stochastic Cahn-Hilliard equation extends directly to the more general situation described in Example 1.1, when R is a polynomial of degree 3 with positive dominant coefficient (which has no constant term in the case of the Dirichlet boundary conditions).

The paper is organized as follows. Section 2 gives necessary and sufficient conditions to ensure that stochastic integrals of a function G which satisfies (1.2) are well defined, and sufficient conditions to ensure Hölder properties of stochastic integrals appearing in (1.6), provided that the process u has bounded moments. In section 3, we prove both the existence of solutions to (1.6) either in $\mathcal{C}([0,T],L^q(Q))$ or $\mathcal{C}([0,T]\times Q)$ when the coefficients are globally Lipschitz functions. We then concentrate on the proof of the a priori estimates which allow to deduce the existence and uniqueness of a solution to the stochastic Cahn-Hilliard equation in dimension 4 and 5. Section 4 establishes Hölder regularity of the trajectories, while section 5 shows the absolute continuity of the law of $u(t,x)$ for $t>0$ and $x\in Q$ and the solution u to the Cahn-Hilliard equation.

All the constants C appearing in the statements can change from one line to the next one. When we want to stress the fact that C depends on some parameter k, we denote it by C_k.

2 Stochastic integrals with respect to a space correlated noise

Let Q be a compact subset of $\mathbb{R}^d$, $\mathcal{D}(\mathbb{R}_+\times Q)$ denote the space of functions $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}_+\times Q)$ with compact support, endowed with the topology defined by the following convergence: $\varphi_n\to\varphi$ if:

(i) There exists a compact subset K of $\mathbb{R}^+\times Q$ such that support$(\varphi_n-\varphi)\subset K$ for all n.
(ii) $\lim_{n\to+\infty} D^a\varphi_n = D^a\varphi$ uniformly on K for every multi-index a.

Let $F=\big(F(\varphi)\,,\,\varphi\in\mathcal{D}(\mathbb{R}_+\times Q)\big)$ be an $L^2(P)$-valued centered Gaussian process, which is white in space but has a space correlation defined as follows: given φ and ψ in $\mathcal{D}(\mathbb{R}_+\times Q)$, the covariance functional of $F(\varphi)$ and $F(\psi)$ is

$$J(\varphi,\psi)=E\big(F(\varphi)\,F(\psi)\big)=\int_0^{+\infty}dt\int_Q dy\int_Q \varphi(t,y)\,f(y-z)\,\psi(t,z)\,dz \tag{2.1}$$

where $(Q-Q)^*=\{y-z\,:\,y,z\in Q,\,y\neq z\}$ and $f:(Q-Q)^*\to[0,+\infty[$ is a continuous function. According to [27], the bilinear form J defined by (2.1)is non-negative definite if and only if f is the Fourier transform of a non-negative tempered distribution μ on Q. Then F defines a martingale-measure (still denoted by F), which allows to define stochastic integrals (see [28]).

In this section, we consider fairly general functions $H:([0,T]\times Q)^2\to\mathbb{R}$, including Green functions associated with parabolic operators in the sense of Petrovskiĭ: more precisely, we suppose that H satisfies the following upper estimate for some $\beta\geq 1$, some strictly positive parameters α and γ and some positive constants c and C: for any $t>0$, $x,y\in Q$:

$$|H(t,x;s,y)|\leq C\,(t-s)^{-\alpha}\,\exp\left(-c\frac{|x-y|^\beta}{(t-s)^\gamma}\right)\,. \tag{2.2}$$

Then the change of variables $u=(y-x)\,(t-s)^{-\frac{\gamma}{\beta}}$ yields

$$\int_Q |H(t,x;s,y)|\,dy\leq C\,(t-s)^{-\alpha+\frac{\gamma}{\beta}d}\,. \tag{2.3}$$

We now give a sufficient integrability condition on the space-correlation function f of the Gaussian noise F to ensure that the stochastic integral of a bounded adapted process multiplied by a kernel satisfying (2.2) is a well-defined stochastic process. Under some additional assumptions on H we prove that this condition is necessary.

Lemma 2.1 . *(i) Let $H : ([0,T]\times Q)^2 \to \mathbb{R}$ satisfy (2.2) and suppose that either*

$$\int_{B_d(0,1)} f(v)\ \ln\left(|v|^{-1}\right)\, dv < \infty, \quad if \quad d = \frac{\beta}{\gamma}(2\alpha-1)\,, \tag{2.4}$$

or

$$\int_{B_d(0,1)} f(v)\,|v|^{-[\frac{\beta}{\gamma}(2\alpha-1)-d]^+}\, dv < \infty, \quad if \quad d \neq \frac{\beta}{\gamma}(2\alpha-1)\,. \tag{2.5}$$

Then for any $t\in[0,T]$,

$$I(t) = \int_0^T \int_Q \int_Q |H(t,x;s,y)|\, f(y-z)\,|H(t,x;s,z)|\, dy\, dz\, dt < +\infty\,. \tag{2.6}$$

(ii) Let $H : ([0,T]\times Q)^2 \to \mathbb{R}$ satisfy

$$\inf\{H(t,x;s,x)\ :\ x\in Q\} \geq C_0(t-s)^{-\alpha} \tag{2.7}$$

and for every multi-index k with $|k|=1$:

$$\sup\left\{\left|D^k_y H(t,x;s,y)\right|\ :\ x,y\in Q\right\} \leq C_1(t-s)^{-(\alpha+\delta)}\,, \tag{2.8}$$

for $\alpha,\delta>0$, some constant $C>0$ and any $t>s>0$. Then if (2.6) holds for every $x\in Q$ and $t\in]0,T]$, one has either (2.9) or (2.10) depending on d, where:

$$\int_{B_d(0,1)} f(v)\ \ln\left(|v|^{-1}\right)\, dv < \infty, \quad if \quad d = \frac{1}{\delta}(2\alpha-1)\,, \tag{2.9}$$

or

$$\int_{B_d(0,1)} f(v)\,|v|^{-[\frac{1}{\delta}(2\alpha-1)-d]^+}\, dv < \infty, \quad if \quad d \neq \frac{1}{\delta}(2\alpha-1)\,. \tag{2.10}$$

Remark 2.2 . *When H is the Green function of the operator $\frac{\delta}{\delta t}+\Delta^2$ on $Q=[0,\pi]^d$ with homogeneous Neumann's or Dirichlet's boundary conditions, condition (2.7) holds with $\alpha=\frac{d}{4}$. We only sketch the proof for Neumann's boundary conditions; in that case, $G(t,x;s;x)\geq \sum_{k\in\mathbb{N}^{*d}}\exp(-|k|^4(t-s))\left(\prod_{1\leq i\leq d}\cos^2(k_ix_i)\right)$. Set $\mathcal{I}=]\frac{\pi}{2}-\theta,\frac{\pi}{2}+\theta[$ for some $\theta>0$ small enough; then for $k\in\mathbb{N}^*$ and $x\in]0,\pi[$, $kx\in\mathcal{I}$ implies that $(k+1)x\notin\mathcal{I}$. Skipping at most every other term and using the monotonicity of $(\exp(-k^4(t-s));k\geq 1)$, we deduce that $G(t,x;s,x)\geq C\sum_{k\in\mathbb{N}^{*d}}\exp(-c|k|^4(t-s)) = C(t-s)^{-\frac{d}{4}}$ for some positive constants c and C (see Remark 3.5 in [4] for a similar argument). Since in the case of Green functions of parabolic operators, $\delta=\frac{\gamma}{\beta}$, the conditions (2.4) and (2.9) (respectively (2.5) and (2.10)) are identical.*

Proof of Lemma 2.1: (i) Since $\beta\geq 1$, $|y-z|^\beta \leq 2^{\beta-1}(|x-y|^\beta+|x-z|^\beta)$, so that for $\lambda\in]0,1[$ and $t_0\in[0,T]$,

$$\begin{aligned}
I(t_0) &\leq C\int_0^T t^{-2\alpha}dt\int_Q \exp\left(-c\,(1-\lambda)\frac{|x-y|^\beta}{t^\gamma}\right)dy \\
&\qquad\qquad \times \int_Q \exp\left(-c\,2^{1-\beta}\,\lambda\frac{|y-z|^\beta}{t^\gamma}\right)\, f(y-z)\,dz \\
&\leq C\int_0^T t^{-2\alpha}dt\int_Q \exp\left(-\bar{c}\frac{|x-y|^\beta}{t^\gamma}\right)dy\int_Q \exp\left(-\bar{c}\,\frac{|y-z|^\beta}{t^\gamma}\right)\, f(y-z)\,dz\,.
\end{aligned} \tag{2.11}$$

Set $t^{\frac{\gamma}{\beta}}\eta = x-y$, $v=y-z$ and then $u=|\eta|$; there exist positive constants C, c and R such that

$$\begin{aligned} I(t_0) &\le C\int_0^T t^{\frac{\gamma}{\beta}d-2\alpha}dt \int_0^{+\infty} \exp\left(-cu^\beta\right) u^{d-1}\,du \int_{B_d(0,R)} \exp\left(-c\frac{|v|^\beta}{t^\gamma}\right) f(v)\,dv \\ &\le C\int_0^T \psi(t)\,dt\,, \end{aligned} \tag{2.12}$$

where for any $0<t\le T$ one sets

$$\psi(t) = t^{\frac{\gamma}{\beta}d-2\alpha}\int_{B_d(0,R)} \exp\left(-c\frac{|v|^\beta}{t^\gamma}\right) f(v)\,dv\,. \tag{2.13}$$

For fixed $v\neq 0$, set $r=|v|^\beta\,t^{-\gamma}$; then Fubini's theorem yields

$$\int_0^T \psi(t)\,dt \le C\int_{B_d(0,R)} f(v)\,|v|^{d+\frac{\beta}{\gamma}(1-2\alpha)}\,dv \int_{|v|^\beta\,T^{-\gamma}}^{+\infty} r^{-1+\frac{1}{\gamma}(2\alpha-1-\frac{\gamma d}{\beta})} \exp(-cr)\,dr\,.$$

We now distinguish three cases:

Case 1 If $2\alpha > 1+\frac{\gamma d}{\beta}$, then the second integral is bounded by a constant independent of $|v|$ and $\int_0^T \psi(t)\,dt \le C\int_{B_d(0,R)} f(v)\,|v|^{d+\frac{\beta}{\gamma}(1-2\alpha)}\,dv$, which yields (2.5).

Case 2 If $2\alpha = 1+\frac{\gamma d}{\beta}$, since for $0<|v|\le R$,

$$\int_{|v|^\beta\,T^{-\gamma}}^{+\infty} r^{-1}\,\exp(-cr)\,dr \le C\,|1-\ln(|v|)1_{\{|v|\le 1\}}|,$$

we have $\int_0^T \psi(t)\,dt \le C\int_{B_d(0,R)} f(v)\,\left[1+\ln\left(|v|^{-1}\right)1_{\{|v|\le 1\}}\right]dv$, which yields (2.4)

Case 3 Finally, if $2\alpha < 1+\frac{\gamma d}{\beta}$, then for $0<|v|\le R$,

$$\int_{|v|^\beta\,T^{-\gamma}}^{+\infty} r^{-1+\frac{1}{\gamma}(2\alpha-1-\frac{\gamma d}{\beta})} \exp(-cr)\,dr \le C\left(1+|v|^{\frac{\beta}{\gamma}(2\alpha-1)-d}\right)\,,$$

so that $\int_0^T \psi(t)\,dt \le C\int_{B_d(0,R)} f(v)\,dv$, which yields (2.5).

Note that for small T, the following computation gives a more precise upper estimate of $\int_0^T \psi(t)\,dt$, which will be used in the sequel. Indeed, for $\nu\in]0,\gamma[$, the decomposition of the integral over $B_d(0,R)$ into $\{|v|^\beta T^{-\gamma}\ge T^{-\nu}\}$ and its complement yields

$$\begin{aligned} \int_0^T \psi(t)\,dt &\le C\int_{B_d\left(0,T^{\frac{\gamma-\nu}{\beta}}\right)} f(v)\,|v|^{d+\frac{\beta}{\gamma}(1-2\alpha)}\,dv \int_{|v|^\beta T^{-\gamma}}^{+\infty} r^{-1+\frac{1}{\gamma}(2\alpha-1-\frac{\gamma d}{\beta})} \exp(-cr)\,dr \\ &\quad +C\,I\,\exp(-\bar{c}T^{-\nu})\,. \end{aligned}$$

Thus for $0<\nu<\gamma$ and $2\alpha\neq 1+\frac{\gamma d}{\beta}$ one has

$$\int_0^T \psi(t)\,dt \le C\left[\exp(-\bar{c}T^{-\nu}) + \int_{B_d\left(0,T^{\frac{\gamma-\nu}{\beta}}\right)} f(v)\,|v|^{\left[-\frac{\beta}{\gamma}(2\alpha-1)-d\right]^+}dv\right]\,, \tag{2.14}$$

while for $0<\nu<\gamma$ and $2\alpha = 1+\frac{\gamma d}{\beta}$ one has

$$\int_0^T \psi(t)\,dt \le C\left[\exp(-\bar{c}T^{-\nu}) + \int_{B_d\left(0,T^{\frac{\gamma-\nu}{\beta}}\right)} f(v)\,\ln\left(|v|^{-1}\right)dv\right]\,. \tag{2.15}$$

(ii) The assumptions (2.7) and (2.8) imply that for $|x-y|\le 2\,C_2\,t^\delta$ with $C_2<\frac{C_0}{4C_1}$ small enough, one has

$$\begin{aligned} H(t,x;s,y) &\ge H(t,x;s,x)-|H(t,x;s,x)-H(t,x;s,y)| \\ &\ge (C_0-2\,C_1\,C_2)\,(t-s)^{-\alpha}\ge \frac{C_0}{2}\,(t-s)^{-\alpha}\,. \end{aligned}$$

Let $a>0$ be such that $Q_a=\{x\in Q\,:\,d(x,\partial Q)>a\}\neq\emptyset$; then for $0<2\,C_2\,t_0^\delta\le a$, $0<s\le t_0\le T$, $x\in Q_{2\,C_2\,t_0^\delta}$, $y\in B_d(x,C_2\,s^\delta)$ and $z\in B_d(y,C_2\,s^\delta)$, one has $y,z\in Q$. Thus Fubini's theorem implies for $x\in Q_{2\,C_2\,t_0^\delta}\neq\emptyset$:

$$\begin{aligned} I(t_0) &\ge \int_0^{t_0} ds\int_{Q\cap B_d(x,C_2s^\delta)} dy\int_{Q\cap B_d(y,C_2s^\delta)} |H(t_0,x;s,y)|\,f(y-z)\,|H(t_0,x;s,z)|dz \\ &\ge C\int_0^{t_0} s^{-2\alpha+d\delta}\,ds\int_{B_d(0,C_2s^\delta)} f(v)\,dv\ge C\int_{B_d(0,R)} f(v)\,dv\int_{\left(\frac{|v|}{C_2}\right)^{\frac{1}{\delta}}}^{t_0} s^{-2\alpha+d\delta}ds \end{aligned}$$

for $R=C_2\,t_0^\delta$. Again we have to study three cases depending on the power of s.

Case 1 If $d\delta+1<2\alpha$, let $\bar R=\frac{R}{2}$; then one has

$$I(t_0)\ge C\int_{B_d(0,\bar R)} f(v)\,dv\int_{\left(\frac{|v|}{C_2}\right)^{\frac{1}{\delta}}}^{\left(\frac{2|v|}{C_2}\right)^{\frac{1}{\delta}}} s^{-2\alpha+d\delta}\,ds\ge C\int_{B_d(0,\bar R)} f(v)\,|v|^{\frac{1-2\alpha}{\delta}+d}\,dv\,, \tag{2.16}$$

which yields (2.10).

Case 2 If $d\delta+1=2\alpha$, let $\nu>0$ and let $\bar R=R\wedge 1\wedge(C_2^{-\nu}R^{1+\nu}))$; then $|v|\le\bar R$ and $t_0\le C_2^{\frac{1}{\delta}}$ imply $\left(\frac{|v|}{C_2}\right)^{\frac{1}{(1+\nu)\delta}}\le t_0$ and $|v|^{\frac12}\,C_2^{-1}\le 1$; hence $C_2\,|v|^{-1}\ge|v|^{-\frac12}$ and

$$I(t_0)\ge C\int_{B_d(0,\bar R)} f(v)\,dv\int_{\left(\frac{|v|}{C_2}\right)^{\frac{1}{\delta}}}^{\left(\frac{|v|}{C_2}\right)^{\frac{1}{(1+\nu)\delta}}} s^{-1}\,ds\ge C\int_{B_d(0,\bar R)} f(v)\,\ln\left(|v|^{-1}\right)dv\,, \tag{2.17}$$

which yields (2.9).

Case 3 Finally, if $d\delta+1>2\alpha$, one has $I(t_0)\ge C\int_{B_d(0,R)} f(v)\;dv$, which yields (2.10). □

The following lemma gives sufficient conditions on the covariance function f to obtain moment estimates of stochastic integrals which yield Hölder regularity of the corresponding process. For this, we impose an upper estimate of the space and time partial derivatives of the kernel H: there exist positive constants δ,η,c,C such that for any $t>0$, $x,y\in Q$ and $k\in\mathbb{N}^d$ with $|k|=1$,

$$\left|D_x^k H(t,x;s,y)\right|\le C(t-s)^{-(\alpha+\delta)}\,,\qquad\left|\frac{\partial}{\partial t}H(t,x;s,y)\right|\le C(t-s)^{-(\alpha+\eta)}\,\exp\left(-c\,\frac{|x-y|^\beta}{(t-s)^\gamma}\right)\,. \tag{2.18}$$

In order to deal with time increments, we impose also that f satisfies the following "monotonicity" condition:

(C1) There exist strictly positive constants C_1 and c_1 such that

$$f(u)\le C_1 f(v)\quad\text{for}\quad |v|\le c_1\,|u|\,. \tag{2.19}$$

Note that (C1) holds if $f(u)=|u|^{-a}$ for some $a>0$.

Lemma 2.3 . *Suppose that Q is convex and let $H : ([0,T]\times Q)^2 \to \mathbb{R}$ satisfy the condition (2.2). Let F be a Gaussian noise with spatial covariance defined by (2.1) such that the correlation function f satisfies (C1) and either (2.4) or (2.5). Fix $p\in[1,+\infty[$, let $u:\Omega\to\mathbb{R}$ be an adapted process such that $\sup_{(t,x)\in[0,T]\times Q} E(|u(t,x)|^{2p}) < +\infty$ and for $t\in[0,T]$ and $x\in Q$, let*

$$I(t,x) = \int_0^t \int_Q H(t,x;s,y)\, u(s,y)\, F(ds,dy)\,. \tag{2.20}$$

(i) Suppose that H satisfies (2.18) and let $a\in]0,1[$; if either

$$\int_{B_d(0,1)} f(v)\, \ln\big(|v|^{-1}\big)\, dv < +\infty \quad \text{for} \quad d = \frac{\beta}{\gamma}\,(2\alpha + a\delta - 1)\,, \tag{2.21}$$

or

$$\int_{B_d(0,1)} f(v)\, |v|^{-\left[\frac{\beta}{\gamma}(2\alpha+a\delta-1)-d\right]^+}\, dv < +\infty \quad \text{for} \quad d \neq \frac{\beta}{\gamma}\,(2\alpha + a\delta - 1)\,; \tag{2.22}$$

then there exists $C_p>0$ such that for every $x,x'\in Q$,

$$A(x,x') = \sup_{t\in[0,T]} E\left(|I(t,x) - I(t,x')|^{2p}\right) \le C_p\, |x-x'|^{ap}\,. \tag{2.23}$$

(ii) Suppose that H satisfies (1.2)and (1.3) and let $b\in]0,1[$; if either

$$\int_{B_d(0,1)} f(v)\, \ln\big(|v|^{-1}\big)\, dv < +\infty \quad \text{for} \quad d = \frac{\beta}{\gamma}\,(2\alpha + b\eta - 1)\,, \tag{2.24}$$

or

$$\int_{B_d(0,1)} f(v)\, |v|^{-[\frac{\beta}{\gamma}(2\alpha+b\eta-1)-d]^+}\, dv < +\infty \quad \text{for} \quad d \neq \frac{\beta}{\gamma}\,(2\alpha + b\eta - 1)\,; \tag{2.25}$$

then there exists $C_p>0$ such that for every $0\le t<t'\le T$,

$$B(t,t') = \sup_{x\in Q} E\left(\left|\int_0^t \int_Q \Big[H(t,x;s,y) - H(t',x;s,y)\Big]\, u(s,y)\, F(ds,dy)\right|^{2p}\right) \le C_p\, |t-t'|^{bp}\,,$$

and

$$C(t,t') = \sup_{x\in Q} E\left(\left|\int_t^{t'} \int_Q H(t',x;s,y)\, u(s,y)\, F(ds,dy)\right|^{2p}\right) \le C_p\, |t-t'|^{bp}\,. \tag{2.26}$$

Proof: (i) Burkholder's inequality implies that for every $p\in[1,+\infty[$,

$$\begin{aligned} A(x,x') \;&\le\; C_p \sup_{t\in[0,T]} E\Big|\int_0^t ds \int_Q dy \int_Q |H(t,x;s,y) - H(t,x';s,y)|\, |u(s,y)| \\ &\qquad \times f(y-z)\, |H(t,x;s,z) - H(t,x';s,z)|\, |u(s,z)|\, dz\Big|^p\,. \end{aligned}$$

We prove that for $\Delta(x,x') = \sup_{t\in[0;T]} \int_0^t ds \int_Q dy \int_Q dz\, |H(t,x;s,y) - H(t,x';s,y)|\, f(y-z)$ $\times |H(t,x;s,z) - H(t,x';s,z)|$, one has

$$\Delta(x,x') \le C_p\, |x-x'|^a\,, \tag{2.27}$$

provided that H satisfies either (2.21) or (2.22). Assuming that (2.27) holds, Hölder's and Schwarz's inequalities yield

$$\begin{aligned} A(x,x') &\le C_p\, \Delta(x,x')^{p-1} \sup_{t\in[0,T]} \int_0^t ds \int_Q dy\, |H(t,x;s,y) - H(t,x';s,y)| \\ &\quad \times \int_Q f(y-z)\, |H(t,x;s,z) - H(t,x';s,z)|\, E\left(|u(s,y)|^p\, |u(s,z)|^p\right) dz \\ &\le C_p\, \Delta(x,x')^p \sup_{(s,y)\in[0,T]\times Q} E(|u(s,y)|^{2p}) \le C_p\, |x-x'|^{ap}\,. \end{aligned}$$

In order to prove (2.27), we use Taylor's formula, the convexity of Q and the inequalities (2.2) and (2.18); thus for any $a\in]0,1[$, $\Delta(x,x') \le |x-x'|^a\, (T_1+T_2)$, where

$$\begin{aligned} T_1 &= \sup_{x\in Q} \int_0^T s^{-(2\alpha+a\delta)}\, ds \int_Q dy \int_Q \exp\left(-c\frac{|x-y|^\beta}{s^\gamma}\right) f(y-z) \exp\left(-c\frac{|x-z|^\beta}{s^\gamma}\right) dz\,, \\ T_2 &= \int_0^T s^{-(2\alpha+a\delta)}\, ds \int_Q dy \int_Q \exp\left(-c\frac{|x-y|^\beta}{s^\gamma}\right) f(y-z) \exp\left(-c\frac{|x'-z|^\beta}{s^\gamma}\right) dz\,. \end{aligned}$$

Replacing 2α by $2\alpha + a\delta$, the arguments used to prove part (i) of Lemma 2.1 show that if either (2.21) or (2.22) holds, then $T_1 < +\infty$. To study T_2, we have to distinguish several cases. Let $0<c_2<1$, $k\ge 1$ be such that $c_2^k < \frac{1}{3}$ and set $\varepsilon = c_2^{k+1}$, $\bar\varepsilon = \frac{c_2^k}{1-c_2^{k+1}}$; then $\frac{1}{1+\varepsilon} > \bar\varepsilon$ and $\varepsilon < 1$. We study three cases:

Case 1 If $|x-y| \ge \varepsilon\, |x'-y|$ or $|x'-z| \ge \varepsilon\, |x-z|$, we have (changing the constant in the exponential functions) $T_2 \le C(\varepsilon)\, T_1$, and the proof is complete.

Case 2 If $|x-y| < \varepsilon\, |x'-y|$, $|x'-z| < \varepsilon\, |x-z|$ and $|y-z| \le \bar\varepsilon\, |x-x'|$, then $|x-z| \ge \frac{|x-x'|}{1+\varepsilon}$ and we have

$$|x-y| \ge \big|\, |x-z| - |y-z|\, \big| \ge \left(\frac{1}{1+\varepsilon} - \bar\varepsilon\right) |x-x'| \ge \left(\frac{1}{\bar\varepsilon(1+\varepsilon)} - 1\right) |y-z|\,.$$

This implies for $\bar c = c \min\left(1, \frac{1}{\bar\varepsilon(1+\varepsilon)} - 1\right)$ which is positive by the choice of k:

$$T_2 \le \int_0^T s^{-(2\alpha+a\delta)}\, ds \int_Q dy \int_Q \exp\left(-\bar c\frac{|y-z|^\beta}{s^\gamma}\right) f(y-z) \exp\left(-\bar c\frac{|x'-z|^\beta}{s^\gamma}\right) dz\,,$$

and since this is the upper estimate of (2.11), the proof is again concluded by an argument similar to that in Lemma 2.1 (i), with $2\alpha + a\delta$ instead of 2α.

Case 3 Suppose finally that $|x-y| < \varepsilon\, |x'-y|$, $|x'-z| < \varepsilon\, |x-z|$ and $|y-z| > \bar\varepsilon |x-x'|$. Then $|x'-y| \le \frac{|x-x'|}{1-\varepsilon}$ and $|x-y| \le \frac{\varepsilon}{1-\varepsilon}\, |x-x'| \le \frac{\varepsilon}{(1-\varepsilon)\,\bar\varepsilon}\, |y-z| = c_2\, |y-z|$, so that $\frac{c_1}{c_2}\, |x-y| \le c_1\, |y-z|$. Since f satisfies (2.19), we have $f(y-z) \le C_1\, f\left(\frac{c_1}{c_2}\,(y-x)\right)$. Set $\bar y = x + \frac{c_1}{c_2}\,(y-x)$ and $\tilde c = c\, \min\left(\frac{c_1}{c_2}, 1\right)$; then

$$T_2 \le C \int_0^T s^{-(2\alpha+a\delta)}\, ds \int_Q d\bar y \int_Q \exp\left(-\tilde c\frac{|x-\bar y|^\beta}{s^\gamma}\right) f(\bar y - x) \exp\left(-\tilde c\frac{|x'-z|^\beta}{s^\gamma}\right) dz\,,$$

and again the proof is complete, since the right hand-side is similar to (2.11).

(ii) For $0 \le t < t' \le T$, set

$$\overline{\Delta}(t,t') = \sup_{x\in Q} \left| \int_0^t ds \int_Q dy\, |H(t,x;s,y) - H(t',x;s,y)| \right.$$

$$\times \int_Q f(y-z)\,|H(t,x;s,z) - H(t',x;s,z)|\,dz \Bigg|^p .$$

Again we prove that under either condition (2.24) or (2.25) we have

$$\overline{\Delta}(t,t') \le C|t-t'|^b . \tag{2.28}$$

If (2.28) holds, using again Burkholder's, Hölder's and Schwarz's inequalities, we deduce that

$$B(t,t') \le C_p\,\overline{\Delta}(t,t')^p \sup_{(t,y)\in[0,T]\times Q} E(|u(s,y)|^{2p}) \le C_p\,|t-t'|^{bp} .$$

We now prove (2.28). Using (2.2), (2.18) and Taylor's formula, we obtain for $h = t'-t$: for any $b \in]0,1[$, $\overline{\Delta}(t,t') \le |t-t'|^b\,(T_1' + T_2')$, where

$$\begin{aligned} T_1' &= \sup_{x\in Q} \int_0^t s^{-(2\alpha+b\eta)}\,ds \int_Q dy \int_Q \exp\left(-c\,\frac{|x-y|^\beta}{s^\gamma}\right) f(y-z)\,\exp\left(-c\,\frac{|x-z|^\beta}{s^\gamma}\right) dz\,, \\ T_2' &= \sup_{x\in Q} \int_0^t s^{-(\alpha+b\eta)}\,(s+h)^{-\alpha}\,ds \int_Q dy\,\exp\left(-c\,\frac{|x-y|^\beta}{s^\gamma}\right) \\ &\qquad \times \int_Q f(y-z)\,\exp\left(-c\,\frac{|x-z|^\beta}{(s+h)^\gamma}\right) dz\,. \end{aligned}$$

Clearly, T_1' is similar to T_1 with $b\eta$ instead of $a\delta$; thus the proof of (i) yields (2.3) if either (2.24) or (2.25) holds. To estimate T_2', we distinguish two cases.

Case 1 If $|x-y| \le c_1\,|y-z|$, condition (2.19) on f yields

$$\begin{aligned} T_2' &\le C_1 \int_0^t s^{-(\alpha+b\eta)}\,(s+h)^{-\alpha}\,ds \int_Q \exp\left(-c\,\frac{|x-y|^\beta}{s^\gamma}\right) f(x-y)\,dy \\ &\qquad \times \int_Q \exp\left(-c\,\frac{|x-z|^\beta}{(s+h)^\gamma}\right) dz\,. \end{aligned}$$

Set $v = x-y$; then since $\alpha \ge \frac{\gamma}{\beta}d$, we have for some $R>0$,

$$\begin{aligned} T_2' &\le C_1 \int_0^t s^{-(\alpha+b\eta)}\,(s+h)^{-\alpha+\frac{\gamma}{\beta}d}\,ds \int_{B_d(0,R)} \exp\left(-c\,\frac{|v|^\beta}{s^\gamma}\right) f(v)\,dv \\ &\le C_1 \int_0^t s^{-(2\alpha+b\eta)+\frac{\gamma}{\beta}d}\,ds \int_{B_d(0,R)} \exp\left(-c\,\frac{|v|^\beta}{s^\gamma}\right) f(v)\,dv\,. \end{aligned}$$

This last upper estimate is similar to the right hand side of (2.12) with $2\alpha+b$ instead of 2α; thus the end of the proof of Lemma 2.1 (i) concludes the proof.

Case 2 If $|x-y| > c_1\,|y-z|$, then for $\bar c = \min(c, c\,c_1^\beta)$, we have

$$\begin{aligned} T_2' &\le \sup_{x\in Q} \int_0^t s^{-(\alpha+b\eta)}\,(s+h)^{-\alpha}\,ds \int_Q dy \\ &\qquad \times \int_Q \exp\left(-\bar c\,\frac{|y-z|^\beta}{s^\gamma}\right) f(y-z)\,\exp\left(-\bar c\,\frac{|x-z|^\beta}{(s+h)^\gamma}\right) dz\,. \end{aligned}$$

Since $(s+h)^{-\alpha+\frac{\gamma}{\beta}d} \le s^{-\alpha+\frac{\gamma}{\beta}d}$, the change of variables $\zeta = x-z$ and $v = y-z$ shows that T_2' is dominated by the right hand side of (2.12) with $2\alpha+b$ instead of 2α and $\bar c$ instead of c; this concludes the proof of (2.3)

Finally, using again Burkholder's inequality and (2.2), we have $C(t,t') \le C_p\, T_3^p \sup_{(s,y)\in[0,T]\times Q} E(|u(s,y)|^{2p})$, where

$$T_3 = \sup_{x\in Q} \int_0^{t'-t} s^{-2\alpha}\, ds \int_Q dy \int_Q \exp\left(-c\, \frac{|x-y|^\beta}{s^\gamma}\right) f(y-z) \exp\left(-c\, \frac{|x-z|^\beta}{s^\gamma}\right) dz\,.$$

Computations similar to those in the proof of Lemma 2.1 (i) imply that for some $R>0$, $T_3 \le \int_0^{t'-t} \psi(s)\, ds$, where ψ is defined by (2.13). Fubini's theorem and Hölder's inequality with respect to ds with the conjugate exponents $\lambda = (b\,\eta)^{-1}$ and μ imply

$$T_3 \le |t'-t|^b \int_{B_d(0,R)} f(v)\, dv \left(\int_0^T \exp\left(-c\, \frac{|v|^\beta}{s^\gamma}\right) s^{\mu(\frac{\gamma}{\beta}d-2\alpha)}\, ds\right)^{\frac{1}{\mu}}\,.$$

For $v\neq 0$ set $r=|v|^\beta\, s^{-\gamma}$; then since $\frac{1}{\mu}=1-b$, we obtain

$$T_3 \le C\,|t'-t|^b \int_{B_d(0,R)} f(v)\, |v|^{d+\frac{\beta}{\gamma}(1-2\alpha-b)}\, dv \left(\int_{|v|^\beta\, T^{-\gamma}}^{+\infty} r^{-1+\frac{\mu}{\gamma}[2\alpha-\frac{\gamma}{\beta}d-1+b]} \exp(-c\,r)\, dr\right)^{\frac{1}{\mu}}\,.$$

As in the proof of Lemma 2.1, we distinguish three cases, according to the power of r in the last integral, with $2\alpha+b$ instead of 2α; this concludes the proof of (2.26). □

3 Existence of solutions

3.1 The case of Lipschitz coefficients

Let $\sigma : [0,T]\times Q\times \mathbb{R}\to\mathbb{R}$ and $b_i : [0,T]\times Q\times\mathbb{R}\to\mathbb{R}$, $1\le i\le N$ be continuous functions such that the following boundness and Lipschitz conditions hold:

(L1) Uniform linear growth with respect to the last variable: for every $y\in Q$

$$\sup_{(t,x)\in[0,T]\times Q} \Big(|\sigma(t,x,y)| + \sum_{i=1}^N |b_i(t,x,y)|\Big) \le C(1+|y|)\,. \tag{3.1}$$

(L2) Uniform Lipschitz condition with respect to the last variable: for any $y,z\in Q$

$$\sup_{(t,x)\in[0,T]\times Q} \Big(|\sigma(t,x,y)-\sigma(t,x,z)| + \sum_{i=1}^N |b_i(t,x,y)-b_i(t,x,z)|\Big) \le C\,|y-z|\,. \tag{3.2}$$

We then consider the following non-linear evolution equation for $t\in[0,T]$ and $x\in Q$:

$$\begin{aligned} u(t,x) \;&=\; \int_Q G(t,x;0,y)\, u_0(y)\, dy + \int_0^t \int_Q G(t,x;s,y)\, \sigma(s,y,u(s,y))\, F(ds,dy) \\ &\quad + \sum_{i=1}^N \int_0^t \int_Q H_i(t,x;s,y)\, b_i(s,y,u(s,y))\, dy\, ds\,, \end{aligned} \tag{3.3}$$

for a function $u_0 : Q\to\mathbb{R}\in L^2(Q)$.

In this section we will make the following assumptions (restricting ourselves to the case G is the Green function of an operator which is parabolic in the sense of Petrovskiĭ, and the functions H_i are partial derivatives of G with respect to the space variable y):

(C2) The continuous function $G : ([0,T] \times Q)^2 \to \mathbb{R}$ (respectively for $i = 1, \cdots, N$ each continuous functions $H_i : ([0,T] \times Q)^2 \to \mathbb{R}$) satisfies (2.2) with constants β, γ and $\alpha = \frac{\gamma}{\beta} d$ (respectively α_i, β and γ).

(C3) The covariance of the Gaussian process F is defined in terms of f by (2.1), and the constants α, β, γ in (C2) satisfy either

$$\int_{B_d(0,1)} f(v) \, \ln\left(|v|^{-1}\right) dv < \infty, \quad if \quad d = \frac{\beta}{\gamma}, \tag{3.4}$$

or

$$\int_{B_d(0,1)} f(v) \, |v|^{-[d-\frac{\beta}{\gamma}]^+} dv < \infty, \quad if \quad d \neq \frac{\beta}{\gamma}. \tag{3.5}$$

(C4) The constants α_i, β, γ in (C2) satisfy $\alpha_i < \alpha + 1$ for every $i \in \{1, \cdots, N\}$.

Condition (C3) will allow to define stochastic integrals of $G(t,x;.)$ with respect to the noise F, while (C4) will allow to define deterministic integrals involving $H_i(t,x;.)$.

We at first study moment estimates of deterministic integrals. Let $\lambda, \rho \in [1, +\infty]$; for $v \in L^\lambda([0,T], L^\rho(Q))$, and $0 \le t_0 \le t \le T$, $x \in Q$ set

$$J(v)(t_0,t,x) = \int_{t_0}^t \int_Q H(t,x;s,y)\, v(s,y) dy ds\,. \tag{3.6}$$

The following lemma provides L^q estimates of $J(v)(t_0,t,.)$ in terms of L^ρ estimates of $v(s,.)$. It extends similar results proved in I. Gyöngy [15] and C. Cardon-Weber [4].

Lemma 3.1 . *Fix $\rho \in [1, +\infty]$, $q \in [\rho, +\infty]$, and let r be defined by $\frac{1}{r} = \frac{1}{q} - \frac{1}{\rho} + 1$. Then for $0 \le t_0 \le t \le T$*

$$\|J(v)(t_0,t,.)\|_q \le C \int_{t_0}^t (t-s)^{-\alpha + \frac{\gamma d}{\beta r}} \, \|v(s,.)\|_\rho \, ds\,. \tag{3.7}$$

Hence given any $\lambda \in [0, +\infty]$, if $\alpha + \frac{1}{\lambda} < \frac{\gamma d}{\beta r} + 1$, then $J(0,.,.)$ is a bounded operator from $L^\lambda([0,T], L^\rho(Q))$ into $L^\infty([0,T], L^q(Q))$.

Proof: Using Minkowski's inequality, (2.2), then Young's inequality with $\frac{1}{q} = \frac{1}{r} + \frac{1}{\rho} - 1$ and (2.3), we obtain

$$\begin{aligned} \|J(v)(t_0,t,.)\|_q &\le C \int_{t_0}^t \left\| \int_Q (t-s)^{-\alpha} \exp\left(-c \frac{|.-y|^\beta}{(t-s)^\gamma}\right) |v(s,y)| \, dy \right\|_q ds \\ &\le C \int_{t_0}^t (t-s)^{-\alpha} \, \|v(s,.)\|_\rho \left\| \exp\left(-c \frac{|.|^\beta}{(t-s)^\gamma}\right) \right\|_r ds \\ &\le C \int_{t_0}^t (t-s)^{-\alpha + \frac{\gamma d}{\beta r}} \, \|v(s,.)\|_\rho \, ds\,. \end{aligned}$$

Finally, Hölder's inequality applied with λ and $\mu = \frac{\lambda}{\lambda - 1}$ yields that

$$\|J(v)(t_0,t,.)\|_q \le C \left(\int_{t_0}^t (t-s)^{\mu(-\alpha + \frac{\gamma d}{\beta r})} \, ds \right)^{\frac{1}{\mu}} \left(\int_{t_0}^t \|v(s,.)\|_\rho^\lambda \, ds \right)^{\frac{1}{\lambda}}.$$

Hence for $t_0 = 0$, the right hand side is finite and bounded with respect to $t \in [0,T]$ if and only if $\mu(-\alpha + \frac{\gamma d}{\beta r}) > -1$; this completes the proof. □

The following result proves that the evolution equation (3.3) has a unique solution with moments of all finite order. However, in order to prove that, when $u_0 \in L^q(Q)$ for $2 \le q < +\infty$, the $\| \; \|_q$-norm of the solution has bounded L^p moments for $q < p < +\infty$, we have to reinforce condition (C3) as follows (clearly when $p = q$, the conditions (C3) and (C'3)(q,p) coincide, while if $p < q$, (C'3)(q,p) implies (C3)):

(C'3)(q,p) Let f define the covariance of the Gaussian noise according to (2.1), $2 \le q \le p < +\infty$; the constants α, β, γ in (C2) satisfy one of the following conditions:

$$\int_{B_d(0,1)} f(v)\; \ln\left(|v|^{-1}\right) dv < \infty, \quad if \quad \frac{\beta}{\gamma}(2\alpha - 1) = \frac{q}{p} d\,, \tag{3.8}$$

or

$$\int_{B_d(0,1)} f(v)\,|v|^{-\left[\frac{\beta}{\gamma}(2\alpha-1)-d\frac{q}{p}\right]^+} dv < \infty, \quad if \quad \frac{\beta}{\gamma}(2\alpha - 1) \neq \frac{q}{p} d\,. \tag{3.9}$$

Theorem 3.2 . *Suppose that the functions G and H_i , $1 \le i \le N$ satisfy the conditions (C2) and (C4) and that the functions σ and b_i, $1 \le i \le N$ satisfy the assumptions (L1) and (L2).*

(i) Let $u_0 \in L^\infty(Q)$, and let F denote either the space-time white noise if $\alpha < 1$, or a Gaussian process with covariance defined by (2.1) such that (C3) holds. Then the evolution equation (3.3) has a unique solution $u \in L^\infty([0,T], L^\infty(Q))$, such that for any $p \in [1, \infty[$,

$$\sup_{(t,x)\in[0,T]\times Q} E\left(|u(t,x)|^p\right) < +\infty\,. \tag{3.10}$$

(ii) Let $u_0 \in L^q(Q)$ for $2 \le q < +\infty$, let $p \in [q, +\infty[$. Suppose that the following assumptions holds:

(a) F is the space-time white noise, $\alpha < 1$ and $p < \frac{2\alpha}{\left[\frac{2\alpha}{q}-1+\alpha\right]^+}$.

(b) F is a Gaussian process with covariance defined by (2.1) such that (C'3)(q,p) holds. Then the evolution equation (3.3) has a unique solution $u \in L^\infty([0,T], L^q(Q))$, such that

$$\sup_{t\in[0,T]} E\left(\|u(t,.)\|_q^p\right) < +\infty. \tag{3.11}$$

Proof: In the case of the space-time white noise, the proof which is easier and more classical is omitted, except that of (3.11) in case (ii). Unless specified otherwise, we assume that F is Gaussian with a space-correlation function f. We use the following Picard iteration scheme; $u_0(t,x) = G_t u_0(x) = \int_Q G(t,x;0,y)\, u_0(y)\, dy$ and for $n > 0$ let

$$\begin{aligned} u_{n+1}(t,x) &= u_0(t,x) + \int_0^t \int_Q G(t,x;s,y)\,\sigma(s,y,u_n(s,y))\,F(ds,dy) \\ &\quad + \sum_{i=1}^N \int_0^t \int_Q H_i(t,x;s,y)\, b_i(s,y,u_n(s,y))\,dy\,ds\,. \end{aligned} \tag{3.12}$$

Case (ii) Let $2 \le q \le p < +\infty$ and suppose that condition (b) holds; set

$$M_n(t) = E\Big(\|u_n(t,.)\|_q^p\Big)\,,$$

and let ψ_p be the function defined by $\psi_p(t) = t^{\alpha\left(-2+\frac{q}{p}\right)} \int_{B_d(0,R)} \exp\left(-c\frac{|v|^\beta}{t^\gamma}\right) f(v)dv$. Using (C'3)(q,p), computations similar to that proving (2.6) from (2.12) using (2.13) show that ψ_p is integrable; set $I_p = \int_0^T \psi_p(s)ds < \infty$ and let

$$\varphi_p(s) = C_p\left(\psi_p(s) + \sum_{i=1}^N s^{-\alpha_i+\alpha}\right);$$

the assumptions (C'3)(q,p), (C4), (2.3) and the proof of lemma 2.1 imply that $\varphi_p \in L^1_+([0,T])$. Let ψ and I be defined as in the proof of lemma 2.1; then for $q \le p$, $\psi_p \ge \psi_q = \psi$ and $I_q \ge I_p = I$. We prove that

$$\sup_{t\in[0,T]} M_0(t) < +\infty\,, \tag{3.13}$$

and for any $n \ge 0$, $t \in [0,T]$,

$$M_{n+1}(t) \le \int_0^t \varphi_p(t-s)\,[1+M_n(s)]\,ds\,; \tag{3.14}$$

then Lemma 15 in [6] shows that

$$\sup_n \sup_{0\le t\le T} E\left(\|u_n(t,.)\|_q^p\right) < +\infty\,. \tag{3.15}$$

Fubini's theorem, (2.2), Hölder's inequality and (2.3) imply that

$$\begin{aligned}
\sup_{0\le t\le T} M_0(t) &\le C \sup_{0\le t\le T}\left[\int_Q \left|\int_Q u_0(y)\, t^{-\alpha} \exp\left(-c\,\frac{|x-y|^\beta}{t^\gamma}\right) dy\right|^q dx\right]^{\frac{p}{q}}\\
&\le C\left[\int_Q |u_0(y)|^q \left(t^{-\alpha}\int_Q \exp\left(-\bar c\,\frac{|x-y|^\beta}{t^\gamma}\right) dx\right) dy\right]^{\frac{p}{q}} \le C\,\|u_0\|_q^p\,.
\end{aligned}$$

We now prove (3.14); for $n\ge 0$, $M_{n+1}(t) \le C_p\left[M_0(t) + T^1_n(t,p) + \sum_{i=1}^N T^2_{n,i}(t,p)\right]$, where for $q \le p < +\infty$,

$$\begin{aligned}
T^1_n(t,p) &= E\left(\left\|\int_0^t\int_Q G(t,.;s,y)\,\sigma(s,y,u_n(s,y))\,F(ds,dy)\right\|_q^p\right)\,,\\
T^{2,i}_n(t,p) &= E\left(\left\|\int_0^t\int_Q H_i(t,.;s,y)\,b_i(s,y,u_n(s,y))\,dy\,ds\right\|_q^p\right)\,.
\end{aligned}$$

Since $\beta\ge 1$, $|y-z|^\beta \le 2^{\beta-1}\left[|x-y|^\beta + |x-z|^\beta\right]$. Therefore, Fubini's theorem, Burkholder's and Hölder's inequalities yield the existence of a constant $\bar c$ such that

$$\begin{aligned}
T^1_n(t,p) &\le C_p \int_Q dx\, E\left(\left|\int_0^t ds(t-s)^{-2\alpha}\int_Q dy\,\exp\left(-\bar c\frac{|x-y|^\beta}{(t-s)^\gamma}\right)\left(1+|u_n(s,y)|\right)\right.\right.\\
&\qquad\left.\left.\times\int_Q f(y-z)\,\exp\left(-\bar c\frac{|x-y|^\beta}{(t-s)^\gamma}\right)\left(1+|u_n(s,z)|\right)dz\right|^{\frac{p}{2}}\right)\\
&\le C_p\, I_q^{\frac{p}{2}-\frac{p}{q}}\int_Q dx\,E\left(\left[\int_0^t (t-s)^{-2\alpha}\int_Q dy\,\exp\left(-\bar c\,\frac{|x-y|^\beta}{(t-s)^\gamma}\right)\left(1+|u_n(s,y)|^{\frac{q}{2}}\right)\right.\right.\\
&\qquad\left.\left.\times\int_Q f(y-z)\,\exp\left(-\bar c\,\frac{|y-z|^\beta}{(t-s)^\gamma}\right)\left(1+|u_n(s,z)|^{\frac{q}{2}}\right)dzds\right]^{\frac{p}{q}}\right)\,.
\end{aligned}$$

Fubini's theorem and Jensen's inequality imply that

$$\begin{aligned}
T^1_n(t,p) &\le C_p\, I_q^{\frac{p}{2}-\frac{p}{q}}\, E\left(\left[\int_0^t (t-s)^{-2\alpha}\left\|\int_Q dy\,\exp\left(-\bar c\,\frac{|.-y|^\beta}{(t-s)^\gamma}\right)\left(1+|u_n(s,y)|^{\frac{q}{2}}\right)\right.\right.\right.\\
&\qquad\left.\left.\left.\times\int_Q f(y-z)\,\exp\left(-\bar c\,\frac{|y-z|^\beta}{(t-s)^\gamma}\right)\left(1+|u_n(s,z)|^{\frac{q}{2}}\right)dz\right\|_{\frac{p}{q}} ds\right]^{\frac{p}{q}}\right)\,.
\end{aligned}$$

Then Young's, Schwarz's inequalities, (2.3), Hölder's inequality with respect to $\psi_p(t-s)\,ds$ and $I_q \le I_p$ yield

$$\begin{aligned} T_n^1(t,p) &\le C_p\, I_q^{\frac{p}{2}-\frac{p}{q}}\, E\Bigg(\Bigg|\int_0^t (t-s)^{-2\alpha}\, \Big\| \exp\Big(-\bar{c}\frac{|.|^\beta}{(t-s)^\gamma}\Big) * \Big[\Big(1+|u_n(s,.)|^{\frac{q}{2}}\Big) \\ &\qquad \times\Big\{\Big(f(.)\,\exp\Big(-\bar{c}\frac{|.|^\beta}{(t-s)^\gamma}\Big)\Big) * \Big(1+|u_n(s,.)|^{\frac{q}{2}}\Big)\Big\}\Big]\Big\|_{L^{\frac{p}{q}}(Q,dx)}\, ds\Bigg|^{\frac{p}{q}}\Bigg) \\ &\le C_p\, I_q^{\frac{p}{2}-\frac{p}{q}}\, E\Bigg(\Bigg|\int_0^t (t-s)^{-2\alpha}\, \Big\|\exp\Big(-\bar{c}\frac{|.|^\beta}{(t-s)^\gamma}\Big)\Big\|_{L^{\frac{p}{q}}(Q,dx)} \Big(1+\|u_n(s,.)\|_q^{\frac{q}{2}}\Big) \\ &\qquad \times\Big\|f(.)\,\exp\Big(-\bar{c}\frac{|.|^\beta}{(t-s)^\gamma}\Big)\Big\|_{L^1(Q,dx)}\Big(1+\|u_n(s,.)\|_q^{\frac{q}{2}}\Big)\, ds\Bigg|^{\frac{p}{q}}\Bigg) \\ &\le C_p\, I_q^{\frac{p}{2}-\frac{p}{q}}\, E\Bigg(\Bigg|\int_0^t (t-s)^{\alpha\left(-2+\frac{q}{p}\right)}\,\big[1+\big(\|u_n(s,.)\|_q^q\big)\big] \\ &\qquad \times \int_Q f(v)\,\exp\Big(-\bar{c}\frac{|v|^\beta}{(t-s)^\gamma}\Big)\, dv ds\Bigg|^{\frac{p}{q}}\Bigg) \\ &\le C_q\, I_p^{\frac{p}{2}-1}\int_0^t \psi_p(t-s)\,[1+M_n(s)]\, ds\,. \end{aligned} \tag{3.16}$$

For every $1\le i\le N$, using (C4), (L1), (3.7) with $\rho=q$, $r=1$, and Hölder's inequality (since $\alpha_i<\alpha+1$) and Fubini's theorem, we deduce that for $q\le p<+\infty$,

$$\begin{aligned} T_{n,i}^2(t,p) &\le C\,E\Bigg(\Bigg|\int_0^t (t-s)^{-\alpha_i+\alpha}\,\big(1+\|u_n(s,.)\|_q\big)\, ds\Bigg|^p\Bigg) \\ &\le C\int_0^t (t-s)^{-\alpha_i+\alpha}\,\big[1+E(\|u_n(s,.)\|_q^p)\big]\, ds\,. \end{aligned}$$

This concludes the proof of (3.14). Let $\Delta_n(t)=E\Big(\|u_{n+1}(t,.)-u_n(t,.)\|_q^p\Big)$; a similar computation using the global Lipschitz property (L2) of the coefficients with respect to the last variable shows that

$$\Delta_{n+1}(t)\le C_p\int_0^t \varphi_p(t-s)\,\Delta_n(s)\,ds \tag{3.17}$$

where the function φ_p is the previous one. Using again Lemma 15 in [6], we conclude that $\sum_{n\ge 0}\Delta_n(t)$ converges uniformly on $[0,T]$. Therefore, usual arguments show that the solution u to (3.3) exists in $L^\infty([0,T],L^q(Q))$ and satisfies (3.11).

We now suppose that condition (a) holds. Set $M_n(t)=E\,(\|u_n(t,.)\|_q^p)$. According to the results proved above, it suffices to check that (using the previous notations), $T_n^1(t,p)\le C_p\int_0^t(t-s)^{-a}\,(1+M_n(s,p))\,ds$ for some $a<1$. Using Hölder's and Burkholder's inequalities, Fubini's theorem, then (3.7) with $\frac{p}{2}$ instead of q, 2α instead of α and $\rho=\frac{q}{2}$, we deduce that for $1+\frac{2}{p}=\frac{2}{q}+\frac{1}{r}$, and $a=(2-\frac{1}{r})\alpha<1$ by the choice of p, we have

$$\begin{aligned} T_n^1(t,p) &\le C_p\,E\Bigg(\int_Q\Bigg|\int_0^t\int_Q (t-s)^{-2\alpha}\,\exp\Big(-c\frac{|x-y|^\beta}{(t-s)^\gamma}\Big)\,(1+|u_n(s,y)|^2)\,dy\,ds\Bigg|^{\frac{p}{2}}\,dx\Bigg) \\ &\le \int_0^t (t-s)^{-a}\,\big(1+E(\|u_n(s,.)\|_q^p)\big)\,ds\,. \end{aligned}$$

The rest of the proof, similar to that of the case (ii)(b), is omitted.

Case (i) Let $u_0 \in L^\infty(Q)$, $p \in [1,+\infty[$ and suppose that the covariance function f satisfies (C3); set

$$M_n(t) = \sup_{x\in Q} E\big(|u_n(t,x)|^{2p}\big)\,.$$

We again prove (3.13) and (3.14). Since $u_0 \in L^\infty(Q)$, the inequality (2.3) proves (3.13). Let ψ be defined by (2.13) and let $\varphi(t) = \psi(t) + \sum_{i=1}^N t^{-\alpha_i+\alpha} \in L^1([0,T])$; then Burkholder's, Hölder's inequalities and (L1) yield

$$\begin{aligned} M_{n+1}(t,x) \;&\leq\; C_p \Bigg[M_0(t) + E\Big(\Big| \int_0^t ds \int_Q dy \int_Q |G(t,x;s,y)|\,|\sigma(s,y,u_n(s,y))|\,f(y-z) \\ &\qquad\qquad \times |G(t,x;s,z)|\,|\sigma(s,y,u_n(s,z))|\,dz\Big|^p\Big) \\ &\qquad + \int_0^t \int_Q \sum_{i=1}^N |H_i(t,x;s,y)|\,(1+M_n(s))dy\,ds \Bigg] \\ &\leq\; C_p \left(M_0(t) + \int_0^t \varphi(t-s)\,[1+M_n(s)]\,ds \right)\,. \end{aligned}$$

This implies (3.14) and again Lemma 15 in [6] shows that

$$\sup_n \sup_{(t,x)\in[0,T]\times Q} E(|u_n(t,x)|^{2p}) < +\infty\,. \tag{3.18}$$

A similar computation for $\Delta_n(t) = \sup_{x\in Q} E\big(|u_{n+1}(t,x) - u_n(t,x)|^{2p}\big)$ and the global Lipschitz property (L2) of the coefficients with respect to the last variable show that (3.17) holds. As in case 2, usual arguments prove that the solution u to (3.3) exists and satisfies (3.10). □

3.2 Cahn-Hilliard equation in dimension $d = 4, 5$

The following stochastic Cahn-Hilliard equation has been studied in dimension 1 up to 3 by C. Cardon-Weber [4] and [5]; see also G. Da Prato and A. Debussche [8]. Let $Q = [0,\pi]^d$ or a compact convex subset of $\mathbb{R}^d$, $(t,x) \in [0,T]\times Q$ and multi-indices $(k_i\,,\,1\leq i\leq N)$ with $|k_i| \leq 3$ which satisfy the conditions in Remark 1.1, the following equation is defined in a weak sense:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + (\Delta^2 u(t,x) - \Delta R(u)(t,x)) \;&=\; \sigma(t,x,u(t,x))\dot F + g(t,x,u(t,x)) \\ &\qquad + \sum_{i=1}^N D_x^{k_i}(b_i(t,x,u(t,x)))\,, \end{aligned} \tag{3.19}$$

with the initial condition $u(0,.) = u_0$ and the homogeneous Neumann boundary conditions:

$$\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \text{ on } \partial Q. \tag{3.20}$$

We will also consider the homogeneous Dirichlet boundary conditions:

$$u = \Delta u = 0 \text{ on } \partial Q. \tag{3.21}$$

In this section, we will suppose that $d = 4, 5$ and make the following assumptions:

(H.1) R is a polynomial of degree 3 with positive dominant coefficient, and such that $R(0) = 0$ if (3.21) holds.

(H.2) $\sigma : [0,T] \times Q \times \mathbb{R} \mapsto \mathbb{R}$ is bounded, and the functions σ and $b_i : [0,T] \times Q \times \mathbb{R} \mapsto \mathbb{R}$ are globally Lipschitz with respect to the last variable, while the function $g : [0,T] \times Q \times \mathbb{R} \mapsto \mathbb{R}$ has quadratic growth with respect to the last variable uniformly with respect to the first ones, and satisfies for $y, z \in Q$:

$$\sup_{(t,x)\in[0,T]\times Q} |g(t,x,y) - g(t,x,z)| \le C\,(1 + |y| + |z|)\,|y - z|\,. \tag{3.22}$$

(H.3) u_0 belongs to $L^q(Q)$ for some $q > d$.

(H.4) Set $H_i(t,x;s,y) = D_x^{k_i} G(t,x;s,y)$ for $|k_i| \le 3$, where G is the Green function associated with the operator $\frac{\partial}{\partial t} + \Delta^2$ on Q with the homogeneous boundary conditions (3.20) or (3.21), and the multi-indices k_i and the functions b_i satisfy the assumptions in Example 1.1.

The upper estimates of G stated in condition (C2) are given in the introduction: $\alpha = \frac{d}{4}$, $\beta = \frac{4}{3}$, $\gamma = \frac{1}{3}$, $\delta = \frac{1}{4}$ and $\eta = 1$. Clearly for $|k_i| \le 3$, $\alpha_i \le \alpha + \frac{3}{4}$ and (C4) holds. As explained in the introduction, the Green formula shows that the weak formulation of (3.19) is equivalent with the evolution formulation: for $x \in Q, t \in [0,T]$:

$$\begin{aligned} u(t,x) &= \int_Q G(t,x;0,y)u_0(y)dy + \int_0^t \int_Q G(t,x;s,y)\sigma(s,y,u(s,y))F(ds,dy) \\ &+ \int_0^t \int_Q \Delta G(t,x;s,y) R(u(s,y))dyds \\ &+ \int_0^t \int_Q \Big[\sum_{i=1}^N H_i(t,x;s,y) b_i(s,y,u(s,y)) + G(t,x;s,y) g(s,y,u(s,y))\Big]\,dyds\,. \end{aligned} \tag{3.23}$$

The following theorem completes the existence and uniqueness of the solution to (3.19) in dimension 4 and 5.

Theorem 3.3 . *Let Q denote either $[0,\pi]^d$ or a compact subset of $\mathbb{R}^d$ with boundary of class $\mathcal{C}^{4+\lambda}$ for $\lambda > 0$, $d = 4, 5$ and assume that (H.1)- (H.4) hold. Let f be the covariance function of the Gaussian noise F defined by (2.1), which satisfies (C1) and such that for $\varepsilon \in]0,1[$,*

$$\int_{B_d(0,1)} f(v)\,|v|^{-d(1+\varepsilon)+4}\,dv < \infty. \tag{3.24}$$

There exists a unique adapted process u in $L^\infty([0,T], L^q(Q))$ that satisfies equation (3.23).

Remark 3.4 *Under assumptions (H.1), (H.3) and (H.4), the existence result proved in [4] in dimension 1-3 (respectively Theorem 3.3) extends to a compact Q with boundary of class $\mathcal{C}^{4+\lambda}$ for $\lambda > 0$, for the differential operator $a(t)\Delta^2$ where the function a is such that $\sup_{0\le t\le T} a(t) < 0$, and when F is the space-time white noise (respectively when (H.2) holds).*

Proof: To prove this theorem we at first prove the existence of a solution when the coefficients are truncated. Let $K_n : \mathbb{R}^+ \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

$$K_n(x) = 1 \;\text{ if }\; x < n, \quad K_n(x) = 0 \;\text{ if }\; x \le n+1, \quad |K_n| \le 1 \;\text{ and }\; |K_n'| \le 2\,. \tag{3.25}$$

We denote by u_n the solution to the following evolution equation with truncated coefficients:

$$u_n(t,x) = \int_Q G(t,x;0,y)u_0(y)dy + \int_0^t \int_Q G(t,x;s,y)\,\sigma(s,y,u_n(s,y))F(ds,dy) \tag{3.26}$$

$$+\int_0^t \int_Q K_n(\|u_n(s,.)\|_q))\big[\Delta G(t,x;s,y) R(u_n(s,y)) + G(t,x;s,y)\, g(s,y,u_n(s,y))\big]\, dyds$$
$$+\sum_{i=1}^N \int_0^t \int_Q H_i(t,x;s,y)\, b_i(s,y,u_n(s,y)) dy\, ds\,.$$

Given an adapted process u, let $L(u)$ be defined by

$$L(u)(t,x) = \int_0^t \int_Q G(t,x;s,y)\, \sigma(s,y,u(s,y))\, F(ds,dy).$$

The arguments used in the proof of Theorem 3.2 show that u_n exists, is unique and that for any $p \in [q, \frac{q}{1-\varepsilon}]$,

$$\sup_{t\in[0,T]} E\big(\|u_n(t,.)\|_q^p\big) < +\infty.$$

Indeed, let $\mathcal{K}$ denote the set of adapted $L^q(Q)$-valued processes such that for $q \le p \le \frac{q}{1-\varepsilon}$, $\|u\|_{\mathcal{K}}^p = \sup_{0\le t\le T} E(\|u(t,.)\|_q^p) < +\infty$. For any $u\in\mathcal{K}$, $(t,x)\in[0,T]\times Q$, set

$$\begin{aligned}
H_n(u)(t,x) &= \int_0^t \int_Q \Delta G(t,x;s,y)\, K_n(\|u(s,.)\|_q)\, R(u(s,y))\, dyds\,,\\
J_n(u)(t,x) &= \int_0^t \int_Q G(t,x;s,y)\, K_n(\|u(s,.)\|_q)\, g(s,y,(u(s,y)) dyds\,,\\
B(u)(t,x) &= \sum_{i=1}^N \int_0^t \int_Q H_i(t,x;s,y)\, b_i(s,y,u(s,y))\, dyds\,.
\end{aligned}$$

Since σ is bounded and (3.24) implies (2.5), Burkholder's inequality and Lemma 2.1 yield that for any adapted process u and $2\le p<+\infty$, $\sup\{E(|L(u)(t,x)|^p)\,:\,(t,x)\in[0,T]\times Q\}<+\infty$, so that $\|L(u)\|_{\mathcal{K}}<\infty$. Furthermore, given $u,v\in\mathcal{K}$, (H.2), the fact that (3.24) implies (C'3)(q,p) for $q\le p\le\frac{q}{1-\varepsilon}$ and the argument used to show (3.16) in the proof of Theorem 3.2 yield

$$E\left(\|L(u)(s,.) - L(v)(s.)\|_q^p\right) \le C\left(\int_0^T \psi_p(s)\, ds\right)^{\frac{p}{2}} \sup_{0\le s\le T} E(\|u(s,.)-v(s,.)\|_q^p)\,.$$

Since ψ_p is integrable, L is a contraction of $\mathcal{K}$ for small enough T. For the polynomial term H_n, we just need to notice that if u and v belong to $L^\infty([0,T],L^q(Q))$

$$\Big\| K_n(\|u(s,.)\|_q) R(u(s,.)) - K_n(\|v(s,.)\|_q) R(v(s,.))\Big\|_{\frac{q}{3}} \le C_n \|u(s,.)-v(s,.)\|_q. \tag{3.27}$$

Using (3.7), (3.27) and Hölder's inequality, we obtain that for $d<q\le p<+\infty$,

$$\begin{aligned}
\sup_{0\le t\le T} \|H_n(u)(t,.)-H_n(v)(t,.)\|_{\mathcal{K}}^p &\le C_n\, T^{(p-1)\left(\frac12-\frac{d}{2q}\right)} \int_0^T (t-s)^{-\frac{d+2}{4}+\frac{d}{4}(1-\frac{2}{q})}\\
&\qquad \times E(\|u(s,)-v(s,.)\|_q^p)\, ds\\
&\le C_n\, T^{p\left(\frac12-\frac{d}{2q}\right)}\, \|u-v\|_{\mathcal{K}}^p\,.
\end{aligned}$$

A similar computation based on the quadratic growth and increments property of g with respect to the third variable shows that for $\frac{d}{4}<q\le p<+\infty$,

$$\sup_{0\le t\le T} \|J_n(u)(t,.)-J_n(v)(t,.)\|_{\mathcal{K}} \le C_n\, T^{1-\frac{d}{4q}}\, \|u-v\|_{\mathcal{K}}\,.$$

Finally, the estimation of T_n^2 made in the proof of Theorem 3.2 shows that

$$\sup_{0\leq t\leq T} \|B(u)(t,.)-B(v)(t,.)\|_{\mathcal{K}} \leq C \sup_{1\leq i\leq N} T^{1+\alpha-\alpha_i} \|u-v\|_{\mathcal{K}}\,.$$

Hence, there exists $T_0>0$, independent of the initial condition u_0, such that for $0<T\leq T_0$, $L+H_n+J_n+B$ is a contraction of $\mathcal{K}$, and hence admits a unique fixed point such that $u(0,.)=u_0$. A concatenation argument implies that (3.26) has a unique solution on $[0,T]$ for an arbitrary terminal time T.

To prove the existence and uniqueness of u we follow the proof in C. Cardon-Weber [4]. Let τ_n be the stopping time defined by: $\tau_n=\inf\{t\geq 0, \|u_n(t,.)\|_q\geq n\}$. By uniqueness of the solution to (3.26), the local property of the stochastic integrals yields for $m>n$, $u_m(t,.)=u_n(t,.)$ if $t\leq\tau_n$, so that we can define a process u by setting $u(t,.)=u_n(t,.)$ on $t\leq\tau_n$. Set $\tau_\infty=\lim_n\tau_n$. Then u is the unique solution of (3.19) on the interval $[0,\tau_\infty)$. We just need to prove that $\tau_\infty=+\infty$ a.s.

Set $v_n=u_n-L(u_n)$; then for every $T>0$, v_n is the weak solution on $[0,T]$ to the SPDE (with the same boundary conditions as (3.19)):

$$\left\{\begin{array}{l} \frac{\partial v_n}{\partial t}(t,x)+\Delta^2 v_n(t,x)-\Delta\Big[K_n(\|v_n(t,.)+L(u_n)(t,.)\|_q)R(v_n(t,x)+L(u_n)(t,x))\Big]= \\ \qquad K_n(\|v_n(t,.)+L(u_n)(t,.)\|_q)\,g(t,x,v_n(t,x)+L(u_n)(t,x)) \\ \qquad +\sum_{i=1}^N D_x^{k_i} b_i(t,x,v_n(t,x)+L(u_n)(t,x)), \\ v_n(0,.)=u_0(.), \\ \frac{\partial v_n}{\partial n}=\frac{\partial\Delta v_n}{\partial n}=0 \text{ (resp. } v_n=\Delta v_n=0) \text{ on } \partial Q. \end{array}\right. \tag{3.28}$$

Since σ is bounded, the Garsia-Rodemich-Ramsay Lemma (cf. eg. [14]), (3.24) and lemma 2.3 yield that for any $p\in[2,+\infty[$,

$$\sup_n E(\|L(u_n)\|_\infty^p)<\infty\,. \tag{3.29}$$

Since u_0 belongs to $L^q(Q)$,

$$\sup_{t\leq T}\|G_t u_0\|_q\leq\|u_0\|_q. \tag{3.30}$$

We need to prove a uniform upper estimate for the drift terms $H_n(u_n)$ and $J_n(u_n)$ (the estimation of the other drift term $B(u_n)$, which is easier, will be omitted and to lighten the notations, we will assume that $b_i=0$, $1\leq i\leq N$). Since the function ΔG has a regularizing effect, we first show that u_n belongs to the sets $L^a([0,T],L^q(Q))$ for some well-chosen a.

Let us introduce some notations: set $\mathcal{A}=-\Delta$, let $<.,.>$ denote the usual scalar product in $L^2(Q)$, let $(e_n\,,\,n\in\mathbb{N}^d)$ be a basis of $L^2(Q)$ made of eigenfunctions of $\mathcal{A}$ (namely $e_n(x)=\Pi_{i=1}^d e_{n_i}(x_i)$ and either $e_0=\frac{1}{\sqrt{\pi}}$, $e_n(x)=\sqrt{\frac{2}{\pi}}\cos(nx)$ for $n>0$ in the case of the Neumann boundary conditions (3.20), or $e_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$ for $n>0$ in the case of the Dirichlet boundary conditions (3.21) if $Q=[0,\pi]^d$). The corresponding eigenvalues are λ_n^2, where $\lambda_n=\sum_{i=1}^d n_i^2$. For $\mu\neq 0$ and $u\in Dom(\mathcal{A}^\mu)$, let

$$\mathcal{A}^\mu u=\sum_{k\in\mathbb{N}^{d\star}}\lambda_k^\mu<\varepsilon_k,u>\varepsilon_k\,;$$

$\mathcal{A}^\mu u$ exists for every u such that $\sum_{k\in\mathbb{N}^{d,\star}}\lambda_k^{2\mu}<e_k,u>^2<\infty$. In the sequel, for a function $u:[0,T]\times Q\to\mathbb{R}$, we will set (if e_0 is a constant eigenfunction):

$$m(u)(t)=<\varepsilon_0,u(t,.)>=\pi^{-\frac{d}{2}}\int_Q u(t,x)dx \text{ and } \tilde{u}(t,y)=u(t,y)-m(u)(t).$$

Apply $\mathcal{A}^{-1}$ to the equation (3.33) and take its scalar product in $L^2(Q)$ with $\tilde{v}_n(t,.)$; this leads to

$$\|\mathcal{A}^{-1/2}\tilde{v}_n(t,.)\|_2^2 - \|\mathcal{A}^{-1/2}\tilde{v}_n(0,.)\|_q^2 + \int_0^t \|\mathcal{A}^{1/2}v_n(s,.)\|_2^2\,ds \tag{3.31}$$
$$+\int_0^t K_n(\|v_n(s,.)+L(u_n)(s,.)\|_q)\int_Q \Big[R\big(v_n(s,x)+L(u_n)(s,x)\big)\tilde{v}_n(s,x)$$
$$+g\big(s,x,v_n(s,x)+L(u_n(s,x)\big)\,\mathcal{A}^{-1}\tilde{v}_n(s,x)\Big]\,dxds = 0.$$

This equation is justified because v_n belongs to $L^\infty([0,T]\times Q)$. Using the properties of the polynomial R, computations made to obtain (2.19)-(2.21) in [4], we obtain that for some $b>0$,

$$\|\mathcal{A}^{-1/2}\tilde{v}_n(t,.)\|_2^2 + \int_0^t \|\mathcal{A}^{1/2}v_n(s,.)\|_2^2\,ds$$
$$+\frac{b}{4}\int_0^t K_n(\|v_n(s,.)+L(u_n)(s,.)\|_q)\|v_n(s,.)+L(u_n)(s,.)\|_4^4\,ds$$
$$\leq \int_0^t C(1+m(u_0)(s)^4 + \|L(v_n)(s,.)\|_4^4)\,ds + \|\mathcal{A}^{-1/2}\tilde{u}_0(.)\|_2^2. \tag{3.32}$$

Let us find a second "a priori" estimate. Denote by v_n^m the Galerkin approximation of v_n and let P_m be the orthogonal projector on Span$\{e_0,...,e_m\}$. For every ω, v_n^m is the "strong" solution to the following PDE:

$$\left\{\begin{array}{l} \frac{\partial v_n^m}{\partial t}(t,x) + \Delta^2 v_n^m(t,x) \\ \quad -\Delta\Big[K_n(\|v_n^m(t,.)+L(u_n)(t,.)\|_q)P_m(R(v_n^m(t,x)+L(u_n)(t,x)))\Big] \\ -K_n(\|v_n^m(t,.)+L(u_n)(t,.)\|_q)P_m(g(t,x,v_n^m(t,x)+L(u_n)(t,x))) = 0\,. \end{array}\right. \tag{3.33}$$

The solution v_n^m to (3.33) is unique on some random time interval $[0,t_n^m[$ and we prove that $t_n^m=+\infty$. The boundary conditions satisfied by v_n^m and the Green Formula yield

$$\int_Q \Delta^2 v_n^m(t,x)\times v_n^m(t,x)\,dx = \|\Delta v_n^m(t,x)\|_2^2.$$

We now take the scalar product in $L^2(Q)$ of (3.33) with v_n^m; using once more the Green formula, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|v_n^m(t,.)\|_2^2 + \int_Q \Delta^2 v_n^m(t,x)\times v_n^m(t,x)\,dx = K_n(\|v_n^m(t,.)+L(u_n)(t,.)\|_q)$$
$$\times\int_Q \Big[R(v_n^m(t,x)+L(u_n)(t,x)))\,\Delta v_n^m(t,x) + g(t,x,v_n^m(t,x)+L(u_n)(t,x))\,v_n^m(t,x)\Big]\,dx.$$

Using the local Lipschitz property of R and $g(t,x,.)$ and the fact that $\int_Q (v_n^m(t,x))^3\Delta v_n^m(t,x)\,dx$ is negative, that the leading coefficient of R is positive and that $\|K_n\|_\infty\leq 1$, we obtain:

$$\|v_n^m(t,.)\|_2^2 + \int_0^t \Big[\|\Delta v_n^m(s,.)\|_2^2 + m(v_n^m(s,.))^2\Big]\,ds \leq \|u_0\|_2^2 + C_T\,(1+\|L(u_n)\|_\infty^6 + m(u_0)^4)$$
$$+C(1+\|L(u_n)\|_\infty^2)\int_0^t \|v_n^m(s,.)\|_4^4 K_n(\|v_n^m(s,.)+L(u_n)(s,.)\|_q)\,ds. \tag{3.34}$$

The norm $(\|\Delta\bullet\|_{L^2(Q)}^2 + m(\bullet)^2)^{\frac{1}{2}}$ is equivalent with the Sobolev norm of $W^{2,2}(Q)$ (cf. eg. Da Prato and Debussche (1996) p. 245). The sequence $(v_n^m)_m$ is bounded in $L^2([0,T],W^{2,2}(Q))$.

Thus, $t_n^m = \infty$ and this sequence converges as $m \to +\infty$ in the weak* topology of $L^2([0,T], W^{2,2}(Q))$. Its weak limit is the weak solution to (3.28) and hence is equal to v_n. Therefore v_n belongs to $L^2([0,T], W^{2,2}(Q))$, and we can repeat the above computation with v_n instead of v_n^m, which yields

$$\|v_n(t,.)\|_2^2 + \int_0^t \Big[\|\Delta v_n(s,.)\|_2^2 + m(v_n(s,.))^2\Big]\, ds \le \|u_0\|_2^2 + C_T(1+\|L(u_n)\|_\infty^6 + m(u_0)^4)$$
$$+C(1+\|L(u_n)\|_\infty^2)\int_0^t \|v_n(s,.)\|_4^4 K_n(\|v_n(s,.)+L(u_n)(s,.)\|_q)\, ds.$$

Thus, (3.32) and Schwarz's inequality imply that

$$\|v_n(t,.)\|_2^2 + \int_0^t \Big[\|\Delta v_n(s,.)\|_2^2 + m(v_n(s,.))^2\Big]\, ds \le \|u_0\|_2^2$$
$$+C_T(1+\|L(u_n)\|_\infty^6) + C_T\,(1+\|L(u_n)\|_\infty^2)\Big[\|\mathcal{A}^{-\frac12}u_0\|_2^2 + m(u_0)^4\Big].$$

Inequality (3.29) yields that for $\beta \in]1,+\infty[$,

$$\sup_n E(\sup_{t\in[0,T]} \|v_n(t,.)\|_2^{2\beta}) < \infty, \tag{3.35}$$
$$\sup_n E\Big(\Big[\int_0^T \Big\{\|\Delta v_n(t,.)\|_2^2 + m(v_n(t,.))^2\Big\}\, dt\Big]^\beta\Big) < \infty. \tag{3.36}$$

Moreover, by Sobolev's embedding theorem (Adams 1975, Corollary 5.16) there exists $C>0$ such that for $d\ge 4$ and $2\le r\le \frac{2d}{d-4}$, if $u\in W^{2,2}(Q)$, $\|u\|_{L^r(Q)} \le C\|u\|_{W^{2,2}(Q)}$. Thus, (3.36) becomes for $2\le r\le \frac{2d}{d-4}$, $\beta\in[1,+\infty[$,

$$\sup_n E\Big(\Big[\int_0^T \|v_n(t,.)\|_r^2\, dt\Big]^\beta\Big) < \infty. \tag{3.37}$$

The inequalities (3.29), (3.35) and (3.37) imply for $2\le r< \frac{2d}{d-4}$, $\beta\in[1,+\infty[$,

$$\sup_n E(\sup_{t\in[0,T]} \|u_n(t,.)\|_2^{2\beta}) < \infty, \tag{3.38}$$
$$\sup_n E\Big(\Big[\int_0^T \|u_n(t,.)\|_r^2\, dt\Big]^\beta\Big) < \infty. \tag{3.39}$$

Let us use the interpolation method to prove that u_n belongs a.s. to $L^a([0,T], L^{\bar R}(Q))$ for $\frac{2d}{d-4} > r \ge \bar R \ge 2$, $a\ge 1\vee \frac{2\bar R}{r}$. Set $\bar R = (1-\lambda)\,2 + \lambda\, r$ for $\lambda\in[0,1]$; Hölder's inequality implies that

$$\int_0^T \|u_n(t,.)\|_{\bar R}^a\, dt \le \int_0^T \|u_n(t,.)\|_2^{\frac{2a(1-\lambda)}{\bar R}}\ \|u_n(t,.)\|_r^{\frac{ar\lambda}{\bar R}}\, dt.$$

Let $\lambda = \frac{2\bar R}{ar}$, we obtain

$$\int_0^T \|u_n(t,.)\|_{\bar R}^a\, dt \le \sup_{t\in[0,T]} \|u_n(t,.)\|_2^{\frac{2}{\bar R}a(1-\lambda)} \times \int_0^T \|u_n(t,.)\|_r^2\, dt;$$

(3.38) and (3.39) imply that for $\bar R\in[2,\frac{2d}{d-4}[$ and $a\ge 2$,

$$\sup_n E\Big(\Big[\int_0^T \|u_n(t,.)\|_{\bar R}^a\, dt\Big]^\beta\Big) < \infty. \tag{3.40}$$

Using lemma 3.1 with $\rho = \frac{\bar{R}}{3}$, so that $\frac{1}{r'} = 1 + \frac{1}{q} - \frac{3}{\bar{R}}$, we obtain

$$\|H_n(u_n)(t,.)\|_q \leq C \int_0^t (t-s)^{-\frac{d+2}{4}+\frac{d}{4r'}} (\|u_n(s,.)\|_{\bar{R}}^3 + 1)\, ds.$$

Let $\gamma, \gamma' \in]1, +\infty[$ be conjugate exponents, with γ close enough to one to ensure $\gamma\,(-\frac{d+2}{4} + \frac{d}{4r'}) > -1$; this is possible (choosing $\bar{R}$ close enough to $\frac{2d}{d-4}$) if $\frac{1}{2} + \frac{d}{4q} > \frac{3}{8}(d-4)$, i.e., $(\frac{3}{8} - \frac{1}{4q})d < 2$. Since $q > d$, this yields $d \leq 5$ for any $q \in]d, +\infty[$. Then Hölder's inequality implies

$$\|H_n(u_n)(t,.)\|_q \leq C \Big[\int_0^t (t-s)^{(-\frac{d+2}{4}+\frac{d}{4r'})\gamma}\, ds \Big]^{\frac{1}{\gamma}} \Big[\int_0^t (\|u_n(s,.)\|_{\bar{R}}^3 + 1)^{\gamma'}\, ds \Big]^{\frac{1}{\gamma'}}.$$

Using (3.40), we obtain

$$\sup_n E\Big(\sup_{t\in[0,T]} \|H_n(u_n)(t,.)\|_q^\beta \Big) < \infty; \tag{3.41}$$

A similar computation (using the quadratic growth of g) yields for $\rho = \frac{\bar{R}}{2}$, $\frac{1}{r"} = 1 - \frac{2}{\bar{R}} + \frac{1}{q}$ and γ close enough to one to ensure that $\gamma d(-\frac{1}{4} + \frac{1}{4r"}) > -1$ (i.e., for $d < 8$ and $\bar{R}$ close to $\frac{2d}{d-4}$) yields

$$\sup_n E\Big(\sup_{t\in[0,T]} \|J_n(u_n)(t,.)\|_q^\beta \Big) < +\infty\,. \tag{3.42}$$

The equations (3.29)-(3.30), (3.41) and (3.42) imply that for $\beta \in [q, +\infty[$, $d < q$:

$$\sup_n E\Big(\sup_{t\in[0,T]} \|u_n(t,.)\|_q^\beta \Big) < \infty.$$

We can now conclude that $\tau_\infty = +\infty$ a.s.; indeed, for every $T > 0$,

$$P(\tau_n \leq T) = P\Big(\sup_{t\leq T} \|u_n(t,.)\|_q \geq n \Big) \leq n^{-\beta}\, E\Big(\sup_{t\leq T} \|u_n(t,.)\|_q^\beta \Big)\,,$$

so that $\lim_{n\to\infty} P(\tau_n \leq T) = 0$. Therefore, we can construct the solution to the SPDE (3.23) on any interval $[0,T]$. □

4 Regularity of the solution

The following lemma studies the Hölder regularity of the term involving the initial condition. There are many possible situations, depending on the boundary conditions, whether $\int_Q G(t,x;0,y)dy = 1$ or not, which requires two different arguments. For $(t,x) \in]0,T] \times Q$, set $G_t u_0(x) = \int_Q G(t,x;0,y)\, u_0(y)\, dy$ and set $G_0 u_0 = u_0$.

Lemma 4.1 . *Suppose that Q is convex and that G satisfies (1.2) with $a, b \in \{0,1\}$.*

1) (i) Let $u_0 \in L^q(Q)$ for some $q \in [1, +\infty[$; then $Gu_0 \in \mathcal{C}(]0,T], L^q(Q))$.

(ii) Let u_0 be bounded; then for $0 < \lambda < 1$ and $0 < t_0 < T$, $Gu_0 \in \mathcal{C}^\lambda([t_0,T] \times Q)$.

2) Assume furthermore that $\int_Q G(t,x;s,y)\, dy = 1$ for all $(s,t,x) \in]0,T]^2 \times Q$ with $s < t$.

(i) Let u_0 be continuous; then $Gu_0 \in \mathcal{C}([0,T] \times Q)$.

(ii) Let $u_0 \in L^q(Q)$; then $Gu_0 \in \mathcal{C}([0,T], L^q(Q))$.

(iii) Let $u_0 \in \mathcal{C}^\lambda(Q)$ for some $\lambda \in]0,1[$; then for $0 \leq s < t \leq T$, $\sup_{x\in Q} |G_t u_0(x) - G_s u_0(x)|$ $\leq C(t-s)^{\frac{\lambda}{4}}$.

3) Let $Q = [0,M]^d$, $u_0 \in \mathcal{C}^\lambda(Q)$ for some $\lambda \in]0,1[$ and suppose that for every $1 \le i \le d$, if $\hat{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d)$, there exists a function $\phi_i : [0,T] \times \mathbb{R} \times [0,M]^{d-1} \times [0,M]^{d-1}$ such that

$$G(t,x;0,y) = \phi_i(t, x_i + y_i, \hat{x}_i, \hat{y}_i) + \varepsilon_i \phi_i(t, x_i - y_i, \hat{x}_i, \hat{y}_i)\,,$$

with $\varepsilon_i \in \{-1,1\}$, with $\sup_{(t,x)\in[0,T]\times Q} \int_Q |\phi_i(t, x_i + \epsilon y_i, \hat{x}_i, \hat{y}_i)|\, dy < +\infty$ for $\epsilon \in \{-1,+1\}$, and suppose that either one of the conditions (a) or (b) holds:

(a) $\varepsilon_i = +1$ and $\phi_i(t, r+2M, \hat{x}_i, \hat{y}_i) = \phi_i(t, r, \hat{x}_i, \hat{y}_i)$ for every $r \in \mathbb{R}$.

(b) $u_0(x) = 0$ for $x \in \partial Q$.

Then for any $x, x' \in Q$ one has $\sup_{t\in[0,T]} |G_t u_0(x) - G_t u_0(x')| \le C\,|x - x'|^\lambda$.

Remark 4.2 . *This proposition can be used for any convex compact subset Q; in (1), G need not to be a semi-group. If $Q = [0,\pi]^d$ and G is the Green function of the operator $\frac{\partial}{\partial t} + \Delta^2 = 0$, for $u_0 \in \mathcal{C}^\lambda(Q)$, $0 < \lambda < 1$, the function $G_.u_0(.) \in \mathcal{C}^{\frac{\lambda}{4},\lambda}([0,T]\times Q)$ under the homogeneous Neumann boundary conditions (3.20), while under the homogeneous Dirichlet boundary conditions (3.21), one has $\sup_{0\le t\le T} |G_t u_0(x) - G_t u_0(x')| \le C\,|x-x'|^\lambda$, and for $0 < t_0 \le s < t \le T$, one has for any $0 < \mu < 1$, $\sup_{x\in Q} |G_t u_0(x) - G_s u_0(x)| \le C(t_0)\,|t-s|^\mu$.*

Proof of Lemma 4.1: 1) (i) Given $0 < t_0 \le s < t \le T$, $0 < \lambda < 1$, $\|G_t u_0 - G_s u_0\|_q^q \le C|t-s|^{\lambda q}\, C(M_1 + M_2)$, where

$$\begin{aligned} M_1 &= \int_Q dx \left| \int_Q \left| \int_0^1 (\theta t + (1-\theta)s)^{-(\alpha+\eta)} \exp\left(-c\frac{|x-y|^\beta}{(\theta t + (1-\theta)s)^\gamma}\right) d\theta \right|^\lambda t^{-\alpha(1-\lambda)}\, u_0(y)\, dy \right|^q \\ M_2 &= \int_Q dx \left| \int_Q \left| \int_0^1 (\theta t + (1-\theta)s)^{-(\alpha+\eta)}\, d\theta \right|^\lambda s^{-\alpha(1-\lambda)} \exp\left(-c\frac{|x-y|^\beta}{s^\gamma}\right) u_0(y)\, dy \right|^q . \end{aligned}$$

Clearly, $M_1 + M_2 \le t_0^{-c}$ for some $c > 0$.

(ii) A similar argument for $q = +\infty$ shows that for $0 < t_0 \le t < t' \le T$, $x, x' \in Q$, $0 < \lambda < 1$ and some $c > 0$,

$$|G_t u_0(x) - G_{t'} u_0(x')| \le C\, t_0^{-c}\, \left(|t'-t|^\lambda + |x-x'|^\lambda\right)$$

2) Since $\int_Q G(t,x;s,y)\, dy = 1$ for every $s < t$, $G_t u_0(x) - u_0(x) = \int_Q G(t,x;0,y)\,[u_0(y) - u_0(x)]\, dy$ and since G is a semi-group, for $t < t'$, $G_{t'} u_0(x) - G_t u_0(x) = \int_Q G(t,x;0,y)\, dy \int_Q G(t',y;t,z)[u_0(z) - u_0(y)]\, dz$, so that the study of the time-regularity is completed by that at 0 .

(i) One has to check that for any $x \in Q$, $G_t u_0(x) - u_0(x)$ converges to 0 as $t \to 0$. The argument, based on the continuity of u_0 at x, is similar to the previous one (see e.g. [4], Lemma 2.1).

(ii) Let $u_0 \in L^q(Q)$, let $(u_0^n)_{n\ge1}$ be a sequence of continuous function converging to u_0 in $L^q(Q)$. According to (i), (Gu_0^n) belong $C([0,T]\times Q)$ and it suffices to check that $\sup_t \|G_t u_0\|_q \le C\|u_0\|_q$. This follows from Hölder's inequality and (2.3).

(iii) Using the Hölder continuity of u_0, one has

$$\begin{aligned} \sup_{x\in Q} |G_t u_0(x) - u_0(x)| &\le \sup_{x\in Q} C \int_Q t^{-\alpha} \exp\left(-c\frac{|x-y|^\beta}{t^\gamma}\right) |u_0(y) - u_0(x)|\, dy \\ &\le \sup_{x\in Q} C \int_Q t^{-\alpha} \exp\left(-c\frac{|x-y|^\beta}{t^\gamma}\right) \left(\frac{|x-y|}{t^{\frac{\gamma}{\beta}}}\right)^\lambda t^{\lambda\frac{\gamma}{\beta}}\, dy \le C\, t^{\lambda\frac{\gamma}{\beta}} . \end{aligned}$$

(3) The proof of the space regularity under condition (a), which is a straightforward extension of that of [2] Lemma A.2 and [4], Lemma 2.2, is omitted.

We suppose that (b) holds and compare the function Gu_0 at points $x = (x_1, \hat{x}_1)$ and $x' = (x'_1, \hat{x}_1)$ with $x_1 < x'_1$; increments of other components are similarly dealt with, and provide the required regularity. Obvious changes of variables yield $G_t u_0(x) - G_t u_0(x') = \sum_{i=1}^4 D_i(t,x,x')$, where if we set $\tilde{Q} = [0,M]^{d-1}$,

$$\begin{aligned}
D_1(t,x,x') &= \int_0^{M-(x'_1-x_1)} \int_{\tilde{Q}} \phi_1(t, x'_1 + y_1, \hat{x}_1, \hat{y}_1)\,[u_0(y) - u_0(y_1 + (x'_1 - x_1), \hat{y}_1)]\,d\hat{y}_1\,dy_1\,, \\
D_2(t,x,x') &= \varepsilon_1 \int_{x'_1-x_1}^{M} \int_{\tilde{Q}} \phi_1(t, x'_1 - y_1, \hat{x}_1, \hat{y}_1)\,[u_0(y) - u_0(y_1 - (x'_1 - x_1), \hat{y}_1)]\,d\hat{y}_1\,dy_1\,, \\
D_3(t,x,x') &= \int_{M-(x'_1-x_1)}^{M} \int_{\tilde{Q}} \phi_1(t, x'_1 + y_1, \hat{x}_1, \hat{y}_1)\,[u_0(y) - u_0(M, \hat{y}_1)]\,d\hat{y}_1\,dy_1 \\
&\quad - \int_{-(x'_1-x_1)}^{0} \int_{\tilde{Q}} \phi_1(t, x'_1 + y_1, \hat{x}_1, \hat{y}_1)\,[u_0(y_1 + (x'_1 - x_1), \hat{y}_1) - u_0(0, \hat{y}_1)]\,d\hat{y}_1\,dy_1\,, \\
D_4(t,x,x') &= \varepsilon_1 \int_0^{x'_1-x_1} \int_{\tilde{Q}} \phi_1(t, x'_1 - y_1, \hat{x}_1, \hat{y}_1)\,[u_0(y) - u_0(0, \hat{y}_1)]\,d\hat{y}_1\,dy_1 \\
&\quad - \varepsilon_1 \int_M^{M+(x'_1-x_1)} \int_{\tilde{Q}} \phi_1(t, x'_1 - y_1, \hat{x}_1, \hat{y}_1)\,[u_0(y_1 - (x'_1 - x_1), \hat{y}_1) - u_0(M, \hat{y}_1)]\,d\hat{y}_1\,dy_1\,.
\end{aligned}$$

The Hölder regularity of u_0 and the integrability property of $\phi_1(t, ., \hat{x}_1, .)$, uniformly with respect to $(t, \hat{x}_1)$, conclude the proof. □

We suppose that $u_0 \in \mathcal{C}^a(Q)$ for some $a \in]0,1[$; then $u_0 \in L^q(Q)$ for any $q > d$, so that by Theorem 3.3, the solution u to (3.23) belongs to $L^\infty([0,T], L^q(Q))$ for every $q \in]d, +\infty[$. Remark 4.2 gives the regularity of Gu_0 depending on the boundary conditions, while Lemma 2.3 gives the regularity of the stochastic integral in (3.23). Thus it suffices to study the regularity of the drift terms of (3.23) with coefficients which may have polynomial growth.

Lemma 4.3 . *Let G be a (non-necessarily time-homogeneous) semi-group satisfying (1.2) with $\alpha = \frac{\gamma}{\beta} d$, let a be a multi-index such that $|a|\delta < 1$, $H(t,x;s,y) = D_x^a G(t,x;s,y)$. Let $b : [0,T] \times Q \times \mathbb{R} \to \mathbb{R}$ be measurable such that* $\sup\{|b(t,x,y)|\,,\,(t,x) \in [0,T] \times Q\} \le C\,|y|^m$ *for some $m \ge 1$, let $0 < \lambda < \frac{1-|a|\delta}{\eta \vee 1}$ and $0 < \mu < \Big[(1 - |a|\delta)\left(\frac{\beta}{\gamma} \wedge \frac{1}{\delta}\right)\Big] \wedge 1$. Then if $u : \Omega \times [0,T] \times Q \to \mathbb{R}$ is a process in $L^\infty([0,T], L^q(Q))$ for q large enough,*

$$B(u)(t,x) = \int_0^t \int_Q H(t,x;s,y)\,b(s,y,u(s,y))\,dy\,ds\,.$$

the map $B(u)$ belongs to $\mathcal{C}^{\lambda,\mu}([0,T] \times Q)$.

Proof: The argument, based on the factorization method (see e.g. G. Da Prato and J. Zabczyk [10]) is similar to that in the proof of section 2.3 in [4]; it is briefly sketched.

Let $\bar{\delta} = |a|\,\delta$, $\varepsilon \in]0,1[$ and set

$$\begin{aligned}
\mathcal{J}(v)(t,x) &= \int_0^t \int_Q G(t,x;s,y)\,(t-s)^{-\varepsilon}\,v(s,y)\,dyds\,, \\
\mathcal{K}(v)(t,x) &= \int_0^t \int_Q H(t,x;s,y)\,(t-s)^{\varepsilon-1}\,b(s,y,v(s,y))\,dyds\,.
\end{aligned}$$

The semi-group property of G implies that for every $(t,x) \in [0,T] \times Q$, $B(u)(t,x) = \frac{\sin(\varepsilon\pi)}{\pi} \times \mathcal{J}(\mathcal{K}(u))(t,x)$. We prove that for $\varepsilon > \bar\delta + \alpha\,\frac{m-1}{q}$, $\mathcal{K}$ maps $L^\infty([0,T], L^q(Q))$ into itself. Indeed, it suffices to use Lemma 3.1 with q and $\frac{q}{m}$, so that $\frac{1}{r} = 1 - \frac{m-1}{q}$, and (2.3). We then prove that for $v \in L^\infty([0,T], L^q(Q))$, the trajectories of $\mathcal{J}(v)$ have the required Hölder regularity. Let $x, x' \in Q$; then $|\mathcal{J}(v)(t,x) - \mathcal{J}(v)(t,x')| \le A_1(t,x,x') + A_2(t,x,x')$, where

$$\begin{aligned}
A_1(t,x,x') &= \int_0^t \int_Q 1_{\{|y-x| \le |x'-x|\}} (t-s)^{-\varepsilon} \Big(|G(t,x;s,y)| + |G(t,x';s,y)| \Big)\, |v(s,y)|\, dy\, ds\,,\\
A_2(t,x,x') &= \int_0^t \int_Q 1_{\{|y-x| > |x'-x|\}} (t-s)^{-\varepsilon}\, |G(t,x;s,y) - G(t,x';s,y)|\, |v(s,y)|\, dy\, ds\,.
\end{aligned}$$

Hölder's inequality and a change of variables yield that for $0 < \mu < \frac{q-1}{q}$,

$$\begin{aligned}
A_1(t,x,x') &\le C \int_0^t (t-s)^{-(\varepsilon+\alpha)}\, \|v(s,.)\|_q \left(\int_{|z| \le |x-x'|\, t^{-\frac{\gamma}{\beta}}} \exp(-c|z|^\beta)\, t^\alpha\, dz \right)^{\frac{q-1}{q}} ds\\
&\le |x-x'|^\mu \int_0^t (t-s)^{-\varepsilon - \frac{\alpha}{q} - \frac{\gamma\mu}{\beta}}\, ds\,. \qquad (4.1)
\end{aligned}$$

The convergence of this last integral requires $\mu < \frac{\beta}{\gamma}\left(1 - \varepsilon - \frac{\alpha}{q}\right)$ and $1 - \varepsilon - \frac{\alpha}{q} > 0$. Furthermore, if $|x-y| > |x'-x|$, and $\tilde{x}$ is a convex combination of x and x', then $|\tilde{x} - y| \ge \frac{1}{\sqrt{2}}\big(|2x - x' - y| \wedge |x'-y|\big)$. Therefore, Taylor's formula and Hölder's inequality imply that for $0 < \mu < 1$,

$$A_2(t,x,x') \le C\,|x-x'|^\mu \int_0^t (t-s)^{-\varepsilon - \mu\delta - \frac{\alpha}{q}}\, \|v(s,.)\|_q\, ds\,, \qquad (4.2)$$

and the last integral converges if $\mu < \frac{1-\varepsilon-\frac{\alpha}{q}}{\delta}$.

Similarly, for $0 < t < t' \le T$ and $x \in Q$, $|\mathcal{J}(v)(t,x) - \mathcal{J}(v)(t',x)| \le B_1(t,t',x) + B_2(t,t',x)$, where

$$\begin{aligned}
B_1(t,t',x) &= \int_0^t \int_Q |(t'-s)^{-\varepsilon}\, G(t',x;s,y) - (t-s)^{-\varepsilon}\, G(t,x;s,y)|\, |v(s,y)|\, dy\, ds\,,\\
B_2(t,t',x) &= \int_t^{t'} \int_Q (t'-s)^{-\varepsilon}\, |G(t',x;s,y)|\, |v(s,y)|\, dy\, ds\,.
\end{aligned}$$

Computations similar to the previous ones yield for $\lambda \in]0,1[$

$$\begin{aligned}
B_1(t,t',x) &\le C\,|t-t'|^\lambda \int_0^t (t-s)^{-\varepsilon - \eta\lambda - \frac{\alpha}{q}}\, \|v(s,.)\|_q\, ds\,, \qquad (4.3)\\
B_2(t,t',x) &\le C \int_t^{t'} (t'-s)^{-\varepsilon - \frac{\alpha}{q}}\, \|v(s,.)\|_q\, ds \le C\,|t-t'|^{1-\varepsilon-\frac{\alpha}{q}}\,. \qquad (4.4)
\end{aligned}$$

The integral in the right hand-side of (4.3) converges if $\lambda < \frac{1-\varepsilon-\frac{\alpha}{q}}{\eta}$, while that in the right hand-side of (4.4) converges for $\varepsilon < 1 - \frac{\alpha}{q}$. Thus for q arbitrary large and ε close enough to $\bar\delta$, we see that the inequalities (4.1)-(4.4) conclude the proof. □

The following theorem summarizes the results proved in Lemmas 2.3, 4.3 and Remark 4.2.

Theorem 4.4 . *Assume (H.1), (H.2) and (H.4), let* $d = 4, 5$, $u_0 \in \mathcal{C}^a(Q)$ *for some* $a \in]0,1[$, F *be a Gaussian process with covariance define in terms of the function* f *by (2.1), such that*

(C1) holds and $\int_{B_d(0,1)} f(v)\,|v|^{-d(1+\varepsilon)+4}\,dv\;<\;+\infty$ *for some* $\varepsilon > 0$*. Then the solution* u *to (3.23) belongs to* $\mathcal{C}^{\bar\lambda,\bar\mu}([0,T]\times Q)$ *under the Neumann boundary conditions (3.20) (resp. the map* $x \to u(t,x) \in \mathcal{C}^{\bar\mu}(Q)$ *uniformly in* $t \in [0,T]$ *while the map* $t \to u(t,x) \in \mathcal{C}^{\lambda}([t_0,T])$ *uniformly in* $x \in Q$ *for* $0 < t_0 < T$ *under the Dirichlet boundary conditions (3.21)), where*

$$0 < \lambda < \left(1 - \frac{\max_i k_i \vee 2}{4}\right) \wedge \frac{\varepsilon d}{8}, \quad \bar\lambda = \lambda \wedge \frac{a}{4} \quad \textit{and} \quad 0 < \bar\mu < a \wedge \frac{\varepsilon d}{2}.$$

Finally, a straightforward extension of the preceding computations, using Lemmas 2.3 and 4.3 (with α_i instead of $\alpha + |a|\delta$), provides Hölder regularity for the solution to (3.3).

Theorem 4.5 *Let* Q *be convex, suppose that* G *satisfies (1.2) and that the assumptions of Theorem 3.2 are fulfilled. Let* u_0 *be a continuous function on* Q*, let* u *be the solution to (3.3) and set* $v(t,x) = u(t,x) - G_t u_0(x)$*. Then*

(i) If $\alpha < 1$ *and* F *is the space-time white noise, then* $v \in \mathcal{C}^{\lambda,\mu}([0,T]\times Q)$ *for* $0 < \lambda < \inf_i(1+\alpha-\alpha_i) \wedge \frac{1-\alpha}{2}$ *and* $0 < \mu < \frac{\inf_i(1+\alpha-\alpha_i)}{\delta} \wedge \frac{1-\alpha}{2\delta} \wedge 1$.

(ii) If $\alpha \geq 1$ *and* F *is a Gaussian process with covariance function* f *defined by (2.1) such that (C1) holds and* $\int_{B_d(0,1)} f(v)\,|v|^{-d(1+\epsilon)+\frac{1}{\delta}}\,dv < +\infty$ *for some* $\epsilon > 0$*, then* $v \in \mathcal{C}^{\lambda,\mu}([0,T]\times Q)$ *for* $0 < \lambda < \inf_i(1+\alpha-\alpha_i) \wedge (\epsilon d\delta) \wedge 1$ *and* $0 < \mu < \frac{\inf_i(1+\alpha-\alpha_i)}{\delta} \wedge (\epsilon d) \wedge 1$.

5 Density of the solution to the stochastic Cahn-Hilliard PDE

In this section, we concentrate on the solution to (3.23) in dimension 4 and 5 under either the homogeneous Neumann or Dirichlet boundary conditions on $Q = [0,\pi]^d$. Thus, we prove that under proper non-degeneracy conditions on the "diffusion" coefficient σ, the law of $u(t,x)$ has a density for $t > 0$ and $x \in Q$. This extends results proved in [4] and [5] to higher dimension. Since the noise F has a space correlation, the setting of the corresponding stochastic calculus of variations is that used in [21]. Let $Q = [0,\pi]^d$, $\mathcal{E}$ denote the inner product space of measurable functions $\varphi : Q \to \mathbb{R}$ such that $\int_Q dx \int_Q dy\, |\varphi(x)|\, f(x-y)\, |\varphi(y)|\; < +\infty$, endowed with the inner product

$$< \varphi, \psi >_{\mathcal{E}} = \int_Q dx \int_Q dy\, \varphi(x)\, f(x-y)\, \psi(y)\,.$$

Let $\mathcal{H}$ denote the completion of $\mathcal{E}$ and set $\mathcal{E}_T = L^2([0,T],\mathcal{E})$ and $\mathcal{H}_T = L^2([0,T],\mathcal{H})$. Note that $\mathcal{H}$ and $\mathcal{H}_T$ need not be spaces of functions, and that $\mathcal{H}_T$ is a Hilbert space which is isomorphic to the reproducing kernel Hilbert space of the Gaussian noise $(F(\varphi)\,;\,\varphi \in \mathcal{D}([0,T]\times Q)$. This noise can be identified with a Gaussian process $(W(h)\,,\, h \in \mathcal{H}_T)$ defined as follows. Let $(e_j\,,\, j \geq 0) \subset \mathcal{E}$ be a CONS of $\mathcal{H}$; then $(W_j(t) = \int_0^t \int_Q e_j(x)\, F(ds,dx)\,,\, j \geq 0)$ is a sequence of independent standard Brownian motions such that

$$F(\varphi) = \sum_{j\geq 0} \int_0^t < \varphi(s,*), e_j >_{\mathcal{H}}\, dW_j(s)\,, \varphi \in \mathcal{D}([0,T]\times Q)\,.$$

For $h \in \mathcal{H}_T$, we set $W(h) = \sum_j \int_0^T < h(s), e_j >_{\mathcal{H}}\, dW_j(s)$, and use the framework of the Malliavin calculus described in [23] to define the Malliavin derivative DX of a random variable X and the corresponding Sobolev spaces $\mathbb{D}^{N,p}$. Given $X \in \mathbb{D}^{1,2}$, $h \in \mathcal{H}_T$, set $D_h X =< DX, h >_{\mathcal{H}_T} = \int_0^T < D_{r,*}X, h(r) >_{\mathcal{H}}\, dr$, where $D_{r,*}X \in \mathcal{H}$ for every $r \in [0,T]$. Finally for $r \in [0,T]$ and $\varphi \in \mathcal{H}$, set $D_{r,\varphi}X =< D_{r,*}, \varphi >_{\mathcal{H}}$.

Since the coefficients R and $g(t,x,.)$ are locally Lipschitz, we need to localize the Sobolev spaces as follows. A random variable X belongs to $\mathbb{D}^{1,p}_{\mathrm{loc}}$ if there exists an increasing sequence

$\Omega_n \subset \Omega$ such that $\lim_n P(\Omega_n) = 1$ and for every n, there exists a random variable $X_n \in \mathbb{D}^{1,p}$ and $X = X_n$ on Ω_n. Let $u_0 \in \mathcal{C}(Q)$ and suppose that the conditions (H1) and (H'2) hold, where

(H'2) The function $\sigma : \mathbb{R} \to \mathbb{R}$ is bounded, globally Lipschitz, the map $g(t, x, .)$ is of class $\mathcal{C}^1$ with quadratic growth and satisfies (3.22), and the maps $b_i(t, x, .), 1 \le i \le N$ are of class $\mathcal{C}^1$ with derivatives bounded uniformly in (t, x).

Let u denote the solution to (3.23) with either the homogeneous Neumann or Dirichlet boundary conditions. Lemmas 2.3 and 4.3 imply that the trajectories of $u - Gu_0$ are almost surely Hölder continuous on $[0, T] \times Q$, while the function Gu_0 is clearly bounded by $\|u_0\|_\infty$. Therefore, $\lim_n P(\Omega_n) = 1$ if for every $n \ge 1$ one sets

$$\Omega_n = \Big\{\omega \in \Omega \,:\, \sup\{|u(t,x)|\,,\,(t,x) \in Q\} \le n\Big\}.$$

We now construct a sequence of processes $u(n) \in \mathbb{D}^{1,p}$ for every $p \in [2, +\infty[$ such that $u = u(n)$ on Ω_n. Let K_n be the sequence defined at the beginning of the proof of Theorem 3.3, which satisfies (3.25); set $\bar{R}_n(x) = K_n(|x|)R(x)$ and $\bar{g}_n(t, x, y) = K_n(|y|)\, g(t, x, y)$. The functions $\bar{R}_n$ and $y \to \bar{g}_n(t, x, y)$ are of class $\mathcal{C}^1$ with bounded derivatives. Hence Theorem 3.2 yields the existence and uniqueness of the process $u(n)$ solution to the evolution equation:

$$\begin{aligned} u(n)(t,x) \;&=\; G_t u_0(x) + \int_0^t \int_Q G(t,x;s,y)\,\sigma(u(n)(s,y))\,F(ds,dy) \qquad (5.1)\\ &\quad + \int_0^t \int_Q \Big[\Delta G(t,x;s,y)\,\bar{R}_n(u(n)(s,y)) + G(t,x;s,y)\,\bar{g}_n(s,y,u(n)(s,y))\\ &\qquad + \sum_{i=1}^N H_i(t,x;s,y)\,b_i(s,y,u(n)(s,y))\Big]\,dyds\,. \end{aligned}$$

The local property of stochastic integrals implies that $u(n) = u$ on Ω_n. The following proposition shows that $u(n) \in \mathbb{D}^{1,p}$ for every $p \in [2, +\infty[$.

Proposition 5.1 . *Let Q be a compact subset of $\mathbb{R}^d$, $u_0 \in \mathcal{C}(Q)$, let $\sigma : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz and for $i \in \{1, \cdots, N\}$, let $b_i : [0, T] \times Q \times \mathbb{R}$ satisfy the conditions (L1) and (L2), and suppose that the maps $y \to b_i(t, x, y)$ are of class $\mathcal{C}^1$ with bounded derivatives, let G and $(H_i\,, 1 \le i \le N)$ satisfy the conditions (C2) and (C4). Let F be a Gaussian process with space covariance defined by (2.1) in terms of a function f which satisfies the condition (C3). Then for every $p \in [2, +\infty[$ and $(t, x) \in [0, T] \times Q$, the solution $v(t, x)$ to (3.3) belongs to $\mathbb{D}^{1,p}$. Furthermore, for every $r \in [0, T]$ and $\varphi \in \mathcal{H}$, $D_{r,\varphi}v(t, x) = 0$ if $r > t$, while there exists a bounded, adapted family of random variables $(S(s, y)\,, (s, y) \in [0, T]\times Q)$ such that for $0 \le r \le t$:*

$$\begin{aligned} D_{r,\varphi}v(t,x) \;&=\; <G(t,x;r,*)\,\sigma(v(r,*)),\varphi>_{\mathcal{H}} + \int_r^t \int_Q \Big[G(t,x;s,y)\,S(s,y) \qquad (5.2)\\ &\times D_{r,\varphi}v(s,y)\,F(ds,dy) + \sum_{i=1}^N H_i(t,x;s,y)\,\partial_3 b_i(s,y,v(s,y))\,D_{r,\varphi}v(s,y)\,dyds\Big]\,, \end{aligned}$$

and for every $p \in [1, +\infty[$,

$$\sup_{(t,x)\in[0,T]\times Q} E\left(\left|\int_0^t \|D_{r,*}v(t,x)\|_{\mathcal{H}}^2\,dr\right|^p\right) = C(p) < +\infty\,. \tag{5.3}$$

Furthermore, given $0 \le s < t \le T$, if ψ denotes the function defined by (2.13),

$$\sup_{x\in Q} E\left(\int_s^t \|D_{r,\varphi}v(t,x)\|_{\mathcal{H}}^2\,dr\right) \le C\left[\int_0^{t-s}\psi(\tau)\,d\tau + \sum_{i=1}^N (t-s)^{2(1+\alpha-\alpha_i)}\right]. \tag{5.4}$$

Proof: Let $(v_k\,,\,k\geq 0)$ be the Picard approximation scheme of v defined by (3.12); then by the proof of Theorem 3.2, the sequence $(v_k(t,x)\,,\,k\geq 0)$ is bounded in $L^p(\Omega)$ uniformly in (t,x) and converges in $L^p(\Omega)$ to $v(t,x)$. Following a classical argument, we prove by induction on k that $v_k(t,x)\in\mathbb{D}^{1,p}$ and that

$$\sup_k \sup_{(t,x)\in[0,T]\times Q} E\Big(\|Dv_k(t,x)\|^p_{\mathcal{H}_T}\Big) < +\infty\,, \tag{5.5}$$

$$\sum_k \sup_{(t,x)\in[0,T]\times Q} E\Big(\|Dv_{k+1}(t,x)-Dv_k(t,x)\|^p_{\mathcal{H}_T}\Big) < +\infty\,. \tag{5.6}$$

Then using [23], Lemma 1.2.3, we conclude that $Dv_k(t,x)$ converges to $Dv(t,x)$ in the weak topology of $L^p(\Omega,\mathcal{H}_T)$; furthermore, this yields (5.3). Since v_0 is deterministic, it belongs to $\mathbb{D}^{1,p}$; suppose that $v_k\in\mathbb{D}^{1,p}$; since σ is globally Lipschitz, Proposition 1.2.3 in [23] implies that $\sigma(v_k(s,y))\in\mathbb{D}^{1,p}$ and that $D_{r,\varphi}(\sigma(v_k(s,y)))=S_k(s,y)\,D_{r,\varphi}v_k(s,y)$, where $S_k(s,y)$ is a bounded adapted process. Furthermore, for every $r\in[0,T]$ and $\varphi\in\mathcal{H}$, $D_{r,\varphi}v_{k+1}(t,x)=0$ if $r>t$ and for $r\leq t$:

$$\begin{aligned} D_{r,\varphi}v_{k+1}(t,x) =&< G(t,x;r,*)\,\sigma(v_k(r,*)),\varphi>_{\mathcal{H}} + \int_r^t\int_Q\Big[G(t,x;s,y)\,S_k(s,y) \\ &\times D_{r,\varphi}v_k(s,y)\,F(ds,dy)+\sum_{i=1}^N H_i(t,x;s,y)\,\partial_3 b_i(s,y,v_k(s,y))\,D_{r,\varphi}v(s,y)\,dyds\Big]\,. \end{aligned}$$

Let ψ be the $L^1([0,T])$ function defined by (2.13) and set $I=\int_0^T\psi(r)\,dr$. The linear growth condition on σ and equations (3.18) and (2.12) imply that for any $p\in[2,+\infty[$, there exists a constant C_p (which does not depend on k) such that for every k

$$\sup_{x\in Q} E\Big(\|G(t,x;.,*)\,\sigma(v_k(.,*))\|^{2p}_{\mathcal{H}_T}\Big)\leq C\,I^{p-1}\int_0^t\psi(t-r)\,\sup_{(r,x)}E(|v_k(r,x)|^{2p})\,dr\leq C_p\,. \tag{5.7}$$

For $t\in[0,T]$, $x\in Q$, let $(Y_\tau(x)\,,\,0\leq\tau\leq T)$ be the $\mathcal{H}_T$-valued martingale defined by

$$Y_\tau(x)=\int_0^{\tau\wedge t}\int_Q G(t,x;s,y)\,S_k(s,y)\,Dv_k(s,y)\,F(ds,dy)\,.$$

Let $(\epsilon_j\,,\,j\geq 0)$ be a CONS of $\mathcal{H}_T$; then Burkholder's inequality for Hilbert-valued martingales (see e.g. [19], p. 212) and Parseval's identity yield

$$\sup_{x\in Q}\|Y_t(x)\|^{2p}_{L^{2p}(\Omega,\mathcal{H}_T)}\leq C_p\,I^{p-1}\int_0^t\psi(t-s)\,\sup_y E(\|Dv_k(s,y)\|^{2p}_{\mathcal{H}_T})\,ds\,. \tag{5.8}$$

Finally, Lemma 3.1 and the conditions (C2) - (C4) imply that the function $\bar\psi(s)=\psi(s)+\sum_{i=1}^N s^{-\alpha_i+\alpha}\in L^1([0,T])$, and together with the inequalities (5.7) and (5.8) this yields that there exists $C_p>0$ such that for every $k\geq 0$,

$$\sup_{x\in Q}\|Dv_k(t,x)\|^{2p}_{L^{2p}(\Omega,\mathcal{H}_T)}\leq C_p+C_p\int_0^t\bar\psi(t-s)\,\sup_{y\in Q}\|Dv_k(s,y)\|^{2p}_{L^{2p}(\Omega,\mathcal{H}_T)}\,ds\,.$$

Thus Lemma 15 in [6] concludes the proof of (5.5). A similar argument shows (5.6). We conclude that each $v_k(t,x)\in\mathbb{D}^{1,p}$, and that (5.2)and (5.3) hold. To prove (5.4), we use (3.10), (5.2) and arguments similar to the previous ones; then for $0\leq s<t\leq T$,

$$E\left(\int_s^t\|G(t,x;r,*)\,\sigma(v(r,*))\|^2_{\mathcal{H}}\,dr\right)\;\leq\; C\left(\int_s^t\||G(t,x;r,*)|\|^2_{\mathcal{H}}\,dr\right)\left[1+\sup_{(r,y)}E(|v(r,y)|^{2p})\right]$$

$$\leq \; C \int_0^{t-s} \psi(\tau)\, d\tau\, .$$

Furthermore, the isometry of Hilbert-spaced valued martingales, Fubini's theorem and (5.3) imply

$$E\left(\int_s^t \left\| \int_r^t \int_Q G(t,x;\tau,y)\, S(\tau,y)\, D_{r,*}v(\tau,y)\, F(d\tau,dy) \right\|_{\mathcal{H}}^2 dr \right)$$
$$\leq C \int_s^t d\tau \int_Q dy \int_Q dz\, G(t,x;\tau,y)\, f(y-z)\, G(t,x;\tau,z)$$
$$\times \int_s^\tau E(< D_{r,*}v(s,y)\, ,\, D_{r,*}v(s,z) >_{\mathcal{H}})\, dr \leq C \int_0^{t-s} \psi(\tau)\, d\tau\, .$$

Finally, conditions (C2) and (C4), Minkowski's and Schwarz's inequalities and Fubini's theorem imply that for every $i \leq N$,

$$E\left(\int_s^t \left\| \int_r^t \int_Q H_i(t,x;\tau,y)\, \partial_3 b_i(\tau,y,v(s,y))\, D_{r,*}v(\tau,y)\, dy d\tau \right\|_{\mathcal{H}}^2 dr \right)$$
$$\leq C\,(t-s)^{\alpha+1-\alpha_i} \int_s^t d\tau \int_Q dy\, |H_i(t,x;\tau,y)| \sup_{(\tau,y)\in[0,T]\times Q} \int_s^\tau E(\|D_{r,*}v(\tau,y)\|_{\mathcal{H}}^2)\, dr$$
$$\leq (t-r)^{2(\alpha+1-\alpha_i)}\, .$$

This completes the proof of (5.4). □

The following theorem, which establishes the absolute continuity of the law to the stochastic Cahn-Hilliard PDE, is the main result of this section.

Theorem 5.2 . *Let $Q=[0,\pi]^d$, suppose that the conditions (H.1), (H'2) and (H.4) hold, let F be a Gaussian noise with covariance defined by (2.1) in terms of a function f which satisfies the conditions (C1) and (3.24). Let $u_0 \in \mathcal{C}(Q)$ and let u be the solution to (3.23) with initial conditions u_0 and the homogeneous Neumann or Dirichlet boundary conditions. Let $t_0 \in]0,T]$, $x_1,\cdots,x_l$ be pairwise distinct points in $]0,\pi[^d$, and set $u(t_0,\underline{x}) = (u(t_0,x_1),\cdots,u(t_0,x_l))$. For any $\tau>0$, let*

$$I(\tau) = \int_{B_d(0,\tau)} f(v)\, |v|^{-d+4}\, \ln\left(|v|^{-1}\right)^{(5-d)^+}\, dv\, . \tag{5.9}$$

(i) Suppose that $|\sigma| \geq C > 0$; then if for some $0<\nu<\frac{1}{4}$

$$\lim_{\tau\to 0} I(\tau^{\frac{1}{4}+\nu})^{-1}\left[\tau + \tau^{\frac{1}{2}}\, I(\tau^{\frac{1}{4}-\nu})\right] = 0\, ,$$

the law of $u(t_0,\underline{x})$ is absolutely continuous with respect to Lebesgue's measure.

(ii) Let $u_0 \in \mathcal{C}^a(Q)$ for some $a>0$ and suppose that for some $\nu>0$:

$$\lim_{\tau\to 0} I(\tau^{\frac{1}{4}+\nu})^{-1}\left[\tau^A + \tau^{\frac{1}{2}}\, I(\tau^{\frac{1}{4}-\nu})\right] = 0 \;for\; A \in]0, \frac{d\varepsilon}{4}+\frac{\mu}{8}[\;and\; \mu \in]0, 1\wedge \frac{d\varepsilon}{2}\wedge a[\, . \tag{5.10}$$

Then the law of $u(t_0,\underline{x})$ is absolutely continuous on $\{\sigma\neq 0\}^l$.

Remark 5.3 . *Let $f(v)=|v|^{-B}$ for some $B>0$; then the condition (C1) holds for any $B>0$, while (3.24) holds if and only if $d\varepsilon + B < 4$. Since for small $\tau>0$, $\int_0^\tau \rho^{d-1-B-d+4}\, d\rho = C\,\tau^{4-B} \leq I(\tau) \leq \int_0^\tau \rho^{d-1-B-d+4-\xi(d-4)^+}\, d\rho = C\,\tau^{4-B-\xi(d-4)^+}$ for any small $\xi>0$, one has $\lim_{\tau\to 0} I(\tau^{\frac{1}{4}+\nu})^{-1}[\tau + \tau^{\frac{1}{2}}\, I(\tau^{\frac{1}{4}-\nu})] = 0$ for every $B>0$ and $0<\nu<\frac{B\wedge 1}{16}$, while (5.10) holds if and only if $B + d\varepsilon > \frac{7}{2} \vee \left(4 - \left(\frac{d\varepsilon}{4}\wedge\frac{a}{2}\right)\right)$.*

Remark 5.4 *The proofs of Theorems 1.5 in [4] and Theorems 1.2-1.4 in [5] extend to the case of the space-time white noise F in dimension $d \le 3$ under the homogeneous Dirichlet boundary conditions on $[0,\pi]^d$.*

Proof of Theorem 5.2: The proof, which is similar to that of Theorem 1.2 in [5] and Theorem 3.1 in [21] is only sketched in case (ii). According to Theorem 4.4, the trajectories of the solution $u(n)$ to (5.1) almost surely belong to $\mathcal{C}^{\lambda,\mu}([\frac{t_0}{2},T]\times Q)$ for $0<\lambda<\frac12\wedge\frac{d\varepsilon}{8}$ and $0<\mu<1\wedge\frac{d\varepsilon}{2}\wedge a$. (Note that according to Theorem 4.4, the trajectories of u have the same Hölder regularity.) Using Theorem 2.1.2 and the following remark in [23], it suffices to show that for every $n\ge 1$ and $M\ge 1$, the $l\times l$ Malliavin covariance matrix $\Gamma(n)$ defined by

$$\Gamma(n)(i,j) = < Du(n)(t_0,x_i), Du_n(t_0,x_j) >_{\mathcal{H}_T}$$

is almost surely invertible on the set $\tilde\Omega(M)=\cap_{i=1}^l\{|\sigma(u(n)(t_0,x_i))|\ge \frac{1}{M}\}$. As usual, this reduces to proving that for any vector $v\in\mathbb{R}^l$, with $|v|=1$, $<\Gamma(n)v,v>_{\mathbb{R}^l}>0$ a.s. on $\tilde\Omega(M)$.

For $1\le i\le l$, $r\le t$, (5.2) in Proposition 5.1 shows that $D_{r,*}u(n)(t_0,x_i)=G(t_0,x_i;r,*)\times\sigma(u(n)(r,*))+U(t_0,r,x_i)$; for a fixed unit vector v of $\mathbb{R}^d$, a usual argument shows that given $\tau\in]0,\frac{t_0}{2}]$,

$$< \Gamma(n)v, v \ge \frac{I_1}{4}+\frac14\sum_{i=1}^l\sum_{j\neq i;j=1}^l v_i\,v_j\,I_2(i,j)-\frac{l}{2}\sum_{i=1}^l v_i^2\,I_3(i)-l\sum_{i=1}^l v_i^2\,I_4(i)\,, \tag{5.11}$$

where

$$\begin{aligned}
I_1 &= \sum_{i=1}^l v_i^2\int_{t_0-\tau}^{t_0}\|G(t_0,x_i;r,y)\,\sigma(u(n)(r,x_i))\|_{\mathcal{H}}^2\,dr\,,\\
I_2(i,j) &= \int_{t_0-\tau}^{t_0} < G(t_0,x_i;r,*)\,\sigma(u(n)(r,x_i)), G(t_0,x_j;r,*)\,\sigma(u(n)(r,x_j)) >_{\mathcal{H}}\ dr\,,\\
I_3(i) &= \int_{t_0-\tau}^{t_0}\Big\|G(t_0,x_i;r,y)\,\Big[\sigma(u(n)(r,y))-\sigma(u(n)(r,x_i))\Big\|_{\mathcal{H}}^2\,dr\,,\\
I_4(i) &= \int_{t_0-\tau}^{t_0}\|U(t_0,r,x_i)\|_{\mathcal{H}}^2\,dr\,.
\end{aligned}$$

Remark 2.2 shows that condition (2.7) is satisfied. Let $\bar c=\inf\{d(x_i,\partial Q)\,,\,1\le i\le l\}$ and suppose that for the constant C_2 defined in the proof of Lemma 2.1 (ii), $2\,C_2\,\tau^{\frac14}\le\bar c$. Then (2.17) and (2.16) imply that on $\tilde\Omega(M)$, for $\nu>0$, τ small enough and $d=4,5$,

$$I_1\ge C\,(\sum_{i=1}^l|v_i|^2)\,\frac{1}{M^2}\int_{B_d\left(0,C_2\,\tau^{\frac14+\nu}\right)}f(v)\,|v|^{-d+4}\,\ln\left(|v|^{-1}\right)^{(5-d)^+}\,dv\,. \tag{5.12}$$

We now prove upper estimates of $I_2(i,j)$ up to $I_4(i)$. Let $m=\inf\{|x_i-x_j|\,,\,1\le i<j\le l\}$, let c_1 and C_1 denote the constants appearing in condition (C1), and let $k\in]0,\frac13[$ be such that $k\,(1+c_1)<\frac12$. Fix $1\le i<j\le l$; then $I_2(i,j)\le C\,\|\sigma\|_\infty^2\,J_2(i,j)$, where

$$J_2(i,j)=\int_0^\tau r^{-\frac d2}\,dr\int_Q dy\int_Q dz\exp\left(-c\frac{|x_i-y|^{\frac43}}{r^{\frac13}}\right)\,f(y-z)\,\exp\left(-c\frac{|x_j-z|^{\frac43}}{r^{\frac13}}\right)\,.$$

We split the integral on $Q\times Q$ in several parts. Indeed, if $|y-z|\ge km$, the continuity of f implies that $f(y-z)\le C<+\infty$. Suppose now that, $|y-z|\le km$. Then if $|y-x_i|\le c_1\,|y-z|$, (C1) implies that $f(y-z)\le C_1\, f(y-x_i)$, while $|z-x_j|\ge \Big| |x_i-x_j| - (|x_i-y|+|y-z|)\Big| \ge m\Big(1-k(1+c_1)\Big) \ge \frac{m}{2}$. Similarly, if $|z-x_j|\le c_1\,|y-z|$, then $f(y-z)\le C_1 f(z-x_j)$ and $|y-x_i|\ge \frac{m}{2}$. Finally, suppose that $|y-z|\le km$, $c_1\,|y-z|\le |y-x_i|\wedge|z-x_j|$; then since $k<\frac13$, one of the norms $|y-x_i|$ or $|z-x_j|$ (say $|z-x_j|$) is larger than $\frac{m}{3}$. Thus, $J_2(i,j)\le J_{2,1}(i,j)+2\,J_{2,2}(i,j)+2\,J_{2,3}(i,j)$, where

$$
\begin{aligned}
J_{2,1}(i,j) &= \int_0^\tau r^{-\frac d2}\int\int_{|y-z|\ge km} \exp\left(-c\frac{|y-x_i|^{\frac43}}{r^{\frac13}}\right) f(y-z)\exp\left(-c\frac{|z-x_j|^{\frac43}}{r^{\frac13}}\right) dydz\le C\,\tau\,,\\
J_{2,2}(i,j) &\le C_1\int_0^\tau r^{-\frac d4}\,dr\int_{B_d(0,R)} f(v)\exp\left(-c\frac{|v|^{\frac43}}{r^{\frac13}}\right) dv\int_{|z|\ge\frac m2 r^{-\frac14}}\exp\left(-c|z|^{\frac43}\right)dz\\
&\le C_1\int_{B_d(0,R)} f(v)\,dv\int_0^\tau r^{-\frac d4}\exp\left(-c\frac{|v|^{\frac43}}{r^{\frac13}}\right)\exp\left(-\tilde c r^{-\frac13}\right)dr\\
&\le C\exp(-\tilde c\tau^{-\frac13})\int_{B_d(0,R)} f(v)\,|v|^{-d+4}\,\ln\left(|v|^{-1}\right)^{(5-d)^+}dv\le C\exp(-\tilde c\tau^{-\frac13})\,,\\
J_{2,3}(i,j) &\le C_1\int_0^\tau r^{-\frac d4}\,dr\int_Q dy\int_{|z-x_j|\ge\frac m3} f(y-z)\exp\left(-\tilde c\frac{|y-z|^{\frac43}}{r^{\frac13}}\right)\exp\left(-\tilde c\frac{|z-x_j|^{\frac43}}{r^{\frac13}}\right)dz\\
&\le C\exp(-\tilde c\tau^{-\frac13})\,.
\end{aligned}
$$

Hence, for τ small enough,

$$
\Big|\sum_{i=1}^{l}\sum_{j\ne i;j=1}^{l} v_i\,v_j\,I_2(i,j)\Big|\le C\,\tau\,. \tag{5.13}
$$

Fubini's theorem, the Lipschitz property of σ, the Hölder regularity of $x\to u(n)(t,x)$ uniformly for $\frac{t_0}{2}\le t\le t_0$, Schwarz's inequality and $|y-z|^{\frac43}\le 2^{\frac13}\Big(|y-x_i|^{\frac43}+|z-x_j|^{\frac43}\Big)$ yield for any $i\le l$, $0<\tau\le\frac{t_0}{2}$:

$$
\begin{aligned}
E(|I_3(i)|) &\le C\int_{t_0-\tau}^{t_0}\int_Q\int_Q dy\,dz\,|G(t,x_i;r,y)|\,f(y-z)\,|G(t,x_i;r,z)|\\
&\qquad\times E\Big(|u(n)(r,y)-u(n)(r,x_i)||u(n)(r,z)-u(n)(r,x_i)|\Big)dr\\
&\le C\int_0^\tau r^{-\frac d2}dr\int_Q\int_Q dydz\exp\left(-\tilde c\frac{|y-z|^{\frac43}}{r^{\frac13}}\right) f(y-z)\,|z-x_i|^{\frac\mu2}\exp\left(-\tilde c\frac{|z-x_i|^{\frac43}}{r^{\frac13}}\right)\\
&\le C\int_{B_d(0,R)} f(v)\,dv\int_0^\tau r^{-\frac d4+\frac\mu8}\exp\left(-\tilde c\frac{|v|^{\frac43}}{r^{\frac13}}\right)dr\int_{B_d(0,R)}|z|^{\frac\mu2}\exp(-\tilde c|z|^{\frac43})\,dz\\
&\le C\int_{B_d(0,R)} f(v)\,|v|^{-d+4+\frac\mu2}\,dv\int_{|v|^{\frac43}\tau^{-\frac13}}^{+\infty} s^{-4+\frac{3d}{4}-\frac{3\mu}{8}}\exp(-\tilde cs)\,ds\,,
\end{aligned}
$$

where to obtain the last integral, we have set $s=|v|^{\frac43}\,r^{-\frac13}$. We split the last integral on $\{|v|\le\tau^{\frac14-\nu}\}$ and its complement for some $\nu>0$. Then using (3.24), a straightforward computation yields for $d=4,5$ and τ small enough:

$$
E(|I_3(i)|)\ \le\ C\int_{B_d(0,\tau^{\frac14-\nu})} f(v)|v|^{4-d+\frac\mu2}\left(1+|v|^{-\frac\mu2}\,\tau^{\frac\mu8}\right)dv
$$

$$
\begin{aligned}
&+C\int_{B_d(0,R)} 1_{\{|v|\geq \tau^{\frac14-\nu}\}}\, f(v)|v|^{4-d+\frac{\mu}{2}}\, \exp(-\tilde c\tau^{-\frac{4\nu}{3}})\, dv \\
\leq\ & C\,\tau^{\frac{d\varepsilon}{4}+\frac{\mu}{8}-\nu(d\varepsilon+\frac{\mu}{2})}\,.
\end{aligned}
$$

Thus, choosing ν close enough to 0 and τ_0 small enough, we deduce that for $0<\tau\leq\tau_0$:

$$
E(|I_3(i)|)\leq C\tau^A\ ,\quad 0<A<\frac{d\varepsilon}{4}+\frac{\mu}{8}\,. \tag{5.14}
$$

Finally, using the decomposition of $U(t_0,r,x_i)$, and Fubini's theorem, we obtain that $E(|I_4(i)|)\leq \sum_{i=1}^4 T_j$, where

$$
\begin{aligned}
T_1 &= \int_{t_0-\tau}^{t_0} drE\left(\left\|\int_r^{t_0}\int_Q G(s,x_i;r,y)\, S_i(n)(s,y)\, D_{r,*}u(n)(s,y)\, F(ds,dy)\right\|_{\mathcal H}^2\right)\ ,\\
T_2 &= \int_{t_0-\tau}^{t_0} drE\left(\left\|\int_r^{t_0}\int_Q \Delta G(s,x_i;r,y)\, R_n'(u(n)(s,y))\, D_{r,*}u(n)(s,y)\, dyds\right\|_{\mathcal H}^2\right)\ ,\\
T_3 &= \int_{t_0-\tau}^{t_0} drE\left(\left\|\int_r^{t_0}\int_Q G(s,x_i;r,y)\, \partial_3 g(s,y,(u(n)(s,y))\, D_{r,*}u(n)(s,y)\, dyds\right\|_{\mathcal H}^2\right)\ ,\\
T_4 &= \sum_{j=1}^N\int_{t_0-\tau}^{t_0} drE\left(\left\|\int_r^{t_0}\int_Q H_j(s,x_i;r,y)\, \partial_3 b_j(s,y,(u(n)(s,y))\, D_{r,*}u(n)(s,y)\, dyds\right\|_{\mathcal H}^2\right)\ .
\end{aligned}
$$

The isometry property for Hilbert-space valued martingales, (5.3) with $p=1$, Schwarz's inequality and (2.14) or (2.15) imply for ν small enough

$$
\begin{aligned}
T_1 &\leq C\int_{t_0-\tau}^{t_0} ds\int_Q\int_Q dydz\, |G(s,x_i;r,y)|\, f(y-z)\, |G(s,x_i;r,z)| \\
&\qquad \times \sup_{(s,y)\in[t_0-\tau,t_0]\times Q} E\left(\int_{t_0-\tau}^s \|D_{r,*}u(n)(s,y)\|_{\mathcal H}^2 dr\right)\\
&\leq C\,\tau\left[I(\tau^{\frac14-\nu})+\exp\left(-\tilde c\tau^{\frac{3\nu}{4}}\right)\right]\ .
\end{aligned} \tag{5.15}
$$

Minkowski's and Schwarz's inequalities, then Fubini's theorem and (5.4) imply

$$
\begin{aligned}
T_2 &\leq C\int_{t_0-\tau}^{t_0}(t_0-r)^{\frac12}\, dr\int_r^{t_0}\int_Q \Delta G(t_0,x_i;s,y)\, R_n'(u(n)(s,y))\, E(\|D_{r,*}u(n)(s,y))\|_{\mathcal H}^2)\, dyds\\
&\leq C_n\,\tau^{\frac12}\int_{t_0-\tau}^{t_0} ds\int_Q dy\, \Delta G(t_0,x_i;s,y)\ \sup_{(s,y)\in[t_0-\tau,t_0]\times Q} E\left(\int_{t_0-\tau}^s \|D_{r,*}u(n)(s,y))\|_{\mathcal H}^2\, dr\right)\\
&\leq C(n)\,\tau\left[I(\tau)+\tau^{\frac12}\right]\ .
\end{aligned} \tag{5.16}
$$

A similar computation, using Minkowski's inequality, Fubini's theorem, Lemma 3.1 with $\rho=\infty$ and $q=1$ and (5.4) yields

$$
\begin{aligned}
T_3 &\leq C_n\,\tau\int_{t_0-\tau}^{t_0} ds\int_Q dy\, G(t_0,x_i;s,y)\ \sup_{(s,y)\in[t_0-\tau,t_0]\times Q} E\left(\int_{t_0-\tau}^s \|D_{r,*}u(n)(s,y))\|_{\mathcal H}^2\, dr\right)\\
&\leq C_n\,\tau^2\left[I(\tau)+\tau^{\frac12}\right]\ ,
\end{aligned} \tag{5.17}
$$

$$
T_4 \leq C_n\tau^{\frac14}\sum_{j=1}^N\int_{t_0-\tau}^{t_0} ds\int_Q dy H_j(t_0,x_i;s,y)\ \sup_{(s,y)\in[t_0-\tau,t_0]\times Q} E\left(\int_{t_0-\tau}^s \|D_{r,*}u(n)(s,y))\|_{\mathcal H}^2 dr\right)
$$

$$\leq \quad C_n\, \tau^{\frac{1}{2}} \left[I(\tau) + \tau^{\frac{1}{2}} \right] , \tag{5.18}$$

The inequalities (5.15)-(5.18) yield that

$$E(|I_4(i)|) \leq C(n) \left[\tau^{\frac{1}{2}}\, I(\tau^{\frac{1}{4}-\nu}) + \tau \right] . \tag{5.19}$$

Finally, the inequalities (5.11)-(5.14) and (5.19) imply that for $\rho > 0$ such that $\tau + \tau^{\rho} \leq \frac{1}{2} I(\tau^{\frac{1}{4}+\nu})$ (which exists because of (5.10)), we have for small enough τ:

$$\begin{aligned} P(< v, \Gamma(n) v >_{\mathbb{R}^l} < \tau^{\rho}) &\leq P\left(I_3 + I_4 \geq C \left[I(\tau^{\frac{1}{4}+\nu}) - \tau - \tau^{\rho} \right] \right) \\ &\leq P\left(I_3 + I_4 \geq \frac{1}{2}\, I(\tau^{\frac{1}{4}+\nu}) \right) \leq C\, I(\tau^{\frac{1}{4}+\nu})^{-1} \left[\tau^{A} + \tau^{\frac{1}{2}}\, I(\tau^{\frac{1}{4}-\nu}) \right] \end{aligned}$$

for some $A < \frac{\mu}{8} + \frac{d\varepsilon}{2}$; then the condition (5.10) concludes the proof of (ii). □

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