EDGEWORTH TYPE EXPANSIONS FOR EULER SCHEMES FOR STOCHASTIC DIFFERENTIAL EQUATIONS. *

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We study the approximation of the density of $Y(1)$ by the density of $Y_n(1)$, where $Y(t)$ is solution of a stochastic differential equation, and where $Y_n(t)$ is defined by Euler discretisation of Y with discretisation step $1/n$. Using the parametrix approach developped in Konakov and Mammen (2000,2001) we obtain an asymptotic expansion in terms of powers of $1/n$, under a uniform ellipticity condition for the infinitesimal generator of Y. The number of terms of this expansion depends only on the smoothness of the drift and diffusion coefficients. The first term of this expansion was obtained in Bally and Talay (1996a,b) under a nondegeneracy condition of Hormander type for the infinitesimal generator of Y.

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Short title: Edgeworth expansions for Euler schemes.

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1 Introduction and main result.

We consider the Euler scheme with equidistant partitions where for $0 \leq k \leq n-1$

$$(1.1) \qquad Y_n\left(\frac{k+1}{n}\right) = Y_n\left(\frac{k}{n}\right) + \frac{1}{n} m\left(Y_n\left(\frac{k}{n}\right)\right) + \frac{1}{\sqrt{n}} \varepsilon_n\left(\frac{k+1}{n}\right),$$

with initial condition $Y_n(0) = x$ and innovations

$$\varepsilon_n\left(\frac{k+1}{n}\right) = \Lambda\left(Y_n(k)\right)\sqrt{n}\left(W\left(\frac{k+1}{n}\right) - W\left(\frac{k}{n}\right)\right)$$

Here W is a d-dimensional Brownian motion. For $z \in I\!R^d$ the $d \times d$ matrix $\Lambda(z)$ is positive definite. The starting point x is a value in $I\!R^d$. The drift function $m(z) = (m_1(z), ..., m_d(z))^t$ is a function from $I\!R^d$ to $I\!R^d$.

By definition the conditional distribution of the innovations $\varepsilon_n\left(\frac{k+1}{n}\right)$ given $Y_n\left(\frac{i}{k}\right) = x_i$ for $1 \leq i \leq k$ only depends on the last value x_k and it is $N(0, \Sigma(x_k))$ where $\Sigma(z) = \Lambda(z)\Lambda(z)^t$ We denote the elements of $\Sigma(z)$ by $\sigma_{ij}(z)$ $(1 \leq i, j \leq d)$.

We make the following assumptions.

(A) There exist constants $0 < c < C$ such that for all $\theta \in I\!R^d$ with norm $\|\theta\| = 1$ and $x \in I\!R^d$

$$c \leq \theta^t \Sigma(x) \theta \leq C.$$

(B) For a positve integer $M \geq 1$ the functions $m_i(x)$ and $\sigma_{ij}(x)$ have partial derivatives up to order $2M$ that are absolutely bounded and continuous.

Under these assumptions the process Y_n converges weakly to a diffusion $Y(t)$. This follows for instance from Theorem 1, p.82 in Skorohod (1987). The diffusion is defined by

$$(1.2) \qquad dY(t) = m\{Y(t)\}\, dt + \Lambda\{Y(t)\}\, dW(t), Y(0) = x, t \in [0,1].$$

Let $p(t, x, y)$ be the transition density of Y and let $p_n(k/n, x, y)$ be the transition density of the chain Y_n. In our main result we will give an asymptotic expansion for the difference $p_n - p$.

Euler approximations for stochastic differential equations have been studied in a series of papers. For an introduction that includes practical aspects we refer to Kloeden and Platen (1992) and Platen (1999). Convergence of the expectations of smooth transformations of $Y_n(1)$ has been shown in Talay and Tubaro (1990) and Bally and Talay (1996a,b). The rate of convergence of the error process $Y_n - Y$ has been discussed in Jacod and Protter (1998), Protter and Talay (1997) and Jacod (2001). These papers also discuss the case of a Lévy driven stochastic differential equation.

For the statement of the theorem we have to introduce the following operators $L, \tilde{L}^*$ and $\tilde{L}_*$

$$
\begin{aligned}
(1.3) \qquad Lf(t,x,y) &= \frac{1}{2}\sum_{i,j=1}^{d}\sigma_{ij}(x)\frac{\partial^2 f(t,x,y)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} m_i(x)\frac{\partial f(t,x,y)}{\partial x_i},\\
(1.4) \qquad \tilde{L}^* f(t,x,y) &= \frac{1}{2}\sum_{i,j=1}^{d}\sigma_{ij}(y)\frac{\partial^2 f(t,x,y)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} m_i(y)\frac{\partial f(t,x,y)}{\partial x_i},\\
(1.5) \qquad \tilde{L}_* f(t,x,y) &= \tilde{L}_x f(t,x,y),
\end{aligned}
$$

where for $\xi \in I\!R^d$

$$
\tilde{L}_\xi f(t,x,y) = \frac{1}{2}\sum_{i,j=1}^{d}\sigma_{ij}(\xi)\frac{\partial^2 f(t,x,y)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} m_i(\xi)\frac{\partial f(t,x,y)}{\partial x_i}.
$$

As usual, the operator that is defined by m times iterative application of an operator K ($K = L, \tilde{L}^*$ or $\tilde{L}_\xi$) is denoted by K^m. By abuse of notation we denote by $(\tilde{L}_*)^m$ the operator that is defined by first applying m times the operator $\tilde{L}_\xi$ to the function $f(t,x,y)$ and then putting $\xi = x$. Note that e.g. in our notation we have that $\tilde{L}_* f(t,x,y) = Lf(t,x,y)$ but that in general $(\tilde{L}_*)^m f(t,x,y)) \neq (\tilde{L}_x)^m f(t,x,y)$ for $m \geq 2$.

We make use of the following convolution type binary operation $\otimes$

$$
(f \otimes g)(t,x,y) = \int_0^t du \int_{R^p} f(u,x,z)\, g(t-u,z,y)\, dz.
$$

By iteration we also define the r fold convolution product for a function f and an operator K:

$$
\tilde{f} \otimes K^{(r)} = \left(\tilde{f} \otimes K^{(r-1)}\right) \otimes K.
$$

By time discretisation we get the following modification of $\otimes$

$$
(f \otimes_n g)\left(\frac{k}{n}, x, y\right) = \frac{1}{n}\sum_{i=0}^{k-1}\int f\left(\frac{i}{n}, x, z\right) g\left(\frac{k}{n} - \frac{i}{n}, z, y\right) dz.
$$

By Lemma 3.1 in Konakov & Mammen (2000) the following representation for the density $p(t,x,y)$ holds

$$
(1.6) \qquad p(t,x,y) = \sum_{r=0}^{\infty}\left(\tilde{p} \otimes H^{(r)}\right)(t,x,y),
$$

where the kernel H is given by $H = (L - \tilde{L}^*)\tilde{p}$ and $\tilde{p}$ is the following Gaussian density

$$
\begin{aligned}
\tilde{p}(t,x,y) &= (2\pi t)^{-d/2}(\det \Sigma(y))^{-1/2}\\
&\qquad \exp\left\{-\frac{1}{2t}(y - x - tm(y))^t \Sigma(y)^{-1}(y - x - tm(y))\right\}.
\end{aligned}
$$

Formula (1.6) can be shown by iterative comparisons of the diffusion Y and purely Gaussian processes ("diffusions with frozen drift and diffusion coefficients"). The operator L is the infinitesimal operator of the diffusion. The operator $\tilde{L}^*$ is the infinitesimal operator of a diffusion frozen at y. The density $\tilde{p}$ is the transition density of the frozen diffusion. This approach has been called parametrix method. For details see Konakov and Mammen (2000).

We are now in the position to state our main result.

Theorem 1.1 *Assume that (A) and (B) hold for some integer $M \geq 1$. Then there exists a function $R(x,y)$ with $|R(x,y)| \leq C_1 \exp[-C_2(x-y)^2]$ for some positive constants C_1 and C_2 such that*

$$\begin{aligned}(1.7)\qquad & p\,(1,x,y) - p_n\,(1,x,y) \\ &= \sum_{k=1}^{M-1} \frac{1}{(k+1)!n^k} \left(p \otimes_n \left(L - \tilde{L}^*\right)^{k+1} p_n^d\right)(1,x,y) \\ &\quad - \sum_{k=1}^{M-1} \frac{1}{(k+1)!n^k} \left(p_n^d \otimes_n \left(\tilde{L}_* - \tilde{L}^*\right)^{k+1} p_n\right)(1,x,y) + \frac{1}{n^M} R\,(x,y)\,,\end{aligned}$$

where

$$p_n^d\,(t,x,y) = \sum_{r=0}^{\infty} \left(\tilde{p} \otimes_n H^{(r)}\right)(t,x,y)\,.$$

For $M=1$ we make the convention that $\sum_{k=1}^{M-1} \ldots = 0$. It holds that

$$(1.8)\qquad \sum_{k=1}^{M-1} \left| \left(p \otimes_n \left(L - \tilde{L}^*\right)^{k+1} p_n^d\right)(1,x,y)\right| \leq C_1 \exp[-C_2(x-y)^2],$$

$$(1.9)\qquad \sum_{k=1}^{M-1} \left| \left(p_n^d \otimes_n \left(\tilde{L}_* - \tilde{L}^*\right)^{k+1} p_n\right)(1,x,y)\right| \leq C_1 \exp[-C_2(x-y)^2].$$

The terms of our expansion depend on n. It is possible to use our expansion to arrive at an expansion with terms that do not depend on n. But the expansion will be less transparent and we are not aware of explicite formulas for the terms. For small M it is easy to get explicite formulas. E.g. for $M=2$ we get by using bounds on $\otimes_n - \otimes$ (see also Theorem 2.1) and iterative application of Theorem 1.1 that

$$\begin{aligned} & p\,(1,x,y) - p_n\,(1,x,y) \\ &= \frac{1}{2n}\left(p \otimes_n \left(L - \tilde{L}^*\right)^2 p_n^d\right)(1,x,y) \\ &\quad - \frac{1}{2n}\left(p_n^d \otimes_n \left(\tilde{L}_* - \tilde{L}^*\right)^2 p_n\right)(1,x,y) + \frac{1}{n^2} R\,(x,y) \\ &= \frac{1}{2n}\left(p \otimes \left(L - \tilde{L}^*\right)^2 p\right)(1,x,y) \\ &\quad - \frac{1}{2n}\left(p \otimes \left(\tilde{L}_* - \tilde{L}^*\right)^2 p\right)(1,x,y) + \frac{1}{n^2} R^*\,(x,y) \\ &= -\frac{1}{2n}\left(p \otimes \left((\tilde{L}_*)^2 - L^2\right) p\right)(1,x,y) + \frac{1}{n^2} R^*\,(x,y)\,,\end{aligned}$$

where R^* is a remainder term that has subgaussian tails (in $x - y$) as R. It is easy to check that in this form the expansion coincides with that of Bally and Talay (1996a). Note that the operator $\mathcal{U}$ in their formula is equal to $\left((\tilde{L}_*)^2 - L^2\right)/2$.

In Konakov and Mammen (2000) a discrete representation is given in their Lemma 3.6 that is analogous to (1.6). There it was shown that

$$p_n\left(\frac{k}{n}, x, y\right) = \sum_{r=0}^{k} \left(\tilde{p} \otimes_n H_n^{(r)}\right)\left(\frac{k}{n}, x, y\right) \tag{1.10}$$

where the kernel H_n is given by

$$H_n = \left(L_n - \tilde{L}_n^*\right)\tilde{p}$$

with

$$\begin{aligned} L_n f\left(\frac{k}{n}, x, y\right) &= n\left\{\int p_n\left(\frac{1}{n}, x, z\right) f\left(\frac{k-1}{n}, z, y\right) dz - f\left(\frac{k-1}{n}, x, y\right)\right\}, \\ \tilde{L}_n^* f\left(\frac{k}{n}, x, y\right) &= n\left\{\int \tilde{p}_n^y\left(\frac{1}{n}, x, z\right) f\left(\frac{k-1}{n}, z, y\right) dz - f\left(\frac{k-1}{n}, x, y\right)\right\}. \end{aligned}$$

Here

$$\begin{aligned} \tilde{p}_n^y\left(\frac{1}{n}, x, z\right) &= (2\pi/n)^{-p/2}\left(\det\Sigma(y)\right)^{-1/2} \\ &\quad \exp\left\{-\frac{n}{2}\left(z - x - \frac{1}{n}m(y)\right)^t \Sigma(y)^{-1}\left(z - x - \frac{1}{n}m(y)\right)\right\}. \end{aligned}$$

denotes the one-step transition density of $\tilde{Y}_n$ "frozen" at the point y. The operators L_n and $\tilde{L}_n^*$ are the infinitesimal operators of the Markov chain Y_n and of a frozen chain, respectively. Formula (1.10) for the transition density can be shown by direct arguments (compared with the more technical derivation of (1.6)). Now a Markov chain is compared with a chain with frozen distributions. In a frozen chain corresponding to an Euler scheme the innovations are i.i.d. Gaussian variables. In particular, therefore the transition densities are Gaussian densities and coincide on the time grid with the transition densities of the frozen diffusions. This is the reason why the same density $\tilde{p}$ appears in the definition of the kernels H_n and H. For details and motivation of formulas (1.10) and (1.6) see Konakov and Mammen (2000).

2 Auxiliary results.

Our first auxiliary result gives an expansion for the difference between the transition density p and the density p_n^d defined in the statement of Theorem 1.1.

Theorem 2.1 *Let the assumptions of Theorem 1.1 hold for some $M \geq 1$. Then the following expansion holds*

$$\begin{aligned} p(1,x,y) - p_n^d(1,x,y) \\ = \sum_{k=1}^{M-1} \frac{1}{(k+1)!n^k} \left(p \otimes_n \left(L - \tilde{L}^* \right)^{k+1} p_n^d \right)(1,x,y) + \frac{1}{n^M} R_1(1,x,y) \end{aligned} \tag{2.1}$$

with

$$\sum_{k=1}^{M-1} \left| \left(p \otimes_n \left(L - \tilde{L}^* \right)^{k+1} p_n^d \right)(1,x,y) \right| + |R_1(1,x,y)| \leq C \exp[-C'(x-y)^2] \tag{2.2}$$

for some constants C and C'.

In our proof of Theorem 1.1 we need bounds on partial derivatives of the transition density p. The following result can be found in Friedman (1964) (Theorem 7 , page 260) where it has been used for studying smoothness of fundamental solutions of uniformly parabolic systems.

Theorem 2.2 *Let the assumptions of Theorem 1.1 hold for some $M \geq 1$. Then the partial derivatives $D_y^{r+a} D_x^b p(t,x,y)$ exist and are continuous for all $0 \leq |a| + |b| \leq 2M, |r| = 0,1,$ and for some positive constants D_1, D_2 it holds that*

$$\left| D_y^{r+a} D_x^b p(t,x,y) \right| \leq \frac{D_1}{t^{(|r|+|a|+|b|+d)/2}} \exp\left\{ -D_2 \frac{\|y-x\|^2}{t} \right\}, \tag{2.3}$$

$$\left| D_x^r D_\xi^b p(t,\xi,\xi+x) \right| \leq \frac{D_1}{t^{(|r|+d)/2}} \exp\left\{ -D_2 \frac{\|x\|^2}{t} \right\}. \tag{2.4}$$

The next theorem states that the same bounds apply for the partial derivatives of $p_n^d(t,x,y)$ for t in the grid $\{\frac{1}{n}, \frac{2}{n}, ..., 1\}$. This result does not follow from the results in Friedman (1964) because p_n^d is not a fundamental solution of a parabolic equation.

Theorem 2.3 *Let the assumptions of Theorem 1.1 hold for some $M \geq 1$. Then the partial derivatives $D_y^{r+a} D_x^b p_n^d(t,x,y)$ exist for t in the grid $\{\frac{1}{n}, \frac{2}{n}, ..., 1\}$ and are continuous functions in x and y for all $0 \leq |a| + |b| \leq 2M, |r| = 0,1,$ and and for some positive constants D_1, D_2 it holds for $t = \frac{1}{n}, \frac{2}{n}, ..., 1$ that*

$$\left| D_y^{r+a} D_x^b p_n^d(t,x,y) \right| \leq \frac{D_1}{t^{(|r|+|a|+|b|+d)/2}} \exp\left\{ -D_2 \frac{\|y-x\|^2}{t} \right\}, \tag{2.5}$$

$$\left| D_x^r D_\xi^b p_n^d(t,\xi,\xi+x) \right| \leq \frac{D_1}{t^{(|r|+d)/2}} \exp\left\{ -D_2 \frac{\|x\|^2}{t} \right\}. \tag{2.6}$$

3 Proof of Theorem 1.1.

It follows from (2.3) and (2.4) that for all s,t , $s<t$, $s,t \in \{0, 1/n, 2/n, ..., 1\}$

$$\frac{1}{n} H_n\left(t-s,u,y\right) = \frac{1}{n}\left(L_n - \tilde{L}_n^*\right)\tilde{p}\left(t-s,u,y\right) = F(1) - F(0)$$

where

$$F(\delta) = \phi\left(y-u; \left(t-s\right)m(y) + \delta\frac{\Delta m^u(y)}{n}, (t-s)\Sigma(y) + \delta\frac{\Delta\Sigma^u(y)}{n}\right).$$

$\phi\left(x;m,\Sigma\right)$ denotes the gaussian density with the mean vector m and the covariance matrix Σ. Furthermore we write $\Delta m^\xi(y) = m(\xi) - m(y) = \left(\Delta m_1^\xi(y), ..., \Delta m_d^\xi(y)\right)$. The matrix with elements $\Delta\sigma_{ij}^\xi(y) = \sigma_{ij}\left(\xi\right) - \sigma_{ij}\left(y\right)$ is denoted by $\Delta\Sigma^\xi(y)$. By Taylor expansion we get

$$\begin{aligned} \frac{1}{n} H_n\left(t-s,u,y\right) &= F(1) - F(0) \\ &= \sum_{k=1}^{M}\frac{1}{k!}F^{(k)}(0) + \frac{1}{(M+1)!}\int_0^1 \left(1-\tau\right)^M F^{(M+1)}(\tau)d\tau. \end{aligned} \tag{3.1}$$

Note that

$$F\left(0\right) = \phi\left(y-u;\left(t-s\right)m(y),(t-s)\Sigma(y)\right) = \tilde{p}\left(t-s,u,y\right).$$

We will make use of the following well-known relations

$$\begin{aligned} \frac{\partial\phi\left(y-u;\tilde{m}^u(y),\tilde{\Sigma}^u(y)\right)}{\partial\tilde{m}_i^u(y)} &= \frac{\partial\phi\left(y-u;\tilde{m}^\xi(y),\tilde{\Sigma}^\xi(y)\right)}{\partial u_i}\Big|_{\xi=u}, \\ \frac{\partial\phi\left(y-u;\tilde{m}^u(y),\tilde{\Sigma}^u(y)\right)}{\partial\tilde{\sigma}_{ij}^u(y)} &= \frac{1}{2}\frac{\partial^2\phi\left(y-u;\tilde{m}^\xi(y),\tilde{\Sigma}^\xi(y)\right)}{\partial u_i\partial u_j}\Big|_{\xi=u}, \end{aligned}$$

where $\tilde{m}^\xi(y) = \left(\tilde{m}_1^\xi(y), ..., \tilde{m}_d^\xi(y)\right)$ with $\tilde{m}_i^\xi(y) = \left(t-s\right)m_i(y) + \delta\frac{\Delta m_i^\xi(y)}{n}$ and $\tilde{\sigma}_{ij}^\xi(y)$ are the elements of the matrix $\tilde{\Sigma}^\xi(y) = (t-s)\Sigma(y) + \delta\frac{\Delta\Sigma^\xi(y)}{n}$.

With these relations we obtain the following formulas for $\frac{1}{k!}\ F^{(k)}(0), k = 1, 2...$:

$$\begin{aligned} F'(\delta) &= \frac{1}{n}\left(\tilde{L}_* - \tilde{L}^*\right)\phi\left(y-u;\left(t-s\right)m(y) + \delta\frac{\Delta m^u(y)}{n}, (t-s)\Sigma(y) + \delta\frac{\Delta\Sigma^u(y)}{n}\right), \\ F'(0) &= \frac{1}{n}\left(\tilde{L}_* - \tilde{L}^*\right)\tilde{p}\left(t-s,u,y\right) = \frac{1}{n}H\left(t-s,u,y\right), \\ \frac{1}{2!}F''(0) &= \frac{1}{2!}\frac{d}{d\delta}\left[F'(\delta)\right]|_{\delta=0} \\ &= \frac{1}{2n}\left(\tilde{L}_* - \tilde{L}^*\right)\frac{d}{d\delta}\left[\phi\left(y-u;\left(t-s\right)m(y) + \delta\frac{\Delta m^u(y)}{n}, (t-s)\Sigma(y) + \delta\frac{\Delta\Sigma^u(y)}{n}\right)\right]|_{\delta=0} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2n^2}\left(\tilde{L}_* - \tilde{L}^*\right)\left(\sum_{i=1}^{p}\left(m_i\left(u\right) - m_i\left(y\right)\right)\frac{\partial\phi}{\partial\tilde{m}_i^u(y)} + \sum_{i,j=1}^{p}\left(\sigma_{ij}\left(u\right) - \sigma_{ij}\left(y\right)\right)\frac{\partial\phi}{\partial\tilde{\sigma}_{ij}^u(y)}\right) \\
&= \frac{1}{2!}\frac{1}{n^2}\left(\tilde{L}_* - \tilde{L}^*\right)\left(\tilde{L}_* - \tilde{L}^*\right)\tilde{p}\left(t-s,u,y\right) \\
&= \frac{1}{2!n^2}\left(\tilde{L}_* - \tilde{L}^*\right)^2\tilde{p}\left(t-s,u,y\right).
\end{aligned}
$$

By iterative application of similar arguments we get

$$
\begin{aligned}
\frac{1}{k!}F^{(k)}(\delta) &= \frac{1}{k!n^k}\left(\tilde{L}_* - \tilde{L}^*\right)^k \\
&\qquad \phi\left(y-u;\left(t-s\right)m(y) + \delta\frac{\Delta m^u(y)}{n}, (t-s)\Sigma(y) + \delta\frac{\Delta\Sigma^u(y)}{n}\right).
\end{aligned}
$$

In particular, we get

$$
\frac{1}{k!}F^{(k)}(0) = \frac{1}{k!n^k}\left(\tilde{L}_* - \tilde{L}^*\right)^k\tilde{p}\left(t-s,u,y\right). \tag{3.2}
$$

By putting (3.2) into (3.1) we get

$$
\begin{aligned}
&H_n\left(t-s,u,y\right) - H\left(t-s,u,y\right) \qquad (3.3)\\
&= \sum_{k=1}^{M-1}\frac{1}{(k+1)!n^k}\left(\tilde{L}_* - \tilde{L}^*\right)^{k+1}\tilde{p}\left(t-s,u,y\right) + \frac{1}{(M+1)!n^M}\int_0^1\left(1-\tau\right)^M \\
&\quad\times\left(\tilde{L}_* - \tilde{L}^*\right)^{M+1}\phi\left(y-u;\left(t-s\right)m(y) + \tau\frac{\Delta m^u(y)}{n}, (t-s)\Sigma(y) + \tau\frac{\Delta\Sigma^u(y)}{n}\right)d\tau.
\end{aligned}
$$

It follows from (3.3) with $M = 1$ that

$$
\begin{aligned}
D_y^a D_u^b H_n\left(\frac{k}{n},u,y\right) &= D_y^a D_u^b H\left(\frac{k}{n},u,y\right) \\
&\quad + \frac{1}{2!n}\int_0^1\left(1-\tau\right)^M D_y^a D_u^b\left[\left(\tilde{L}_*\right)^2 - 2\tilde{L}_*\tilde{L}^* + \left(\tilde{L}^*\right)^2\right]\phi d\tau.
\end{aligned} \tag{3.4}
$$

Note that under our assumptions for some positive constants C and b_1 and for $f(s,u,y) = \phi\left(y-u;\left(t-s\right)m(y) + \tau\frac{\Delta m^\xi(y)}{n}, (t-s)\Sigma(y) + \tau\frac{\Delta\Sigma^\xi(y)}{n}\right)$ with some fixed ξ

$$
\begin{aligned}
&\left|D_y^a D_u^b H\left(\frac{k}{n},u,y\right)\right| \le C\left(\frac{k}{n}\right)^{-(d+1+|a|+|b|)/2}\exp\left\{-b_1\frac{\left\|y-u\right\|^2}{k/n}\right\}, \\
&\left[\left(\tilde{L}_*\right)^2 - 2\tilde{L}_*\tilde{L}^* + \left(\tilde{L}^*\right)^2\right]f\left(s,u,y\right) \\
&= \frac{1}{4}\sum_{l,k,q,r}\sigma_{lk}\left(u\right)\left(\sigma_{qr}\left(u\right) - \sigma_{qr}\left(y\right)\right) - \sigma_{qr}\left(y\right)\left(\sigma_{lk}\left(u\right) - \sigma_{lk}\left(y\right)\right)\frac{\partial^4 f\left(s,u,y\right)}{\partial u_l\partial u_k\partial u_q\partial u_r} \\
&\quad + \sum_{l,k,q}\sigma_{lk}\left(u\right)\left(m_q\left(u\right) - m_q\left(y\right)\right) - m_q\left(y\right)\left(\sigma_{lk}\left(u\right) - \sigma_{lk}\left(y\right)\right)\frac{\partial^3 f\left(s,u,y\right)}{\partial u_l\partial u_k\partial u_q} \\
&\quad + \sum_{l,k}m_l\left(u\right)\left(m_k\left(u\right) - m_k\left(y\right)\right) - m_k\left(y\right)\left(m_l\left(u\right) - m_l\left(y\right)\right)\frac{\partial^2 f\left(s,u,y\right)}{\partial u_l\partial u_k}.
\end{aligned}
$$

By the last two equations (with $\xi = u$ and by using Lipschitz conditions on the drift function and diffusion coefficient) and (3.4) we get

$$\left| D_y^a D_u^b H_n \left(\frac{k}{n}, u, y \right) \right| \le C \left(\frac{k}{n} \right)^{-(d+1+|a|+|b|)/2} \exp \left\{ -b_1 \frac{\|y-u\|^2}{k/n} \right\}.$$

We now argue that the following two bounds hold with some positive constants D_1 and D_2.

$$\begin{aligned} \left| D_y^{r+a} D_x^b p_n \left(\frac{i}{n}, x, y \right) \right| &\le D_1 \left(\frac{i}{n} \right)^{-(|r|+|a|+|b|+d)/2} \left\{ -D_2 \frac{\|y-x\|^2}{i/n} \right\} \\ \left| D_x^r D_\xi^b p_n \left(\frac{i}{n}, \xi, \xi + x \right) \right| &\le D_1 \left(\frac{i}{n} \right)^{-(|r|+d)/2} \exp \left\{ -D_2 \frac{\|x\|^2}{i/n} \right\} \end{aligned}$$

. For the proof of these claims one can proceed as in the proof of Theorem 2.3. Note that

$$p_n \left(\frac{k}{n}, x, y \right) = \tilde{p} \left(\frac{k}{n}, x, y \right) + \sum_{i=0}^{k-1} \frac{1}{n} \int \tilde{p} \left(\frac{i}{n}, x, z \right) \Phi_n \left(\frac{k}{n} - \frac{i}{n}, z, y \right) dz$$

with

$$\Phi_n \left(\frac{j}{n}, z, y \right) = \sum_{r=1}^{\infty} H_n^{(r)} \left(\frac{j}{n}, z, y \right).$$

We now use the following identity for $r \ge 1$

$$\begin{aligned} &\left(\tilde{p} \otimes_n H^{(r)} \right)(t, x, y) - \left(\tilde{p} \otimes_n H_n^{(r)} \right)(t, x, y) \\ &= \left(\left(\tilde{p} \otimes_n H^{(r-1)} \right) \otimes_n (H - H_n) \right)(t, x, y) \\ &\quad + \left(\left(\tilde{p} \otimes_n H^{(r-1)} - \tilde{p} \otimes_n H_n^{(r-1)} \right) \otimes_n H_n \right)(t, x, y). \end{aligned} \tag{3.5}$$

Remind that

$$p_n^d(t, x, y) - p_n(t, x, y) = \sum_{r=0}^{\infty} \left[\left(\tilde{p} \otimes_n H^{(r)} \right)(t, x, y) - \left(\tilde{p} \otimes_n H_n^{(r)} \right)(t, x, y) \right].$$

Hence, by summing from $r = 0$ to $r = \infty$ in (3.5) and by taking into account the linearity of the operation $\otimes_n$ we get

$$p_n^d - p_n = p_n^d \otimes_n (H - H_n) + \left[p_n^d - p_n \right] \otimes_n H_n$$

By iterative application of the last identity we obtain

$$p_n^d - p_n = \sum_{r=0}^{\infty} \left[p_n^d \otimes_n (H - H_n) \right] \otimes_n H_n^{(r)}. \tag{3.6}$$

We get from (3.3)

$$\begin{aligned} &\left(p_n^d \otimes_n (H - H_n) \right)(t, x, y) \\ &= -\sum_{k=1}^{M-1} \frac{1}{(k+1)! n^k} \left[p_n^d \otimes_n \left(\tilde{L}_* - \tilde{L}^* \right)^{k+1} \tilde{p} \right](t, x, y) \\ &\quad - \frac{1}{(M+1)! n^M} \int_0^1 (1-\tau)^M \left[p_n^d \otimes_n \left(\tilde{L}_n - \tilde{L}^* \right)^{M+1} \tilde{p}_\tau \right](t, x, y)\, d\tau, \end{aligned} \tag{3.7}$$

where $\tilde{p}_0 = \tilde{p}$ and for $\tau > 0$

$$\tilde{p}_\tau(t,u,y) = \phi\left(y-u; tm(y) + \tau\frac{\Delta m^u(y)}{n}, t\Sigma(y) + \tau\frac{\Delta\Sigma^u(y)}{n}\right).$$

From (3.6) and (3.7)

$$\begin{aligned}
&\left(p_n^d - p_n\right)(t,x,y) \\
&= -\sum_{k=1}^{M-1} \frac{1}{(k+1)!n^k} p_n^d \otimes_n \left(\tilde{L}_* - \tilde{L}^*\right)^{k+1} p_n(t,x,y) + \frac{1}{n^M} R_2(t,x,y),
\end{aligned}$$

with

$$\begin{aligned}
R_2(t,x,y) &= \frac{1}{(M+1)!n^M}\int_0^1 (1-\tau)^M \left[p_n^d \otimes_n \left(\tilde{L}_* - \tilde{L}^*\right)^{M+1} \tilde{p}_\tau^\Delta\right](t,x,y)\,d\tau, \\
p_\tau^\Delta(t,x,y) &= \sum_{r=0}^{\infty} \tilde{p}_\tau \otimes_n H_n^{(r)}(t,x,y), \\
p_0^\Delta &= p_n.
\end{aligned}$$

From the last identity and Theorem 2.1 we obtain the first statement (1.7) of Theorem 1.1 with $R(x,y) = R_1(1,x,y) + R_2(1,x,y)$. It remains to show that R has subgaussian tails and that (1.8)-(1.9) hold. We only show that the first summand of the left hand side of (1.8) has subgaussian tails. Subgaussianity for the other terms can be shown by similar arguments. For this term we get

$$\left[p \otimes_n \left(L - \tilde{L}^*\right)^2 p_n^d\right](1,x,y) = \sum_{i=0}^{n-1} \frac{1}{n} \int p\left(\frac{i}{n}, x, z\right)\left(L - \tilde{L}^*\right)^2 p_n^d\left(1 - \frac{i}{n}, z, y\right) dz.$$

We split the last sum into two sums

$$\sum_{i=0}^{n-1} \ldots = \sum_{\{i: i/n > 1/2\}} \ldots + \sum_{\{i: i/n \le 1/2\}} \ldots = I + II.$$

For the first sum I the density $p\left(\frac{i}{n}, x, z\right)$ is non-singular and by (2.3) and (2.4) we get

$$\begin{aligned}
&\left|\sum_{\{i: i/n > 1/2\}} \frac{1}{n} \int p\left(\frac{i}{n}, x, z\right)\left(L - \tilde{L}^*\right)^2 p_n^d\left(1 - \frac{i}{n}, z, y\right) dz\right| \\
&= \left|\sum_{\{i: i/n > 1/2\}} \frac{1}{n} \int \left(\left(\tilde{L}^*\right)^t - L^t\right)^2 p\left(\frac{i}{n}, x, z\right) p^d\left(1 - \frac{i}{n}, z, y\right) dz\right| \\
&\le C \exp\left(-C'\|y - x\|^2\right)
\end{aligned}$$

for some positive constants C, C'. Again, here the adjoint of an operator K is denoted by K^t.

For the second sum II the density $p_n^d\left(1-\frac{i}{n},z,y\right)$ is non-singular and by (2.5) and (2.6) we get

$$\begin{aligned}&\left|\sum_{\{i:i/n\le 1/2\}}\frac{1}{n}\int p\left(\frac{i}{n},x,z\right)\left(L-\tilde{L}^*\right)^2 p_n^d\left(1-\frac{i}{n},z,y\right)dz\right|\\&\le C\exp\left(-C'\|y-x\|^2\right)\end{aligned}$$

for some positive constants C, C'.

4 Proof of Theorem 2.1.

We start from the recurrence relation for $r=1,2,3,...$

$$\begin{aligned}\tilde{p}\otimes H^{(r)}-\tilde{p}\otimes_n H^{(r)} &= \left[\left(\tilde{p}\otimes H^{(r-1)}\right)\otimes H-\left(\tilde{p}\otimes H^{(r-1)}\right)\otimes_n H\right]\\&\quad+\left[\left(\tilde{p}\otimes H^{(r-1)}\right)-\left(\tilde{p}\otimes_n H^{(r-1)}\right)\right]\otimes_n H,\end{aligned}$$

By summing up these terms from $r=1$ to ∞ and by using the linearity of the operations $\otimes$ and $\otimes_n$ we get

$$p-p^d=p\otimes H-p\otimes_n H+\left(p-p^d\right)\otimes_n H.$$

By iterative application of the last identity we obtain

$$p-p^d=\sum_{r=0}^{\infty}\left[p\otimes H-p\otimes_n H\right]\otimes_n H^{(r)}. \tag{4.1}$$

Clearly we have

$$\begin{aligned}\left(p\otimes H-p\otimes_n H\right)(t,x,y) &= \sum_{i=0}^{n-1}\int_{i/n}^{(i+1)/n}du\int[p(u,x,z)H(t-u,z,y)\\&\qquad p(i/n,x,z)H(t-i/n,z,y)]dz\end{aligned} \tag{4.2}$$

By Taylor expansion of the function $\lambda(u)=p(u,x,z)H(t-u,z,y)$ in the interval $\left[\frac{i}{n},\frac{i+1}{n}\right]$ we get

$$\begin{aligned}&\int\left[p(u,x,z)H(t-u,z,y)-p(i/n,x,z)H(t-i/n,z,y)\right]dz\\&=\sum_{k=1}^{M-1}\frac{1}{k!}\left(u-\frac{i}{n}\right)^k\int\frac{\partial^k}{\partial s^k}\lambda(s)\mid_{s=i/n}dz\\&\quad+\frac{1}{M!}\left(u-\frac{i}{n}\right)^M\int_0^1(1-\delta)^{M-1}\int\frac{\partial^M}{\partial s^M}\lambda(s)\mid_{s=s_i(u,\delta)}dzd\delta,\end{aligned} \tag{4.3}$$

where $s_i(u,\delta)=i/n+\delta(u-i/n)$. We now use the following properties of p and $\widetilde{p}$

$$-\frac{\partial p(t-s,x,y)}{\partial s}=Lp(t-s,x,y),\ \frac{\partial p(t-s,x,y)}{\partial t}=L^tp(t-s,x,y),\ Lp=L^tp,$$

where L^t is the adjoint operator of L. By application of these equalities we get

$$
\begin{aligned}
\int \frac{\partial}{\partial s}\lambda(s)\Big|_{s=i/n} dz &= \int \frac{\partial}{\partial s}[p(s,x,z)]_{s=i/n} H(t-\frac{i}{n},z,y)dz \\
&\quad + \int p(\frac{i}{n},x,z)\frac{\partial}{\partial s}[H(t-s,z,y)]_{s=i/n}dz \\
&= \int L^t p\left(\frac{i}{n},x,z\right)\left(L-\tilde{L}^*\right)\tilde{p}(t-\frac{i}{n},z,y)dz \qquad (4.4)\\
&\quad - \int p(\frac{i}{n},x,z)\left(L-\tilde{L}^*\right)\tilde{L}^*\tilde{p}\left(t-\frac{i}{n},z,y\right)dz \\
&= \int p\left(\frac{i}{n},x,z\right)\left(L-\tilde{L}^*\right)^2\tilde{p}(t-\frac{i}{n},z,y)dz.
\end{aligned}
$$

By iterative application of similar arguments we get

$$
\int \frac{\partial^k}{\partial s^k}\lambda(s)\,|_{s=i/n}\,dz = \int p\left(\frac{i}{n},x,z\right)\left(L-\tilde{L}^*\right)^k\tilde{p}(t-\frac{i}{n},z,y)dz. \qquad (4.5)
$$

By plugging (4.4) and (4.3) into (4.2) we get

$$
\left(p\otimes H - p\otimes_n H\right)(t,x,y) = \sum_{k=1}^{M-1}\frac{1}{(k+1)!n^k}p\otimes_n\left(L-\tilde{L}^*\right)^{k+1}\tilde{p}(t,x,y)+\frac{1}{n^M}R_M(t,x,y), \qquad (4.6)
$$

where

$$
\begin{aligned}
R_M(\frac{k}{n},x,y) &= \frac{1}{M!}\sum_{i=0}^{k-1}\int_{i/n}^{(i+1)/n}\left[n\left(u-\frac{i}{n}\right)\right]^M\int_0^1(1-\delta)^{M-1} \\
&\qquad \int\frac{\partial^M}{\partial s^M}\left[p(s,x,z)H(\frac{k}{n}-s,z,y)\right]_{s=s_i(u,\delta)}dzd\delta du. \qquad (4.7)
\end{aligned}
$$

By plugging (4.6) and (4.7) into (4.1) we get

$$
\begin{aligned}
p(1,x,y)-p_n^d(1,x,y) &= \sum_{k=1}^{M-1}\frac{1}{(k+1)!n^k}\sum_{r=0}^{\infty}p\otimes_n\left(L-\tilde{L}^*\right)^{k+1}\tilde{p}\otimes_n H^{(r)}(1,x,y) \\
&\qquad + \frac{1}{n^M}R_1(1,x,y), \qquad (4.8)
\end{aligned}
$$

with

$$
R_1(1,x,y) = \sum_{r=0}^{\infty}\left(R_M\otimes_n H^{(r)}\right)(1,x,y).
$$

Now we apply that for an operator S and its adjoint operator S^t we have $p\otimes_n S\tilde{p} = S^t p\otimes_n\tilde{p}$. This gives

$$
\begin{aligned}
\sum_{r=0}^{\infty}p\otimes_n\left(L-\tilde{L}^*\right)^{k+1}\tilde{p}\otimes_n H^{(r)}(1,x,y) &= \left[\left(L-\tilde{L}^*\right)^{k+1}\right]^t p\otimes_n\sum_{r=0}^{\infty}\left(\tilde{p}\otimes_n H^{(r)}\right)(1,x,y) \\
&= p\otimes_n\left(L-\tilde{L}^*\right)^{k+1}p_n^d.
\end{aligned}
$$

By putting this into (4.8) we get the expansion (2.1). The bound (2.2) follows by application of the estimates for the partial derivatives of p and p_n^d given in Theorems 2.2 and 2.3.

5 Proof of Theorem 2.3.

By definition of p_n^d

$$p_n^d\left(\frac{k}{n},x,y\right)=\tilde{p}\left(\frac{k}{n},x,y\right)+\sum_{i=0}^{k-1}\frac{1}{n}\int\tilde{p}\left(\frac{i}{n},x,z\right)\Phi\left(\frac{k}{n}-\frac{i}{n},z,y\right)dz \tag{5.1}$$

with

$$\begin{aligned}\Phi\left(\frac{j}{n},z,y\right) &= \sum_{r=1}^{\infty}H^{(r)}\left(\frac{j}{n},z,y\right),\\ H^{(1)}(t,x,y) &= H(t,x,y)=\left(L-\tilde{L}\right)\tilde{p}(t,x,y),\\ H^{(r)}\left(\frac{k}{n},x,y\right) &= H^{(1)}\otimes_n H^{(r-1)}\left(\frac{k}{n},x,y\right)\\ &= \sum_{i=0}^{k-1}\frac{1}{n}\int H^{(1)}\left(\frac{i}{n},x,z\right)H^{(r-1)}\left(\frac{k}{n}-\frac{i}{n},z,y\right)dz. \qquad (5.2)\end{aligned}$$

From our assumptions (A) and (B) we get for a constant b_1

$$\left|D_y^a D_x^b H^{(1)}\left(\frac{k}{n},x,y\right)\right| \le C\left(\frac{k}{n}\right)^{-(d+1+|a|+|b|)/2}\exp\left\{-b_1\frac{\|y-x\|^2}{k/n}\right\}, \tag{5.3}$$

$$\left|D_x^b H^{(1)}\left(\frac{k}{n},x,x+v\right)\right| \le C\left(\frac{k}{n}\right)^{-(d+1)/2}\exp\left\{-b_1\frac{\|v\|^2}{k/n}\right\}. \tag{5.4}$$

We now treat the kernel $H^{(2)}$.

$$\begin{aligned}H^{(2)}\left(\frac{k}{n},x,y\right) &= \sum_{\left\{i:\frac{i}{n}\le\frac{k}{2n}\right\}}\frac{1}{n}\int H^{(1)}\left(\frac{i}{n},x,z\right)H^{(1)}\left(\frac{k}{n}-\frac{i}{n},z,y\right)dz\\ &\quad+\sum_{\left\{i:\frac{i}{n}>\frac{k}{2n}\right\}}\frac{1}{n}\int H^{(1)}\left(\frac{i}{n},x,z\right)H^{(1)}\left(\frac{k}{n}-\frac{i}{n},z,y\right)dz\\ &= H_{21}\left(\frac{k}{n},x,y\right)+H_{22}\left(\frac{k}{n},x,y\right).\end{aligned}$$

For the summands in H_{21}, we have $\frac{k}{n}-\frac{i}{n}\ge\frac{k}{2n}>0$. Hence

$$D_y^a H_{21}\left(\frac{k}{n},x,y\right)=\sum_{\left\{i:\frac{i}{n}\le\frac{k}{2n}\right\}}\frac{1}{n}\int H^{(1)}\left(\frac{i}{n},x,z\right)D_y^a H^{(1)}\left(\frac{k}{n}-\frac{i}{n},z,y\right)dz.$$

Substituting $z=x+v$ we get

$$D_y^a H_{21}\left(\frac{k}{n},x,y\right)=\sum_{\left\{i:\frac{i}{n}\le\frac{k}{2n}\right\}}\frac{1}{n}\int H^{(1)}\left(\frac{i}{n},x,x+v\right)D_y^a H^{(1)}\left(\frac{k}{n}-\frac{i}{n},x+v,y\right)dv$$

Applying D_x^b to both sides of the last equality we obtain because of (5.3) and (5.4) with some positive constants c and C

$$
\begin{aligned}
&\left|D_y^a D_x^b H_{21}\left(\frac{k}{n}, x, y\right)\right| \\
&= \left| \sum_{\left\{i: \frac{i}{n} \le \frac{k}{2n}\right\}} \frac{1}{n} \int D_x^b H^{(1)}\left(\frac{i}{n}, x, x+v\right) D_y^a H^{(1)}\left(\frac{k}{n}-\frac{i}{n}, x+v, y\right) dv \right. \\
&\qquad \left. + \sum_{\left\{i: \frac{i}{n} \le \frac{k}{2n}\right\}} \frac{1}{n} \int H^{(1)}\left(\frac{i}{n}, x, x+v\right) D_y^a D_x^b H^{(1)}\left(\frac{k}{n}-\frac{i}{n}, x+v, y\right) dv \right| \\
&\le C \sum_{\left\{i: \frac{i}{n} \le \frac{k}{2n}\right\}} \frac{1}{n} \int \left(\frac{k}{n}-\frac{i}{n}\right)^{-(d+1+|a|+|b|)/2} \left(\frac{i}{n}\right)^{-(d+1)/2} \exp\left\{-b_1 \frac{\|z-x\|^2}{i/n}\right\} \\
&\qquad \exp\left\{-b_1 \frac{\|y-z\|^2}{k/n-i/n}\right\} dz \\
&\le C \sum_{\left\{i: \frac{i}{n} \le \frac{k}{2n}\right\}} \frac{1}{n} \left(\frac{k}{n}-\frac{i}{n}\right)^{-(1+|a|+|b|)/2} \left(\frac{i}{n}\right)^{-1/2} \phi_{c,\sqrt{k/n}}(y-x) \\
&\le C \sum_{\left\{i: \frac{i}{n} \le \frac{k}{2n}\right\}} \frac{1}{n} \left(\frac{k}{n}-\frac{i}{n}\right)^{-(1+|a|+|b|)/2} \left(\frac{i}{n}\right)^{-1/2} \phi_{c,\sqrt{k/n}}(y-x) \\
&\le C \left(\frac{k}{n}\right)^{-(|a|+|b|)/2} B\left(\frac{1}{2}, \frac{1}{2}\right) \phi_{c,\sqrt{k/n}}(y-x),
\end{aligned}
$$

where $\phi_{c,s}(u) = s^d \exp(-c\|u\|^2/s^2)$ and $B(u,v)$ is the Beta function.

Since H_{22} can be treated similarly to H_{21}, we obtain

$$
\left|D_y^a D_x^b H^{(2)}\left(\frac{k}{n}, x, y\right)\right| \le C \left(\frac{k}{n}\right)^{-(|a|+|b|)/2} B\left(\frac{1}{2}, \frac{1}{2}\right) \phi_{c,\sqrt{k/n}}(y-x). \tag{5.5}
$$

This bound will be used for the proof of claim (2.5). We now derive a similar bound that will be used for the proof of (2.6). If we substitute $z = x + z'$ into (5.2) for $r = 2$ we get

$$
H^{(2)}\left(\frac{k}{n}, x, x+v\right) = \sum_{i=0}^{k-1} \frac{1}{n} \int H^{(1)}\left(\frac{i}{n}, x, x+z'\right) H^{(1)}\left(\frac{k}{n}-\frac{i}{n}, x+z', x+v\right) dz'.
$$

By applying D_x^b to both sides and by using (5.4), we get

$$
\begin{aligned}
\left|D_x^b H^{(2)}\left(\frac{k}{n}, x, x+v\right)\right| &\le C \sum_{i=0}^{k-1} \frac{1}{n} \left(\frac{i}{n}\right)^{-1/2} \left(\frac{k}{n}-\frac{i}{n}\right)^{-1/2} \phi_{c,\sqrt{k/n}}(v) \\
&\le C B\left(\frac{1}{2}, \frac{1}{2}\right) \phi_{c,\sqrt{k/n}}(v).
\end{aligned} \tag{5.6}
$$

The inequalities (5.5), (5.6) are analogous to (5.3) and (5.4). By iterative application of similar arguments we get the following bound for partial derivatives of $H^{(r)}$ for all $1 \leq r < \infty$

$$\left| D_y^a D_x^b H^{(r)} \left(\frac{k}{n}, x, y \right) \right| \leq C^r B \left(\frac{r-1}{2}, \frac{1}{2} \right) \times \ldots \times B \left(\frac{1}{2}, \frac{1}{2} \right) \left(\frac{k}{n} \right)^{-(2-r+|a|+|b|)/2} \phi_{c,\sqrt{k/n}} (y-x) \, ,$$

$$\left| D_x^b H^{(r)} \left(\frac{k}{n}, x, x+v \right) \right| \leq C^r B \left(\frac{r-1}{2}, \frac{1}{2} \right) \times \ldots \times B \left(\frac{1}{2}, \frac{1}{2} \right) \left(\frac{k}{n} \right)^{-(2-r)/2} \phi_{c,\sqrt{k/n}} (v) \, .$$

By summing up we get the bounds

$$\left| D_y^a D_x^b \Phi \left(\frac{k}{n}, x, y \right) \right| \leq C \left(\frac{k}{n} \right)^{-(1+|a|+|b|)/2} \phi_{c,\sqrt{k/n}} (y-x) \, , \tag{5.7}$$

$$\left| D_x^b \Phi \left(\frac{k}{n}, x, x+v \right) \right| \leq C \left(\frac{k}{n} \right)^{-1/2} \phi_{c,\sqrt{k/n}} (v) \tag{5.8}$$

We now make use of the bounds (5.7), (5.8) and the estimate

$$D_y^s D_x^r \tilde{p} \left(\frac{k}{n}, x, y \right) \leq C_r \left(\frac{k}{n} \right)^{-|r|/2} \phi_{c,\sqrt{k/n}} (y-x)$$

for $0 \leq |r|$ and $0 \leq |s| \leq 2M$ and a constant $C_r$depending on r. Using these bounds we can treat the integral in the identity (5.1) in the same manner that we have treated H_2 above and we get the conclusion of the Theorem.

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