STANDING WAVE SOLUTIONS OF THE NONLINEAR SCHRODINGER EQUATION IN $\mathbb{R}^N$

Boyan SIRAKOV
Laboratoire MODALX, UFR SEGMI
Université Paris 10, Bâtiment G
92001 Nanterre Cedex, FRANCE

Abstract. We study the existence of standing wave solutions of the Schrödinger equation in $\mathbb{R}^N$. We give conditions on the potential under which such notrivial waves exist, allowing external forces which depend on the space variable. This problem leads us to the ellitpic equation $-\varepsilon^2 \Delta u + V(x)u = f(x,u)$, where ε is a small parameter.

1 Introduction

This paper is devoted to the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_x \psi + V(x)\psi - \bar{f}(x, \psi), \tag{1}$$

where m and $\hbar$ are positive constants, $\psi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, $V \in C(\mathbb{R}^N, \mathbb{R})$, and $\bar{f} \in C(\mathbb{R}^N \times \mathbb{C}, \mathbb{C})$. We suppose that the potential $V(x)$ is bounded from below and

$$\bar{f}(x, s\xi) = f(x, s)\xi \tag{2}$$

for $s \in \mathbb{R}, \xi \in \mathbb{C}$, with $|\xi| = 1$, and some function $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. We study the existence of standing waves of (1), more precisely, of solutions of the form

$$\psi(t, x) = e^{-\frac{iEt}{\hbar}} v(x), \tag{3}$$

where E is some real constant and $v : \mathbb{R}^N \to \mathbb{R}$. Substituting (3) into (1), we obtain the real elliptic equation for v

$$-\hbar^2 \Delta v + (V(x) - E)\, v = f(x, v) \tag{4}$$

(for simplicity we have taken $m = \frac{1}{2}$).

One of the classical problems in this framework is to investigate equation (1) (respectively (4)) for small values of $\hbar$. This question has been studied extensively in recent years. In [4] Floer and Weinstein showed that (4) has a nontrivial solution for small $\hbar$, when $N = 1$, $f(x, s) = |s|^2 s$, $V \in L^\infty(\mathbb{R}^N)$, and V has a non-degenerate critical point. Their work was generalised by Oh (see [7], [8] and [9]) who studied the case $N \geq 1, f(x,s) = |s|^{p-1}s$, with $1 < p < \frac{N+2}{N-2}$, and replaced the hypothesis $v \in L^\infty(\mathbb{R}^N)$ by "V belongs to a Kato class" (see ([7]) for details). These works used a Lyapunov-Schmidt reduction type method.

In [10] Rabinowitz developed a variational method which permitted him to show that (4) has a nontrivial solution for small values of $\hbar$, provided that

$$\inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x), \tag{5}$$

$$E < \inf_{x\in\mathbb{R}^N} V(x). \tag{6}$$

In addition, Rabinowitz assumes that the nonlinearity f does not depend on x, satisfies some "mountain pass" type assumptions, and

$$\frac{z}{s}f(sz) \text{ is an increasing function of } s > 0, \text{ for all } z \in \mathbb{R}\setminus\{0\}. \tag{7}$$

In a number of works (see [1], [2], [3], [5], [6] and the references in these papers), the hypothesis (5) was replaced by assumptions on the local behaviour of V, namely existence of different types of critical points of the potential. In all these works it was assumed that the nonlinearity does not depend on x. Furthermore, in addition to the standard "mountain pass" conditions on f, it was supposed either that $f(s) = |s|^{p-1}s$, or that the equation

$$-\Delta u + u = f(u) \tag{8}$$

has a unique positive solution in $H^1(\mathbb{R}^N)$, or that (7) holds for $z = 1$.

In this work we give a global condition on the potential V, which includes (5) as a particular case, and under which the Schrödinger equation (1) has a nontrivial standing wave solution for any general nonlinearity $f(x,s)$ of "mountain pass" type. In particular, none of the above conditions on f is imposed.

We suppose that the function $b(x) = V(x) - \inf_{x\in\mathbb{R}^N} V(x)$ satisfies the following assumptions :

(b1) there exists $x_0 \in \mathbb{R}^N$ such that $b(x_0) = 0$;

(b2) there exists $A > 0$ such that the level set $G_A = \left\{x \in \mathbb{R}^N : b(x) < A\right\}$ has finite Lebesgue measure.

We make the following assumptions on f.

(f1) $\lim_{s\to 0} \frac{f(x,s)}{s} = 0$ for all $x \in \mathbb{R}^N$;

(f2) There exists a constant $C_0 > 0$ such that

$$|f(x,s)| \leq C_0(1+|s|^p),$$

for some $p \in \left(1, \frac{N+2}{N-2}\right)$ and for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$;

(f3) There exists a constant $\mu > 2$ such that

$$sf(x,s) \geq \mu F(x,s) > 0,$$

for all $x \in \mathbb{R}^N$, $s \in \mathbb{R} \setminus \{0\}$; here, as usual, $F(x,s) = \int_0^s f(x,s)\, ds$.

The following theorem contains our main result.

Theorem 1 *If (2), (b1)-(b2) and (f1)-(f3) are satisfied then (1) has a nontrivial standing wave solution for small $\hbar$.*

Clearly, condition (5) implies that (b1) and (b2) are satisfied. The hypothesis (b2) is strongly related to, but more general than (5). Indeed, observe that (5) is equivalent to the existence of a constant $A > 0$ such that G_A is bounded. However, (b2) allows for richer geometry of the potential; for instance, the set of minima of V can be unbounded – a case which is not included in any of the earlier works.

Another important remark is that in all previous works the constant E in (3) was taken to satisfy (6), so that the function space in which a solution is sought for (see section 2) is embedded into $H^1(\mathbb{R}^N)$, with an embedding constant independent of $\hbar$. This hypothesis on E led to the supplementary assumptions on f, since the authors needed an additional knowledge on the properties of the autonomous equation (8).

The main point in our work is that we make a precise choice of E, namely $E = \inf V$. In this situation we solve the elliptic equation (4) under (f1)–(f3) only, obtaining as a consequence a general existence result for nontrivial standing wave solutions of the Schrödinger equation (1).

As we can see from (3), the time period of the standing wave solutions we obtained in Theorem 1 depends on $\hbar$. Under an additional hypothesis on $\inf V$ we can show that, for any *a priori* fixed period, we can find a solution of (1) with that period, for *all* sufficiently small $\hbar$. We do not know of any other result in this direction.

Theorem 2 *In addition to the hypotheses of Theorem 1, we assume that $\inf_{x\in\mathbb{R}^N} V(x) = 0$. Then for any $T > 0$ there exists $\hbar_1 = \hbar_1(T) > 0$ such that (1) has a standing wave solution with time period T, for all $\hbar < \hbar_1$.*

Theorems 1 and 2 will be proved in Section 2. Both of these theorems are derived from a somewhat more general result (see Theorem 3 below).

Acknowledgement. The author wishes to thank Prof. H.Berestycki for important remarks on this work.

2 Proof of the Theorems

Without loss of generality, in the sequel we suppose that $x_0 = 0,\ 0 < \hbar \leq 1$, and $A \leq 1$.

Consider the equation

$$-\Delta u + b_\hbar(\hbar x)u = f(\hbar x, u), \qquad x \in \mathbb{R}^N, \tag{9}$$

where $b_\hbar$ are continuous functions, satisfying the following assumptions

(bh1) $b_\hbar(y) \geq 0 \quad$ for $\ y \in \mathbb{R}^N$;

(bh2) $b_\hbar(0) \to 0 \quad$ as $\ \hbar \to 0$;

(bh3) there exists $A > 0$ such that $|G_{A,\hbar}| < \infty$ for all $\hbar > 0$, where

$$G_{A,\hbar} = \left\{ y \in \mathbb{R}^N \ : \ b_\hbar(y) < A \right\}.$$

We have the following result.

Theorem 3 *If (bh1)-(bh3) and (f1)-(f3) are satisfied then (9) possesses a nontrivial strong solution for sufficiently small* $\hbar$*.*

First, let us observe that this statement covers Theorems 1 and 2. If u solves (9) then $v(x) = u(\frac{x}{\hbar})$ is a solution of

$$-\hbar^2 \Delta u + b_\hbar(x)v = f(x, v).$$

Therefore, to prove Theorem 1 it suffices to take $b_\hbar(x) = V(x) - \inf V$. By taking $E = E(\hbar) = -\frac{2\pi}{T}\hbar$ we see that Theorem 2 is a consequence of Theorem 3, for $b_\hbar(x) = V(x) + \frac{2\pi}{T}\hbar$.

The rest of Section 2 is devoted to the proof of Theorem 3. We use a variational approach.

We start with the description of the variational setting. For each $\hbar > 0$ we introduce the space

$$\mathcal{E}_\hbar = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} b_\hbar(\hbar x)|u|^2 \, dx < \infty \right\},$$

and the quantity

$$\|u\|_\hbar^2 = \|u\|_{\mathcal{E}_\hbar}^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + b_\hbar(\hbar x)u^2 \, dx.$$

It follows from the following lemma that $\mathcal{E}_\hbar$ is a Hilbert space and $\|.\|_\hbar^2$ is a norm on it.

Lemma 1 *For each $\hbar > 0$ there exists a constant $\nu_\hbar > 0$ such that*

$$\|u\|_\hbar^2 \geq \nu_\hbar \|u\|_{H^1(\mathbb{R}^N)}^2,$$

for every $u \in \mathcal{E}_\hbar$.

Proof. First we show that there exists $\theta_\hbar > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 + b_\hbar(x)u^2 \, dx \geq \theta_\hbar \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx, \tag{10}$$

for every u for which the quantity in the left-hand side of (10) is finite. Indeed, since $G_{A,\hbar}$ has finite measure, there exists a constant $C_\hbar > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u|^2 + b_\hbar(x)u^2 \, dx \quad &\geq \quad \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + C_\hbar \int_{G_{A,\hbar}} |u|^2 \, dx \\
&\quad + A \int_{\mathbb{R}^N \setminus G_{A,\hbar}} |u|^2 \, dx \\
&\geq \quad \min\left\{\frac{1}{2},\, C_\hbar,\, A\right\} \|u\|_{H^1(\mathbb{R}^N)},
\end{aligned}$$

where we used the Hölder and the Sobolev inequalities, which give

$$\begin{aligned}
\int_{G_{A,\hbar}} |u|^2 \, dx \quad &\leq \quad |G_{A,\hbar}|^{\frac{2}{N}} \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \\
&\leq \quad C_\hbar^{-1} \|\nabla u\|_{L^2(\mathbb{R}^N)}^2.
\end{aligned}$$

Take $u \in \mathcal{E}_\hbar$ and put $v(x) = u(\frac{x}{\hbar})$. Then

$$\begin{aligned}
\|u\|_\hbar^2 \quad &= \quad \hbar^{-N} \int_{\mathbb{R}^N} \hbar^2 |\nabla v|^2 + b_\hbar(x)v^2 \, dx \\
&\geq \quad \hbar^{2-N}\theta_\hbar \int_{\mathbb{R}^N} |\nabla v|^2 + v^2 \, dx \\
&= \quad \hbar^2\theta_\hbar \int_{\mathbb{R}^N} \hbar^{-2}|\nabla u|^2 + u^2 \, dx \\
&\geq \quad \hbar^2\theta_\hbar \|u\|_{H^1(\mathbb{R}^N)}^2,
\end{aligned}$$

which is the statement of Lemma 1. □

Lemma 1, together with (f1) and (f2), implies that the functional

$$\Phi_\hbar(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + b_\hbar(\hbar x)u^2\,dx - \int_{\mathbb{R}^N} F(\hbar x, u)\,dx$$

is well defined and C^1 on $\mathcal{E}_\hbar$, for each $\hbar > 0$. It is well-known that (9) is the Euler-Lagrange equation of the functional $\Phi_\hbar$.

Let us prove that $\Phi_\hbar$ has a "mountain pass" geometry on $\mathcal{E}_\hbar$. By using (f1) and (f2) we see that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$F(x,s) \le \varepsilon |s|^2 + C_\varepsilon |s|^{p+1}.$$

For each $\hbar > 0$ we fix $\varepsilon_\hbar < \frac{\nu_\hbar}{4}$ and obtain, by Lemma 1,

$$\begin{aligned} \Phi_\hbar(u) &\ge \frac{1}{2}\|u\|_\hbar^2 - \varepsilon_\hbar \|u\|^2_{L^2(\mathbb{R}^N)} - C_{\varepsilon_\hbar}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\ &\ge \frac{1}{4}\|u\|_\hbar^2 - \frac{C(N,p)C_{\varepsilon_\hbar}}{\nu_\hbar^{\frac{p+1}{2}}}\|u\|_\hbar^{p+1}. \end{aligned}$$

We set $\lambda_\hbar = \frac{C(N,p)C_{\varepsilon_\hbar}}{\nu_\hbar^{\frac{p+1}{2}}}$ and choose $r_\hbar > 0$ such that $\frac{1}{4} - \lambda_\hbar r_\hbar^{p-1} > \frac{1}{8}$. Then

$$\Phi_\hbar(u) \ge \frac{1}{8}\|u\|_\hbar^2 \quad \text{provided that } \|u\|_\hbar \le r_\hbar.$$

Next, from (f2) and (f3) it follows that there exists a positive function $d(x) \in L^\infty(\mathbb{R}^N)$ such that

$$F(x,s) \ge d(x)|s|^\mu \quad \text{for all } x \in \mathbb{R}^N, s \in \mathbb{R}.$$

Hence for all $u \in \mathcal{E}_\hbar \setminus \{0\}$ we have $\Phi_\hbar(tu) \to -\infty$ as $t \to \infty$.

Set

$$\Gamma_\hbar := \{\gamma \in C([0,1],\mathcal{E}_\hbar) \,:\, \gamma(0) = 0, \gamma(1) \ne 0, \Phi_\hbar(t\gamma(1)) \le 0 \text{ for all } t \ge 1\}$$

and

$$c_\hbar = \inf_{\gamma\in\Gamma_\hbar} \max_{t\in[0,1]} \Phi_\hbar(\gamma(t)) > 0.$$

Standard critical point theory (see for example [11]) implies the existence of sequences $\{u_n^\hbar\}_{n=1}^\infty \subset \mathcal{E}_\hbar$ such that

$$\Phi_\hbar(u_n^\hbar) \to c_\hbar \qquad \text{as } n \to \infty \tag{11}$$

and

$$\Phi'_\hbar(u_n^\hbar) \to 0 \qquad \text{in } \ \mathcal{E}'_\hbar \ \text{ as } \ n \to \infty, \tag{12}$$

for all $\hbar > 0$.

The rest of the paper will be devoted to the proof of the fact that, for sufficiently small values of $\hbar$, each of these sequences possesses an accumulation point, which is a nontrivial solution of (9).

Lemma 2 *There exists a constant $\alpha > 0$ such that*

$$\limsup_{n\to\infty} \|u_n^\hbar\|_\hbar^2 \le \alpha c_\hbar,$$

for all $\hbar > 0$.

Proof. Using (11), (12) and (f3) we obtain

$$\begin{aligned} \left(\frac{\mu}{2} - 1\right) \|u_n^\hbar\|_\hbar^2 &\le \mu \Phi_\hbar(u_n^\hbar) - \Phi'_\hbar(u_n^\hbar) u_n^\hbar \\ &\le \mu c_\hbar + o(1) + o(1)\|u_n^\hbar\|_\hbar, \end{aligned}$$

and the lemma follows. Here, as everywhere in the sequel, $o(1)$ denotes a quantity that tends to zero as $n \to \infty$. □

Using Lemma 2, for each $\hbar > 0$ we extract a subsequence of $\{u_n^\hbar\}$ which converges weakly in $\mathcal{E}_\hbar$ to a function $u_0^\hbar$. It follows from Lemma 1 and the Rellich-Kondrachov theorem that these subsequences converge to $u_0^\hbar$ weakly in $H^1(\mathbb{R}^N)$, strongly in $L^s_{loc}(\mathbb{R}^N)$, $2 \le s < \frac{2N}{N-2}$, and almost everywhere in $\mathbb{R}^N$. Then it is easy to see that $u_0^\hbar$ is a weak solution of (9). By standard elliptic theory, $u_0^\hbar$ is actually a strong solution of (9).

The only point is to show that $u_0^\hbar$ is not identically zero. We claim that for small $\hbar$ this is the case.

We shall derive our claim from the following two lemmas.

Lemma 3 *There exists a constant $\beta > 0$ such that for all $\hbar > 0$ we can find $R(\hbar) > 0$ for which*

$$\limsup_{n\to\infty} \|u_n^\hbar\|^2_{H^1(\mathbb{R}^N \setminus B_{R(\hbar)})} \le \beta c_\hbar$$

(B_R denotes the open ball with center zero and radius R).

Lemma 4 *We have*

$$\lim_{\hbar\to 0} c_\hbar = 0.$$

Before proving the lemmas, let us show how Theorem 3 follows from them. By using (f1) we find a constant $C > 0$ such that

$$|f(x,s)| \le \frac{1}{\beta}|s| + 2C|s|^p \quad \text{for all } \; x \in \mathbb{R}^N, s \in \mathbb{R}.$$

Then, using (11), (12), (f3) and Lemma 2, we get

$$\begin{aligned}
c_\hbar &= \lim_{n\to\infty} \Phi_\hbar(u_n^\hbar) - \frac{1}{2}\Phi'_\hbar(u_n^\hbar)u_n^\hbar \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{1}{2} u_n^\hbar f(\hbar x, u_n^\hbar) - F(\hbar x, u_n^\hbar)\, dx \\
&\le \frac{1}{2} \liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^\hbar f(\hbar x, u_n^\hbar)\, dx \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{1}{2\beta}|u_n^\hbar|^2 + C|u_n^\hbar|^{p+1}\, dx \\
&\le \frac{1}{2\beta}\|u_0^\hbar\|^2_{L^2(B_{R(\hbar)})} + \frac{1}{2\beta}\limsup_{n\to\infty}\|u_n^\hbar\|^2_{L^2(\mathbb{R}^N\setminus B_{R(\hbar)})} \\
&\quad + C\|u_0^\hbar\|^{p+1}_{L^{p+1}(B_{R(\hbar)})} + C\limsup_{n\to\infty}\|u_n^\hbar\|^{p+1}_{L^{p+1}(\mathbb{R}^N\setminus B_{R(\hbar)})}.
\end{aligned}$$

This implies, by Lemma 3,

$$c_\hbar \le \frac{1}{2\beta}\|u_0^\hbar\|^2_{L^2(B_{R(\hbar)})} + C\|u_0^\hbar\|^{p+1}_{L^{p+1}(B_{R(\hbar)})} + \frac{c_\hbar}{2} + C\beta^{\frac{p+1}{2}} c_\hbar^{\frac{p+1}{2}}.$$

Hence

$$\frac{1}{2\beta}\|u_0^\hbar\|^2_{L^2(B_{R(\hbar)})} + C\|u_0^\hbar\|^{p+1}_{L^{p+1}(B_{R(\hbar)})} \ge c_\hbar\left(\frac{1}{2} - C\beta^{\frac{p+1}{2}} c_\hbar^{\frac{p-1}{2}}\right).$$

The last inequality shows that $u_0^\hbar$ is not identically zero for all $\hbar < \hbar_1$, where $\hbar_1$ is such that

$$c_\hbar < \left(\frac{1}{4C\beta^{\frac{p+1}{2}}}\right)^{\frac{2}{p-1}},$$

provided $\hbar < \hbar_1$. Such a choice of $\hbar_1$ is possible via Lemma 4.

Proof of Lemma 3. By Lemma 2, it is sufficient to show that for each $\hbar > 0$ we can find $R = R(\hbar) > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R} |u_n^\hbar|^2\, dx \le A^{-1}\left(\int_{\mathbb{R}^N} |\nabla u_n^\hbar|^2 + b_\hbar(\hbar x)|u_n^\hbar|^2\, dx\right).$$

Set $B_R^c = \mathbb{R}^N \backslash B_R$, $G_{A,\hbar}^\hbar = \frac{1}{\hbar} G_{A,\hbar} = \{x \in \mathbb{R}^N : \hbar x \in G_{A,\hbar}\}$ and $v_n^\hbar(x) = u_n^\hbar(\frac{x}{\hbar})$. Then

$$\int_{B_R^c \backslash G_{A,\hbar}^\hbar} |u_n^\hbar|^2 \, dx \le A^{-1} \int_{B_R^c \backslash G_{A,\hbar}^\hbar} b_\hbar(\hbar x) |u_n^\hbar|^2 \, dx.$$

By using the Hôlder and the Sobolev inequalities, we obtain, as in the proof of Lemma 1,

$$\begin{aligned} \int_{B_R^c \cap G_{A,\hbar}^\hbar} |u_n^\hbar|^2 \, dx &= \hbar^{-N} \int_{B_{R\hbar}^c \cap G_{A,\hbar}} |v_n^\hbar|^2 \, dx \\ &\le \hbar^{-N} |B_{R\hbar}^c \cap G_{A,\hbar}|^{\frac{2}{N}} \int_{\mathbb{R}^N} |\nabla v_n^\hbar|^2 \, dx \\ &= \hbar^{-2} |B_{R\hbar}^c \cap G_{A,\hbar}|^{\frac{2}{N}} \int_{\mathbb{R}^N} |\nabla u_n^\hbar|^2 \, dx. \end{aligned}$$

To prove Lemma 3 it suffices to take $R = R(\hbar)$ such that

$$\left| B_{\hbar R(\hbar)}^c \cap G_{A,\hbar} \right| < A^{-\frac{2}{N}} \hbar^2.$$

This is possible, since $|G_{A,\hbar}| < \infty$ implies

$$\lim_{R \to \infty} |G_{A,\hbar} \setminus B_R| = 0. \qquad \square$$

Proof of Lemma 4. It is clear that any ray in $\mathcal{E}_\hbar$ can be parametrised so that a part of it belongs to $\Gamma_\hbar$. Hence

$$c_\hbar \le \inf_{u \in \mathcal{E}_\hbar \backslash \{0\}} \max_{t \ge 0} \Phi_\hbar(tu).$$

We have, for all $u \in \mathcal{E}_\hbar$,

$$\Phi_\hbar(u) \le \Psi_\hbar(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + b_\hbar(\hbar x) u^2 \, dx - \int_{\mathbb{R}^N} d(\hbar x) |u|^\mu \, dx,$$

so

$$c_\hbar \le \inf_{u \in \mathcal{E}_\hbar \backslash \{0\}} \max_{t \ge 0} \Psi_\hbar(tu).$$

An explicit computation shows that for any $u \in \mathcal{E}_\hbar \setminus \{0\}$

$$\max_{t \ge 0} \Psi_\hbar(tu) = \text{const.} \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + b_\hbar(\hbar x) u^2 \, dx}{\left(\int_{\mathbb{R}^N} d(\hbar x) |u|^\mu \, dx \right)^{\frac{2}{\mu}}} \right)^{\frac{\mu}{\mu - 2}}.$$

We set

$$\bar{c}_\hbar := \inf_{u \in \mathcal{M}_\hbar} \int_{\mathbb{R}^N} |\nabla u|^2 + b_\hbar(\hbar x) u^2 \, dx,$$

where $\mathcal{M}_\hbar = \left\{ u \in \mathcal{E}_\hbar \; : \; \int_{\mathbb{R}^N} d(\hbar x) |u|^\mu \, dx = 1 \right\}$. We have proved that

$$c_\hbar \leq \text{const.} \bar{c}_\hbar^{\frac{\mu}{\mu - 2}}.$$

We finish the proof of Lemma 4 with the help of the following claim.

Claim $\bar{c}_\hbar$ tends to 0 as $\hbar \to 0$.

Proof of the claim. Suppose that for some sequence $\hbar_m \to 0$ we have $\bar{c}_{\hbar_m} \geq \bar{c}_0 > 0$. For simplicity we drop the subscript m and write $\bar{c}_\hbar$ instead of $\bar{c}_{\hbar_m}$.

It is well known that

$$\inf_{\substack{u \in C_c^\infty(\mathbb{R}^N) \\ \int_{\mathbb{R}^N} |u|^\mu = 1}} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 0$$

(here we use (f2) which yields $\mu \leq p + 1 < \frac{2N}{N-2}$). So we can take a sequence $\{u_n\} \subset C_c^\infty(\mathbb{R}^N)$ such that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^\mu = 1. \tag{13}$$

We set

$$v_{n,\hbar} = \frac{u_n}{\left(\int_{\mathbb{R}^N} d(\hbar x) |u_n|^\mu \, dx \right)^{\frac{1}{\mu}}},$$

so that $v_{n,\hbar} \in \mathcal{M}_\hbar$ for all n and all $\hbar$.

For simplicity we suppose that $d(0) = 1$. By using (13) we see that for every n we can find $\hbar_n > 0$ such that

$$\int_{\mathbb{R}^N} d(\hbar x) |u_n|^\mu \, dx > \frac{1}{2}$$

for $\hbar < \hbar_n$. Hence for every n and every $\hbar < \hbar_n$

$$\int_{\mathbb{R}^N} |\nabla v_{n,\hbar}|^2 \, dx \leq 2^{\frac{2}{\mu}} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx.$$

It follows from (13) that we can find n_0 such that for every $\hbar < \hbar_{n_0}$

$$\int_{\mathbb{R}^N} |\nabla v_{n_0,\hbar}|^2 \, dx < \frac{\bar{c}_0}{2}.$$

On the other hand,

$$\bar{c}_0 \leq \int_{\mathbb{R}^N} |\nabla v_{n,\hbar}|^2 + b_\hbar(\hbar x)|v_{n,\hbar}|^2 \, dx$$

for all n and $\hbar$. Hence, for $\hbar < \hbar_{n_0}$,

$$\begin{aligned}\int_{\mathbb{R}^N} b_\hbar(\hbar x)|u_{n_0}|^2 \, dx &\geq \frac{\int_{\mathbb{R}^N} b_\hbar(\hbar x)|u_{n_0}|^2 \, dx}{2^{\frac{2}{\mu}} \left(\int_{\mathbb{R}^N} d(\hbar x)|u_{n_0}|^\mu \, dx\right)^{\frac{2}{\mu}}} \\ &= \frac{1}{2^{\frac{2}{\mu}}} \int_{\mathbb{R}^N} b_\hbar(\hbar x)|v_{n_0,\hbar}|^2 \, dx \\ &\geq \frac{\bar{c}_0}{2^{\frac{2}{\mu}+1}},\end{aligned}$$

which contradicts $b_\hbar(0) \to 0$, for $\hbar$ sufficiently small.

The proof of Theorem 3 is complete.

Remark 2 In order to obtain a positive solution of (9) we replace $f(x,s)$ by

$$f^+(x,s) = \begin{cases} f(x,s) & \text{if} \quad s > 0 \\ 0 & \text{if} \quad s \leq 0. \end{cases}$$

Remark 3 Lemmas 1 and 4 show that the particular choice $E = \inf V$ in (3) is really essential for Theorem 1. Lemma 4 – a key point in our proof – fails if we take $E < \inf V$. On the other hand, if $E > \inf V$ we would not be able to use the functional spaces $\mathcal{E}_\hbar$, since for small $\hbar$ there will be functions for which $\|u\|_\hbar$ is negative. Indeed, if $b(x) \leq -\delta$ for $x \in B_r$ and $\delta, r > 0$, and if we suppose that for all functions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ $\|.\|_\hbar$ is non-negative, we obtain

$$\int_{B_{\frac{r}{\hbar}}} |\nabla u|^2 \, dx \geq \delta \int_{B_{\frac{r}{\hbar}}} |u|^2 \, dx$$

for all $\hbar > 0$ and all $u \in H_0^1(B_{\frac{r}{\hbar}})$. This contradicts the well known fact that the first eigenvalue of the Laplacian in B_R tends to zero as $R \to \infty$.

References

[1] Ambrosetti A., Badiale M. and Cingolani S. Semiclassical states of nonlinear Schrodinger equations. *Arch. Rat. Mech. Anal.* **140**:285-300, **(1997)**.

[2] delPino M. and Felmer P. Local mountain passes for semilinear elliptic problems in unbounded domains. *Calc. Var. PDE* **4**:121-137, **(1996)**.

[3] delPino M. and Felmer P. Semiclassical states for nonlinear Schrodinger equations. *J.Funct. Anal.* **149**:245-265, **(1997)**.

[4] Floer A. and Weinstein A. Non-spreading wave packets for the cubic Schrödinger equation with a bounded potential. *J. Funct. Anal.* **69**:397-408, **(1986)**.

[5] Gui C. Existence of multi-bump solutions for nonlinear Schrödinger equations via variational methods. *Comm. Part. Diff. Eq.* **21**:787-820, **(1996)**.

[6] Li Y. On a singularly perturbed equation. *Adv. Diff. Eq.* **2**(6):955-980, **(1997)**.

[7] Oh Y.-G. Existence of semi-classical bound states of nonlinear Schrödinger equations with potential of the class $(V)_a$. *Comm. Part. Diff. Eq.* **13**:1499-1519, **(1988)**.

[8] Oh Y.-G. Corrections to "Existence of semi-classical bound states of nonlinear Schrödinger equations with potential of the class $(V)_a$." *Comm. Part. Diff. Eq.* **14**:833-834, **(1989)**.

[9] Oh Y.-G. On positive multi-bump bound states of nonlinear Schrödinger equations under multiple well potential. *Comm. Math. Phys.* **131**:223-253, **(1990)**.

[10] Rabinowitz P.S. On a class of non-linear Schrödinger equations. *Z. Angew. Math. Phys.* **43**:270-291, **(1992)**.

[11] Willem M. Minimax methods, Springer 1996.

e-mail : sirakov@ann.jussieu.fr