Overdetermined elliptic problems in physics

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Abstract. The purpose of this survey is to describe several physically motivated problems in which the corresponding mathematical models lead to overdetermined elliptic boundary-value problems. Such situations appear for instance in fluid mechanics and capillarity theory, theory of elasticity, electrostatics. Our goal here is to give an idea of how they have been treated mathematically.

1 The physical problems

1.1 Interior overdetermined problems. Examples

In this section we describe three physical problems which were first given a mathematical consideration and solved by Serrin in his classical paper [19]. They lead to overdetermined elliptic boundary-value problems in bounded domains.

Fluid moving in a straight pipe. We begin with a simple example which will help us to introduce the kind of problems we consider. Suppose we have a viscous incompressible fluid moving in a straight pipe with a given cross section. Fix rectangular coordinates (x, y, z) in space with the $z-$axis directed along the pipe. Then the cross section of the pipe containing the origin is a domain in the (x, y)-plane, which we denote by Ω. It is a standard result from fluid mechanics that the flow velocity does not depend on z and therefore can be regarded as a function of x and y, defined in Ω. Furthermore, it is known (see for example [10]) that u satisfies the Poisson equation

$$\Delta u = -\frac{\delta p}{\eta l} \qquad \text{in } \Omega, \tag{1}$$

where η denotes the dynamic viscosity, l is the length of the pipe and δp is the change of pressure between the two ends of the pipe. All these quantities are constants in this model.

The adherence condition on the wall of the pipe is expressed by the Dirichlet boundary condition

$$u = 0 \qquad \text{on } \partial\Omega. \tag{2}$$

Finally, the tangential stress on the wall is $\eta\frac{\partial u}{\partial n}$, where n denotes the interior normal to $\partial\Omega$. The precise determination of the point of maximal tangential stress is an important but mathematically very difficult problem. Here we are interested in the following question : *when is the tangential stress the same at each point of a cross section of the wall* ? In other words, can we have a solution of (1)–(2) which satisfies the Neumann type boundary condition

$$\frac{\partial u}{\partial n} = \text{const} \qquad \text{on } \partial\Omega \ ? \tag{3}$$

It is very standard and classical to consider the Poisson equation (1) with either of the boundary conditions (2) or (3) and there is a huge literature on both problems (1)-(2) and (1)-(3). However, the question we asked above requires that both of these conditions be satisfied by the solution of (1) – this is what we call overdetermined. Problem (1)-(2)-(3) can be viewed as a free boundary problem, in the sense that the domain is a part of the problem.

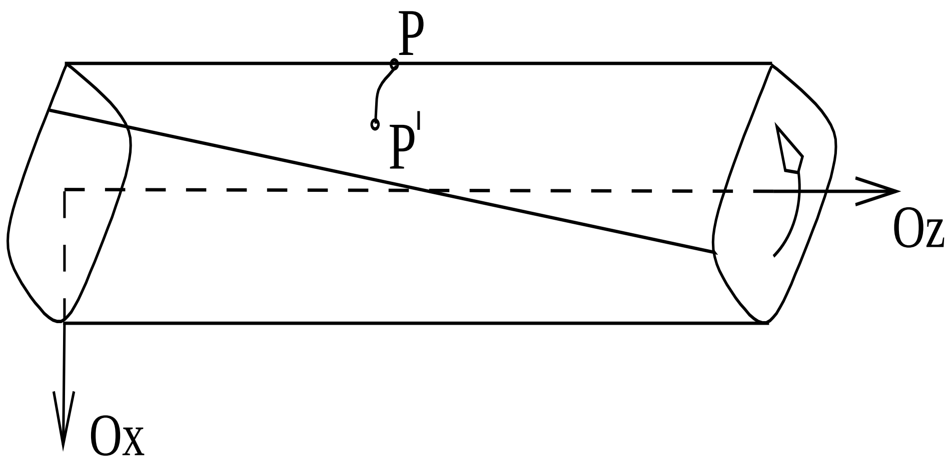
It is intuitively clear that (2) and (3) together is too much to ask, and in most cases the answer to our question will be no, that is, problem (1)-(2)-(3) will not be solvable. On the other hand, if Ω is a ball with radius R then the unique solution of (1)-(2) is

$$u(x) = \frac{\eta l}{4\delta p}(R^2 - x^2 - y^2),$$

and hence satisfies (3). Thus a natural question to ask is whether there exists a non-circular pipe such that a fluid moving inside it has the same tangential stress on all points of its wall.

The torsion problem. An equation of type (1), together wtih the boundary condtions (2) and (3), arises when we model the torsion of a solid cylindrical bar. We follow the presentation in [21]. Suppose we have a cylindrical body of arbitrary (simply connected) cross section, one end of which is fixed, while the other is twisted by a couple of given magnitude. We fix the coordinate system as in the previous example, with the $z-$axis along the axis of the cylinder, and the plane Oxy containing the fixed cross section. It is known that, in general, after the bar is twisted its cross sections do not remain plane but are warped. Actually, any point $P(x, y, z)$ of the body occupies a new position $P'(x + r, y + s, z + t)$ after the twisting, where

$$r = -\alpha z y\,, \qquad s = \alpha z x\,, \qquad t = \alpha\varphi(x, y), \tag{4}$$



where α is the twist per unit length of the bar and φ denotes the torsion function. Let us note that the torsion is zero (that is cross sections of the bar do remain plane) if and only if the cylinder is circular. This result turns out to be a very particular case of the symmetry theorems we present in Section 2.

We shall consider the torsion in terms of L. Prandtl's "stress function"

$$\Psi(x,y) = \psi(x,y) - \frac{1}{2}(x^2+y^2),$$

where $\psi(x,y)$ stands for the complex conjugate of φ. It can be checked ([21]) that Ψ satisfies the equation

$$\left\{ \begin{array}{rcll} \Delta\Psi & = & -2 & \text{in} \quad \Omega \\ \Psi & = & \text{const} & \text{on} \quad \partial\Omega. \end{array} \right. \tag{5}$$

The function Ψ has the following important property : at each point of a level curve of Ψ (these curves, defined by $\Psi = $ const, are called lines of shearing stress) the stress vector is directed along the tangent to the curve. In particular, since all tangential derivatives of Ψ are zero on $\partial\Omega$,

$$\frac{\partial\Psi}{\partial n} = |\nabla\Psi| = \frac{1}{\mu\alpha}.\tau \qquad \text{on } \partial\Omega\,,$$

(see [21]) ; here μ denotes the modulus of rigidity of the bar and τ is the magnitude of tangential stress (τ is called shearing stress).

Like in the previous example, the shearing stress is maximal on the lateral boundary, so elastic failure of the material is to be expected on that boundary. Here we address the following two questions :

- Can the shearing stress be constant on the lateral surface ?

- If the bar is invariant under some group action, in particular, if the bar is symmetric with respect to a hyperplane, then do lines of shearing stress have the same property ?

The interior capillarity problem. A more complicated example is provided by the equation of equilibrium shapes of the surface of a homogeneous and incompressible liquid contained in a straight tube, subject to a gravitational field. Our presentation follows [13], Chapter 2.

We again consider a rectangular coordinate system with the $z-$axis directed along the axis of the vessel and denote with Ω the cross section of the tube containing the origin. For each $(x, y) \in \Omega$ we define $u(x, y)$ to be the height, with respect to the level of Ω, to which the liquid rises above or below the point (x, y).

In this situation, the first two conditions for hydrostatic equlibrium (Euler's condition and Laplace's condition) reduce to the following equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - bu = q \qquad \text{in } \Omega, \tag{6}$$

where $b = \dfrac{\rho g}{\sigma}$ and q is some constant depending only on the height at which we fix the origin. As usual, ρ denotes the density of the fluid, σ is the surface tension, and g is the intensity of the gravitational field.

In this setting the Dupré-Young condition for hydrostatic equilibrium becomes

$$\frac{\partial u}{\partial n} = -\cos\alpha\sqrt{1+|\nabla u|^2} \qquad \text{on } \partial\Omega, \tag{7}$$

where n is the interior normal to the boundary of $\partial\Omega$ and α is the contact (or wetting) angle between the liquid surface and the wall of the vessel.

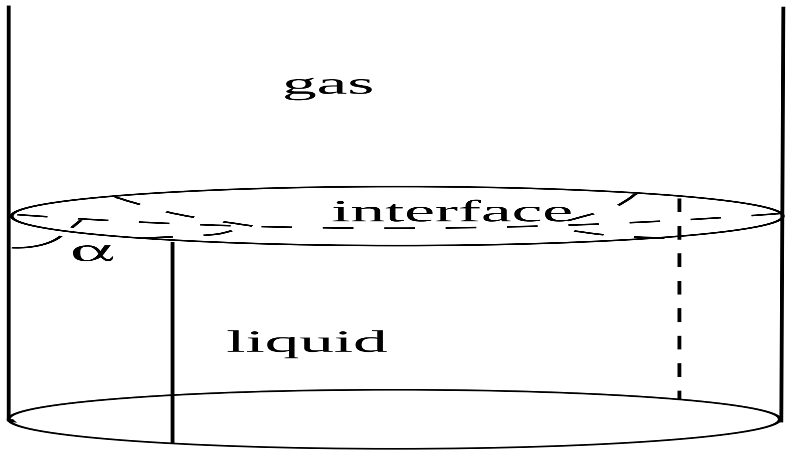
The question we are interested in is : *when does the liquid rise to the same height at each point of the wall* ?

If $u = \text{const}$ on $\partial\Omega$ then the normal derivative of u is equal to the length of the gradient of u on $\partial\Omega$, so (7) transforms into

$$\frac{\partial u}{\partial n} = -\text{cotg}\alpha \qquad \text{on } \partial\Omega. \tag{8}$$

We shall exclude the two limiting cases $\alpha = 0$ and $\alpha = \frac{\pi}{2}$ ($\alpha = 0$ is clearly irrealistic, while for $\alpha = \frac{\pi}{2}$ the only solution of (6)-(7) is $u \equiv \dfrac{q}{b}$, independently of the form of the vessel).

Note that the maximum principle (see Section 3), applied to (6), says u attains its maximum in $\overline{\Omega}$ on the boundary $\partial\Omega$. By the strong maximum



principle, if $u = \text{const}$ on $\partial\Omega$ then u attains its maximum only on $\partial\Omega$, except it is constant in Ω. The latter is excluded by $\alpha \neq \frac{\pi}{2}$.

Finally, to answer the question we asked above, we have to study the solvability of the following problem

$$\left\{ \begin{array}{rcll} \mathrm{div}\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} - bu & = & \text{const} & \text{in} \quad \Omega \\ u & > & 0 & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \\ \dfrac{\partial u}{\partial n} & = & \text{const} & \text{on} \quad \partial\Omega \end{array} \right. \tag{9}$$

(we have fixed the reference level to be the surface level on the vessel wall, and have replaced u by $-u$).

1.2 Exterior elliptic problems

The theory of elliptic partial differential equations is far less advanced when these equations are considered on *unbounded* domains. We give below some examples of physical problems whose mathematical representation leads to free boundary problems in exterior domains (we recall that an exterior domain is the complement of a bounded domain).

An important feature of exterior problems is that they permit us to consider systems of many bodies interacting with each other.

The electrostatics problem. Consider a smooth conducting body G in $\mathbb{R}^N$ ($N = 2$ or 3) with a charge distribution on its boundary. We recall that a charge distribution $\rho \in C(\partial G)$ is called an equilibrium charge distribution if the single-layer potential induced by ρ

$$\psi(x) = \int_{\partial G} \rho(y)\gamma(|x-y|)\, d\sigma_y \tag{10}$$

is constant in G ; here $\gamma(t) = -\frac{1}{2\pi}\log t$ if $N=2$ and $\gamma(t) = -\frac{1}{4\pi t}$ if $N=3$. Note that the potential ψ is harmonic and smooth in G and $\Omega := \mathbb{R}^N \setminus G$.

We are interested in constant equilibrium charge distributions. First, if G is a ball and $\rho = \text{const}$ then ψ, being rotationally invariant and harmonic, is constant in G. Hence, a natural question is : do non-circular conductors admit constant equilibrium charge distributions ? This question was given a negative answer by Martensen ([12]) and Reichel ([17]), respectively for $N=2$ and $N=3$.

However, the exterior nature of the problem permits an important generalisation, namely, we can ask the same question for m conducting bodies in the space. More precisely, suppose we have $m, m \geq 2$, simply connected conductors, with possibly different (but constant !) charge distributions on their boundaries. Can such a system be an equilibrium one ? The negative answer is contained in Theorem 4.

The mathematical formulation of the problem is as follows. Suppose we have $C^{2,\alpha}$-regular mutually disjoint bounded domains $G_1, \ldots, G_m$, such that $\mathbb{R}^N \setminus G$ is connected, where

$$G = \bigcup_{i=1}^{k} G_i. \tag{11}$$

Suppose each body G_i has a constant equilibrium charge distribution ρ_i on its boundary. This means that the single-layer potential defined by (10), with $\rho(y) = \rho_i$ for $y \in \partial G_i$, is constant in each G_i.

Then

$$\frac{\partial \psi}{\partial n} = -\rho_i \qquad \text{on } \partial G_i,$$

by the jump condition for single-layer potentials ; here n is the exterior normal to ∂G_i (interior to $\partial\Omega$). Furthermore, ψ is always above its value ψ_∞ at infinity ; indeed, we have $\psi_\infty = -\infty$ for $N=2$ and $\psi_\infty = 0$ for $N=3$.

Hence, if the system is in equilibrium, then the function $\psi \in C^{2,\alpha}(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathbb{R}^N \setminus G \\ \psi \geq \psi_\infty & \text{in } \mathbb{R}^N \setminus G \\ \psi = a_i > 0 & \text{on } \partial G_i,\ i = 1, \ldots, m \\ \dfrac{\partial u}{\partial n} = -\rho_i & \text{on } \partial G_i,\ i = 1, \ldots, m. \end{cases} \tag{12}$$

The exterior capillarity problem. Here we consider a large (mathematically speaking : infinite) reservoir full of a homogeneous and incompressible liquid, into which we dip a straight solid cylinder. We study the contact surface between the liquid and the cylinder's wall. This problem is dual to the third problem considered in Section 1.1 and leads to the same equation in the exterior of the cylinder.

More generally, consider m solid cylindrical bodies of arbitrary (smooth) cross sections $G_i, i = 1, \ldots, m$, dipped into a large reservoir without touching each other. They make the liquid rise around their walls to some level higher than the (reference) level at the walls of the reservoir. We want to know if the points on the contact surfaces between the liquid and the walls of the cylinders can be at the same height, allowing different heights for the different contact lines. Another way of putting the question is : if we have a set of cylinders dipped into a infinite reservoir, can we build another set of cylinders which, added to the first, will make a system in equilibrium, with each contact surface being at a constant height ?

The mathematical problem to which the above question reduces is the following. As in the previous example, suppose we have m $C^{2,\alpha}$-regular mutually disjoint bounded domains $G_1, \ldots, G_m$, such that $\mathbb{R}^N \setminus G$ is connected. We need to investigate the solvability of the problem

$$\left\{\begin{array}{rcll} \operatorname{div}\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} - bu & = & \text{const} & \text{in} \quad \mathbb{R}^N \setminus G \\ u & \geq & u_\infty & \text{in} \quad \mathbb{R}^N \setminus G \\ u & \to & u_\infty & \text{as} \quad |x| \to \infty \\ u & = & a_i > 0 & \text{on} \quad \partial G_i, \ i = 1, \ldots, m. \\ \dfrac{\partial u}{\partial n} & = & -\text{cotg}\alpha_i & \text{on} \quad \partial G_i, \ i = 1, \ldots, m. \end{array}\right. \tag{13}$$

Both (12) and (13) are particular cases of the equation considered in Theorem 4 of Section 2. This theorem says (12) and (13) do not have a solution, more precisely, if they do, then $m = 1$ and $G = G_1$ is a ball.

2 The symmetry theorems

In this section we present the mathematical results which answer the questions posed in Section 1. They are all based on the famous "moving planes" method of Alexandrov (1962) which has proved to be by far the most powerful tool for establishing symmetry properties of positive solutions of elliptic partial differential equations.

We begin with the case of a bounded domain $\Omega \subset \mathbb{R}^N, N \geq 2$. We consider classical (i.e. C^2-regular) solutions of the problem

$$\begin{cases} Qu + f(u) &= 0 \quad \text{in} \quad \Omega \\ u &> 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{14}$$

where $f \in C^1(\mathbb{R}^+, \mathbb{R})$ and Q is a regular strictly elliptic operator, that is,

(q) $Qu = \mathrm{div}(g(|\nabla u|)\nabla u)$, where $g \in C^2([0,\infty))$, $g(s) > 0$ and $(sg(s))' > 0$ for all $s \geq 0$.

This assumption is satisfied by the Laplace operator ($Q = \Delta$), by the mean curvature operator ($Qu = \mathrm{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$). Let us note that all results in this paper remain true for another physically important operator, the Monge-Ampère operator ($Qu = \det(D^2 u)$), and also for any linear strictly elliptic operator. We also note the function f can be allowed to depend on $|\nabla u|$.

We shall often consider (14) together with the following boundary condition (n will always denote the interior normal to $\partial\Omega$)

$$\frac{\partial u}{\partial n} = \text{const} \qquad \text{on } \partial\Omega. \tag{15}$$

We are interested in the following two questions :

- if the domain is symmetric with respect to a hyperplane, do solutions of (14) have the same property ?
- if problem (14)-(15) is solvable, then is Ω a ball ?

These two questions were answered in the affirmative in two classical and very well-known today papers, by Serrin (1971) and Gidas-Ni-Nirenberg (1981).

In the context of elliptic partial differential equations the moving planes method was first used by Serrin in [19], where he proved the following theorem

Theorem 1 (Serrin, 1971) *Suppose Ω is a bounded C^2-domain and $u \in C^2(\overline{\Omega})$ is a solution of (14)-(15). Then Ω must be a ball.*

The result of Gidas, Ni and Nirenberg states the following

Theorem 2 (Gidas-Ni-Nirenberg, 1981) *Suppose Ω is bounded and convex with respect to some direction $\gamma \in \mathbb{R}^N \setminus \{0\}$ and is symmetric with respect to a hyperplane perpendicular to γ. Then any solution of (14) is symmetric with respect to this hyperplane. In addition, the solution is a strictly decreasing function along any segment which links the hyperplane and $\partial\Omega$.*

In particular, if Ω is a ball, then any solution of (14) is radial and decreasing, that is, it depends only on, and decreases with, the distance to the center of the ball.

In a paper of 1991 Berestycki and Nirenberg gave an alternative proof of this theorem. In their paper they greatly simplified the moving planes method and showed Theorem 2 can be extended to only Liptscitz continuous f's, and to a large class of non-smooth domains.

The Berestycki-Nirenberg improved moving planes method became very popular during the last ten years and was used in many different contexts, where symmetry of solutions of elliptic PDE's was studied.

We next turn to unbounded domains. In 1991 C. Li adapted the moving planes method to the case when equation (14) is defined in the whole space. He proved the following theorem.

Theorem 3 (C. Li, 1991) *Suppose we have a classical solution of*

$$\begin{cases} Qu + f(u) & = & 0 & in & \mathbb{R}^N \\ u \geq 0, \ u & \not\equiv & 0 & in & \mathbb{R}^N \\ u & \rightarrow & 0 & as & |x| \rightarrow \infty, \end{cases} \tag{16}$$

and, in addition, that f is decreasing in a right neighbourhood of zero. Then the solution is radial with respect to some point $x^0 \in \mathbb{R}^N$, that is, u is a function of $|x - x^0|$ alone, and

$$\frac{du}{dr} < 0 \quad for \ r = |x - x^0| \in (0, \infty).$$

In a series of papers Berestycki, Caffarelli and Nirenberg studied symmetry properties of positive solutions of elliptic equations in various types of unbounded domains, including a half-space, a cylinder, a domain bounded by a Lipschitz graph.

In view of the applications exterior and annuli-like domains were extensively studied too. In particular, a number of important partial results were obtained by Alessandrini, Willms-Gladwell-Siegel, Phillipin and Reichel. None of these results could apply to the problems of many bodies discussed in Section 1.2.

Recently the author solved this problem in [20]. Here is the precise statement of the main result in this paper. We make the following hypotheses.

(i) We are in the context of Section 1.2, that is, we have a set G which is the union of m mutually disjoint bounded $C^{2,\alpha}$-domains $G_1, \ldots, G_m$;

(ii) f is a Lipschitz continuous function in $[0, \infty)$ and is decreasing in a right neighbourhood of zero ;

(iii) there exists a C^2-regular solution of the problem

$$\begin{cases} Qu + f(u) &= 0 & \text{in} & \mathbb{R}^N \setminus G \\ u &\geq 0 & \text{in} & \mathbb{R}^N \setminus G \\ u &\to 0 & \text{as} & |x| \to \infty \\ u &= a_i > 0 & \text{on} & \partial G_i, \ i = 1, \ldots, m. \\ \dfrac{\partial u}{\partial n} &= b_i \leq 0 & \text{on} & \partial G_i, \ i = 1, \ldots, m, \end{cases} \tag{17}$$

where a_i and b_i, $i = 1, \ldots, m$ are constants and n denotes the exterior normal to the boundary of G (exterior to $\partial\Omega$).

Remark The constant which appears in the right-hand side of the equation in (13) is necessarily zero, since the solvability of (17) implies $f(0) = 0$.

Theorem 4 *Suppose (i), (ii) and (iii) hold. Then $m = 1$, $G = G_1$ is a ball centered at some point $x^0 \in \mathbb{R}^N$, the solution of (17) u is radial, that is $u = u(|x - x^0|)$, and*

$$\frac{du}{dr} < 0 \quad \text{for } r = |x - x^0| \in (\rho_1, \infty),$$

where ρ_1 denotes the radius of G.

There is a version of this theorem for multiply connected bounded domains, more specifically, domains of the type $\Omega \setminus G$, where Ω is a smooth bounded domain and G is as above. Furthermore, we have a Gidas-Ni-Nirenberg type result in the exterior of a ball. More precisely, if G is a ball and we have a solution of (17), with the last boundary condition relaxed to $\dfrac{\partial u}{\partial n} \leq 0$ on ∂G, then this solution depends only on and decreases with the distance to the center of G.

3 Maximum principles, Serrin's lemma and the moving planes method in bounded domains

3.1 The improved moving planes method.

In this section we describe the moving planes method of Berestycki and Nirenberg, and give their proof of Theorem 2.

Before going to the method itself we recall two very classical results in elliptic theory, *the maximum principle* and *Hopf's lemma* (known also as the strong maximum principle). For these, see for example [15]. Recent results are contained in [6].

Theorem 5 (The maximum principle) *Let* Ω *be a bounded domain and*

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{n} b_i(x)\partial_i + c(x)$$

be a linear uniformly elliptic operator (that is, the matrix $(a_{ij}(x))$ *is positive definite uniformly in* x*), with bounded coefficients. Suppose* $c(x) \leq 0$ *in* Ω*. Then for any* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$$\begin{cases} Lu \leq 0 & in \quad \Omega \\ u \geq 0 & on \quad \partial\Omega \end{cases}$$

implies $u \geq 0$ *in* Ω*. We say that* L *satisfies the maximum principle in* Ω*.*

The same result holds if Ω *is unbounded,* $u \geq 0$ *on* $\partial\Omega$ *meaning also that* $\liminf_{|x|\to\infty} u(x) \geq 0$.

Theorem 6 (Hopf's lemma) *Let* Ω *be a bounded domain and* L *be a linear uniformly elliptic operator with bounded coefficients. Then*

$$\begin{cases} Lu \leq 0 & in \quad \Omega \\ u \geq 0 & in \quad \Omega \end{cases}$$

implies that either $u \equiv 0$ *in* Ω *or* $u > 0$ *in* Ω *and, in addition,* $\dfrac{\partial u}{\partial n} > 0$ *on any point of* $\partial\Omega$ *at which* $\partial\Omega$ *admits an interior tangent ball and* u *vanishes.*

Note that in Theorem 5 we supposed that the zero-order coefficient of the elliptic operator is non-positive. In general, the maximum principle is false if we do not make a hypothesis on $c(x)$. However, it is possible to give conditions on the domain Ω which ensure the validity of the maximum principle, for *any* bounded $c(x)$. In particular, the following maximum principle "in small domains" holds.

Theorem 7 *Let* Ω *be a bounded domain, with* $\operatorname{diam}\Omega \leq d$*. Let* L *be a linear uniformly elliptic operator with coefficients bounded in the uniform norm by a constant* A*. Then the maximum principle is satisfied by* L *in* Ω*, provided* $\operatorname{vol}(\Omega) < \delta$*, where* δ *is a constant depending only on* d, A*, and the ellipticity constant of* L*.*

Proof of Theorem 2 Suppose for simplicity Ω is convex in the direction of the vector $e_1 = (1, 0, \ldots, 0)$ and is symmetric with respect to the hyperplane $T_0 = \{x \mid x_1 = 0\}$. We want to show that

$$u(-x_1, x_2, \ldots, x_N) = u(x_1, x_2, \ldots, x_N) \qquad \text{for any } \ x \in \Omega.$$

For any $\lambda \in \mathbb{R}$ we define

$$\begin{aligned} T_\lambda &= \{x \mid x_1 = \lambda\}, \qquad D_\lambda = \{x \mid x_1 > \lambda\}, \qquad \Sigma_\lambda = D_\lambda \cap \Omega, \\ x^\lambda &= (2\lambda - x_1, x_2, \ldots, x_n) \ - \ \text{the reflexion of } x \text{ with respect to } T_\lambda, \\ w_\lambda(x) &= u(x^\lambda) - u(x), \qquad \text{for any } \ x \in \Sigma_\lambda \\ d &= \inf\{\lambda \in \mathbb{R} \mid T_\mu \cap \overline{\Omega} = \emptyset \ \text{ for all } \ \mu > \lambda\} \end{aligned}$$

(see Fig. 1). With this notation, our goal is to show that $w_0 \equiv 0$ in Σ_0.

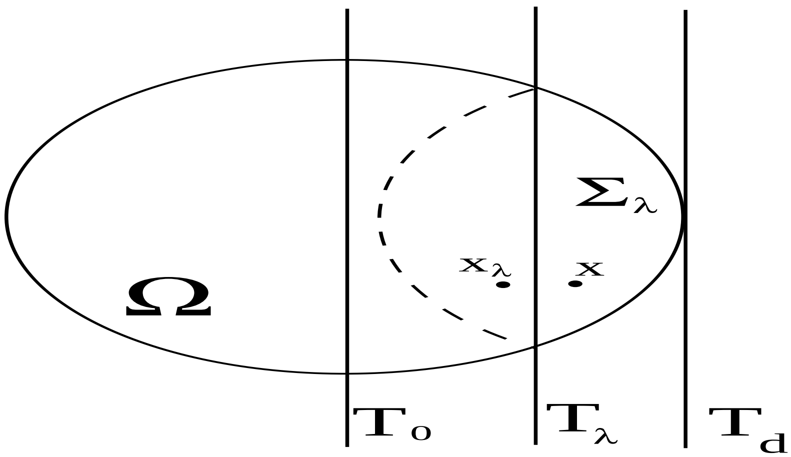


Figure 1: The moving planes method

Lemma 3.1 *The function $w_\lambda, \lambda \in [0, d)$, satisfies a linear uniformly elliptic equation of the form (summing over repeting indices)*

$$\left\{ \begin{array}{rcll} L_0 w_\lambda = a_{ij}(x)\partial_{ij} w_\lambda + b_i(x)\partial_i w_\lambda + c(x) w_\lambda & = & 0 & \text{in} \quad \Sigma_\lambda \\ w_\lambda & \geq & 0 & \text{on} \quad \partial\Sigma_\lambda. \end{array} \right. \tag{18}$$

The ellipticity constant and the coefficients of L are bounded independently of λ.

This lemma is obvious for $Q = \Delta$ and requires some computations in the case of a more general operator (see for example [18]). Note that we take

$$c(x) = \left\{ \begin{array}{ll} \dfrac{f(u(x^\lambda)) - f(u(x))}{u(x^\lambda) - u(x)} & \text{if} \quad u(x^\lambda) \neq u(x) \\ 0 & \text{if} \quad u(x^\lambda) = u(x). \end{array} \right. \tag{19}$$

We say the hyperplane T_λ has reached a position $\lambda < d$ provided w_μ is non-negative in Σ_μ, for all $\mu \in [\lambda, d)$. The plane T_λ "starts" at $\lambda = d$ and "moves" to the left as λ decreases. If we prove that T_λ reaches position zero we are done, since then we can take a hyperplane coming from the other side, that is, starting from -d and moving to the right. The situation is totally symmetric so the second hyperplane would reach position zero too. This means that $w_0 \geq 0$ and $w_0 \leq 0$ in Σ_0, hence $w_0 \equiv 0$ in Σ_0.

Step 1 *The above procedure can begin, that is, there exists $\overline{\lambda} < d$ such that $w_\mu \geq 0$ in Σ_μ, for all $\mu \in [\overline{\lambda}, d)$.*

Proof By using Theorem 7 we can find a number δ such that the operator L_0 defined in Lemma 3.1 satisfies the maximum principle in any subdomain $\Omega' \subset \Omega$, with $\mathrm{vol}(\Omega') < \delta$. We fix $\overline{\lambda} < d$ so close to d that $\mathrm{vol}(\Sigma_\lambda) < \delta$, for any $\lambda \in [\overline{\lambda}, d)$. Hence, by Theorem 7, equation (18) implies that $w_\mu \geq 0$ in Σ_μ, for all $\mu \in [\overline{\lambda}, d)$. □

Note that, by the definition of w_λ, we have $w_\lambda > 0$ on $\partial\Sigma_\lambda \cap \partial\Omega$, for any $\lambda \in (0, d)$ (since u vanishes on $\partial\Omega$ and is strictly positive in Ω). Hence, by Hopf's lemma, $w_\lambda > 0$ in Σ_λ, for $\lambda \in (\overline{\lambda}, d)$.

Step 1 permits us to define the number

$$\lambda_0 = \inf\{\lambda \in (0, d) \mid w_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for all } \mu \geq \lambda\}.$$

Note that, by continuity with respect to λ, $w_{\lambda_0} \geq 0$ in Σ_{λ_0}. By Hopf's lemma, if $\lambda_0 > 0$ then $w_{\lambda_0} > 0$ in Σ_{λ_0}.

Step 2 $\dfrac{\partial u}{\partial x_1} < 0$ *in* Σ_{λ_0}.

Proof Let x be an arbitrary point in Σ_{λ_0}, with $x_1 = \lambda$. Then, by the preceding remarks, $w_\lambda > 0$ in Σ_λ. Since $w_\lambda = 0$ on T_λ, Hopf's lemma implies

$$0 < \frac{\partial w_\lambda}{\partial x_1}(x) = -2\frac{\partial u}{\partial x_1}(x)$$

(recall that $w_\lambda(x) = u(x^\lambda) - u(x)$). □

Step 3 $\lambda_0 = 0$.

Proof Suppose for contradiction $\lambda_0 > 0$. We are going to "push" the moving plane to the left of λ_0. Let K be a compact subset of Σ_{λ_0} such that $\mathrm{vol}\,(\Sigma_{\lambda_0} \setminus K) < \frac{\delta}{2}$ (δ is the number from Theorem 7). Since w_{λ_0} is continuous and strictly positive in Σ_{λ_0}, there exists a number $\varepsilon > 0$ such that $w_{\lambda_0} \geq \varepsilon$ in K. Fix a number $\lambda_1, 0 < \lambda_1 < \lambda_0$, such that $\mathrm{vol}\,(\Sigma_\lambda \setminus K) < \delta$, for

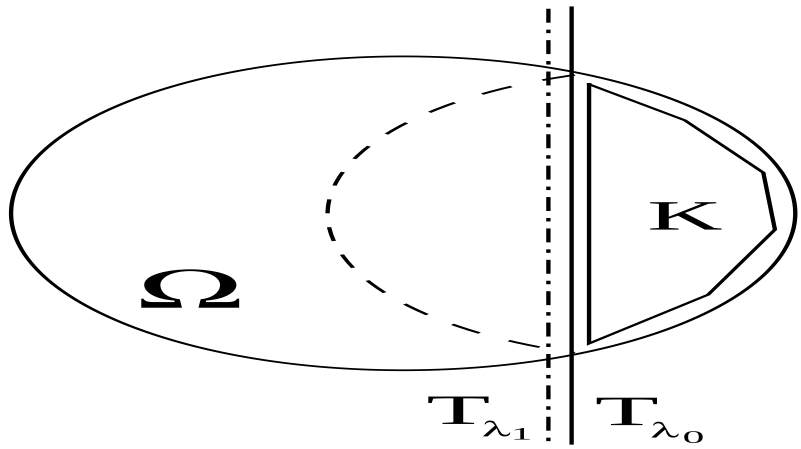


Figure 2: The contradiction in Step 3

$\lambda \in [\lambda_1, \lambda_0)$ (see figure 3.1). By continuity, if λ_1 is sufficiently close to λ_0 we have $w_\lambda \geq \frac{\varepsilon}{2} > 0$ in K, for any $\lambda \in [\lambda_1, \lambda_0)$. In the remaining part of Σ_λ the function $w_\lambda, \lambda \in [\lambda_1, \lambda_0)$, satisfies the equation

$$\begin{cases} L_0 w_\lambda = 0 & \text{in} \quad \Sigma_\lambda \setminus K \\ w_{\lambda_1} \geq 0 & \text{on} \quad \partial(\Sigma_\lambda \setminus K). \end{cases}$$

By Theorem 7, $w_\lambda \geq 0$ in $\Sigma_\lambda \setminus K$. Hence $w_\lambda \geq 0$ in Σ_λ, for any $\lambda \in [\lambda_1, \lambda_0)$. This contradicts the definition of λ_0. □

3.2 Overdetermined Problems. Serrin's lemma

In this section we give a summary of the proof of Serrin's result (Theorem 1). The idea is to show that for any direction $\gamma \in \mathbb{R}^N \setminus \{0\}$ there exists $\lambda = \lambda(\gamma) \in \mathbb{R}$ such that the domain and the solution are symmetric with respect to the hyperplane $T_\lambda = \{x \in \mathbb{R}^N \mid <x, \gamma>= \lambda\}$; here $<\cdot, \cdot>$ denotes the scalar product in $\mathbb{R}^N$.

We fix for instance $\gamma = e_1 = (1, 0, \ldots, 0)$. By using the moving planes method described in the previous section we can show, in exactly the same way, that a hyperplane starting from position d and moving to the left will move as long as the function w_λ is defined in Σ_λ, that is, as long as the reflexion of Σ_λ with respect to T_λ is contained in Ω.

In any case the moving plane reaches position $\lambda_\star$ (called the critical position), where

$$\lambda_\star = \inf\{\lambda \leq d \mid (\Sigma_\mu)^\mu \subset \Omega \;\text{ and }\; <n(z), e_1> < 0 \text{ for all } \mu > \lambda,\; z \in T_\mu \cap \partial\Omega\}.$$

(here, and in the sequel, an upper index means reflexion with respect to the hyperlane with the same index). In other words, the reflexion of Σ_λ with respect to T_λ stays in Ω until at least one of the following two events occurs

(i) the reflexion of $\partial\Omega \cap \partial\Sigma_\lambda$ wih respect to T_λ becomes internally tangent to $\partial\Omega$ at some point P ;

(ii) T_λ becomes orthogonal to $\partial\Omega$ at some point Q.

(see figure 3.2).

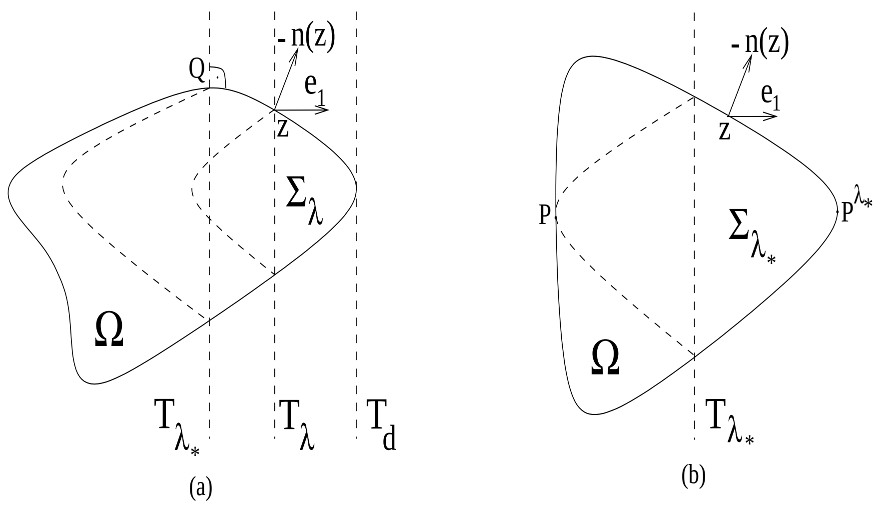


Figure 3: *Two types of domains Ω : the critical position $\lambda_\star$ is attained at a point of orthogonality (a), or at a point of tangency (b). For all $\mu > \lambda_\star$ the part of $\overline{\Omega}$ to the right of T_μ has its reflection inside Ω and the outward normal to $\partial\Omega$ at each point of the boundary of this part makes an acute angle with the direction e_1.*

We are going to show that $w_{\lambda_\star} \equiv 0$ in $\Sigma_{\lambda_\star}$. Since $u > 0$ in Ω, this implies that the reflexion of $\partial\Sigma_{\lambda_\star} \cap \partial\Omega$ with respect to $T_{\lambda_\star}$ lies on $\partial\Omega$, that is, Ω is symmetric with respect to $T_{\lambda_\star}$.

In case (i) Hopf's lemma (applied to (18) for $w_{\lambda_\star}$) immediately yields $w_{\lambda_\star} \equiv 0$ in $\Sigma_{\lambda_\star}$. Indeed, the function $w_{\lambda_\star}$ vanishes at $P^{\lambda_\star} \in \partial\Sigma_{\lambda_\star} \cap \partial\Omega$ (since $u = 0$ on $\partial\Omega$) and so does its normal derivative, because of condition (15).

Case (ii) is more difficult to treat, since Hopf's lemma does not apply at Q ($\partial\Sigma_{\lambda_\star}$ does not admit an interior tangent ball at this point). Serrin proved the following refinement of Hopf's lemma (see [19], p. 313-314).

Lemma 3.2 *Let D^* be a C^2-domain and let T be a plane containing the normal to ∂D^* at some point Q. Let D then denote the portion of D^* lying* *on some particular side of T. Suppose $w \in C^2(\overline{D})$ satisfies a linear uniformly elliptic inequation with bounded coefficients of the form*

$$\left\{ \begin{array}{rcl} Lw = a_{ij}(x)\partial_{ij}w + b_i(x)\partial_i w & \leq & 0 \quad in \quad D \\ w & \geq & 0 \quad in \quad D. \end{array} \right. \tag{20}$$

Suppose also that

$$|a_{ij}(x)\xi_i\eta_j| \leq K(|<\xi,\eta>| + |\xi||d(x)|), \qquad K = const.$$

where $\xi \in \mathbb{R}^N, \eta \perp T, d(x) = \mathrm{dist}(x,T)$. Under these hypotheses, if w and all its first and second order derivatives vanish at Q then $w \equiv 0$ in D.

Let us show that Serrin's lemma applies to $w_{\lambda_\star}$, for $Q = \Delta$ and $N = 2$ (for the general case we refer the reader to [19], p. 315-316 and [18], Appendix 1). First, $w_{\lambda_\star}$ satisfies an equation of type (20), by Lemma 3.1 (we can always achieve $c(x) \geq 0$ in (18), by making the change of functions $\overline{w} = \exp(\beta x_1)w$, with β sufficiently large). Recall that

$$w_{\lambda_\star}(x) = u(2\lambda_\star - x_1, x_2) - u(x_1, x_2), \qquad \text{for} \ \ (x_1, x_2) \in \Sigma_{\lambda_\star}.$$

This trivially yields

$$\frac{\partial w}{\partial x_2}(Q) = \frac{\partial^2 w}{\partial x_1 x_1}(Q) = \frac{\partial^2 w}{\partial x_2 x_2}(Q) = 0.$$

Since $\tau = -e_1$ is tangent to $\partial\Omega$ at Q and $n = -e_2$ is normal to $\partial\Omega$ at Q, we get

$$\begin{array}{rcl} \dfrac{\partial w}{\partial x_1}(Q) = -2\dfrac{\partial u}{\partial x_1}(Q) & = & 2\dfrac{\partial u}{\partial \tau}(Q) \\ \dfrac{\partial^2 w}{\partial x_1 x_2}(Q) = -2\dfrac{\partial^2 u}{\partial x_1 x_2}(Q) & = & -2\dfrac{\partial}{\partial \tau}\left(\dfrac{\partial u}{\partial n}\right)(Q). \end{array}$$

Since both u and its normal derivative are constant on $\partial\Omega$, the last two quantities vanish. By Serrin's lemma $w_{\lambda_\star} \equiv 0$ in $\Sigma_{\lambda_\star}$. □

4 Exterior domains. Proof of Theorem 4

In this section we give a sketch of the proof of Theorem 4. The complete proof is rather lengthy and can be found in [20]. Our goal here is to outline its main ideas.

The proof is based on the moving planes method. The principal difficulties to overcome are the following.

- The domain Ω is unbounded ;
- The domain Σ_λ can be very complex in nature (in particular, not connected) ;
- We do not know a priori whether the solution is below its value on the boundary ∂G.

We shall suppose that $Q = \Delta, m = 1, f \in C^1(\mathbb{R}^+)$ and $f'(0) < 0$, for simplicity. With the notations from the preceding sections, our goal is again to show that the domain $\Omega = \mathbb{R}^N \setminus G$ and the solution u are symmetric with respect to a hyperplane T_λ, for some λ. The function w_λ is now defined in the set $\Sigma_\lambda = D_\lambda \setminus \overline{G^\lambda}$, for $\lambda \geq \lambda_\star$ (here $\lambda_\star$ denotes the critical position for G ; note that $G \subset G^\lambda$, for $\lambda \geq \lambda_\star$).

We divide the proof into ten steps. The first step is again "initializing", in the sense that it permits us to start the moving plane process.

Step 1 *There exists $\overline{\lambda} \in \mathbb{R}$ such that $w_\lambda \geq 0$ in Σ_λ for all $\lambda \geq \overline{\lambda}$.*

Proof. This step is based on the idea of the proof of C.Li's result (see [11]). We shall take the opportunity to explain this idea. In order to use Li's argument we notice that, since u tends to zero at infinity, we can take $\tilde{\lambda} \in \mathbb{R}$ such that $\tilde{\lambda} > d$ and

$$u(x) < \frac{a_1}{2} \qquad \text{for} \quad |x| > \tilde{\lambda},$$

so that $w_\lambda > \dfrac{a_1}{2} > 0$ on ∂G^λ for all $\lambda \geq \tilde{\lambda}$.

Suppose the claim in Step 1 is false, that is, there exists a sequence $\{\lambda_m\}_{m=1}^\infty$ such that

$$\lim_{m \to \infty} \lambda_m = \infty\,,\ \lambda_m \ \geq\ \tilde{\lambda},$$

and w_{λ_m} takes negative values in Σ_{λ_m}. Since w_λ is zero on T_λ and tends to zero at infinity for a fixed λ, we see that w_{λ_m} attains its negative minimum inside Σ_{λ_m}, say at a point $x^{(m)}$. Then

$$\nabla w_{\lambda_m}(x^{(m)}) = 0 \quad \text{and} \quad \Delta w_{\lambda_m}(x^{(m)}) \geq 0.$$

Recall that w_λ satisfies a linear equation of the type

$$\Delta w_\lambda + c(x) w_\lambda = 0 \quad \text{in } \Sigma_\lambda \tag{21}$$

for all $\lambda \in \mathbb{R}$, where $c(x) = f'(d(\lambda, x))$ with

$$d(\lambda, x) \in [\min\{u(x^\lambda), u(x)\}, \max\{u(x^\lambda), u(x)\}].$$

Since $w_{\lambda_m}(x^{(m)}) < 0$ we see that $0 \le d(\lambda_m, x^{(m)}) \le u(x^{(m)})$ and therefore $\lim_{m\to\infty} d(\lambda_m, x^{(m)}) = 0$. It follows that $b_{\lambda_m}(x^{(m)})$ is strictly negative for large m. Hence we obtain

$$0 \le \Delta w_{\lambda_m}(x^{(m)}) = -b_{\lambda_m}(x^{(m)}) w_{\lambda_m}(x^{(m)}) < 0,$$

a contradiction. □

Step 1 shows that the number

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} \mid w_\mu \ge 0 \ \text{in} \ \Sigma_\mu \ \text{for all} \ \mu > \lambda\}$$

is well defined. It is obvious that λ_0 is finite. Notice that, by continuity, $w_{\lambda_0} \ge 0$ in Σ_{λ_0}.

Step 2 *We have* $\dfrac{\partial u}{\partial x_1} < 0$ *in the set* $\left\{x \in \mathbb{R}^N \mid x_1 > \max\{\lambda_0, d\}\right\}$.

The proof of this step is similar in idea to the proof of Step 2 in Section 3.2.

Step 3 $\lambda_0 \le d$.

The ideas of the proof of this step appear in the proof of Step 7 below.

Step 4 *For any* $z \in \partial G$ *and any unit vector* η*, for which* $<\eta, n(z)> > 0$*, we can find a sufficiently small ball* $B_\delta(z)$ *such that*

$$\frac{\partial u}{\partial \eta}(\zeta) < 0 \quad \textit{for all} \ \zeta \in B_\delta(z) \setminus \overline{G}.$$

Proof. This statement says the solution is strictly decreasing in a neighbourhoood of $\partial\Omega$, along any outward direction. If $\alpha_1 < 0$, Step 4 is obvious, by continuity. Hence we can assume $\alpha_1 = 0$, or equivalently $\nabla u \equiv 0$ on ∂G. Notice that this implies $|D^2 u| = \left|\dfrac{\partial^2 u}{\partial n^2}\right|$ on ∂G.

Fix a point $z^0 \in T_d \cap \partial G$, so that

$$\frac{\partial u}{\partial x_1}(z^0) = \frac{\partial u}{\partial n}(z^0) = 0$$

(note that $n = e_1$ at z_0). Steps 2 and 3, together with the assumption $\lambda_0 \le d$, imply

$$\frac{\partial u}{\partial x_1}\left(z^0 + te_1\right) < 0,$$

for positive t. We conclude that

$$\frac{\partial^2 u}{\partial n^2}(z^0) = \frac{\partial^2 u}{\partial x_1^2}(z^0) \leq 0. \tag{22}$$

On the other hand, it is easy to compute that $u = \text{const}$ and $\nabla u = 0$ on ∂G imply

$$\Delta u|_{\partial G} = \left.\frac{\partial^2 u}{\partial n^2}\right|_{\partial G}.$$

Hence

$$\frac{\partial^2 u}{\partial n^2} \equiv -f(a_1) = \text{const} \quad \text{ on } \partial G.$$

By (22), $f(a_1) \geq 0$. If $f(a_1) > 0$, Step 4 follows easily, since

$$\frac{\partial^2 u}{\partial \eta^2} = <\eta, n>^2 \frac{\partial^2 u}{\partial n^2} \quad \text{ on } \partial G.$$

If $f(a_1) = 0$, we see that all first and second order derivatives of u vanish on ∂G. This implies that the function

$$\overline{u}(x) = \begin{cases} u(x) & \text{for } \ x \in \mathbb{R}^N \setminus G \\ a_1 & \text{for } \ x \in \overline{G} \end{cases}$$

belongs to $C^2(\mathbb{R}^N)$ and solves the equation

$$\begin{cases} \Delta \overline{u} + f(\overline{u}) = 0 & \text{in} \quad \mathbb{R}^N \\ \overline{u} \geq 0\,, \quad \overline{u} \not\equiv 0 & \text{in} \quad \mathbb{R}^N \\ \overline{u}(x) \to 0 & \text{as} \quad x \to \infty. \end{cases} \tag{23}$$

However, the shape of $\overline{u}$ contradicts C. Li's result for equations on $\mathbb{R}^N$ (Theorem 3). □

Step 5 *$w_\lambda > 0$ in Σ_λ, for any $\lambda \in [\lambda_0, \infty) \cap (\lambda_\star, \infty)$.*

Step 6 *We have*

$$\frac{\partial u}{\partial x_1} < 0 \quad \textit{ in } D_{\lambda^\star} \setminus \overline{G},$$

where $\lambda^\star = \max\{\lambda_0, \lambda_\star\}$.

The last two steps are a consequence from Hopf's lemma. However, we have to be careful about the fact that the domain Σ_λ may have many connected components (see Figure 4), so we have to exclude $w_\lambda \equiv 0$ in each of them. Suppose for contradiction that $w_\lambda \equiv 0$ in a connected component Z

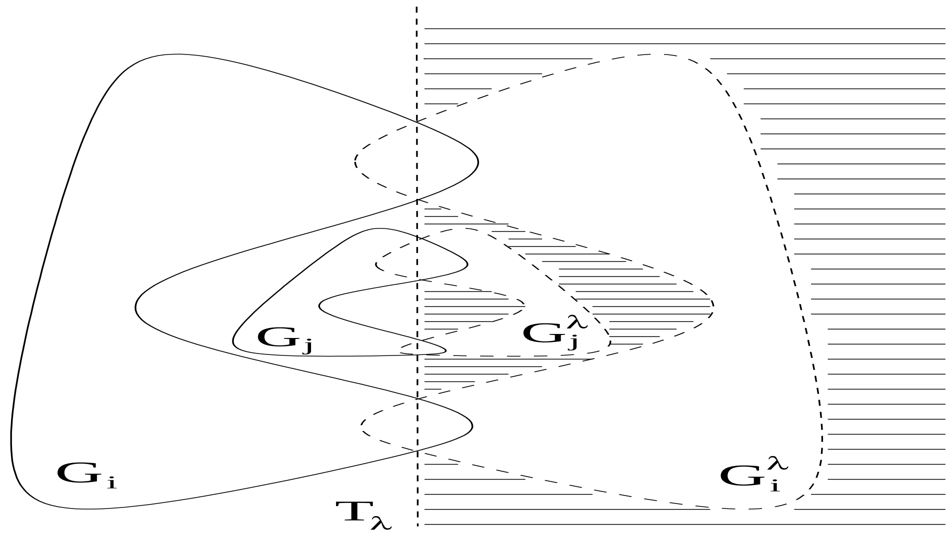


Figure 4: *The shaded regions are the connected components of* Σ_λ.

of Σ_λ. We observe that the boundary of each connected component of Σ_λ contains a point of intersection between ∂G and T_λ. We take a point $y \in Z$ so close to such a point of intersection $z \in \overline{Z}$, that it belongs to the ball $B_\delta(z)$ defined in Step 4. Then, by Step 4, u decreases strictly from y^λ to y. Hence $w_\lambda(y) > 0$, a contradiction. The next step is the key point in the proof of Theorem 4.

Step 7 $\lambda_0 \le \lambda_\star$.

Proof. Suppose $\lambda_0 > \lambda_\star$. By Step 5 we know that $w_{\lambda_0} > 0$ in Σ_{λ_0}. Proceeding as in Step 1, we find two sequences $\{\lambda_m\}_{m=1}^{\infty}$, $\{x^{(m)}\}_{m=1}^{\infty}$, such that

$$\lim_{m\to\infty} \lambda_m = \lambda_0 \, , \; \lambda_\star < \lambda_m < \lambda_0 \, , \; x^{(m)} \in \overline{\Sigma_{\lambda_m}} \setminus T_{\lambda_m}$$

and w_{λ_m} attains its *negative* minimum in $\overline{\Sigma_{\lambda_m}}$ at $x^{(m)}$.

A number of different situations may arise. We obtain a contradiction in each of them.

Case 1 There is a subsequence of $\{x^{(m)}\}$, such that $x^{(m)} \in \mathrm{int}\Sigma_{\lambda_m}$.

As in Step 1 we see that $\Delta w_{\lambda_m}(x^{(m)}) \ge 0$ and $\nabla w_{\lambda_m}(x^{(m)}) = 0$. If $\lim\limits_{m\to\infty} |x^{(m)}| = \infty$ we obtain a contradiction as in Step 1. If a subsequence of $\{x^{(m)}\}$ converges to a point x^0 which belongs to the regular part of $\partial\Sigma_{\lambda_0}$, by passing to the limit we obtain $x^0 \in \overline{\Sigma_{\lambda_0}}$, $w_{\lambda_0}(x^0) \le 0$ and $\nabla w_{\lambda_0}(x^0) = 0$. This means that $x^0 \in \partial\Sigma_{\lambda_0}$ and $w_{\lambda_0}(x^0) = 0$, so Hopf's lemma implies $\nabla u(x^0) \ne 0$, a contradiction. If a subsequence of $\{x^{(m)}\}$ converges to a point x^0 which

belongs to the singular part of $\partial\Sigma_{\lambda_0}$ (which is nothing else but $\partial G \cap T_{\lambda_0}$), then, as in the proof of Steps 5-6, we get $w_{\lambda_m}(x^{(m)}) > 0$ for large m, which is a contradiction.
Case 2 There is a subsequence of $\{x^{(m)}\}$, such that $x^{(m)} \in \partial\Sigma_{\lambda_m}$.

This is the most involved part of the proof. We shall avoid being too rigorous here, and try to concentrate on the main ideas. The following lemma plays a crucial role.

Lemma 4.1 *Suppose $\lambda \geq \lambda_\star$. Any $z \in \partial G^\lambda \cap D_\lambda$, $i = 1, \ldots, k$, has one of the following properties (exclusively)*

(I) *If we move along direction $-e_1$, from z to the left, we enter Σ_λ ;*

(II) *If we move along direction $-e_1$, from z to the left, we enter G^λ and, in addition, we meet ∂G^λ again before or on reaching T_λ;*

(III) *If we move along direction $-e_1$, from z to the left, we enter G^λ and, in addition, we meet ∂G before or on reaching T_λ;*

(IV) $\lambda = \lambda_\star$ *and $z \in \partial G^\lambda \cap \partial G_i$ (the symmetry case).*

The four cases of Lemma 4.1 are shown on Fig. 5. In this way we obtain four types of points on $\partial G^\lambda \cap D_\lambda$.

Step 8 $w_{\lambda_\star} \equiv 0$ *in at least one connected component of* $\Sigma_{\lambda_\star}$.

Step 9 *Let Z be a connected component of $\Sigma_{\lambda_\star}$ such that $w_{\lambda_\star} \equiv 0$ in Z. Then*

$$\partial Z \setminus T_{\lambda_\star} \subset \partial G.$$

Step 10 *Conclusion.*

Proof. Once we have proved Step 9, the conclusion is obtained via a topological argument, due to Fraenkel and used in this setting by Reichel.

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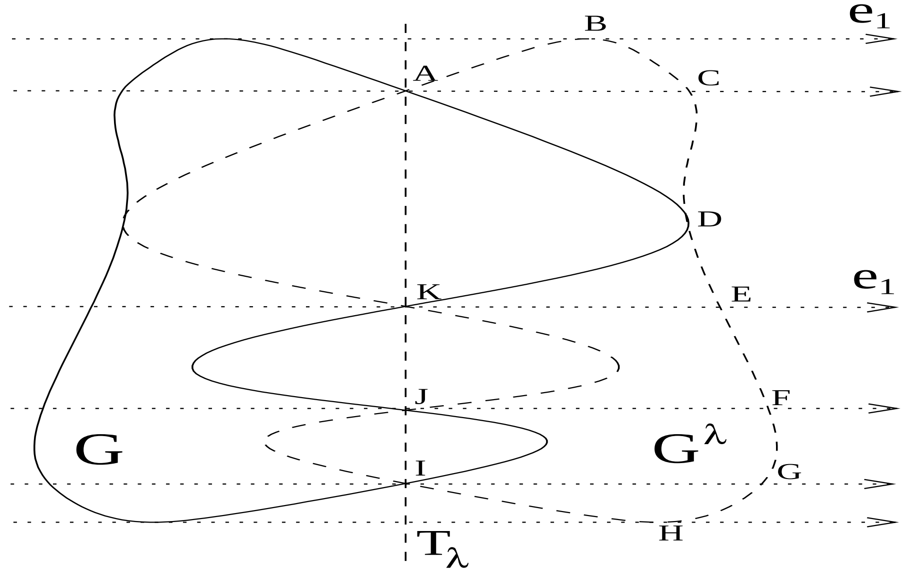

Figure 5: *Four types of points on $\partial G^{\lambda} \cap D_{\lambda}$: the arcs (A,B], [H,I), (J,K) are of type (I), the arcs (B,C], [E,F], [G,H) are of type (II), the arcs (C,D), (D,E), (F,G) are of type (III), and the point D is of type (IV).*

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