# Existence and regularity of a weak function-solution for some Landau equations with a stochastic approach

Hélène Guérin*

March 13, 2001

**Abstract**

Using the Malliavin Calculus, this paper proves the existence of a weak function-solution of class $\mathcal{C}^{\infty}$ of the Landau equation for a generalization of Maxwellian molecules when the initial data is a probability measure.

**Key Words** : Landau Equation, Malliavin Calculus, White Noise, Nonlinear Stochastic Differential Equation.

**MSC 2000** : 60H07, 60H30, 82C40

## 1 Introduction

The Landau equation, also called the Fokker-Planck-Landau equation, is obtained as limit of the Boltzmann equation when all the collisions become grazing. Its expression, in the spatially homogeneous case, is:

$$\frac{\partial f}{\partial t}(v,t) = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d} dv_* a_{ij}(v-v_*)\left[f(v_*,t)\frac{\partial f}{\partial v_j}(v,t) - f(v,t)\frac{\partial f}{\partial v_{*j}}(v_*,t)\right]\right\} \qquad (1)$$

where $f(v,t) \geq 0$ is the density of particles with velocity $v \in \mathbb{R}^d$ at time $t \in \mathbb{R}^+$, and $(a_{ij}(z))_{1\leq i,j\leq d}$ is a nonnegative symmetric matrix depending on the interaction between the particles.

In this paper, we study the Landau equation for a generalization of Maxwell gas. We consider a matrix $a$ of the form

$$a_{ij}(z) = h\left(|z|^2\right)\left(|z|^2\delta_{ij} - z_i z_j\right) \qquad (2)$$

where $h$ is a positive continuous function on $\mathbb{R}_+$ such that there exist $m, M > 0$ with $\forall z \in \mathbb{R}^d$

$$m \leq h\left(|z|^2\right) \leq M \qquad (3)$$

When $h$ is a constant, we recognize the coefficient of the Landau equation for Maxwellian molecules.

We define the vector $b$ by

$$\begin{aligned} b_i(z) &= \sum_{j=1}^{d}\partial_j a_{ij}(z) \\ &= -(d-1)\,h\left(|z|^2\right)z_i \end{aligned} \qquad (4)$$

Then, by integration by parts, we can give a weak formulation of the equation (1), and consequently we define the notion of weak function-solution:

*Université Paris 10, MODALX, UFR SEGMI, 200 avenue de la République, 92000 Nanterre, hguerin@ccr.jussieu.fr

**Definition 1** *Let $f(.,0)$ be a nonnegative function on $\mathbb{R}^d$ with finite mass and energy. A nonnegative function $f$ on $\mathbb{R}^d \times \mathbb{R}^+$ is a* weak function-solution *of the Landau equation with initial data $f(.,0)$, if $f$ satisfies the following equation for any test function $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$*

$$\begin{aligned} \frac{d}{dt}\int \varphi(v) f(v,t)\, dv &= \frac{1}{4}\sum_{i,j=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(v,t) f(v_*,t)\, a_{ij}(v-v_*)\left(\partial_{ij}\varphi(v)+\partial_{ij}\varphi(v_*)\right) \\ &+\frac{1}{2}\sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(v,t) f(v_*,t)\, b_i(v-v_*)\left(\partial_{i}\varphi(v)-\partial_{i}\varphi(v_*)\right) \quad (5) \end{aligned}$$

*where $\partial_i\varphi = \frac{\partial\varphi}{\partial v_i}$ and $\partial_{ij}\varphi = \frac{\partial^2\varphi}{\partial v_i \partial v_j}$.*

The equation (5) conserves mass, momentum and energy. Thus, if there exists a weak function-solution $f$ of (5) with an initial data satisfying $\int_{\mathbb{R}^d} f(v,0)\,dv = 1$, the measure $P_t$ on $\mathbb{R}^d$ given by $P_t(dv) = f(v,t)\,dv$ is a probability measure, for any $t \geq 0$. Thus, we define a probabilistic notion of solutions of the Landau equation :

**Definition 2** *Let $P_0$ be a probability measure on $\mathbb{R}^d$ with a finite two-order moment (i.e. $\int_{\mathbb{R}^d} |v|^2 P_0(dv) < \infty$). A* measure-solution *of the Landau equation (6) with initial data $P_0$ is a probability flow $(P_t)_{t\geq 0}$ on $\mathbb{R}^d$ satisfying*

$$\begin{aligned} \frac{d}{dt}\int \varphi(v) P_t(dv) &= \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\, a_{ij}(v-v_*)\right)\partial_{ij}\varphi(v) \\ &+\sum_{i=1}^d \int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\, b_i(v-v_*)\right)\partial_{i}\varphi(v) \quad (6) \end{aligned}$$

*for any function $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$.*

This approach allows us to have weaker conditions on the initial data, i.e. we can assume that the initial data is a probability measure and not necessarily a density of probability.

**Remark 3** *With an abuse of notation, we will still say that a probability measure $P$ on $\mathcal{C}([0,T], \mathbb{R}^d)$ is a measure-solution of the Landau equation when its time-marginals flow is a measure-solution in the sense of Definition 2.*

We have already proved, in [3], the existence of a probability measure-solution of the Landau equation (6). We are here interested in proving with a stochastic approach the existence of a weak function-solution of (5) of class $\mathcal{C}^\infty$.

We recall briefly the main results of [3]. We have associated with the Landau equation (6) a nonlinear stochastic differential equation driven by a space-time white noise. We highlight the nonlinearity using two probability spaces: let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space and $([0,1], \mathcal{B}([0,1]), d\alpha)$ be an auxiliary probability space, where $d\alpha$ is the Lebesgue measure on $[0,1]$. In order to avoid any confusion, we will denote by $E$ the expectation and $\mathcal{L}$ the distribution of a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $E_\alpha$, $\mathcal{L}_\alpha$ for a random variable on $([0,1], \mathcal{B}([0,1]), d\alpha)$.

For $k \geq 2$, we define $\mathcal{P}_k$ the space of continuous adapted processes $X = (X_t)_{t\geq 0}$ from $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ to $\mathbb{R}^d$, such that $E\left[\sup_{0\leq t\leq T} |X_t|^k\right] < \infty$ $\forall T > 0$, and $\mathcal{P}_{k,\alpha}$ the space of continuous processes $Y = (Y_t)_{t\geq 0}$ from $([0,1], \mathcal{B}([0,1]), d\alpha)$ to $\mathbb{R}^d$, such that $E_\alpha\left[\sup_{0\leq t\leq T} |Y_t|^k\right] < \infty$ $\forall T > 0$.

$a$ is a nonnegative symmetric matrix, then there exists a matrix $\sigma$ of order $d \times d'$ such that

$$a = \sigma.\sigma^* \tag{7}$$

where $\sigma^*$ is the adjoint matrix of $\sigma$.

We define a $d'$- dimensional space-time white noise on $[0,1] \times [0,\infty)$ by

$$W^{d'} = \begin{pmatrix} W_1 \\ \vdots \\ W_{d'} \end{pmatrix} \tag{8}$$

where the $W_i$ are independent space-time white noises with covariance measure $d\alpha dt$ on $[0,1] \times [0,\infty)$ (according to Walsh's Definition, [8]).

Let $X_0$ be a random vector on $\mathbb{R}^d$, independent of $W^{d'}$, with a finite moment of order 2.

We consider the following nonlinear stochastic differential equation:

**Definition 4** *Let $X_0$ and $W^{d'}$ be defined as below. A couple of process $(X,Y)$ on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right) \times ([0,1], \mathcal{B}([0,1]), d\alpha)$ is solution of the nonlinear stochastic differential equation (NSDE$(\sigma, b)$) if for any $t \geq 0$*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma(X_s - Y_s(\alpha)).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b(X_s - Y_s(\alpha))\, d\alpha ds$$

*and $\mathcal{L}(X) = \mathcal{L}_\alpha(Y)$.*

We notice, using Ito's Formula, that the distribution of a solution of $(NSDE(\sigma, b))$ is a weak measure-solution of the Landau equation (6) with initial data $P_0 = \mathcal{L}(X_0)$.

In [3] (Theorem 10), we have proved the following theorem for $k = 2$, but, adapting the proofs, it is still true for any $k \geq 2$.

**Theorem 5** *Assume that $W^{d'}$ is a $d'$-dimensional space-time white noise and assume that $X_0$ is an independent random vector on $\mathbb{R}^d$ with finite moment of order $k$. If the functions $\sigma$ and $b$, defined by (7), (2) and (4), are Lipschitz continuous, there exists a couple $(X,Y)$, unique in law, solution of the nonlinear equation $(NSDE(\sigma, b))$ with $(X,Y) \in \mathcal{P}_k \times \mathcal{P}_{k,\alpha}$.*

**Corollary 6** *Assume that $P_0$ is a probability measure with a finite moment of order 2. There exists a measure-solution $(P_t)_{t\geq 0}$ with initial data $P_0$ to the Landau equation (6) when $\sigma$ and $b$ are Lipschitz continuous functions.*

**Corollary 7** *Assume that $P_0$ is a probability measure with a finite moment of order 2. There is uniqueness of the measure-solution $(P_t)_{t\geq 0}$ with initial data $P_0$ to the Landau equation (6) when $\sigma$ and $b$ are Lipschitz continuous functions.*

**Proof.**

- We just have to prove the uniqueness of the solution $(Q_t^\mu)_{t\geq 0}$ of the linear Landau equation, i.e.

$$\begin{aligned} \frac{d}{dt}\int \varphi(v)\, Q_t(dv) &= \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} Q_t(dv) \left(\int_{\mathbb{R}^d} \mu_t(dv_*)\, a_{ij}(v - v_*)\right) \partial_{ij}\varphi(v) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}^d} Q_t(dv) \left(\int_{\mathbb{R}^d} \mu_t(dv_*)\, b_i(v - v_*)\right) \partial_i\varphi(v) \end{aligned} \tag{9}$$

  where $(\mu_t)_{t\geq 0}$ is a probability flow on $\mathbb{R}^d$. In fact, as a measure-solution $(P_t)_{t\geq 0}$ to the *(nonlinear)* Landau equation is also a solution of the linear equation (9) with $\mu_t = P_t$ for any $t \geq 0$, by uniqueness of the measure-solution of (9), $(P_t)_{t\geq 0}$ is uniquely determined.

- The linear Landau equation (9) satisfies the assumptions of [2] Theorem 5.2, then there is uniqueness of the measure-solution of (9).

■

**Remark 8** *We notice that we can choose*

$$\sigma(z) = \sqrt{h\left(|z|^2\right)} \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} \tag{10}$$

*in dimension two, and*

$$\sigma(z) = \sqrt{h\left(|z|^2\right)} \begin{bmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{bmatrix} \tag{11}$$

*in dimension three. Then, if $h$ is a bounded function of class $\mathcal{C}^1$ with $h'(x) = O\left(\frac{1}{x^2}\right)$ when $x \to +\infty$, $\sigma$ and $b$ are Lipschitz continuous functions of class $\mathcal{C}^1$ on $\mathbb{R}^d$, for $d = 2, 3$. This property can be generalized in higher dimension.*

The aim of this article is to find a weak function-solution of the Landau equation when the initial data is a probability measure. To state the existence of a weak function-solution of (5) from a measure-solution, it is enough to show that the measure-solution is absolutely continuous with respect the Lebesgue measure. Indeed, if $(P_t)_{t\geq 0}$ is a measure-solution of (6) with initial data $P_0$ and if there exists a nonnegative function $f_t$ on $\mathbb{R}^d$ such that $P_t(dv) = f_t(v)\, dv$ for any $t > 0$, then the function $f$ defined by $f(v,t) = f_t(v)$ for any $v \in \mathbb{R}^d$, $t > 0$, is a weak function-solution of (5) with initial data $P_0$.

The idea consists in using the relation between the Landau equation and the nonlinear differential equation $(NSDE(\sigma, b))$. In fact, we develop a Malliavin Calculus for the value $X_t$, $t > 0$, of the solution $X$ of $(NSDE(\sigma, b))$ obtained in Theorem 5, inspired by the methods used by V. Bally and E. Pardoux in [1] and by D. Nualart in [6].

The Maxwellian case (i.e., when the function $h$ is a constant) is studied in detail with an analytic approach by C. Villani in [7]. When the initial data is a nonnegative function $f_0$ with finite mass and energy, C. Villani has proved the existence and the uniqueness of a solution of (1) of class $\mathcal{C}^\infty$.

We prove here the existence of a weak function-solution of the Landau equation (5) of class $\mathcal{C}^\infty$ when the initial data is a probability measure with finite moments for some bounded functions $h$.

## 1.1 About the Malliavin calculus for a white noise

The Malliavin calculus is almost the same for the white noise as for the Brownian Motion. We use in the following the same notation as D. Nualart, [6]. We just recall the definitions of the main spaces for the Malliavin calculus.

Let $W^{d'}$ be a $d'$-dimensional space-time white noise.

Let $\mathcal{S}$ be the class of random variables $F$ having the following form

$$F = f\left(W^{d'}(g_1), .., W^{d'}(g_n)\right)$$

where $f$ is a $\mathcal{C}^\infty$ real values function on $\mathbb{R}^n$ with partial derivatives having polynomial growth, $g = (g_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq d'}}$ is a matrix with components in $\mathbb{L}^2([0,1] \times [0,\infty), d\alpha ds)$, and

$$W^{d'}(g_i) = \sum_{j=1}^{d'} \int_0^\infty \int_0^1 g_{ij}(s,\alpha)\, W_j(d\alpha, ds)$$

Assuming that $(r,z)\in[0,\infty)\times[0,1]$, we define the first order Malliavin derivative $D^l_{(r,z)}F$ of $F$ in relation to the $l^{th}$ white noise $W_l$ at point $(r,z)$, with $l\in\{1,...,d'\}$, by

$$D^l_{(r,z)}F=\sum_{i=1}^{n}\partial_i f\left(W^{d'}(g_1),..,W^{d'}(g_n)\right)g_{il}(r,z)$$

We will say that $F$ is differentiable when all its first derivatives exist.

For $k\geq 1$, $\lambda_k=((r_1,z_1),...,(r_k,z_k))$ with $r_m\in[0,\infty)$ and $z_m\in[0,1]$, $m=1,..,k$. We define by iteration the derivatives of order $k$. Let $(l_1,..,l_k)$ be a $k$-uplet of $\{1,..,d'\}$, we denote by

$$D^{l_1,..,l_k}_{\lambda_k}F=D^{l_k}_{(r_k,z_k)}D^{l_{k-1}}_{(r_{k-1},z_{k-1})}..D^{l_1}_{(r_1,z_1)}F$$

We will say that $F$ has a derivative of order $k$ when all its derivatives of order $k$ exist.

We denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}=\left[E\left(|F|^p\right)+\sum_{m=1}^{k}\sum_{l_1,..,l_m=1}^{d'}E\left(\left\|D^{l_1,..,l_m}F\right\|^p_{\mathbb{L}^2(\Lambda_m)}\right)\right]^{\frac{1}{p}}$$

where

$$\left\|D^{l_1,..,l_m}F\right\|^2_{\mathbb{L}^2(\Lambda_m)}=\int_{\Lambda_m}\left|D^{l_1,..,l_m}_{\lambda_m}F\right|^2d\lambda_m$$

with $\Lambda_m=([0,\infty)\times[0,1])^m$ and $\lambda_m=((r_1,z_1),...,(r_m,z_m))\in\Lambda_m$.

We also denote by $\mathbb{D}^\infty$ the subspace of the infinitely differentiable variables:

$$\mathbb{D}^\infty=\underset{p\geq1}{\cap}\,\underset{k\geq1}{\cap}\,\mathbb{D}^{k,p}$$

When $F$ is a random vector in $\mathbb{R}^d$, we derive component by component and we denote by $DF$ the matrix $(DF)_{i,l}=D^lF_i$, $1\leq i\leq d$,$1\leq l\leq d'$. The Malliavin matrix is defined by

$$I=\int_0^T\int_0^1 D_{(r,z)}F.\left(D_{(r,z)}F\right)^*dzdr$$

In this paper, under suitable assumptions on $\sigma$ and $b$ and integrability conditions on the initial data $X_0$, we show that for any $t>0$ the value $X_t$ of $X$ obtained in Theorem 5 satisfies the conditions of one of those two following theorems.

**Theorem (a)** (see [6] Theorem 2.1.2)

*Let $F=(F_1,..,F_d)$ be a random vector on $(\Omega,\mathcal{F},\mathbb{P})$ satisfying the following conditions :*

*$(i)$ $F_i$ belongs to the space $\mathbb{D}^{1,p}$, $p>1$, for any $i=1,..,d$.*

*$(ii)$ The Malliavin matrix $I=\int_0^\infty\int_0^1 D_{(r,z)}F.\left(D_{(r,z)}F\right)^*dzdr$ is invertible a.s..*

*Then the distribution of $F$ is absolutely continuous with respect the Lebesgue measure on $\mathbb{R}^d$.*

**Theorem (b)** (see [6] Corollary 2.1.2)

*Let $F=(F_1,..,F_d)$ be a random vector on $(\Omega,\mathcal{F},\mathbb{P})$ satisfying the following conditions :*

*$(i)$ $F_i$ belongs to $\mathbb{D}^\infty$, for any $i=1,..,d$.*

*$(ii)$ The Malliavin matrix $I=\int_0^\infty\int_0^1 D_{(r,z)}F.\left(D_{(r,z)}F\right)^*dzdr$ satisfies*

$$(\det I)^{-1}\in\underset{p>1}{\cap}\mathbb{L}^p(\Omega)$$

*Then $F$ has an infinitely differentiable density.*

### 1.2 Notations

- $\mathcal{C}\left([0,T],\mathbb{R}^d\right)$ is the space of continuous functions from $[0,T]$ to $\mathbb{R}^d$, and for $k\in\mathbb{N}$, $\mathcal{C}_b^k\left([0,T],\mathbb{R}^d\right)$ is the space of functions of class $\mathcal{C}^k$ with all its derivatives bounded up to order $k$.
- $\mathcal{M}_{d,d'}(\mathbb{R})$ is the set of $d\times d'$ matrix on $\mathbb{R}$.
- For $k\geq 2$, a random variable $Z$ on $(\Omega,\mathcal{F},\mathbb{P})$ belongs to $\in\mathbb{L}^k$ if $Z$ has a finite moment of order $k$, i.e. $E\left[|Z|^k\right]<\infty$.
- $K$ is an arbitrary notation for a positive constant ($K$ can change from line to line).

## 2 Computation of the derivatives of $X$

### 2.1 The first derivative

**Assumption $\left(H^1\right)$:** $\sigma$ and $b$ are Lipschitz continuous functions of class $\mathcal{C}^1$ from $\mathbb{R}^d$ to $\mathcal{M}_{d,d'}(\mathbb{R})$ and $\mathbb{R}^d$ respectively.

We denote by $K_\sigma$ and $K_b$ their Lipschitz constants.

**Theorem 9** *We assume that $X_0$ has a finite 2-order moment. Let $(X,Y)$ be the solution of the nonlinear stochastic differential equation $(NSDE(\sigma,b))$ obtained in theorem 5. ($Y$ will play a parameter role in the following.)*

*Under Assumption $\left(H^1\right)$, $\forall t\in[0,T]$ $\forall i=1,..,d$, $X_{i,t}\in\mathbb{D}^{1,2}$. The $i^{th}$ component of its derivative in relation to the $l^{th}$ white noise at point $(r,z)\in[0,\infty)\times[0,1]$ is given by*

$$\begin{aligned} D^l_{(r,z)}X_{i,t} &= \sigma_{i,l}\left(X_r-Y_r(z)\right) \\ &\quad +\int_r^t\int_0^1\sum_{k=1}^{d'}\sum_{m=1}^{d}\partial_m\sigma_{i,k}\left(X_s-Y_s(\alpha)\right)D^l_{(r,z)}X_{m,s}W_k(d\alpha,ds) \\ &\quad +\int_r^t\int_0^1\sum_{m=1}^{d}\partial_m b_i\left(X_s-Y_s(\alpha)\right)D^l_{(r,z)}X_{m,s}d\alpha ds \end{aligned}$$

*if $t\geq r$ and $D^l_{(r,z)}X_{i,t}=0$ if $t<r$.*

**Proof.** We consider the Picard sequence of $\mathcal{P}_2$-processes defined by

$$\begin{aligned} X_t^0 &= X_0 \\ X_t^{n+1} &= X_0+\int_0^t\int_0^1\sigma\left(X_s^n-Y_s(\alpha)\right).W^{d'}(d\alpha,ds)+\int_0^t\int_0^1 b\left(X_s^n-Y_s(\alpha)\right)d\alpha ds \end{aligned} \tag{12}$$

Then, the $i^{th}$ component writes

$$X_{i,t}^{n+1}=X_{i,0}+\int_0^t\int_0^1\sum_{k=1}^{d'}\sigma_{i,k}\left(X_s^n-Y_s(\alpha)\right)W_k(d\alpha,ds)+\int_0^t\int_0^1 b_i\left(X_s^n-Y_s(\alpha)\right)d\alpha ds$$

According to [3] Theorem 8, the sequence $(X^n)$ satisfies

$$\sup_n E\left[\sup_{0\leq r\leq T}\left|X_r^n\right|^2\right]<\infty \tag{13}$$

and converges for the norm $\|U\|=\left\|\sup_{0\leq t\leq T}U_t\right\|_{\mathbb{L}^2}$ toward $X$.

Let $T>0$ be arbitrary fixed. Let $t\in[0,T]$ and $(r,z)\in[0,T]\times[0,1]$ be fixed.

We show firstly by recurrence that for any $n\geq 0$ $X_t^n$ is differentiable at point $(r,z)$ in the Malliavin sense.

Recurrence Hypothesis:

$(i)$ $X_{i,t}^n\in\mathbb{D}^{1,2}$ $\forall t\in[0,T]$ $\forall i=1,...,d.$

$(ii)$ $\sup_{t\in[0,T]}\sum_{l=1}^{d'}E\left(\int_0^\infty\int_0^1\left|D_{(r,z)}^lX_t^n\right|^2dzdr\right)<\infty$ where $\left|D_{(r,z)}^lX_t^n\right|^2=\sum_{i=1}^d\left(D_{(r,z)}^lX_{i,t}^n\right)^2.$

For $n=0$, Recurrence Hypothesis is satisfied.

We assume that it is true at rank $n$. Since $\sigma$ and $b$ are functions of class $\mathcal{C}_b^1$, according to [6] proposition 1.2.2, $\forall i=1,..,d$ $\forall k=1,..,d'$, we have

$$\begin{aligned}\sigma_{i,k}\left(X_t^n-Y_t\left(\alpha\right)\right)&\in\mathbb{D}^{1,2}\\ b_i\left(X_t^n-Y_t\left(\alpha\right)\right)&\in\mathbb{D}^{1,2}\end{aligned}$$

As for the Brownian Motion, we can show that derivative and integral commute (see [6]), then $X_{i,t}^{n+1}\in\mathbb{D}^{1,2}$ $\forall t\in[0,T]$ $\forall i=1,...,d.$ Moreover, its derivative at point $(r,z)\in[0,T]\times[0,1]$ in relation to the $l^{th}$ white noise $W_l$ is given by

$$\begin{aligned}D_{(r,z)}^lX_{i,t}^{n+1}&=\sigma_{i,l}\left(X_r^n-Y_r\left(z\right)\right)\\&+\int_r^t\int_0^1\sum_{k=1}^{d'}\sum_{m=1}^d\partial_m\sigma_{i,k}\left(X_s^n-Y_s\left(\alpha\right)\right)D_{(r,z)}^lX_{m,s}^nW_k\left(d\alpha,ds\right)\\&+\int_r^t\int_0^1\sum_{m=1}^d\partial_mb_i\left(X_s^n-Y_s\left(\alpha\right)\right)D_{(r,z)}^lX_{m,s}^nd\alpha ds\end{aligned}$$

if $r\leq t$, and $D_{(r,z)}^lX_{i,t}^{n+1}=0$ else.

We still have to check that

$$\sup_{t\in[0,T]}\sum_{l=1}^{d'}E\left[\int_0^\infty\int_0^1\left|D_{(r,z)}^lX_t^{n+1}\right|^2dzdr\right]<\infty$$

We define

$$\begin{aligned}S_n&=\sup_{t\in[0,T]}\sum_{l=1}^{d'}E\left[\int_0^\infty\int_0^1\left|D_{(r,z)}^lX_t^n\right|^2dzdr\right]\\&=\sup_{t\in[0,T]}\sum_{l=1}^{d'}E\left[\int_0^t\int_0^1\left|D_{(r,z)}^lX_t^n\right|^2dzdr\right]\end{aligned}$$

According to Recurrence Hypothesis, $S_n<\infty$. Let us study $S_{n+1}$.

Let $l\in\{1,..,d'\}$ be arbitrary fixed. We divide in three parts the expectation $E\left[\int_0^t\int_0^1\left|D_{(r,z)}^lX_{i,t}^{n+1}\right|^2dzdr\right]$.

We define

$$\begin{aligned}E_1&=E\left[\int_0^t\int_0^1\left|\sigma_{i,l}\left(X_r^n-Y_r\left(z\right)\right)\right|^2dzdr\right]\\&\leq 2K_\sigma^2E\left[\int_0^t\int_0^1\left|X_r^n\right|^2+\left|Y_r\left(z\right)\right|^2dzdr\right]+T\left|\sigma\left(0\right)\right|\\&\qquad\text{since }\sigma\text{ is Lipschitz continuous}\\&\leq 2K_\sigma^2T\left[\sup_nE\left[\sup_{0\leq r\leq T}\left|X_r^n\right|^2\right]+E_\alpha\left[\sup_{0\leq r\leq T}\left|Y_r\right|^2\right]\right]+T\left|\sigma\left(0\right)\right|\end{aligned}$$

According to (13), we have $\sup_{0\le t\le T} E_1 < \infty$.
We define

$$\begin{aligned}
E_2 &= E\left[\int_0^t\int_0^1 \left|\int_r^t\int_0^1 \sum_{k=1}^{d'}\sum_{m=1}^{d} \partial_m\sigma_{i,k}\left(X_s^n - Y_s(\alpha)\right) D_{(r,z)}^l X_{m,s}^n W_k(d\alpha, ds)\right|^2 dzdr\right] \\
&= \int_0^t\int_0^1 E\left[\int_r^t\int_0^1 \sum_{k=1}^{d'}\left(\sum_{m=1}^{d}\partial_m\sigma_{i,k}\left(X_s^n - Y_s(\alpha)\right) D_{(r,z)}^l X_{m,s}^n\right)^2 d\alpha ds\right] dzdr \\
&\quad \text{since } W_k \text{ are independent} \\
&\le d\sum_{k=1}^{d'}\sum_{m=1}^{d}\int_0^t\int_0^1 E\left[\int_r^t\int_0^1 \left(\partial_m\sigma_{i,k}\left(X_s^n - Y_s(\alpha)\right) D_{(r,z)}^l X_{m,s}^n\right)^2 d\alpha ds\right] dzdr \\
&\quad \text{using Hölder's Inequality}
\end{aligned}$$

Since the partial derivatives of $\sigma$ are bounded by $K_\sigma$,

$$\begin{aligned}
E_2 &\le dK_\sigma^2 \sum_{k=1}^{d'}\sum_{m=1}^{d}\int_0^t\int_0^1 E\left[\int_r^t\int_0^1 \left[D_{(r,z)}^l X_{m,s}^n\right]^2 d\alpha ds\right] dzdr \\
&= d'dK_\sigma^2 \int_0^t E\left[\int_0^s\int_0^1 \left|D_{(r,z)}^l X_s^n\right|^2 dzdr\right] ds \\
&\quad \text{using Fubini's Theorem} \\
&\le d'dK_\sigma^2 \int_0^t \sup_{s\in[0,T]} E\left[\int_0^s\int_0^1 \left|D_{(r,z)}^l X_s^n\right|^2 dzdr\right] ds \\
&\le d'dK_\sigma^2 T S_n
\end{aligned}$$

Then, by Recurrence Hypothesis, we have $\sup_{0\le t\le T} E_2 < \infty$.
We consider now

$$E_3 = E\left[\int_0^t\int_0^1 \left|\int_r^t\int_0^1 \sum_{m=1}^{d}\partial_m b_i\left(X_s^n - Y_s(\alpha)\right) D_{(r,z)}^l X_{m,s}^n d\alpha ds\right|^2 dzdr\right]$$

Using the same method as for integral $E_2$, we also have $\sup_{0\le t\le T} E_3 < \infty$.
Finally, we have proved that for any $l \in \{1, .., d'\}$

$$\sup_{t\in[0,T]} E\left(\int_0^t\int_0^1 \left|D_{(r,z)}^l X_t^{n+1}\right|^2 dzdr\right) \le C_0 + C_1 T S_n < \infty$$

with

$$\begin{aligned}
C_0 &= 6dK_\sigma^2 T\left(\sup_n E\left(\sup_{0\le r\le T}|X_r^n|^2\right) + E_\alpha\left(\sup_{0\le r\le T}|Y_r|^2\right)\right) + 3dT\,|\sigma(0)| \\
C_1 &= 6d^2 \max\left(d'K_\sigma^2, K_b^2 T\right)
\end{aligned}$$

Thus, Recurrence Hypothesis is satisfied for any $n \ge 0$.
We notice that we have in fact a stronger result concerning property $(ii)$:

**Lemma 10** *The sequence of the first derivatives of $(X^n)_{n\ge 0}$ satisfies*

$$\sup_{n\ge 0}\sup_{t\in[0,T]}\sum_{l=1}^{d'} E\left(\int_0^t\int_0^1 \left|D_{(r,z)}^l X_t^n\right|^2 dzdr\right) \le C < \infty$$

**Proof.** We have already checked the following estimate

$$E\left(\int_0^t\int_0^1\left|D^l_{(r,z)}X^{n+1}_t\right|^2dzdr\right) \leq C_0+C_1\int_0^t S_n ds = C_0+C_1tS_n$$

Since $S_0=0$,

$$E\left(\int_0^t\int_0^1\left|D^l_{(r,z)}X^1_t\right|^2dzdr\right) \leq C_0$$

$$\vdots$$

$$E\left(\int_0^t\int_0^1\left|D^l_{(r,z)}X^n_t\right|^2dzdr\right) \leq C_0+C_0C_1t+...+C_0{C_1}^{n-1}\frac{t^{n-1}}{(n-1)!}$$

If we define $C=d'C_0e^{C_1T}$, we have

$$\sup_{n\geq 0}\sup_{t\in[0,T]}\sum_{l=1}^{d'}E\left(\int_0^t\int_0^1\left|D^l_{(r,z)}X^n_t\right|^2dzdr\right)\leq C$$

■

Finally, we have proved

$$\forall n\geq 0\ \forall t\in[0,T]\quad \forall i=1,...,d\ \ X^n_{i,t}\in\mathbb{D}^{1,2} \tag{14}$$

$$\sup_{n\geq 0}\sup_{t\in[0,T]}\sum_{l=1}^{d'}E\left(\int_0^t\int_0^1\left|D^l_{(r,z)}X^n_t\right|^2dzdr\right)<\infty \tag{15}$$

Since the sequence $(X^n)$ converges uniformly on $[0,T]$ in $\mathbb{L}^2$ toward $X$ and thanks to (14) and (15), we deduce that $X$ is differentiable (see [6] lemma 1.2.3). Moreover, the sequence of derivatives $(DX^n)$ converges toward $DX$ for the weak topology on $\mathbb{L}^2([0,T]\times[0,1]\times\Omega)$. Thus, the theorem is proved. ■

## 2.2 The upper order derivatives

We state that $X$ belongs to $\mathbb{D}^\infty$ under a stronger assumption on $\sigma$ and $b$.

Assumption $(H^\infty)$: $\sigma$ and $b$ are Lipschitz continuous functions of class $\mathcal{C}^\infty$ with bounded derivatives from $\mathbb{R}^d$ to $\mathcal{M}_{d,d'}(\mathbb{R})$ and $\mathbb{R}^d$ respectively.

**Notations**: Let $k\geq 1$. We define $\lambda_k=((r_1,z_1),...,(r_k,z_k))$ and

$$\hat{\lambda}_m=((r_1,z_1),...,(r_{m-1},z_{m-1}),(r_{m+1},z_{m+1}),...,(r_k,z_k))$$

with $r_m\in[0,t]$ and $z_m\in[0,1]$ for $m=1,..,k$.

Let us now define $l(E)=l_{\varepsilon_1},...,l_{\varepsilon_\eta}$ and $\lambda(E)=\left((r_{\varepsilon_1},z_{\varepsilon_1}),...,(r_{\varepsilon_\eta},z_{\varepsilon_\eta})\right)$ for any subspace $E=\{\varepsilon_1,...,\varepsilon_\eta\}$ of $\{1,...,k\}$. We consider

$$\Sigma^i_{j,(l_1,...,l_k)}((s,\alpha),\lambda_k) = \sum\sum_{k_1,...,k_\nu=1}^{d}(\partial_{k_1}...\partial_{k_\nu}\sigma_{i,j})(X_s-Y_s(\alpha))\times D^{l(E_1)}_{\lambda(E_1)}X_{k_1,s}...\times D^{l(E_\nu)}_{\lambda(E_\nu)}X_{k_\nu,s}$$

$$\beta^i_{(l_1,...,l_k)}((s,\alpha),\lambda_k) = \sum\sum_{k_1,...,k_\nu=1}^{d}(\partial_{k_1}...\partial_{k_\nu}b_i)(X_s-Y_s(\alpha))\times D^{l(E_1)}_{\lambda(E_1)}X_{k_1,s}...\times D^{l(E_\nu)}_{\lambda(E_\nu)}X_{k_\nu,s}$$

where the first sum is taken on all partitions $E_1\cup...\cup E_\nu=\{1,...,k\}$.

We define at last,

$$\Sigma_j^i((s,\alpha)) = \sigma_{ij}(X_s - Y_s(\alpha))$$

We denote by $r_1 \vee ... \vee r_k = \sup\{r_1, ..., r_k\}$.

**Theorem 11** *Assume that $X_0 \in \mathbb{L}^p$, for any $p \geq 1$. Under Assumption $(H^\infty)$, $\forall t \geq 0$ $X_t \in \mathbb{D}^\infty$. Moreover, the $i^{th}$ component of one of its derivative of order $k$ at point $\lambda_k = ((r_1,z_1), ..., (r_k,z_k))$ is given by the following equation*

$$\begin{aligned} D_{\lambda_k}^{l_1,..,l_k} X_{i,t} &= \sum_{m=1}^k \Sigma^i_{l_m,(l_1,..l_{m-1},l_{m+1},..,l_k)}\left((r_m,z_m),\hat{\lambda}_m\right) + \int_{r_1\vee..\vee r_k}^t \int_0^1 \sum_{j=1}^{d'} \Sigma^i_{j,(l_1,..,l_k)}((s,\alpha),\lambda_k) W_j(d\alpha,ds) \\ &+ \int_{r_1\vee..\vee r_k}^t \int_0^1 \beta^i_{(l_1,..,l_k)}((s,\alpha),\lambda_k)\, d\alpha ds \end{aligned} \quad (16)$$

*if $t \geq r_1 \vee ... \vee r_k$ and $D_{\lambda_k}^{l_1,...,l_k} X_{i,t} = 0$ if $t < r_1 \vee .. \vee r_k$.*

**Remark 12** *In expression (16) of the $k^{th}$ derivative, the terms in the first sum with $r_m < r_1 \vee ... \vee r_k$ are equal to* 0.

**Proof.** We use again the Picard sequence $(X^n)_{n\geq 0}$ defined by (12). For any $p \geq 2$, $n \geq 0$, $X^n \in \mathbb{L}^p$ and $(X^n)$ converges uniformly toward $X$ in $\mathbb{L}^p$. As $\sigma$ and $b$ satisfy Assumption $(H^\infty)$, using the same method as in the previous paragraph, we prove that $X_t^n \in \mathbb{D}^{1,p}$ $\forall p \geq 1$ for any $t \geq 0$. By recurrence, we prove that $\forall t \geq 0$, $\forall n \geq 0$ $X_t^n \in \mathbb{D}^\infty$.

Let us fix $T > 0$.

Recurrence Hypothesis $(h_n)$:

$(i)$ $X_{i,t}^n \in \mathbb{D}^\infty$, $\forall t \in [0,T]$, $\forall i = 1, ..., d$.

$(ii)$ $\sup_{t\in[0,T]} \sum_{l_1,..,l_k=1}^{d'} E\left(\int_{([0,t]\times[0,1])^k} \left|D_{\lambda_k}^{l_1,..,l_k} X_{i,t}^n\right|^p d\lambda_k\right) < \infty$ $\forall p \geq 1, \forall k \geq 1$.

$(iii)$ the derivatives of order $k$ have the following expression:

$$\begin{aligned} D_{\lambda_k}^{l_1,..,l_k} X_{i,t}^{n+1} &= \sum_{m=1}^k \Sigma^{n,i}_{l_m,(l_1,..l_{m-1},l_{m+1},..,l_k)}\left((r_m,z_m),\hat{\lambda}_m\right) + \int_{r_1\vee..\vee r_k}^t \int_0^1 \sum_{j=1}^{d'} \Sigma^{n,i}_{j,(l_1,..,l_k)}((s,\alpha),\lambda_k) W_j(d\alpha,ds) \\ &+ \int_{r_1\vee..\vee r_k}^t \int_0^1 \beta^{n,i}_{(l_1,..,l_k)}((s,\alpha),\lambda_k)\, d\alpha ds \end{aligned} \quad (17)$$

if $t \geq r_1 \vee .. \vee r_k$ and $D_{\lambda_k}^{l_1,...,l_k} X_{i,t}^n = 0$ else, where $\Sigma^n$ and $\beta^n$are defined as $\Sigma$ and $\beta$ replacing $X$ with $X^n$.

Hypothesis $(h_0)$ is satisfied.

Let us assume that Hypothesis $(h_n)$ is true, and let us study $(h_{n+1})$. According to Assumption $(H^\infty)$ and adapting the computation of the first derivative, it is easy to state that the two first properties are satisfied. We just check the expression of the $k^{th}$ derivative by recurrence on $k$.

For $k = 1$, we have

$$\begin{aligned} D_{(r,z)}^l X_{i,t}^{n+1} &= \Sigma_l^{n,i}((r,z)) + \int_r^t \int_0^1 \sum_{j=1}^{d'} \Sigma^{n,i}_{j,(l)}((s,\alpha),(r,z)) W_j(d\alpha,ds) \\ &+ \int_r^t \int_0^1 \beta^{n,i}_{(l)}((s,\alpha),(r,z))\, d\alpha ds \end{aligned}$$

then the expression (17) is satisfied.

We assume that the expression (17) of the $k^{th}$ derivative is true, and we now compute the derivative of order $k+1$

$$\begin{aligned}
D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(D^{l_1,..,l_k}_{\lambda_k}X^{n+1}_{i,t}\right) &= D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(\sum_{m=1}^{k}\Sigma^{n,i}_{l_m,(l_1,..l_{m-1},l_{m+1},..,l_k)}\left((r_m,z_m),\hat{\lambda}_m\right)\right) \\
&+\Sigma^{n,i}_{l_{k+1},(l_1,..,l_k)}\left((r_{k+1},z_{k+1}),\lambda_k\right) \\
&+\int_{r_1\vee..\vee r_k\vee r_{k+1}}^{t}\int_0^1\sum_{j=1}^{d'}D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(\Sigma^{n,i}_{j,(l_1,..,l_k)}\left((s,\alpha),\lambda_k\right)\right)W_j(d\alpha,ds) \\
&+\int_{r_1\vee..\vee r_k\vee r_{k+1}}^{t}\int_0^1 D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(\beta^{n,i}_{(l_1,..,l_k)}\left((s,\alpha),\lambda_k\right)\right)d\alpha ds
\end{aligned}$$

Using some elementary computations, we obtain

$$\begin{aligned}
D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(D^{l_1,..,l_k}_{\lambda_k}X^{n+1}_{i,t}\right) &= \sum_{m=1}^{k}\Sigma^{n,i}_{l_m,(l_1,..l_{m-1},l_{m+1},..,l_k,l_{k+1})}\left((r_m,z_m),\hat{\lambda}_m\right) \\
&+\Sigma^{n,i}_{l_{k+1},(l_1,..,l_k)}\left((r_{k+1},z_{k+1}),\hat{\lambda}_{k+1}\right) \\
&+\int_{r_1\vee..\vee r_{k+1}}^{t}\int_0^1\sum_{j=1}^{d'}\Sigma^{n,i}_{j,(l_1,..,l_k,l_{k+1})}\left((s,\alpha),\lambda_{k+1}\right)W_j(d\alpha,ds) \\
&+\int_{r_1\vee..\vee r_{k+1}}^{t}\int_0^1 \beta^{n,i}_{(l_1,..,l_k,l_{k+1})}\left((s,\alpha),\lambda_k\right)d\alpha ds
\end{aligned}$$

So by recurrence, the property $(iii)$ of $(h_{n+1})$ is proved and consequently for any $n\geq 0$ Recurrence Hypothesis $(h_n)$ is satisfied.

Moreover, as in the computation of the first derivative, we have a stronger property than $(ii)$ in $(h_n)$:

**Lemma 13** *If we denote by*

$$\begin{aligned}
S_{n,k}(t) &= \sum_{l_1,..,l_k=1}^{d'}E\left(\int_{([0,t]\times[0,1])^k}\left|D^{l_1,..,l_k}_{\lambda_k}X^n_t\right|^p d\lambda_k\right) \\
M_k &= \sup_{0\leq q\leq k}\sup_{n\geq 0}\sup_{t\in[0,T]}S_{n,q}(t)
\end{aligned}$$

*then for any* $k\geq 1$

$$M_k<\infty$$

**Proof.** The proof is similar to the proof of Lemma 10. ■

As $(X^n)$ converges toward $X$ in $\mathbb{L}^p$ uniformly on $[0,T]$ for any $T>0$, the process $X$ satisfies the conditions of lemma 1.5.4 in [6]. Then, the theorem is proved. ■

# 3 Existence of a weak function-solution of the Landau equation

Under some suitable conditions on the function $h$, the Landau coefficients satisfy Assumption $\left(H^1\right)$ (see Remark 8). Consequently, if $X_0$ belongs to $\mathbb{L}^2$, the process $X$ solution of $(NSDE(\sigma,b))$ is differentiable in the Malliavin sense. Let us now study the Malliavin matrix $I_t=\int_0^T\int_0^1 D_{(r,z)}X_t.\left(D_{(r,z)}X_t\right)^* dzdr$ for any $t>0$ to state the following theorem.

**Theorem 14** *Assume that $X_0$ is a $\mathbb{R}^d$-valued random vector with a finite 2-order moment. Let $\sigma$ and $b$ be the coefficients of the Landau equation defined respectively by (7), (2) and (4). We assume that $\sigma$ and $b$ are Lipschitz continuous of class $\mathcal{C}^1$. If the distribution of $X_0$ is not a Dirac mass and if we denote by $(X,Y)$ the solution of the nonlinear stochastic differential equation*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma(X_s - Y_s(\alpha)) .W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b(X_s - Y_s(\alpha)) d\alpha ds \qquad (NSDE(\sigma, b))$$

*then, for any $t > 0$ the distribution $P_t$ of $X_t$ is absolutely continuous with respect the Lebesgue measure.*

**Corollary 15** *Let $P_0$ be a probability measure such that $\int |x|^2 P_0(dx) < \infty$. Let $\sigma$ and $b$ be the coefficients of the Landau equation defined respectively by (7), (2) and (4). We assume that $\sigma$ and $b$ are Lipschitz continuous of class $\mathcal{C}^1$. If $P_0$ is not a Dirac measure, there exists a unique weak function-solution of the Landau equation with initial data $P_0$.*

**Proof.** (Corollary 15)
Let $X_0$ be a random vector with distribution $P_0$ and $X$ be a solution of $(NSDE(\sigma, b))$ with initial data $X_0$. If we denote by $f_t$ the density of the distribution of $X_t$, then, using Itô's Formula, the function $f$, defined by $f(x,t) = f_t(x)$ for $t > 0$, is a weak function solution of the Landau equation (5) with initial data $P_0$.
The uniqueness is given by Corollary 7. ■

**Remark 16** *Without any restriction, we can assume that* $\mathbf{E}[\mathbf{X_0}] = \mathbf{0}$ *to simplify the computations.*

**Proof.** (Remark 16)
By conservation of momentum, if we define for any $t \geq 0$, $X'_t = X_t - E[X_0]$, the expectation of $X'$ is equal to 0 and $X'$ satisfies the following equation

$$X'_t = X'_0 + \int_0^t \int_0^1 \sigma(X'_s - Y'_s(\alpha)) .W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b(X'_s - Y'_s(\alpha)) d\alpha ds$$

with $Y'_s(\alpha) = Y_s(\alpha) - E[X_0]$.
As $\mathcal{L}(X) = \mathcal{L}_\alpha(Y)$, we also have $\mathcal{L}(X') = \mathcal{L}_\alpha(Y')$.
If we prove that the distribution of $X'_t$ has a density $f'_t$ with respect the Lebesgue measure, then $X_t$ has a density given by $f_t(z) = f'_t(z - E[X_0])$. ■

**Proof.** (Theorem 14)
We recall the expression of the first Malliavin derivative at point $(r,z) \in [0,\infty) \times [0,1]$ of $X$:

$$\begin{aligned} D^l_{(r,z)} X_{i,t} &= \sigma_{i,l}(X_r - Y_r(z)) \\ &\quad + \int_r^t \int_0^1 \sum_{k=1}^{d'} \sum_{m=1}^{d} \partial_m \sigma_{i,k}(X_s - Y_s(\alpha)) D^l_{(r,z)} X_{m,s} W_k(d\alpha, ds) \\ &\quad + \int_r^t \int_0^1 \sum_{m=1}^{d} \partial_m b_i(X_s - Y_s(\alpha)) D^l_{(r,z)} X_{m,s} d\alpha ds \end{aligned}$$

if $t \geq r$ and $D^l_{(r,z)} X_{i,t} = 0$ else.
We fix $(r,z) \in [0,\infty) \times [0,1]$ and we define

$$\begin{aligned} S_k(.) &= (\partial_m \sigma_{i,k}(.))_{1 \leq i,m \leq d} \\ B(.) &= (\partial_m b_i(.))_{1 \leq i,m \leq d} \end{aligned}$$

Thus we give a matricial expression of the derivative of $X$

$$D_{(r,z)}X_t \underset{\text{if } t\geq r}{=} \sigma\left(X_r - Y_r(z)\right) + \int_r^t \int_0^1 \sum_{k=1}^{d'} S_k\left(X_s - Y_s(\alpha)\right).D_{(r,z)}X_s W_k(d\alpha, ds)$$
$$+ \int_r^t \int_0^1 B\left(X_s - Y_s(\alpha)\right).D_{(r,z)}X_s d\alpha ds$$
$$D_{(r,z)}X_t \underset{\text{if } t<r}{=} 0$$

Let us define the semimartingale $Z^r$, for any $t \geq r$,

$$Z_t^r = \int_r^t \int_0^1 \sum_{k=1}^{d'} S_k\left(X_s - Y_s(\alpha)\right) W_k(d\alpha, ds) + \int_r^t \int_0^1 B\left(X_s - Y_s(\alpha)\right) d\alpha ds$$

As $S$ and $B$ are bounded, $(Z_t^r)_{t\geq r}$ is a continuous semimartingale and the first derivative satisfies the equation

$$D_{(r,z)}X_t = \sigma\left(X_r - Y_r(z)\right) + \int_r^t dZ_s^r.D_{(r,z)}X_s \tag{18}$$

Using the results of [4], there is a unique solution of (18) defined almost surely, for any $t \geq r$ by

$$D_{(r,z)}X_t = \mathcal{E}(Z)_t^r.\sigma\left(X_r - Y_r(z)\right)$$

with $\mathcal{E}(Z)_t^r$ invertible for any $t \geq r$.

We fix now $t > 0$.
We want to apply Theorem $(a)$, thus study if the Malliavin matrix $I_t$ is invertible a.s..

$$\begin{aligned}
I_t &= \int_0^\infty \int_0^1 D_{(r,z)}X_t.\left(D_{(r,z)}X_t\right)^* drdz \\
&= \int_0^t \int_0^1 D_{(r,z)}X_t.\left(D_{(r,z)}X_t\right)^* drdz \\
&= \int_0^t \int_0^1 \mathcal{E}(Z)_t^r.\sigma\left(X_r - Y_r(z)\right).\sigma^*\left(X_r - Y_r(z)\right).\left(\mathcal{E}(Z)_t^r\right)^* drdz \\
&= \int_0^t \mathcal{E}(Z)_t^r.\left(\int_0^1 a\left(X_r - Y_r(z)\right)dz\right).\left(\mathcal{E}(Z)_t^r\right)^* dr
\end{aligned}$$

$I_t$ is a nonnegative symmetric matrix, then $I_t$ is invertible if and only if $V^*.I_t.V > 0$ for any $V \in \mathbb{R}^d \setminus \{0\}$.
We define $\Gamma_r = \int_0^1 a\left(X_r - Y_r(z)\right)dz$.

We prove the theorem by contradiction.
**Assumption** : let us suppose that $I_t$ is not "invertible a.s.".
Then, there exists a subset $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) > 0$, such that $\forall\omega \in \Omega_1$ $I_t(\omega)$ is not invertible.
Let $\Omega_2$ be such that $\mathbb{P}(\Omega_2) = 1$ and $\forall\omega \in \Omega_2$ $\forall r \leq t$, $\mathcal{E}(Z)_t^r(\omega)$ is invertible.
We define $\Omega_0 = \Omega_1 \cap \Omega_2$, and we notice that $\mathbb{P}(\Omega_0) > 0$.
We fix $\omega \in \Omega_0$.
As $I_t(\omega)$ is not invertible, there exists a vector $V_\omega \in \mathbb{R}^d \setminus \{0\}$ such that

$$\begin{aligned}
V_\omega^*.I_t(\omega).V_\omega &= \int_0^t V_\omega^*.\mathcal{E}(Z)_t^r(\omega).\Gamma_r(\omega).\left(\mathcal{E}(Z)_t^r(\omega)\right)^*.V_\omega dr \\
&= 0
\end{aligned}$$

As for any $r \leq t$, $\mathcal{E}(Z)_t^r(\omega).\Gamma_r(\omega).(\mathcal{E}(Z)_t^r(\omega))^*$ is a nonnegative symmetric matrix, we notice that

$$V_\omega^*.\mathcal{E}(Z)_t^r(\omega).\Gamma_r(\omega).(\mathcal{E}(Z)_t^r(\omega))^*.V_\omega \geq 0$$

Then, on a subset $J_\omega$ of full measure in $[0,t]$, $V_\omega^*.\mathcal{E}(Z)_t^r(\omega).\Gamma_r(\omega).(\mathcal{E}(Z)_t^r(\omega))^*.V_\omega = 0$. This implies that $\forall r \in J_\omega$, $\mathcal{E}(Z)_t^r(\omega).\Gamma_r(\omega).(\mathcal{E}(Z)_t^r(\omega))^*$ is a non invertible matrix. However, since $\Omega_0 \subset \Omega_2$, $\mathcal{E}(Z)_t^r(\omega)$ is invertible for any $r \leq t$, and consequently $\Gamma_r(\omega)$ is not invertible for $r \in J_\omega$.

Let us now study if the situation "$\Gamma_r(\omega)$ non invertible for almost all $r$" is possible.

Using Lebesgue's Theorem, we notice that the mapping $r \to \Gamma_r(\omega)$ is continuous. Consequently, $\Gamma_r(\omega)$ non invertible for almost all $r$ implies that $\Gamma_r(\omega)$ is non invertible for any $r \in [0,t]$.

Let $V = (V_i)_{1\leq i\leq d}$ be a vector in $\mathbb{R}^d \setminus \{0\}$,

$$\begin{aligned} V^*.\Gamma_r(\omega).V &= \int_0^1 h\left(|X_r(\omega) - Y_r(z)|^2\right)\left[|V|^2|X_r(\omega) - Y_r(z)|^2 - \left(\sum_{i=1}^d V_i\left(X_{i,r}(\omega) - Y_{i,r}(z)\right)\right)^2\right]dz \\ &\geq m\left[|V|^2|X_r(\omega)|^2 - \left(\sum_{i=1}^d V_i X_{i,r}(\omega)\right)^2 + |V|^2 E|X_r|^2 - \sum_{i=1}^d V_i^2 E\left[X_{i,r}^2\right] + \sum_{i\neq j} V_i V_j E\left[X_{i,r}X_{j,r}\right]\right] \\ &\quad \text{using the lower bound (3) of } h,\ E[X_t] = 0 \text{ and } \mathcal{L}(X_t) = \mathcal{L}_\alpha(Y_t)\ \forall t \geq 0 \end{aligned}$$

Using Cauchy-Schwarz's inequality, we notice that $|V|^2|X_r(\omega)|^2 - \left(\sum_{i=1}^d V_i X_{i,r}(\omega)\right)^2 \geq 0$. Then,

$$V^*.\Gamma_r(\omega).V \geq m\left(E\left[|X_r|^2\right]|V|^2 - \sum_{i=1}^d V_i^2 E\left[X_{i,r}^2\right] - \sum_{i\neq j} V_i V_j E\left[X_{i,r}X_{j,r}\right]\right) \tag{19}$$

$\Gamma_r(\omega)$ non invertible means that for any $r \in [0,t]$ there exists $V_r(\omega) \in \mathbb{R}^d \setminus \{0\}$ such that $V_r(\omega)^*.\Gamma_r(\omega).V_r(\omega) = 0$. Nevertheless, using expression (19), this implies that there is equality in Cauchy-Schwarz, i.e. for any $i, j$ such that $V_{i,r}(\omega) \neq 0$ and $V_{j,r}(\omega) \neq 0$

$$E\left[X_{i,r}X_{j,r}\right]^2 = E\left[X_{i,r}^2\right]E\left[X_{j,r}^2\right] \tag{20}$$

We notice that the equality in Cauchy-Schwarz (20) is in fact satisfied for any $i, j \in \{1, ..., d\}$. Indeed, if one of the components of $V_r(\omega)$ is equal to 0, for example $V_{1,r}(\omega) = 0$, then

$$\begin{aligned} V_r(\omega)^*.\Gamma_r(\omega).V_r(\omega) &\geq m\left[E\left[|X_r|^2\right]|V_r(\omega)|^2 - \left(\sum_{i=1}^d |V_{i,r}(\omega)|\sqrt{E\left[X_{i,r}^2\right]}\right)^2\right] \\ &= m\left[E\left[(X_{1,r})^2\right]|V_r(\omega)|^2 + |V_r(\omega)|^2\sum_{i=2}^d E\left[(X_{i,r})^2\right] - \left(\sum_{i=2}^d |V_{i,r}(\omega)|\sqrt{E\left[X_{i,r}^2\right]}\right)^2\right] \\ &\geq mE\left[(X_{1,r})^2\right]|V_r(\omega)|^2 \end{aligned}$$

As $V_r(\omega)^*.\Gamma_r(\omega).V_r(\omega) = 0$ and $|V_r(\omega)| \neq 0$, $E\left[(X_{1,r})^2\right] = 0$. Consequently, for any $j$, we still have $E\left[X_{1,r}X_{j,r}\right]^2 = E\left[X_{1,r}^2\right]E\left[X_{j,r}^2\right]$.

Finally, for any $r \in [0,t]$, there exists a vector $\lambda_r \in \mathbb{R}^d$ and a real-valued random variable $U_r$ such that, for any $\omega \in \Omega$,

$$X_r(\omega) = U_r(\omega)\lambda_r \tag{21}$$

As $E\left[|X_0|^2\right] \neq 0$ *(the distribution of $X_0$ is not a Dirac mass)*, using the conservation of energy, $X_r \neq 0$ a.s., then $\lambda_r \neq 0$ and $U_r \neq 0$ a.s.. We can suppose $|\lambda_r| = 1$ for any $r \geq 0$. Then, by conservation of momentum and energy, we notice that $E[U_r] = 0$ and $E\left[{U_r}^2\right] = E\left[|X_0|^2\right]$.

The distribution of a solution of $(NSDE(\sigma, b))$ is a measure-solution of the Landau equation (6). Then, we will now study if the distribution of a process defined by (21) can be a solution of the Landau equation. We denote by $Q$ the distribution of $U$ and by $Q_t$ the distribution on $\mathbb{R}$ of $U_t$. Using (21), the equation (6) writes

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \varphi(\lambda_t x)\, Q_t(dx) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}\times\mathbb{R}} a_{ij}(\lambda_t(x-y))\,\partial_{ij}\varphi(\lambda_t x)\, Q_t(dx)\, Q_t(dy) \\ &\quad + \sum_{i=1}^{d}\int_{\mathbb{R}\times\mathbb{R}} b_i(\lambda_t(x-y))\,\partial_i\varphi(\lambda_t x)\, Q_t(dx)\, Q_t(dy)\end{aligned}$$

for any test function $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$.

As $|\lambda_t| = 1$, for $i, j \in \{1, .., d\}$

$$\begin{aligned}a_{ij}(\lambda_t(x-y)) &= (x-y)^2\, h\left((x-y)^2\right)(\delta_{ij} - \lambda_{i,t}\lambda_{j,t}) \\ b_i(\lambda_t(x-y)) &= -(d-1)(x-y)\, h\left((x-y)^2\right)\lambda_{i,t}\end{aligned}$$

Then,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \varphi(\lambda_t x)\, Q_t(dx) &= \frac{1}{2}\sum_{i,j=1}^{d}(\delta_{ij} - \lambda_{i,t}\lambda_{j,t})\int_{\mathbb{R}\times\mathbb{R}} (x-y)^2\, h\left((x-y)^2\right)\partial_{ij}\varphi(\lambda_t x)\, Q_t(dx)\, Q_t(dy) \\ &\quad - (d-1)\sum_{i=1}^{d}\lambda_{i,t}\int_{\mathbb{R}\times\mathbb{R}} (x-y)\, h\left((x-y)^2\right)\partial_i\varphi(\lambda_t x)\, Q_t(dx)\, Q_t(dy)\end{aligned}$$

We now explicit the equation satisfied by the 2-order moments of $X$: let $k, l \in \mathbb{N}$, $k \neq l$. Using $\varphi(v) = v_k^2$ or $\varphi(v) = v_k v_l$, $v \in \mathbb{R}^d$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \lambda_{k,t}^2 x^2 Q_t(dx) &= E\left[|X_0|^2\right]\frac{d}{dt}\lambda_{k,t}^2 \\ &= \left(1 - d\lambda_{k,t}^2\right)\int_{\mathbb{R}\times\mathbb{R}} (x-y)^2\, h\left((x-y)^2\right) Q_t(dx)\, Q_t(dy) \\ \frac{d}{dt}\int_{\mathbb{R}} \lambda_{k,t}\lambda_{l,t} x^2 Q_t(dx) &= E\left[|X_0|^2\right]\frac{d}{dt}\lambda_{k,t}\lambda_{l,t} \\ &= -d\lambda_{k,t}\lambda_{l,t}\int_{\mathbb{R}\times\mathbb{R}} (x-y)^2\, h\left((x-y)^2\right) Q_t(dx)\, Q_t(dy)\end{aligned}$$

Let us define $f(t) = \int_{\mathbb{R}\times\mathbb{R}} (x-y)^2\, h\left((x-y)^2\right) Q_t(dx)\, Q_t(dy)$. As $h$ satisfies (3) and $E\left[|X_0|^2\right] \neq 0$, for any $t \geq 0$, we notice that $f(t) > 0$.

Let us now compute $\frac{d}{dt}\left(\lambda_{k,t}^2\lambda_{l,t}^2\right)$, using two different ways :

$$\begin{aligned}E\left[|X_0|^2\right]\frac{d}{dt}\left(\lambda_{k,t}^2\lambda_{l,t}^2\right) &= \lambda_{k,t}^2 E\left[|X_0|^2\right]\frac{d}{dt}\lambda_{l,t}^2 + \lambda_{l,t}^2 E\left[|X_0|^2\right]\frac{d}{dt}\lambda_{k,t}^2 \\ &= \lambda_{k,t}^2 f(t) + \lambda_{l,t}^2 f(t) - 2d\lambda_{k,t}^2\lambda_{l,t}^2 f(t) \\ E\left[|X_0|^2\right]\frac{d}{dt}\left(\lambda_{k,t}^2\lambda_{l,t}^2\right) &= 2\lambda_{k,t}\lambda_{l,t} E\left[|X_0|^2\right]\frac{d}{dt}(\lambda_{k,t}\lambda_{l,t}) \\ &= -2d\lambda_{k,t}^2\lambda_{l,t}^2 f(t)\end{aligned}$$

Then $\lambda_t = 0$, consequently $X_t = 0$ which is impossible.

Finally, $I_t$ is invertible a.s. for any $t > 0$ and according to theorem $(a)$, the theorem is proved. ■

**Remark 17** *We notice that the matrix* $\Gamma_r = \int_0^1 a(X_r - Y_r(z))\,dz$ *is invertible a.s., whereas* $\det(a(X_r - Y_r(z))) = 0$ *for any* $r, z$*. In fact, thanks to the nonlinearity of equation* $(NSDE(\sigma, b))$*, we can conclude that the Malliavin matrix has an inverse a.s..*

**Remark 18** *A consequence of Theorem 14 is, for any* $i, j \in \{1, .., d\}$*,* $i \neq j$*, for any* $t > 0$

$$E[X_{i,t}X_{j,t}]^2 < E[X_{i,t}^2]\,E[X_{j,t}^2]$$

*if the random vector* $X_0$ *is not a constant.*

# 4 Regularity of the weak function-solution

**Theorem 19** *Let* $X_0$ *be a random vector such that* $E[|X_0|^p] < \infty$ *for any* $p \geq 1$*. Let* $\sigma$ *and* $b$ *be the coefficients of the Landau equation respectively defined by (7), (2) and (4). We assume that* $\sigma$ *and* $b$ *are Lipschitz continuous and infinitely differentiable with bounded derivatives. If the distribution of* $X_0$ *is not a Dirac mass and if we denote by* $X$ *the solution of the nonlinear stochastic differential equation*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma(X_s - Y_s(\alpha)).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b(X_s - Y_s(\alpha))\,d\alpha ds \qquad (NSDE(\sigma, b))$$

*then for any* $t > 0$ *the distribution of* $X_t$ *has a density of class* $\mathcal{C}^\infty$ *with respect the Lebesgue measure on* $\mathbb{R}^d$*.*

**Corollary 20** *Let* $P_0$ *be a probability measure such that* $\int |x|^p P_0(dx) < \infty$ *for any* $p \geq 1$*. Let* $\sigma$ *and* $b$ *be the coefficients of the Landau equation defined respectively by (7), (2) and (4). We assume that* $\sigma$ *and* $b$ *are Lipschitz continuous and infinitely differentiable with bounded derivatives. If* $P_0$ *is not a Dirac measure, there exists a unique weak function solution of the Landau equation of class* $\mathcal{C}^\infty$ *with initial data* $P_0$*.*

**Remark 21** *Using the expressions (10) or (11), we notice that, if* $h$ *is a bounded function of class* $\mathcal{C}^\infty$ *such that* $h^{(l)}(x) = O\left(\frac{1}{x^{l+1}}\right)$ *when* $x \to +\infty$ *for any* $l \geq 1$*,* $\sigma$ *and* $b$ *are Lipschitz continuous functions of class* $\mathcal{C}^\infty$ *with bounded derivatives.*

**Proof.** As in the previous part, we assume that $E[X_0] = 0$ to simplify the computations. As $\sigma$ and $b$ satisfy Assumption $(H^\infty)$, the process $X$ is infinitely differentiable in the Malliavin sense. We need to study the moments of the inverse of the determinant of the Malliavin matrix $I_t$ at time $t$, for any $t > 0$, to apply Theorem $(b)$. The expression of the determinant is complex, nevertheless we can notice that in dimension $d$,

$$(\det I_t)^{1/d} \geq \inf_{|V|=1} \langle I_t V, V \rangle$$

where $\langle ., . \rangle$ is the euclidean scalar product in $\mathbb{R}^d$.

Moreover, see P. L. Morien [5] lemma 10.5.1, the property $(ii)$ of theorem $(b)$ is satisfied as soon as for any $k \in \mathbb{N}$

$$\lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{P}\left((\det I_t)^{1/d} < c\varepsilon\right) = 0 \qquad (22)$$

where $c$ is a positive constant which will be computed later. Indeed, let us fix $p \geq 1$ and $k \in \mathbb{N}$ such that $k > d\,(p+1)$. If (22) is satisfied, there exists $\varepsilon_0 > 0$ such that $\varepsilon^{-k}\mathbb{P}\left((\det I_t)^{1/d} < c\varepsilon\right) < 1$ for any $\varepsilon < \varepsilon_0$. Let $m_0 \in \mathbb{N}$ such that $\forall m > m_0$, $\frac{1}{m} < (c\varepsilon_0)^d$, then

$$
\begin{aligned}
E\left[\frac{1}{(\det I_t)^p}\right] &\leq 1 + \sum_{m=1}^{\infty} (m+1)^p\, \mathbb{P}\left(m \leq \frac{1}{\det I_t} \leq m+1\right) \\
&\leq 1 + 2^p \sum_{m=1}^{\infty} m^p \mathbb{P}\left(\det I_t \leq \frac{1}{m}\right) \\
&\leq 1 + 2^p \sum_{m=1}^{m_0} m^p \mathbb{P}\left(\det I_t \leq \frac{1}{m}\right) + 2^p c^{-k} \sum_{m=m_0+1}^{\infty} \frac{1}{m^{\frac{k}{d}-p}}
\end{aligned}
$$

since $\frac{k}{d} - p > 1$, we obtain $E\left[\frac{1}{(\det I_t)^p}\right] < \infty$.

Let $t > 0$ be fixed.

As $(\det I_t)^{\frac{1}{d}} \geq \inf_{|V|=1}\langle I_t V, V\rangle$, we want to find a lower bound for $\inf_{|V|=1}\langle I_t V, V\rangle$.

Let $\varepsilon$ be such that $0 < \varepsilon < \frac{t}{2}$. We consider $V = (V_i)_{1\leq i\leq d} \in \mathbb{R}^d$ such that $|V| = 1$.

$$
\begin{aligned}
\langle I_t V, V\rangle &= \int_0^t \int_0^1 \sum_{l=1}^{d'} \left(\sum_{i=1}^{d} D^l_{r,z} X_{i,t} V_i\right)^2 dz dr \\
&\geq \int_{t-\varepsilon}^t \int_0^1 \sum_{l=1}^{d'} \left(\sum_{i=1}^{d} D^l_{r,z} X_{i,t} V_i\right)^2 dz dr \\
&\geq \frac{2}{3} I_1 - 2 I_2
\end{aligned}
$$

with

$$
\begin{aligned}
I_1 &= \int_{t-\varepsilon}^t \int_0^1 \sum_{l=1}^{d'} \left(\sum_{i=1}^{d} \sigma_{i,l}\left(X_r - Y_r(z)\right) V_i\right)^2 dz dr \\
I_2 &= \int_{t-\varepsilon}^t \int_0^1 \sum_{l=1}^{d'} \Bigg[\sum_{i=1}^{d} V_i \int_r^t \int_0^1 \sum_{k=1}^{d'} \sum_{m=1}^{d} \partial_m \sigma_{i,k}\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)} X_{m,s} W_k(d\alpha, ds) \\
&\quad + \sum_{i=1}^{d} V_i \int_r^t \int_0^1 \sum_{m=1}^{d} \partial_m b_i\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)} X_{m,s} d\alpha ds\Bigg]^2 dz dr
\end{aligned}
$$

Then

$$
\inf_{|V|=1}\langle I_t V, V\rangle \geq \frac{2}{3}\inf_{|V|=1} I_1 - 2\sup_{|V|=1} I_2
$$

We want to minimize the first integral:

$$
\begin{aligned}
I_1 &= \int_{t-\varepsilon}^{t}\int_0^1 \sum_{l=1}^{d'}\left(\sum_{i=1}^{d}\sigma_{i,l}\left(X_r - Y_r\left(z\right)\right)V_i\right)^2 dzdr \\
&= \int_{t-\varepsilon}^{t}\int_0^1 \sum_{i,j=1}^{d}\sum_{l=1}^{d'}\sigma_{i,l}\left(X_r - Y_r\left(z\right)\right)\sigma_{j,l}\left(X_r - Y_r\left(z\right)\right)V_iV_jdzdr \\
&= \int_{t-\varepsilon}^{t}\int_0^1 \sum_{i,j=1}^{d} a_{i,j}\left(X_r - Y_r\left(z\right)\right)V_iV_jdzdr \\
&= \int_{t-\varepsilon}^{t} V^*.\Gamma_r.V dr
\end{aligned}
$$

Using the results of Section 3, we obtain

$$
I_1 \geq m\int_{t-\varepsilon}^{t}\left(E\left[\left|X_r\right|^2\right]\left|V\right|^2 - \sum_{i=1}^{d}V_i^2E\left[X_{i,r}^2\right] - \sum_{i\neq j}V_iV_jE\left[X_{i,r}X_{j,r}\right]\right)dr
$$

We define the function

$$
f\left(V,r\right) = E\left[\left|X_r\right|^2\right]\left|V\right|^2 - \sum_{i=1}^{d}V_i^2E\left[X_{i,r}^2\right] - \sum_{i\neq j}V_iV_jE\left[X_{i,r}X_{j,r}\right]
$$

We notice that $f$ is a positive continuous function (see Remark 18) on the compact subset $D = \left\{V \in \mathbb{R}^d : |V| = 1\right\} \cup \left\{r : \frac{t}{2} \leq r \leq t\right\}$, then $f$ reaches its minimum. So, if we denote by

$$
\tilde{c} = \inf\left\{f\left(V,r\right) : |V| = 1 \text{ and } \frac{t}{2} \leq r \leq t\right\}
$$

we notice that $\tilde{c}$ is independent of $\omega \in \Omega$, $\tilde{c} > 0$ and

$$
I_1 \geq m.\tilde{c}.\varepsilon
$$

Let us now study $E\left[\sup_{|V|=1} I_2^p\right]$ for $p \geq 1$.

$$
\begin{aligned}
I_2 &= \int_{t-\varepsilon}^{t}\int_0^1 \sum_{l=1}^{d'}\Bigg[\sum_{i=1}^{d}V_i\int_r^t\int_0^1\sum_{k=1}^{d'}\sum_{m=1}^{d}\partial_m\sigma_{i,k}\left(X_s - Y_s\left(\alpha\right)\right)D_{(r,z)}^{l}X_{m,s}W_k\left(d\alpha,ds\right) \\
&\quad + \sum_{i=1}^{d}V_i\int_r^t\int_0^1\sum_{m=1}^{d}\partial_m b_i\left(X_s - Y_s\left(\alpha\right)\right)D_{(r,z)}^{l}X_{m,s}d\alpha ds\Bigg]^2 dzdr
\end{aligned}
$$

Using Burkholder-Davis-Gundy's and Hölder's inequalities, and the fact that $|V|=1$, we notice that

$$
\begin{aligned}
E\left[\sup_{|V|=1} I_2^p\right] &\leq K\varepsilon^{p-1}\left\{\int_{t-\varepsilon}^t\int_0^1\sum_{l,i,m}E\left[\left|\sum_{k=1}^{d'}\int_r^t\int_0^1\partial_m\sigma_{i,k}\left(X_s-Y_s\left(\alpha\right)\right)D^l_{(r,z)}X_{m,s}W_k\left(d\alpha,ds\right)\right|^{2p}\right]dzdr\right.\\
&\quad\left.+\int_{t-\varepsilon}^t\int_0^1\sum_{l,i,m}E\left[\left|\int_r^t\partial_m b_i\left(X_s-Y_s\left(\alpha\right)\right)D^l_{(r,z)}X_{m,s}d\alpha ds\right|^{2p}\right]dzdr\right\}\\
&\leq K\varepsilon^{p-1}\left\{\int_{t-\varepsilon}^t\int_0^1\sum_{l,i,k,m}E\left[\left|\sum_{k=1}^{d'}\int_r^t\int_0^1\left(\partial_m\sigma_{i,k}\left(X_s-Y_s\left(\alpha\right)\right)D^l_{(r,z)}X_{m,s}\right)^2d\alpha ds\right|^{p}\right]dzdr\right.\\
&\quad\left.+\int_{t-\varepsilon}^t\int_0^1\varepsilon^{2p-1}\sum_{l,i,m}E\left[\left|\int_r^t\partial_m b_i\left(X_s-Y_s\left(\alpha\right)\right)D^l_{(r,z)}X_{m,s}d\alpha ds\right|^{2p}\right]dzdr\right\}
\end{aligned}
$$

As the derivatives of $\sigma$ and $b$ are bounded, using Hölder's inequality, we obtain

$$
\begin{aligned}
E\left[\sup_{|V|=1} I_2^p\right] &\leq K\varepsilon^{2p-2}\left\{\int_{t-\varepsilon}^t\int_0^1\sum_l E\left[\int_r^t\left|D^l_{(r,z)}X_s\right|^{2p}ds\right]dzdr\right\}\\
&\quad\text{using Fubini's Theorem}\\
&\leq K\varepsilon^{2p-2}\int_r^t E\left[\int_{t-\varepsilon}^s\int_0^1\left|D_{(r,z)}X_s\right|^{2p}dzdr\right]ds
\end{aligned}
$$

Then, for any $p\geq 2$ there exists a constant $K=K\left(p,d,d',t\right)$ such that

$$
E\left[\sup_{|V|=1} I_2^p\right]\leq K\varepsilon^{2p-1}\sup_{0\leq s\leq t}E\left[\int_0^s\int_0^1\left|D_{(r,z)}X_s\right|^{2p}dzdr\right]
$$

Let us now check that $\lim_{\varepsilon\to 0}\varepsilon^{-k}\mathbb{P}\left(\left(\det I_t\right)^{\frac{1}{d}}<c\varepsilon\right)=0$, where $k\in\mathbb{N}$ is fixed and $c=\frac{1}{3}m\tilde{c}$ with $\tilde{c}$ the constant built in the study of the first integral $I_1$.

Let $p\in\mathbb{N}$ such that $p>k+1$.

$$
\begin{aligned}
\mathbb{P}\left(\left(\det I_t\right)^{\frac{1}{d}}<c\varepsilon\right) &\leq \mathbb{P}\left(\inf_{|V|=1}\langle I_tV,V\rangle<c\varepsilon\right)\\
&\leq \mathbb{P}\left(\frac{2}{3}\inf_{|V|=1}I_1-2\sup_{|V|=1}I_2<c\varepsilon\right)\\
&\leq \mathbb{P}\left(\sup_{|V|=1}I_2>\frac{c\varepsilon}{2}\right)\\
&\quad\text{using Tchebychev's Inequality}\\
&\leq \left(\frac{2}{c}\right)^p\varepsilon^{-p}E\left[\sup_{|V|=1}I_2^p\right]\\
&\leq K\varepsilon^{p-1}
\end{aligned}
$$

Then, $\lim_{\varepsilon\to 0}\varepsilon^{-k}\mathbb{P}\left(\left(\det I_t\right)^{\frac{1}{d}}<c\varepsilon\right)=0$ and we can apply Theorem ($b$). So the theorem is proved. ■

# References


[1] V. Bally, E. Pardoux, *Malliavin calculus for white noise driven parabolic SPDEs*, Potential Analysis **9** : 27-64 (1998)

[2] A. G. Bhatt, R. L. Karandikar, *Invariant measures and evolution equations for Markov processes characterized via martingale problems*, The Ann. of Prob **21** : 2246-2268 (1993)

[3] H. Guérin, *Solving Landau equation for some soft potentials with a probabilistic approach*, Prépublication de l'Univ. Paris 10 **00/12** (2000)

[4] J. Jacod, *Equations différentielles stochastiques linéaires : la méthode de variation des constantes*, Séminaire de Proba. XVI, Lecture Notes in Math. **920** : 442-448 (1982)

[5] P. L. Morien, *Autour des équations aux dérivées partielles stochastiques: systèmes en interaction et problème non-linéaire, régularité et approximation des densités*, Thèse de Doctorat de l'Univ. Paris 6 (1996)

[6] D. Nualart, *The Malliavin calculus and related topics*, Springer-Verlag (1995)

[7] C. Villani, *On the spatially homogeneous Landau equation for Maxwellian Molecules*, Math. Mod. Meth. Appl. Sci. **8** : 957-983 (1998)

[8] J. B. Walsh, *An introduction to the stochastic partial differential equations*, Ecole d'été de Proba. de Saint Flour XIV, Lecture Notes in Math. **1180** : 265-437 (1984)