# Convergence from Boltzmann to Landau processes with soft potential and particle approximations

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### Abstract

In the present paper, we firstly extend the probabilistic interpretation of spatially homogeneous Boltzmann equations without angular cutoff due firstly to Tanaka and generalized by Fournier-Méléard, to some soft potential cases for a large class of initial data. We relate a measure solution of the Boltzmann equation to the solution of a Poisson-driven stochastic differential equation. Then we consider renormalized such equations which make prevail the grazing collisions, and we prove the convergence of the associated Boltzmann processes to a process related to the Landau equation initially introduced by Guérin. The convergence is pathwise and also implies a convergence at the level of the partial differential equations. An approximation of a solution of the Landau equation with soft potential via colliding stochastic particle systems is derived from this result. We then deduce a Monte-Carlo algorithm of simulation by a conservative particle method following the asymptotics of grazing collisions. Numerical results are given.

*Key words* : Boltzmann equations without cutoff and soft potential, Landau equation with soft potential, Nonlinear stochastic differential equations, Interacting particle systems, Monte-Carlo algorithm.

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## 1 Introduction.

The spatially homogeneous Boltzmann equation deals with the distribution of the velocities $P_t(dv)$ at the instant $t$, in a rarefied gas. In the Maxwell molecules case and with $L^1$-hypotheses, Tanaka [24] built a process $V_t$, which can be seen as the velocity of the "mean particle", of which the law is given by $P_t(dv)$. Horowitz, Karandikar [18] generalized this approach to the $L^2$-case. This probabilistic representation has proved very usefull. Firstly, it did allow to extend the work of Graham, Méléard, [14] concerning numerical Monte-Carlo methods for Boltzmann equations with cutoff, to the case of Boltzmann equations without cutoff, see Desvillettes, Graham, Méléard, [7], and Fournier, Méléard, [12]. Secondly, the

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use of recent tools of stochastic analysis did allow to prove, via Tanaka's representation, the existence of positive smooth solutions to the Boltzmann equation, at least in dimension two, hence generalizing the analytical results, see the works of Graham, Méléard, [15], Fournier [9].

In 1936, Landau, [20], derived from the Boltzmann equation a new equation called the Fokker-Planck-Landau equation, usually considered as an approximation of the homogeneous Boltzmann equation in the limit of grazing collisions. Many authors have been interested in proving rigourously this convergence, in different cases of scaterring cross-section and initial data.

Firstly Arsen'ev and Buryak [1] proved the convergence of solutions of the Bolzmann equation towards solutions of the Landau equation under very restrictive assumptions. Then, Desvillettes [5] gave a mathematical framework for more physical situations, but excluding the main case of Coulomb potential which has been studied by Degond and Lucquin, [4]. Indeed, the Boltzmann equation is not realistic for Coulomb molecules (see [26]) and the Landau equation appears naturally. More recently, Goudon [13] and Villani [26] proved the existence of a solution of the Landau equation for soft potentials using the convergence of the Bolzmann equation toward the Landau equation, with a bounded entropy and energy function as initial data. All the techniques used until now are analytical techniques using convergence theorems or spectral analysis, and asking at least for a bounded entropy and energy initial condition.

In the grazing collision asymptotics, the cross-section in the Boltzmann operator is a function of a small parameter depending on the nature of the collisions. In Buet-Cordier-Lucquin [3], these asymptotics are described for the Coulombian case and the non-Coulombian one.

In this paper, we extend Tanaka's probabilistic interpretation to a case of soft potential spatially homogeneous Boltzmann equation without angular cutoff. We state a nonlinear stochastic differential equation of Poisson type, related to the Boltzmann equation with soft potential. Using the usual tools of the convergence in law on the set of càdlàg functions, we prove the existence of a solution to this stochastic differential equation called a Boltzmann process. As a corollary, we obtain a new result of existence of weak solutions of Boltzmann equations for every fourth-order moment probability measure initial data.

Then we consider renormalized such Boltzmann processes which make prevail the grazing collisions. Our main result in this paper consists in proving the convergence in law, in the Skorohod space, of sequences of such processes to a Landau process for a large class of initial data. The Landau process, introduced by Guérin [16], is related to the Fokker-Planck-Landau equation with soft potential, and can also be obtained as a solution of a nonlinear stochastic differential equation driven by a space-time white noise. We immediatly deduce a convergence result at the level of the partial differential equations for general initial data. The asymptotics of grazing collisions we consider include those of Degond and Lucquin, [4], and Desvillettes, [5]. Unhappily, the probabilistic tools oblige us to a $L^2$-framework, which necessitates to consider the potential $\gamma \in (-1, 0]$. In particular, our theorical approach do not recover the interesting coulombian case, even if our numerical approximations work in the same way.

As in the analytical framework, uniqueness is an open problem for all the equations we consider. All the convergence results we prove are obtained by a compactness method and we only obtain existence of converging subsequences.

The pathwise interpretation of the equations (in the probabilistic framework) provides a natural approximation by interacting particle systems. We build in Section 4 some Monte-Carlo approximations of solutions of the Fokker-Planck-Landau equations, via stochastic interacting particles with grazing collision asymptotics related with the size of the particle system. We prove the convergence of the empirical laws of the particle systems to a weak solution of the Landau equation. We deduce from this theorical result a very simple simulation algorithm, based upon particles conserving momentum and kinetic energy.

We finally discuss about numerical results. At our knowledge, in one hand, no numerical resolution of the Landau equation in 3D seen as limit of grazing collisions Boltzmann equations has been developed by deterministic methods, and in another hand, the deterministic particle methods do not work for the 3D Landau equation. Some numerical Monte-Carlo algorithms for the Landau equation exist, but without convergence proofs, see Takizuka and Abe [25] and Wang, Okamoto, Nakajima, Murakami [27], and they are inspired by the diffusion structure of the Landau equation and do not follow the asymptotics of grazing collisions. An interest of our approach is to see in the simulations the convergence from the renormalized Boltzmann equations to the Landau equation (see Section 6 figure 1).

The paper is organized as follows : in Section 2, we explain the pathwise interpretation of the Boltzmann equation with soft potential, and solve the nonlinear Poisson-driven stochastic differential equation. In Section 3, we study the convergence in law of the renormalized Boltzmann processes to a Landau process and deduce the convergence of solutions of the Boltzmann equations to the ones of the Landau equation when the grazing collisions prevail. In Section 4, we study the approximating particle systems. We describe the pathwise Monte-Carlo algorithm in Section 5. Numerical results are discussed in Section 6.

**Notations**

- $I\!\!D_T$ will denote the Skorohod space $I\!\!D([0, T], I\!\!R^3)$ of càdlàg functions from $[0, T]$ into $I\!\!R^3$. The space $I\!\!D_T$ endowed with the Skorohod topology is a Polish space.
- $\mathcal{C}_T$ is the space $C([0, T], I\!\!R^3)$ of continuous functions from $[0, T]$ into $I\!\!R^3$ and $C_b^2(I\!\!R^3)$ is the space of real bounded functions of class $\mathcal{C}^2$ with bounded derivatives.
- $\mathcal{P}(I\!\!R^3)$ is the set of probability measures on $I\!\!R^3$ and $\mathcal{P}_2(I\!\!R^3)$ the subset of probability measures with a finite second order moment. Similarly, $\mathcal{P}(I\!\!D_T)$ denotes the space of probability measures on $I\!\!D_T$ and $\mathcal{P}_2(I\!\!D_T)$ is the subset of probability measures with a finite second order moment: $q \in \mathcal{P}_2(I\!\!D_T)$ if $\int_{x \in I\!\!D_T} \sup_{t \in [0, T]} |x(t)|^2 q(dx) < \infty$.
- Let $A$ and $B$ be two matrices with same dimensions. The symbol $A : B$ denotes the real $\sum_{i,j} A_{ij} B_{ij}$ and $A^t$ is the transpose matrix of the matrix $A$.
- $K$ will denote a real positive constant of which the value may change from line to line.

**General Properties**

Let us remark that in the soft potential cases, the coefficients in the equation we consider are unbounded. The study is then really harder than in the Maxwell case, developed for example in Fournier [9], Guérin [17]. The probabilistic tools oblige us to work in a $L^2$-framework and we are able to deal with moderately soft potentials, i.e. $\gamma \in (-1, 0]$, thanks to the following estimates:

**Lemma 1.1** *For each $\gamma \in (-1, 0]$, for each $z \in I\!\!R^3$,*

$$|z|^{2+\gamma} \leq |z|^2 + 1 \; ; \; |z|^{2+2\gamma} \leq |z|^2 + 1. \tag{1.1}$$

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**Proof.** One has

$$|z|^{2+\gamma} = |z|^{2+\gamma} \mathbf{1}_{|z|\geq 1} + |z|^{2+\gamma} \mathbf{1}_{|z|<1} \qquad (1.2)$$

But $\gamma \in (-1,0]$, then $2+\gamma > 0$ and $|z|^{2+\gamma} \mathbf{1}_{|z|<1} \leq 1$. Moreover, $|z|^{\gamma} \mathbf{1}_{|z|\geq 1} \leq 1$, and we have the first result. The second one is proved in a similar way. $\triangle$

## 2  The Boltzmann Process

### 2.1  The equation

The Boltzmann equation we consider describes the evolution of the density $f(t,v)$ of particles with velocity $v \in \mathbb{R}^3$ at time $t$ in a rarefied homogeneous gas:

$$\frac{\partial f}{\partial t} = Q_B(f,f), \qquad (2.1)$$

where $Q_B$ is a quadratic collision kernel preserving momentum and kinetic energy, of the form

$$Q_B(f,f)(t,v) \;=\; \int_{v_*\in\mathbb{R}^3} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \left( f(t,v')f(t,v'_*) - f(t,v)f(t,v_*) \right) B(|v-v_*|,\theta)\,d\theta\,d\varphi\,dv_* \qquad (2.2)$$

with $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$ and $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$, the unit vector $\sigma$ having colatitude $\theta$ and longitude $\varphi$ in the spherical coordinates in which $v - v_*$ is the polar axis. The nonnegative function $B$ is called the cross-section.
The unit vector $\sigma$ writes

$$\sigma = \cos\theta \frac{v-v_*}{|v-v_*|} + \sin\theta \frac{\Gamma(v-v_*,\varphi)}{|v-v_*|} \qquad (2.3)$$

where for $x \in \mathbb{R}^3$, $\varphi \in [0,2\pi[$,

$$\Gamma(x,\varphi) = \cos\varphi I(x) + \sin\varphi J(x) \qquad (2.4)$$

and $\frac{1}{|x|}(x, I(x), J(x))$ is an orthonormal basis of $\mathbb{R}^3$. One can choose, for example,

$$I(x) = \begin{cases} \dfrac{|x|}{\sqrt{x_1^2 + x_2^2}} (-x_2, x_1, 0) & \text{if } \quad x_1^2 + x_2^2 > 0 \\[2ex] (x_3, 0, 0) & \text{if } x_1^2 + x_2^2 = 0 \end{cases} \quad ; \quad J(x) \;=\; \frac{x}{|x|} \wedge I(x)$$

Then

$$v' = \frac{v+v_*}{2} + \cos\theta \frac{v-v_*}{2} + \frac{\sin\theta}{2}\Gamma(v-v_*,\varphi) \qquad (2.5)$$

and we set

$$a(v,v_*,\theta,\varphi) = v' - v = \frac{\cos\theta - 1}{2}(v-v_*) + \frac{\sin\theta}{2}\Gamma(v-v_*,\varphi). \qquad (2.6)$$

The main problem here is that the jump amplitude $a$ does not depend regularly on the velocities $v$ and $v_*$. The "almost"-Lipschitz property of $a$ is recalled in the following lemma. Its proof can be found in [24] or in its "fine" version in [12].

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**Lemma 2.1** *1. There exists a measurable function $\varphi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0, 2\pi[$, such that for all $X, Y, \varphi$,*

$$|\Gamma(X, \varphi) - \Gamma(Y, \varphi + \varphi_0(X, Y))| \leq 3|X - Y| \quad \text{(all the angles are modulo } 2\pi) \quad (2.7)$$

*2. This implies that for all $v, v_*, w, w_*$ in $\mathbb{R}^3$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$,*

$$|a(v, v_*, \theta, \varphi) - a(w, w_*, \theta, \varphi + \varphi_0(v - v_*, w - w_*))| \leq 3\theta \left( |v - w| + |v_* - w_*| \right) \quad (2.8)$$

*and in particular that for all $v, v_*, \theta, \varphi$,*

$$|a(v, v_*, \theta, \varphi)| \leq 2 \left| \sin\left(\frac{\theta}{2}\right) \right| (|v - v_*|) \quad (2.9)$$

We are interested in cases for which the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s \geq 2$. The physical cross-sections $B(z, \theta)$ tend to infinity when $\theta$ goes to zero, but satisfy $\int_0^\pi |\theta|^2 B(z, \theta) d\theta < \infty$ for each $z$. Physically, this explosion near 0 comes from the accumulation of grazing collisions.

In this general (spatially homogeneous) setting, the Boltzmann equation is difficult to study. A large literature deals with the non physical equation with angular cutoff, namely under the assumption $\int_0^\pi B(z, \theta) d\theta < \infty$. More recently, the case of Maxwell molecules, for which the cross-section $B(z, \theta) = \beta(\theta)$ only depends on $\theta$, has been studied without the cutoff assumption. In the Maxwell context, Tanaka, [24] was considering the case where $\int_0^\pi \theta \beta(\theta) d\theta < \infty$, and Horowitz, Karandikar [18], Desvillettes [6], and Fournier, Méléard [9], [12], have worked under the physical assumption $\int_0^\pi \theta^2 \beta(\theta) d\theta < +\infty$. In the non Maxwell case, by analytical methods, Goudon [13] and Villani [26] obtain existence results. With a probabilistic approach, Fournier-Méléard [11] obtain such results in dimension 2 and for cross-sections bounded as velocity function. We generalize here this approach in dimension 3 and for unbounded (as velocity field) soft potential cross-sections of the following type:

**Hypothesis:** the cross-section is of the form

$$B(z, \theta) = h(|z|)|z|^\gamma \beta(\theta), \quad (2.10)$$

with $\gamma \in (-1, 0]$ and $h$ a bounded nonnegative locally Lipschitz continuous function and $\beta$ from $]0, \pi] \to \mathbb{R}+$ such that $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$.

Let us first recall the probabilistic approach.

Equation (2.1) has to be understood in a weak sense, i.e. $f$ is a solution of the equation if for test functions $\phi$, $\frac{\partial}{\partial t} < f, \phi > = < Q_B(f, f), \phi >$ where $< ., . >$ denotes the duality bracket between $L^1$ and $L^\infty$ functions. By a standard integration by parts, we define a solution $f$ as satisfying for each $\phi \in C_b^2(\mathbb{R}^3)$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} f(t, v) \phi(v) dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_0^{2\pi} \int_0^\pi (\phi(v') - \phi(v)) B(v - v_*, \theta) d\theta d\varphi f(t, v) dv f(t, v_*) dv_*.$$

Since the function $\beta$ may have an infinite mass on $[0, \pi]$, the RHS term may explode. Thus we have to compensate it, and taking into account the conservation of mass, we obtain finally the following definition of probability measure solutions of (2.1).

**Definition 2.2** *We say that a probability measure family $(P_t)_{t \geq 0}$ is a measure-solution of the Boltzmann equation (2.1) if for each $\phi \in C_b^2(\mathbb{R}^3)$*

$$\langle \phi, P_t \rangle = \langle \phi, P_0 \rangle + \int_0^t \langle K_{\beta,\gamma}^\phi(v, v_*), P_s(dv)\, P_s(dv_*) \rangle\, ds, \qquad (2.11)$$

*where $K_{\beta,\gamma}^\phi$ is defined in the compensated form*

$$K_{\beta,\gamma}^\phi(v, v_*) = -bh(|v - v_*|)|v - v_*|^\gamma (v - v_*).\nabla \phi(v) \qquad (2.12)$$

$$+ \int_0^{2\pi} \int_0^\pi \left( \phi(v + a(v, v_*, \theta, \varphi)) - \phi(v) - a(v, v_*, \theta, \varphi).\nabla \phi(v) \right) h(|v - v_*|)|v - v_*|^\gamma \beta(\theta) d\theta d\varphi$$

*and where*

$$b = \pi \int_0^\pi (1 - \cos \theta)\beta(\theta)d\theta. \qquad (2.13)$$

We consider (2.11) as the evolution equation for the marginals of a Markov process the law of which is defined by a martingale problem.

**Definition 2.3** *Let $\beta$ be a cross section such that $\int_0^\pi \theta^2 \beta(\theta)d\theta < +\infty$ and $Q_0$ in $\mathcal{P}_2(\mathbb{R}^3)$. We say that $Q \in \mathcal{P}(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^3))$ solves the nonlinear martingale problem $(BMP)$ starting at $Q_0$ if under $Q$, the canonical process $V$ satisfies for any $\phi \in C_b^2(\mathbb{R}^3)$*

$$\phi(V_t) - \phi(V_0) - \int_0^t \langle K_{\beta,\gamma}^\phi(V_s, v_*), Q_s(dv_*) \rangle ds \qquad (2.14)$$

*is a square-integrable martingale and the law of $V_0$ is $Q_0$. Here, the nonlinearity appears through $Q_s$ which denotes the marginal of $Q$ at time $s$.*

**Remark 2.4** *Taking expectations in (2.14), we remark that if $Q$ is a solution of $(BMP)$, then its time-marginal family $(Q_t)_{t \geq 0}$ is a measure-solution of the Boltzmann equation, in the sense of Definition 2.2.*

Our first aim is to prove the existence of a solution to the martingale problem (2.14) and then to obtain the existence of a measure-solution to the Boltzmann equation. Our method gives no hope to obtain an uniqueness result. We generalize here the results of Tanaka and Horowitz-Karandikar [18] to soft potential cases.
We will introduce a specific nonlinear stochastic differential equation giving a pathwise version of the probabilistic interpretation. We obtain the existence of weak solutions of this equation under Hypothesis (2.10), as limits in law of solutions of regularized equations.

## 2.2 The pathwise approach

Let us now consider two probability spaces : the first one is the abstract space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$ and the second one is $([0,1], \mathcal{B}([0,1]), d\alpha)$. In order to avoid any confusion, the processes on $([0,1], \mathcal{B}([0,1]), d\alpha)$ will be called $\alpha$-processes, the expectation under $d\alpha$ will be denoted by $E_\alpha$, and the laws $\mathcal{L}_\alpha$.

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**Definition 2.5** *We say that $(V, W, N, V_0)$ is a solution of $(SDE)$ if*
*(i) $(V_t)$ is an adapted càdlàg $I\!D_T$-valued process such that $E(\sup_{t\in[0,T]}|V_t|^2) < +\infty$,*
*(ii) $(W_t)$ is a $\alpha$-process such that $E_\alpha(\sup_{t\in[0,T]}|W_t|^2) < +\infty$,*
*(iii) $N(\omega, dt, d\alpha, dx, d\theta, d\varphi)$ is a $\{\mathcal{F}_t\}$-Poisson point measure on $[0,T] \times [0,1] \times I\!R_+ \times [0,\pi] \times [0,2\pi]$ with intensity $m(dt, d\alpha, dx, d\theta, d\phi) = dt d\alpha dx \beta(\theta) d\theta d\varphi$ and associated compensated martingale $\tilde{N}$,*
*(iv) $V_0$ is a square integrable variable independent of $N$,*
*(v) $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$,*
*(vi)*

$$V_t = V_0 - b \int_0^t \int_0^1 h(|V_s - W_s(\alpha)|)|V_s - W_s(\alpha)|^\gamma (V_s - W_s(\alpha)) d\alpha ds$$

$$+ \int_0^t \int_0^1 \int_{I\!R_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}, W_{s-}(\alpha), \theta, \varphi)\mathbf{1}_{\{x \le h(|V_{s-}-W_{s-}(\alpha)|)|V_{s-}-W_{s-}(\alpha)|^\gamma\}} \tilde{N}(ds, d\alpha, dx, d\theta, d\varphi)$$

This definition can be understood through the following remark.

**Remark 2.6** *If $(V, W, N, V_0)$ is a solution of $(SDE)$, using Itô's formula, one easily proves that $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$ is a solution of $(BMP)$ with initial law $Q_0 = \mathcal{L}(V_0)$, and thus $\{\mathcal{L}(V_s)\}_{s\in[0,T]}$ is a measure-solution of the Boltzmann equation (2.11) with initial data $Q_0$.*

We are now able to state our existence theorem, which is the main result of this section.

**Theorem 2.7** *Assume that $Q_0$ is a probability measure on $I\!R^3$ with a forth order moment, and that $B(z,\theta) = h(|z|)|z|^\gamma \beta(\theta)$ is a cross-section satisfying Hypothesis (2.10). Then*
*1) The nonlinear martingale problem $(BMP)$ with initial data $Q_0$ admits a solution $Q \in \mathcal{P}_2(I\!D_T)$.*
*The mesure $Q$ is also a weak solution of $(SDE)$: if $W$ is an $\alpha$-process such that $\mathcal{L}_\alpha(W) = Q$, then on an enlarged probability space from the canonical space $(I\!D_T, \mathcal{D}_T, Q)$ there exists a Poisson measure $N$ with intensity $m$ and an independent square integrable variable $V_0$ with law $Q_0$ such that $(X, W, N, V_0)$ is solution of $(SDE)$, where $X$ is the canonical process.*
*2) Moreover, $E_Q(\sup_{t\le T}|X_t|^4) < +\infty$.*

**Remark 2.8** *There is no assumption on $Q_0$, except the existence of a forth order moment. This allows us in particular to consider degenerate initial data, as Dirac measures. The point 1) in Theorem 2.7 exhibits in particular a measure-solution to the Boltzmann equation (2.1) for each initial data $Q_0 \in \mathcal{P}_4(I\!R^3)$.*

The proof consists in many steps. The first one generalizes the result of Fournier-Méléard [11] given in dimension 2. The specific difficulty in dimension 3 is the non Lipschitz continuity of $a$ described in lemma 2.1. We will prove

**Proposition 2.9** *Assume that $B(z,\theta) = \psi(z)\beta(\theta)$ with $\psi$ a nonnegative bounded and locally Lipschitz continuous function, and $\beta$ integrating $\theta$. Assume that $V_0$ is a forth-order moment random variable. Then the nonlinear stochastic differential equation $(SDE)$ which can be rewritten in this case*

$$V_t = V_0 + \int_0^t \int_0^1 \int_{I\!R_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}, W_{s-}(\alpha), \theta, \varphi)\mathbf{1}_{\{x \le \psi(V_{s-}-W_{s-}(\alpha))\}} N(ds, d\alpha, dx, d\theta, d\varphi)$$

$$(2.15)$$

*admits a weak solution, and moreover, for every $T > 0$,*

$$E(\sup_{t \leq T} |V_t|^4) < +\infty. \tag{2.16}$$

**Proof.** The proof follows essentially the proof of Theorem 3.4 in [11]. Let us assume that the function $\psi$ is bounded by $M$. Let us define

$$\tilde{a}(v, w, \theta, \varphi, x) = a(v, w, \theta, \varphi) \mathbf{1}_{\{x \leq \psi(v-w)\}}$$

and its cutoff versions

$$\tilde{a}_n(v, w, \theta, \varphi, x) = \tilde{a}(v \wedge n \vee (-n), w \wedge n \vee (-n), \theta, \varphi, x).$$

We remark that

$$\int |\tilde{a}_n(v, w, \theta, \varphi, x)| dx \leq M\theta |v - w| \tag{2.17}$$

$$\int |\tilde{a}_n(v, w, \theta, \varphi, x) - \tilde{a}_n(v', w', \theta, \varphi, x)| dx \leq K_n(|v - v'| + |w - w'|) \tag{2.18}$$

Thanks to these properties, we are able to construct, by a sophisticated Picard iteration mixing results of [12] and [11], a solution of

$$V_t^n = V_0 + \int_0^t \int_0^1 \int_{\mathbb{R}_+} \int_0^\pi \int_0^{2\pi} \tilde{a}_n(V_{s-}^n, W_{s-}^n(\alpha), \theta, \varphi, x) N(ds, d\alpha, dx, d\theta, d\varphi) \tag{2.19}$$

satisfying moreover that

$$\sup_n E(\sup_{s \leq t} |V_s^n|^4) < +\infty. \tag{2.20}$$

Then, if $Q^n$ denotes the law of $V^n$ on the path space, the sequence $(Q^n)$ is clearly uniformly tight.

Now we have to prove that each limit point $Q$ of this sequence is solution of the nonlinear martingale problem associated with (2.15). We consider the canonical process $(X_t)$ on $\mathbb{D}_T$ and for $\phi \in C_b^1(\mathbb{R}^3)$, $t > 0$, we set

$$H_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t \int_0^M \int_0^\pi \int_0^{2\pi} \left( \phi(X_u + \tilde{a}(X_u, w, \theta, \varphi, x)) - \phi(X_u) \right) Q_u(dw) \beta(\theta) d\theta d\varphi dx du$$

and $H_t^{n,\phi}$ denotes a similar quantity with $\tilde{a}_n$ instead of $\tilde{a}$ and $Q^n$ instead of $Q$. If $G$ is a continuous bounded function on $(\mathbb{R}^3)^p$, one has to prove that

$$< (H_t^\phi - H_s^\phi) G(X_{s_1}, ..., X_{s_p}), Q >= 0$$

knowing that

$$< (H_t^{n,\phi} - H_s^{n,\phi}) G(X_{s_1}, ..., X_{s_p}), Q^n >= 0$$

for $0 \leq s_1 < ... < s_p < s < t \leq T$. The only new difficulty in dimension 3 consists in proving that the function

$$K(X, Y) = \int_s^t \int_0^M \int_0^\pi \int_0^{2\pi} \left( \phi(X_u + \tilde{a}(X_u, Y_u, \theta, \varphi, x)) - \phi(X_u) \right) \beta(\theta) d\theta d\varphi dx du$$

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is continuous on $I\!\!D_T \times I\!\!D_T$, although $a$ is not regular. Using the translation invariance of the Lebesgue's measure $d\varphi$ , we write

$$
|K(X,Y) - K(X',Y')| \leq M \int_s^t \int_0^\pi \int_0^{2\pi} \left( \left| \phi(X_u) - \phi(Y_u) \right| \right.
$$
$$
\left. + \left| \phi(X_u + a(X_u, Y_u, \theta, \varphi) - \phi(X'_u + a(X'_u, Y'_u, \theta, \varphi + \varphi_0(X_u, Y_u)))) \right| \right) \beta(\theta) d\theta d\varphi du
$$

and thanks to Lemma 2.1, we see that the RHS term tends to 0 when the uniform distance between $X$ and $Y$ tends to 0.

A standard proof allows us to conclude that $Q$ is solution of the nonlinear martingale problem $(BMP)$ associated with (2.15) and using a representation theorem, we can exhibit an enlarged probability space, on which the canonical process is solution of (2.15) (a similar argument is more detailed in the end of the proof of Theorem 2.7. The property (2.16) follows easily from (2.20). $\triangle$

Let us now prove Theorem 2.7

**Proof**. In order to apply Proposition 2.9, we consider some cutoff of the cross-section in both variables.

We introduce the following **approximating model:**
Let $l, k \in I\!\!N^*$ and define

$$
\beta_l(\theta) = \beta(\theta) \mathbf{1}_{|\theta| \geq \frac{1}{l}} \ ; \ \psi_k(r) = h(r)(r^\gamma \wedge k), \forall r \in I\!\!R_+.
$$

Each function $\psi_k$ is locally Lipschitz continuous and is bounded by $kH$, where $H$ is a bound of the function $h$. Thanks to Proposition 2.9 and for each $(k, l)$, there exists a weak solution to the nonlinear stochastic differential equation $(SDE_{kl})$:

$$
V_t^{k,l} = V_0 + \int_0^t \int_0^1 \int_{I\!\!R_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}^{k,l}, W_{s-}^{k,l}(\alpha), \theta, \varphi) \mathbf{1}_{\{x \leq \psi_k(V_{s-}^{k,l} - W_{s-}^{k,l}(\alpha))\}} N_{k,l}(ds, d\alpha, dx, d\theta, d\varphi)
$$

$$
\tag{2.21}
$$

where $N_{k,l}$ is a point Poisson measure with intensity $dsd\alpha dx \beta_l(\theta) d\theta d\varphi$ on $[0, T] \times [0, 1] \times [0, kH] \times [0, \pi] \times [0, 2\pi]$. So the associated nonlinear martingale problem $(BMP_{k,l})$ admits a solution $P^{k,l}$. The aim is now to prove that the sequence $(P^{k,l})$ of probability measures on the path space $I\!\!D_T$ is uniformly tight and that each limit point is solution of the initial nonlinear martingale problem $(BMP)$.

Since the limit case has sense only in a compensated form, we write each equation (2.21) in its compensated form:

$$
V_t^{k,l} = V_0 - b_l \int_0^t \int_0^1 \psi_k(V_s^{k,l} - W_s^{k,l}(\alpha))(V_s^{k,l} - W_s^{k,l}(\alpha)) d\alpha ds
$$
$$
+ \int_0^t \int_0^1 \int_{I\!\!R_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}^{k,l}, W_{s-}^{k,l}(\alpha), \theta, \varphi) \mathbf{1}_{\{x \leq \psi_k(V_{s-}^{k,l} - W_{s-}^{k,l}(\alpha))\}} \tilde{N}_{k,l}(ds, d\alpha, dx, d\theta, d\varphi)
$$

where

$$
b_l = \pi \int_0^\pi (1 - \cos\theta) \beta_l(\theta) d\theta.
$$

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**Lemma 2.10**

$$\sup_{k,l} E(\sup_{t \leq T} |V_t^{k,l}|^4) < +\infty. \tag{2.22}$$

**Proof.** (of the Lemma) Thanks to Lemma 1.1, we obtain easily that

$$
\begin{aligned}
E(\sup_{s \leq t} |V_s^{k,l}|^4) &\leq K\left( E(|V_0|^4) + \int_0^t \int_0^1 E(\sup_{u \leq s} |V_u^{k,l} - X_u^{k,l}(\alpha)|^4 + 1)d\alpha ds \right) \\
&\leq K\left( 1 + \int_0^t E(\sup_{u \leq s} |V_u^{k,l}|^4)ds \right) \tag{2.23}
\end{aligned}
$$

and the constant number $K$ does not depend on $k$ and $l$. By Proposition 2.9, $E(\sup_{s \leq T} |V_s^{k,l}|^4)$ is finite for each $k, l$ and the proof is obtained by Gronwall's lemma. $\triangle$

It is thus classical to show that the Aldous criterion is satisfied.
Hence the sequence $(P^{k,l})$ is tight.
Let us now identify each limit point of $(P^{k,l})$. Let $Q$ be a limit value of this sequence. We consider the compensated martingale problems. Let $(X_t)_t$ be the canonical process on $\mathbb{D}_T$ and for $\phi \in C_b^2(\mathbb{R}^3)$, $t > 0$, we set

$$
\begin{aligned}
H_t^\phi &= \phi(X_t) - \phi(X_0) + b\int_0^t \int_{w \in \mathbb{R}^2} \nabla\phi(X_u).(X_u - w)h(|X_u - w|)|X_u - w|^\gamma Q_u(dw)du \\
&\quad - \int_0^t \int_0^\pi \int_0^{2\pi} \int_{\mathbb{R}^3} (\phi(X_u + a(X_u, w, \theta, \varphi, x)) - \phi(X_u) - a(X_u, w, \theta, \varphi, x)(X_u - w).\nabla\phi(X_u)) \\
&\quad\quad h(|X_u - w|)|X_u - w|^\gamma Q_u(dw)\beta(\theta)d\theta d\varphi du
\end{aligned}
$$

and $H_t^{k,l,\phi}$ denotes a similar quantity with $\psi_k(X_u - w)$ instead of $h(|X_u - w|)|X_u - w|^\gamma$ and $\beta_l$ instead of $\beta$ and $b_l$ instead of $b$ and $P_u^{k,l}$ instead of $Q_u$. The probability measure $Q$ will be a solution of the nonlinear martingale problem $(BMP)$ with initial law $Q_0$ if it satisfies for each $0 \leq s_1 < ... < s_p < s < t \leq T$, each $G \in C_b((\mathbb{R}^3)^p)$,

$$< (H_t^\phi - H_s^\phi)G(X_{s_1}, ..., X_{s_p}), Q >= 0. \tag{2.24}$$

Since $P^{k,l}$ is solution of $(MP)_{k,l}$, we already know that

$$< (H_t^{k,l,\phi} - H_s^{k,l,\phi})G(X_{s_1}, ..., X_{s_p}), P^{k,l} >= 0.$$

Since the sequence $(P^{k,l})$ satisfies the Aldous criterion, the law $Q$ is the law of a quasi-càg process (cf. [19] p. 321). Then the mapping $F : x \mapsto (\phi(x_t) - \phi(x_s))G(x_{s_1}, ..., x_{s_p})$ is $Q$-a.e. continuous and bounded from $\mathbb{D}_T$ to $\mathbb{R}$. Thus $< F, P^{k,l} >$ tends to $< F, Q >$ as $k, l$ tend to infinity.
Now, let us successively prove that

$$
\begin{aligned}
T_1 &= <\int_s^t \int_0^\pi \int_0^{2\pi} \int_{\mathbb{R}^3} (\phi(X_u + a(X_u, w, \theta, \varphi)) - \phi(X_u) - a(X_u, w, \theta, \varphi).\nabla\phi(X_u)) \\
&\quad h(|X_u - w|)(|X_u - w|^\gamma - (|X_u - w|^\gamma) \wedge k)P_u^{k,l}(dw)\beta_l(\theta)d\theta d\varphi du \Big) G(X_{s_1}, ..., X_{s_p}), P^{k,l} >,
\end{aligned}
$$

$$T_2 \;=\; < \int_s^t \int_0^\pi \int_0^{2\pi} \int_{\mathbb{R}^3} \big( \phi(X_u + a(X_u, w, \theta, \varphi)) - \phi(X_u) - a(X_u, w, \theta, \varphi).\nabla\phi(X_u) \big)$$

$$h(|X_u - w|)|X_u - w|^\gamma (\beta_l(\theta) - \beta(\theta)) P_u^{k,l}(dw) d\theta d\varphi du \Big) G(X_{s_1}, ..., X_{s_p}), P^{k,l} >,$$

$$T_3 \;=\; < G(X_{s_1}, ..., X_{s_p}) \int_s^t \int_{\mathbb{R}^3} K_{\beta,\gamma}^\phi(X_u, Y_u), P^{k,l}(dX) \otimes P^{k,l}(dY) >$$

$$- < G(X_{s_1}, ..., X_{s_p}) \int_s^t \int_{\mathbb{R}^3} K_{\beta,\gamma}^\phi(X_u, Y_u), Q(dX) \otimes Q(dY) >,$$

and the term $T_4$ similar to $T_1$ corresponding to the drift term, tend to $0$ as $k, l$ tend to infinity.

Term $T_1$:

$$
\begin{aligned}
|T_1| \;\leq\;& K \int_0^\pi \theta^2 \beta_l(\theta) d\theta < \int_s^t \int_{\mathbb{R}^3} |X_u - w|^2 (|X_u - w|^\gamma - (|X_u - w|^\gamma) \wedge k) \\
& \qquad\qquad P_u^{k,l}(dw) du, P^{k,l} > \\
\leq\;& K < \int_s^t \int_{\mathbb{R}^3} |X_u - w|^{2+\gamma} \mathbf{1}_{\{|X_u - w|^\gamma \geq k\}} P_u^{k,l}(dw) du, P^{k,l} > \\
\leq\;& K < \int_s^t \int_{\mathbb{R}^3} |X_u - w|^{2+\gamma} \mathbf{1}_{\{|X_u - w| \leq (k)^{\frac{1}{\gamma}}\}} P_u^{k,l}(dw) du, P^{k,l} > \\
\leq\;& K(k)^{\frac{2+\gamma}{\gamma}}
\end{aligned}
$$

and $T_1$ tends to zero when $k$ tends to infinity, uniformly in $l$ since $\int_0^\pi \theta^2 \beta_l(\theta) d\theta \leq \int_0^\pi \theta^2 \beta(\theta) d\theta < +\infty$, and since $\frac{2+\gamma}{\gamma} < 0$.

Term $T_4$: By a similar study with the drift term, we obtain

$$
\begin{aligned}
|T_4| \;\leq\;& K < \int_s^t \int_{\mathbb{R}^3} |X_u - w|(|X_u - w|^\gamma - (|X_u - w|^\gamma) \wedge k), P_u^{k,l}(dw) du, P^{k,l} > \\
\leq\;& K(k)^{\frac{1+\gamma}{\gamma}}
\end{aligned}
$$

and $T_4$ tends to zero when $k$ tends to infinity, since $\gamma \in (-1, 0]$. We observe here the necessity to choose $\gamma$ more than $-1$, and we can not assume more generally $\gamma \in (-2, 0]$ as in [13] or [26]. These authors use a symmetry of the drift term we loose here, because of the functions $G(X_{s_1}, ..., X_{s_p})$ giving the martingale property.

Term $T_2$:

$$
\begin{aligned}
|T_2| \;\leq\;& K \int_0^\pi \theta^2 |\beta_l(\theta) - \beta(\theta)| d\theta < \int_s^t \int_{\mathbb{R}^3} (|X_u - w|^{2+\gamma}) P_u^{k,l}(dw) du, P^{k,l} > \\
\leq\;& K \left( \sup_{k,l} E_{P^{k,l}} (\sup_{u \leq T} |X_u|^2) + 1 \right) \int_{-\pi}^{\pi} \theta^2 |\beta_l(\theta) - \beta(\theta)| d\theta
\end{aligned}
$$

which tends to $0$ as $l$ tends to infinity, uniformly in $k$ thanks to Lemma 2.10.

Term $T_3$: Let us define the function $F(x, y)$ on $I\!\!D_T \times I\!\!D_T$ by $F(x, y) = \int_s^t K_{\beta, \gamma}^\phi(x_u, y_u) du$. The function $F$ is $Q \otimes Q$-a.e. continuous by a similar argument as in the proof of Proposition 2.9 and not bounded. Lemma 1.1 implies that

$$
\begin{aligned}
|F(x, y)| &\leq K \int_0^\pi \theta^2 \beta(\theta) d\theta \left( \sup_{s \leq u \leq t} (|x_u - y_u|^{2+\gamma} + |x_u - y_u|^{1+\gamma}) \right) \\
&\leq K \left( \sup_{u \leq T} |x_u|^2 + \sup_{u \leq T} |y_u|^2 + 1 \right).
\end{aligned}
$$

Now, the measure $P^{k,l} \otimes P^{k,l}$ converges obviously to $Q \otimes Q$. Then, for each fixed real positive number $C$, the sequence $< F \wedge C, P^{k,l} \otimes P^{k,l} >$ converges to $< F \wedge C, Q \otimes Q >$. We remark that

$$
|F(x, y)| \mathbf{1}_{\{|F(x,y)| \geq C\}} \leq K \left( \sup_{u \leq T} |x(u)|^2 + \sup_{u \leq T} |y(u)|^2 + 1 \right) \mathbf{1}_{\{\sup_{u \leq T} |x(u)|^2 + \sup_{u \leq T} |y(u)|^2 \geq C/K - 1\}}
$$

$$
\leq K \left( \sup_{u \leq T} |x(u)|^2 + \sup_{u \leq T} |y(u)|^2 + 1 \right) \left( \mathbf{1}_{\{\sup_{u \leq T} |x(u)|^2 \geq C/2K - 1/2\}} + \mathbf{1}_{\{\sup_{u \leq T} |y(u)|^2 \geq C/2K - 1/2\}} \right)
$$

and it is easy to prove thanks to Lemma 2.10 that

$$
\sup_{k,l} < \left( \sup_{u \leq T} |x(u)|^2 + \sup_{u \leq T} |y(u)|^2 + 1 \right) \left( \mathbf{1}_{\{\sup_{u \leq T} |x(u)|^2 \geq C/2K - 1/2\}} + \mathbf{1}_{\{\sup_{u \leq T} |y(u)|^2 \geq C/2K - 1/2\}} \right), P^{k,l} \otimes P^{k,l} >
$$

tends to 0 as $C$ tends to infinity.

We have thus proved that each limit law of the sequence $(P^{k,l})$ is solution of the martingale problem $(BMP)$. Since such limits exist thanks to the Aldous criterion, we deduce obviously from this approach the existence of at least one solution to $(BMP)$.

Let us now show that each solution $Q$ of $(BMP)$ is a weak solution of $(SDE)$.

The canonical process $X$ is a semimartingale under $Q$. Then a comparison between the Itô formula and the martingale problem proves that $X$ is a pure jump process and that its Lévy measure is the image measure of the measure $m$ on $[0, T] \times [0, 1] \times I\!\!R_+ \times [0, \pi] \times [0, 2\pi]$ by the mapping $(s, \alpha, x, \theta, \varphi) \mapsto a(X_{s-}, W_{s-}(\alpha), \theta, \varphi) \mathbf{1}_{\{x \leq h(|X_{s-} - W_{s-}(\alpha)|)|X_{s-} - W_{s-}(\alpha)|^\gamma\}}$. Then by a representation theorem for point measures [8], there exists on an enlarged probability space a square integrable variable $V_0$ and a point Poisson measure $N$ with intensity $m$ such that $(X, W, N, V_0)$ is a solution of $(SDE)$. $\triangle$

# 3 Convergence of renormalized Boltzmann Processes towards a Landau Process

## 3.1 A probabilistic interpretation of the Landau equation

The Landau equation, also called the Fokker-Planck-Landau equation, describes the collisions of particles in a plasma and is obtained as limit of Boltzmann equations when the collisions become grazing. In the spatially homogeneous case, it writes in $I\!\!R^3$:

$$
\frac{\partial f}{\partial t} = Q_L(f, f) \tag{3.1}
$$

with

$$Q_L \left(f, f\right) \left(t, v\right) = \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial}{\partial v_i} \left\{ \int_{I\!\!R^3} dv_* A_{ij} \left(v - v_*\right) \left[ f \left(t, v_*\right) \frac{\partial f}{\partial v_j} \left(t, v\right) - f \left(t, v\right) \frac{\partial f}{\partial v_{*j}} \left(t, v_*\right) \right] \right\}$$

where $f\left(t, v\right) \geq 0$ is the density of particles having velocity $v \in I\!\!R^3$ at time $t \in I\!\!R^+$, and $\left(A_{ij}\left(z\right)\right)_{1 \leq i,j \leq 3}$ is a nonnegative symmetric matrix depending on the interaction between the particles, of the form

$$
\begin{aligned}
A\left(z\right) &= \Lambda \left|z\right|^{\gamma+2} \Pi\left(z\right) h\left(\left|z\right|\right) \quad\quad\quad\quad (3.2) \\
&= \Lambda \left|z\right|^{\gamma} h\left(\left|z\right|\right) \begin{bmatrix} z_2^2 + z_3^2 & -z_1 z_2 & -z_1 z_3 \\ -z_1 z_2 & z_1^2 + z_3^2 & -z_2 z_3 \\ -z_1 z_3 & -z_2 z_3 & z_1^2 + z_2^2 \end{bmatrix}
\end{aligned}
$$

where $\Pi\left(z\right)$ is the orthogonal projection on $\left(z\right)^{\perp}$, $\Lambda$ is a positive constant and $h$ is a nonnegative locally Lipschitz continuous bounded function.
By integration by parts, see [26], a weak formulation of the equation (3.1) writes, at least formally, for any test function $\phi \in C_b^2 \left(I\!\!R^3\right)$,

$$
\begin{aligned}
\frac{d}{dt} \int \phi\left(v\right) f\left(t, v\right) dv &= \frac{1}{4} \sum_{i,j=1}^{3} \int_{I\!\!R^3 \times I\!\!R^3} dv dv_* f(t, v) f\left(t, v_*\right) A_{ij}\left(v - v_*\right) \left(\partial_{ij}^2 \phi\left(v\right) + \partial_{ij}^2 \phi\left(v_*\right)\right) \\
&\quad + \frac{1}{2} \sum_{i=1}^{3} \int_{I\!\!R^3 \times I\!\!R^3} dv dv_* f(t, v) f\left(t, v_*\right) b_i\left(v - v_*\right) \left(\partial_i \phi\left(v\right) - \partial_i \phi\left(v_*\right)\right) \quad (3.3)
\end{aligned}
$$

where $b_i\left(z\right) = \sum_{j=1}^{3} \partial_j A_{ij}\left(z\right) = -2\Lambda h\left(\left|z\right|\right) \left|z\right|^{\gamma} z_i$.

As for the Boltzmann equation, the equation (3.3) conserves the mass, thus we give a definition of probability-measure solutions of the Landau equation :

**Definition 3.1** Let $P_0$ belong to $\mathcal{P}_2\left(I\!\!R^3\right)$. A probability measure solution of the Landau equation (3.4) with initial data $P_0$ is a probability measure family $\left(P_t\right)_{t \geq 0}$ on $I\!\!R^3$ satisfying

$$\langle \phi, P_t \rangle = \langle \phi, P_0 \rangle + \int_o^t \langle L^{\phi}\left(v, v_*\right), P_s\left(dv\right) P_s\left(dv_*\right) \rangle ds \quad\quad\quad (3.4)$$

for any function $\phi \in C_b^2\left(I\!\!R^3\right)$ where $L^{\phi}$ is the Landau kernel defined on $I\!\!R^3 \times I\!\!R^3$ by :

$$
\begin{aligned}
L^{\phi}\left(v, v_*\right) &= \frac{1}{2} \sum_{i,j=1}^{3} \partial_{ij}^2 \phi\left(v\right) A_{ij}\left(v - v_*\right) + \sum_{i=1}^{3} \partial_i \phi\left(v\right) b_i\left(v - v_*\right) \\
&= \frac{1}{2} J_{\phi}\left(v\right) : A\left(v - v_*\right) + b\left(v - v_*\right) . \nabla \phi\left(v\right)
\end{aligned}
$$

with $J_{\phi} = \left(\partial_{ij}^2 \phi\right)_{1 \leq i,j \leq 3}$.

We consider now the martingale problem associated with the Landau equation and defined as follows.

**Definition 3.2** Let $P_0$ belong to $\mathcal{P}_2\left(I\!\!R^3\right)$.

Let $(Y_s)_{s\geq 0}$ be the canonical process on $\mathcal{C}_T$. $P \in \mathcal{P}\left(\mathcal{C}_T\right)$ is a solution of the martingale problem $\overline{(LMP)}$ with initial data $P_0$ if the law of $Y_0$ is $P_0$ and for any $\phi \in C^2(I\!\!R^3)$,

$$\phi\left(Y_t\right) - \phi\left(Y_0\right) - \int_0^t < L^\phi\left(Y_s, v_*\right), P_s\left(dv_*\right) > ds$$

is a $P$-martingale, where $P_s = P \circ Y_s^{-1}$.

**Remark 3.3** The time-marginals family of a solution of the martingale problem $(LMP)$ is also a measure-solution of the Fokker-Planck-Landau equation.

Guérin already built in [16] by a direct probabilistic approach a Landau process solution of a nonlinear stochastic differential equation driven by a white noise and deduced the existence of a measure-solution of the Landau equation for any dimension $\geq 2$ and for $\gamma \in (-1, 0]$. We obtain here a new proof of the existence of a solution to the Landau process (and then of a solution to the Landau equation) as limit of Boltzmann processes.

## 3.2 Asymptotic of Boltzmann processes towards a Landau process

Arsen'ev and Buryak (see [1]) have shown the convergence of the Boltzmann equation towards the Landau equation under restrictive assumptions on the cross section and on the initial data. In a more physical situation, Degond and Lucquin, [4], study the convergence of the Boltzmann operator $Q_B(f, f)$ for a Coulomb potential ($\gamma = -3$) toward the Landau operator $Q_L(f, f)$ using the following approximation

$$\beta^\varepsilon\left(\theta\right) = \frac{1}{|\log \varepsilon|} \frac{\cos(\theta/2)}{\sin^3(\theta/2)} \mathbb{I}_{\theta \geq \varepsilon}$$

Desvillettes, [5], proves the convergence for non Coulomb potentials using another asymptotic :

$$\beta^\varepsilon\left(\theta\right) = \frac{1}{\varepsilon^3} \beta\left(\frac{\theta}{\varepsilon}\right)$$

Here we are interested in stating the convergence in law of the Boltzmann process, obtained in Section 2, towards a Landau process when the collisions become grazing for "moderately soft potentials", i.e. $\gamma \in (-1, 0]$. With this aim in view, we use a general approximation introduced by Villani in [26]. We consider $\beta^\varepsilon$ a function from $[0, \pi]$ to $\mathbb{R}^+$ satisfying

$$\forall \theta_0 > 0 \ \beta^\varepsilon\left(\theta\right) \xrightarrow[\varepsilon \to 0]{} 0 \text{ uniformly on } \theta \geq \theta_0 \tag{3.5}$$

$$\Lambda^\varepsilon = \pi \int_0^\pi \sin^2\left(\frac{\theta}{2}\right) \beta^\varepsilon\left(\theta\right) d\theta \xrightarrow[\varepsilon \to 0]{} \Lambda > 0 \tag{3.6}$$

Let us notice the following properties of $\beta^\varepsilon$:

**Lemma 3.4** *1)* $\qquad\qquad \int_0^\pi \beta^\varepsilon\left(\theta\right) d\theta \xrightarrow[\varepsilon \to 0]{} +\infty,$

*2) For $k \geq 3$,*

$$\int_0^\pi \sin^k\left(\frac{\theta}{2}\right) \beta^\varepsilon\left(\theta\right) d\theta \xrightarrow[\varepsilon \to 0]{} 0.$$

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**Proof.** 1) An easy computation shows that if the limit of the masses of $\beta^\varepsilon$ is finite, then necessarily, $\Lambda = 0$.

2) Let $k \geq 3$, let $\eta > 0$, we consider $\theta_0 > 0$ such that $\sin^{k-2}\left(\frac{\theta}{2}\right) \leq \eta$ for any $\theta \leq \theta_0$.

$$\int_0^\pi \sin^k\left(\frac{\theta}{2}\right) \beta^\varepsilon(\theta)\, d\theta \leq \eta \int_0^\pi \sin^2\left(\frac{\theta}{2}\right) \beta^\varepsilon(\theta)\, d\theta + \int_{\theta_0}^\pi \sin^k\left(\frac{\theta}{2}\right) \beta^\varepsilon(\theta)\, d\theta$$

Thanks to (3.5), there exists $\varepsilon_0 > 0$ such that $\int_{\theta_0}^\pi \sin^k\left(\frac{\theta}{2}\right) \beta^\varepsilon(\theta)\, d\theta \leq \eta$ for any $\varepsilon \leq \varepsilon_0$. Moreover, using the convergence (3.6), there exists $K > 0$ such that

$$\int_0^\pi \sin^k\left(\frac{\theta}{2}\right) \beta^\varepsilon(\theta)\, d\theta \leq K\eta + \eta$$

Then $\int_0^\pi \sin^k\left(\frac{\theta}{2}\right) \beta^\varepsilon(\theta)\, d\theta \to 0$ as $\varepsilon$ tends to zero. $\triangle$

For each $\varepsilon > 0$, for $\gamma \in (-1, 0]$ we define the Boltzmann kernel $K_{\beta^\varepsilon,\gamma}^\phi$ on $\mathbb{R}^3 \times \mathbb{R}^3$, as in (2.12), by

$$K_{\beta^\varepsilon,\gamma}^\phi(v, v_*) = -b^\varepsilon h(|v - v_*|)|v - v_*|^\gamma (v - v_*).\nabla\phi(v) \tag{3.7}$$
$$\int_0^{2\pi} \int_0^\pi \left( \phi(v + a(v, v_*, \theta, \varphi)) - \phi(v) - a(v, v_*, \theta, \varphi).\nabla\phi(v) \right) h(|v - v_*|)|v - v_*|^\gamma \beta^\varepsilon(\theta) d\theta d\varphi$$

with $b^\varepsilon = \pi \int_0^\pi (1 - \cos\theta) \beta^\varepsilon(\theta)\, d\theta$.

We notice that the Boltzmann kernel converges towards the Landau kernel when $\varepsilon \to 0$, for any $v, v_* \in \mathbb{R}^3$ and $\phi \in C_b^2\left(\mathbb{R}^3\right)$ ( for more details, see the convergence of the term $E_1$ in Section 3.4).

We denote by $(B^\varepsilon MP)$ the martingale problem associated with the Boltzmann equation defined as in Definition 2.3 replacing $K_{\beta,\gamma}^\phi$ by $K_{\beta^\varepsilon,\gamma}^\phi$. In the previous section, we have proved the existence of a solution $Q^\varepsilon$ of $(B^\varepsilon MP)$ for $\gamma \in (-1, 0]$. We are now interested in the asymptotic behaviour of the sequence $(Q^\varepsilon)_{\varepsilon>0}$ when $\varepsilon$ tends to 0.

We state the following main theorem.

**Theorem 3.5** *Consider a bounded locally Lipschitz continuous nonnegative function $h$, $\gamma \in (-1, 0]$, $\beta^\varepsilon$ satisfying (3.5) and (3.6) and $Q_0$ a finite forth-order moment probability measure. Let $Q^\varepsilon \in \mathcal{P}\left(\mathbb{D}_T\right)$ be a solution of the nonlinear martingale problem $(B^\varepsilon MP)$ with kernel $K_{\beta^\varepsilon,\gamma}$ defined by (3.7) and initial data $Q_0$.*
*Then the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is tight when $\varepsilon$ tends to 0, and any of its subsequences converges toward a solution $P \in \mathcal{P}\left(\mathcal{C}_T\right)$ of the nonlinear martingale problem $(LMP)$, associated with the Landau equation (3.4) having diffusion matrix defined by (3.2), with initial data $Q_0$.*

**Remark 3.6** When $\gamma = 0$ and under some regularity assumptions on $h$, Guérin has proved in [17] Corollary 7 the uniqueness of a solution $P$ to the martingale problem $(LMP)$. Then, in this case, the sequence $(Q^\varepsilon)_{\varepsilon>0}$ converges towards this unique solution $P$.

Let us notice that Villani [26], and Goudon [13] prove, in two independent articles, the existence of weak function solutions of the Landau equation for soft potentials using the convergence of the solutions of the Boltzmann equation towards the solutions of the Landau equation. The interest of our approach is the understanding of this convergence at the

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microscopic level of processes. These ones jump more and more often with smaller jumps, when $\beta^\varepsilon$ decreases and then converge finally to a (continuous) diffusion process. Moreover, our convergence result is true for general (even degenerated) initial datas and leads naturally to particle approximations.

## 3.3   C-Tightness of the sequence $(Q^\varepsilon)_{\varepsilon>0}$

We assume that $Q_0$ has a finite fourth-order moment.
Let $Q^\varepsilon$ be a solution of the martingale problem $(B^\varepsilon MP)$ obtained in Theorem 2.7.
Thanks to the point 2) of Theorem 2.7, for any $\varepsilon > 0$, the probability $Q^\varepsilon$ satisfies

$$E_{Q^\varepsilon}\left[\sup_{0\leq t\leq T}|X_t|^4\right]\leq K^\varepsilon$$

with $K^\varepsilon$ a positive constant depending on $\varepsilon$ only through $\int_{-\pi}^{\pi}\sin^4\frac{\theta}{2}\beta^\varepsilon(\theta)\,d\theta$, $\int_{-\pi}^{\pi}\sin^2\frac{\theta}{2}\beta^\varepsilon(\theta)\,d\theta$ and $b^\varepsilon$ according to Lemma 2.1. Using Lemma 3.4 and the asymptotic (3.6) we notice that the sequence $(K^\varepsilon)_{\varepsilon>0}$ converges as $\varepsilon$ tends to 0. Then there exists $K > 0$ such that

$$\sup_{\varepsilon>0}E_{Q^\varepsilon}\left[\sup_{0\leq t\leq T}|X_t|^4\right]\leq K \tag{3.8}$$

Thanks to Aldous criterion, we deduce, with similar arguments as in Section 2, that the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is tight in $\mathcal{P}(I\!\!D_T)$, then a cluster point $P$ of $(Q^\varepsilon)_{\varepsilon>0}$ belongs a priori to $\mathcal{P}(I\!\!D_T)$.
We now prove that the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is moreover $C$-tight, in the sense of Jacod-Shiryaev [19] Definition 3.25, p. 315, and then $P$ will belong to $\mathcal{P}(\mathcal{C}_T)$.
As the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is tight and according to [19] Proposition 3.26 (iii), we just have to prove that for any $\eta > 0$, for $\Delta X_t = X_t - X_{t^-}$,

$$\lim_{\varepsilon\to 0}Q^\varepsilon\left(\sup_{t\leq T}|\Delta X_t| > \eta\right) = 0.$$

We use the stochastic differential equation $(SDE)$ introduced in Section 2.2. Let $V^\varepsilon$ be a process with distribution $Q^\varepsilon$ such that

$$V_t^\varepsilon = V_0 - b^\varepsilon\int_0^t\int_0^1 h(|V_s^\varepsilon - W_s^\varepsilon(\alpha)|)|V_s^\varepsilon - W_s^\varepsilon(\alpha)|^\gamma(V_s^\varepsilon - W_s^\varepsilon(\alpha))d\alpha ds$$

$$+ \int_0^t\int_0^1\int_{I\!\!R_+}\int_0^\pi\int_0^{2\pi} a(V_{s-}^\varepsilon, W_{s-}^\varepsilon(\alpha), \theta, \varphi)\mathbf{1}_{\{x\leq h(|V_{s-}^\varepsilon - W_{s-}^\varepsilon(\alpha)|)|V_{s-}^\varepsilon - W_{s-}^\varepsilon(\alpha)|^\gamma\}}\tilde{N}^\varepsilon(ds, d\alpha, dx, d\theta, d\varphi)$$

with $\mathcal{L}_\alpha(W^\varepsilon) = \mathcal{L}(V^\varepsilon) = Q^\varepsilon$ and $\tilde{N}^\varepsilon(ds, d\alpha, dx, d\theta, d\varphi)$ is the compensated martingale of a Poisson measure with intensity $m^\varepsilon(dt, d\alpha, dx, d\theta, d\varphi) = dtd\alpha dx\beta^\varepsilon(\theta)d\theta d\varphi$.
Then, by Tchebychev and Burkholder-Davis-Gundy inequalities for jump semimartingales

and Lemma 2.1,

$$
\begin{aligned}
Q^\varepsilon \left( \sup_{t \le T} |\Delta X_t| > \eta \right) &\le \frac{1}{\eta^4} E \left[ \sup_{t \le T} |\Delta V_t^\varepsilon|^4 \right] \le \frac{1}{\eta^4} E \left[ \sum_{t \le T} |\Delta V_t^\varepsilon|^4 \right] \\
&\le \frac{1}{\eta^4} E \left[ \int_0^T \int_0^1 \int_0^\pi \int_0^{2\pi} \left| a(V_{s-}^\varepsilon, W_{s-}^\varepsilon(\alpha), \theta, \varphi) \right|^4 h(|V_{s-}^\varepsilon - W_{s-}^\varepsilon(\alpha)|) \right. \\
&\quad \left. |V_{s-}^\varepsilon - W_{s-}^\varepsilon(\alpha)|^\gamma \beta^\varepsilon(\theta) \, d\theta d\varphi d\alpha ds \right] \\
&\le \frac{K}{\eta^4} \left( \int_0^T \int_0^1 E \left[ |V_{u-}^\varepsilon - W_{u-}^\varepsilon(\alpha)|^{\gamma+4} \right] d\alpha du \right) \int_0^\pi \left| \sin \frac{\theta}{2} \right|^4 \beta^\varepsilon(\theta) \, d\theta
\end{aligned}
$$

with $K$ independant of $\varepsilon$. Thanks to estimates (1.1) and (3.8), we obtain

$$
Q^\varepsilon \left( \sup_{t \le T} |\Delta X_t| > \eta \right) \le \frac{KT}{\eta^4} \int_0^\pi \left| \sin \frac{\theta}{2} \right|^4 \beta^\varepsilon(\theta) \, d\theta
$$

As $\int_0^\pi \left| \sin \frac{\theta}{2} \right|^4 \beta^\varepsilon(\theta) \, d\theta$ tends to 0 as $\varepsilon$ tends to 0, the sequence $(Q^\varepsilon)_{\varepsilon > 0}$ is $C$-tight.

## 3.4 Identification of the limit point values $P$

Let $P$ be a limit value of the sequence $(Q^\varepsilon)$. Then $P$ is the limit of a subsequence $(Q^\varepsilon)$ that we will still denote by $(Q^\varepsilon)$ for simplicity. We wish to prove that $P$ is the law of a Landau process, that means is solution of the martingale problem $(LMP)$. Let $\phi \in C_b^2 \left( I\!\!R^3 \right)$. We define the two following processes on $I\!\!D_T$

$$
M_t^\varepsilon = \phi(X_t) - \phi(X_0) - \int_0^t < K_{\beta^\varepsilon, \gamma}^\phi(X_s, v_*), Q_s^\varepsilon(dv_*) > ds \tag{3.9}
$$

$$
M_t = \phi(X_t) - \phi(X_0) - \int_0^t < L^\phi(X_s, v_*), P_s(dv_*) > ds \tag{3.10}
$$

The probability measure $P$ will be a solution of the nonlinear martingale problem $(LMP)$ with initial law $Q_0$ if it satisfies, for any $0 \le s_1 < ... < s_p < s < t \le T$ and $G \in C_b((I\!\!R^3)^p)$,

$$
< (M_t - M_s) G(X_{s_1}, ..., X_{s_p}), P >= 0
$$

However, $Q^\varepsilon$ is a solution of $(B^\varepsilon MP)$, then we know that, for any $0 \le s_1 < ... < s_p < s < t \le T$ and $G \in C_b((I\!\!R^3)^p)$,

$$
< (M_t^\varepsilon - M_s^\varepsilon) G(X_{s_1}, ..., X_{s_p}), Q^\varepsilon >= 0
$$

Thus, we want to state the following convergence

$$
E_{Q^\varepsilon} \left[ (M_t^\varepsilon - M_s^\varepsilon) G(X_{s_1}, ..., X_{s_p}) \right] \xrightarrow[\varepsilon \to 0]{?} E_P \left[ (M_t - M_s) G(X_{s_1}, ..., X_{s_p}) \right]
$$

1. Since $(Q^\varepsilon)$ is C-tight, the distribution $P$ charges only the set $\mathcal{C}_T$, then the mapping $F : x \mapsto (\phi(x_t) - \phi(x_s)) G(x_{s_1}, ..., x_{s_p})$ is $P$-continuous and bounded from $I\!\!D_T$ to $I\!\!R$. Thus $< F, Q^\varepsilon >$ tends to $< F, P >$ as $\varepsilon$ tends to zero.

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2. We now study the convergence of $E_{Q^\varepsilon}\left[\left\{\int_s^t < K^\phi_{\beta^\varepsilon,\gamma}(X_u, v_*), Q^\varepsilon_u(dv_*) > du\right\} G(X_{s_1}, ..., X_{s_p})\right]$

to $E_P\left[\left\{\int_s^t < L^\phi(X_u, v_*), P_u(dv_*) > du\right\} G(X_{s_1}, ..., X_{s_p})\right]$.

If we denote by $(X, Y)$ the canonical process on $\mathbb{D}_T \times \mathbb{D}_T$, we can write

$$E_{Q^\varepsilon}\left[\left\{\int_s^t < K^\phi_{\beta^\varepsilon,\gamma}(X_u, v_*), Q^\varepsilon_u(dv_*) > du\right\} G(X_{s_1}, ..., X_{s_p})\right]$$

$$- E_P\left[\left\{\int_s^t < L^\phi(X_u, v_*), P_u(dv_*) > du\right\} G(X_{s_1}, ..., X_{s_p})\right]$$

$$= E_1 + E_2$$

with

$$E_1 = E_{Q^\varepsilon \otimes Q^\varepsilon}\left[\left\{\int_s^t \left(K^\phi_{\beta^\varepsilon,\gamma}(X_u, Y_u) - L^\phi(X_u, Y_u)\right) du\right\} G(X_{s_1}, ..., X_{s_p})\right]$$

$$E_2 = E_{Q^\varepsilon \otimes Q^\varepsilon}\left[\left\{\int_s^t L^\phi(X_u, Y_u) du\right\} G(X_{s_1}, ..., X_{s_p})\right]$$

$$- E_{P \otimes P}\left[\left\{\int_s^t L^\phi(X_u, Y_u) du\right\} G(X_{s_1}, ..., X_{s_p})\right]$$

(a) <u>Study of $E_1$</u>:

$$|E_1| \le K E_{Q^\varepsilon \otimes Q^\varepsilon}\left[\int_s^t \left| K^\phi_{\beta^\varepsilon,\gamma}(X_u, Y_u) - L^\phi(X_u, Y_u)\right| du\right] \qquad (3.11)$$

The Taylor development of $\phi$ writes

$$\phi(v + u) = \phi(v) + u.\nabla\phi(v) + \frac{1}{2} u^t.J_\phi(v).u + O\left(|u|^3\right)$$

We notice that $u^t.J_\phi(v).u = J_\phi(v) : u.u^t$. Then we can divide the expectation of the right term in (3.11) in three parts :

$$E_{Q^\varepsilon \otimes Q^\varepsilon}\left[\int_s^t \left| K^\phi_{\beta^\varepsilon,\gamma}(X_u, Y_u) - L^\phi(X_u, Y_u)\right| du\right] \le E_{11} + E_{12} + E_{13}$$

with

$$E_{11} = K(|-2\Lambda + b^\varepsilon|) E_{Q^\varepsilon \otimes Q^\varepsilon}\left[\int_s^t |h(|X_u - Y_u|)|X_u - Y_u|^\gamma(X_u - Y_u).\nabla\phi(X_u)| du\right]$$

$$E_{12} = K E_{Q^\varepsilon \otimes Q^\varepsilon}\left[\int_s^t (h(|X_u - Y_u|)|X_u - Y_u|^\gamma) \left| J_\phi(X_u) : \left(\Lambda |X_u - Y_u|^2 \Pi(X_u - Y_u)\right.\right.\right.$$

$$\left.\left.\left. - \int_0^{2\pi}\int_0^\pi a(X_u, Y_u, \theta, \varphi).a^t(X_u, Y_u, \theta, \varphi)\beta^\varepsilon(\theta)d\theta d\varphi\right)\right| du\right]$$

$$E_{13} = K E_{Q^\varepsilon \otimes Q^\varepsilon}\left[\int_s^t h(|X_u - Y_u|)|X_u - Y_u|^\gamma \left[\int_0^{2\pi}\int_0^\pi |a(X_u, Y_u, \theta, \varphi)|^3 \beta^\varepsilon(\theta)d\theta d\varphi\right] du\right]$$

- We consider the first expectation $E_{11}$ :
  Using estimates (1.1) and thanks to (3.8), we have

  $$E_{11} \leq K \left| -2\Lambda + b^{\varepsilon} \right|$$

  As $b^{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} 2\Lambda$, $E_{11}$ converges towards 0 as $\varepsilon$ tends to 0.
- Let us now study $E_{12}$ :
  After some computations, we prove that

  $$\int_0^{2\pi} a(X_u, Y_u, \theta, \varphi).a^t(X_u, Y_u, \theta, \varphi) d\varphi$$
  $$= \frac{\pi}{4} \left[ \Pi(X_u - Y_u) \sin^2 \theta + 2(I - \Pi(X_u - Y_u))(\cos\theta - 1)^2 \right] |X_u - Y_u|^2$$

  Then

  $$\int_0^{2\pi} \int_0^{\pi} a(X_u, Y_u, \theta, \varphi).a^t(X_u, Y_u, \theta, \varphi) \beta^{\varepsilon}(\theta) d\theta d\varphi \underset{\varepsilon \to 0}{\longrightarrow} \Lambda \Pi(X_u - Y_u) |X_u - Y_u|^2$$

  Thanks to (3.8), we conclude that $E_{11}$ converges towards 0 as $\varepsilon$ tends to 0.
- Using similar arguments and Lemma 2.1, we prove that $E_{13}$ also converges towards 0 as $\varepsilon$ tends to 0.

Finally, we have proved that $E_1 \underset{\varepsilon \to 0}{\longrightarrow} 0$.

(b) Study of $E_2$

$$E_2 = E_{Q^{\varepsilon} \otimes Q^{\varepsilon}} \left[ \left\{ \int_s^t L^{\phi}(X_s, Y_s) ds \right\} G(X_{s_1}, ..., X_{s_p}) \right] - E_{P \otimes P} \left[ \left\{ \int_s^t L^{\phi}(X_s, Y_s) ds \right\} G(X_{s_1}, ..., X_{s_p}) \right]$$

The functions $f_{ij}^A : I\!D_T \times I\!D_T \to I\!R, (x, y) \longmapsto G(X_{s_1}, ..., X_{s_p}) \int_s^t A_{ij}(x_u - y_u) \partial_{ij} \phi(x_u) du$

and $f_i^b : I\!D_T \times I\!D_T \to I\!R, (x, y) \longmapsto G(X_{s_1}, ..., X_{s_p}) \int_s^t b_i(x_u - y_u) \partial_i \phi(x_u) du$

are continuous functions ($\gamma \in (-1, 0]$), but not necessarily bounded. Nevertheless, using similar arguments as in the proof of Theorem 2.7 in Section 2, we obtain $E_2 \underset{\varepsilon \to 0}{\longrightarrow} 0$.

**Conclusion** For any $(t, s, s_1, ..., s_p) \in (I\!R_+)^{p+2}$, with $0 \leq s_1 \leq ... \leq s_p \leq s < t$, we have proved

$$E_{Q^{\varepsilon}} \left[ (M_t - M_s) G(X_{s_1}, ..., X_{s_p}) \right] \underset{n \to \infty}{\longrightarrow} E_P \left[ (M_t - M_s) G(X_{s_1}, ..., X_{s_p}) \right]$$

which implies that

$$E_P \left[ (M_t - M_s) G(X_{s_1}, ..., X_{s_p}) \right] = 0$$

So, $(M_t)_{t \geq 0}$ is a $P$-martingale and $P$ satisfies the martingale problem $(LMP)$.

# 4    A stochastic particle approximation

In this part, we introduce some stochastic colliding particle systems and prove the convergence of their empirical measures to a solution of (3.4) when the number of particles tends to infinity and the grazing collision parameter of the cross-section tends to 0. This approximation is natural in a probabilistic point of view and is the theorical fundation of a very simple Monte-Carlo algorithm described in the next section, based upon particle systems which conserve momentum and kinetic energy.

In order to define the particle system, we consider a sequence of cutoff cross-sections

$$B_{k,l}(z,\theta) = h(|z|)(|z|^\gamma \wedge k)\beta^{\varepsilon_l}(\theta) \tag{4.1}$$

where $h$ is a locally Lipschitz function bounded by $H$, $\gamma \in (-1,0]$, $(\varepsilon_l)_l$ a sequence of parameters tending to 0 as $l$ tends to infinity and $\beta^\varepsilon$ is a $L^1([0,\pi])$-function satisfying (3.5) and (3.6), $k$ is a positive integer. As before we denote by $\psi_k(z)$ the function $h(|z|)(|z|^\gamma \wedge k)$.

The natural interpretation of the nonlinearity in (3.4) plus a physical interpretation of the equation lead naturally to binary mean field interacting particle systems which conserve momentum and kinetic energy. These $n$-particle systems are $(I\!R^3)^n$-valued pure-jump Markov processes with generators defined for $\phi \in C_b((I\!R^3)^n)$ by

$$\frac{1}{n-1}\sum_{1\leq i,j\leq n}\int_0^{2\pi}\int_0^\pi\int_0^{kH}\frac{1}{2}\bigg(\phi(v^n + \mathbf{e_i}.a(v_i,v_j,\theta,\varphi)\mathbf{1}_{\{x\leq\psi_k(v_i-v_j)\}}$$

$$+\mathbf{e_j}.a(v_j,v_i,\theta,\varphi)\mathbf{1}_{\{x\leq\psi_k(v_i-v_j)\}}) - \phi(v^n)\bigg)dx\beta^{\varepsilon_l}(\theta)d\theta d\varphi. \tag{4.2}$$

Here $v^n = (v_1,...,v_n)$ denotes the generic point of $(I\!R^3)^n$ and $\mathbf{e_i} : h \in I\!R^3 \mapsto \mathbf{e_i}.h = (0,...,0,h,0,...,0) \in (I\!R^3)^n$ with $h$ at the $i$-th place.

We denote by

$$V^{kl,n} = (V^{kl,1n},...,V^{kl,nn})$$

the Markov process defined by (4.2).

We consider as in Section 2 a pathwise representation of such processes using the family of independent Poisson point measures $(N^{l,ij})_{1\leq i<j\leq n}$ on $[0,\pi]\times[0,2\pi]\times[0,kH]\times[0,T]$ with intensities $\frac{1}{n-1}\beta^{\varepsilon_l}(\theta)d\theta d\varphi dx dt$. For $i>j$, we set $N^{l,ij} = N^{l,ji}$. We define the process $(V^{kl,in})_{1\leq i\leq n}$ solution of the following stochastic differential equation:

$$V_t^{kl,in} = V_0^i + \sum_{j\neq i,j=1}^n \int_0^t\int_0^{kH}\int_0^{2\pi}\int_0^{2\pi} a(V_{s-}^{kl,in}, V_{s-}^{kl,jn},\theta,\varphi)\mathbf{1}_{\{x\leq\psi_k(V_{s-}^{kl,in}-V_{s-}^{kl,jn})\}}N^{l,ij}(d\theta,d\varphi,dx,ds).$$

$$(4.3)$$

We construct it easily by working recursively on each interjump interval of the point process $(N^{l,ij})_{1\leq i,j\leq n}$. It is a $n$-dimensional Markov process with the generator (4.2).

Let us denote by

$$\mu^{kl,n} = \frac{1}{n}\sum_{i=1}^n \delta_{V^{kl,in}}$$

the empirical measure of this system and by $(\pi^{kl,n})_n$ the sequence of laws of $\mu^{kl,n}$, which are probability measures on $\mathcal{P}(I\!D([0,T],I\!R^3))$.

**Theorem 4.1** *Assume $Q_0 \in \mathcal{P}_4(I\!R^3)$. Let $(V_0^i)_{i \geq 1}$ be independent $Q_0$-distributed random variables. Then the sequence $(\pi^{kl,n})_{k,l,n}$ is uniformly tight for the weak convergence and any limit point charges only probability measures which are solutions of $(LMP)$. Thus any limit point (for the convergence in law) of the sequence $(\mu^{kl,n})$ is a solution of $(LMP)$.*

**Proof.** To prove this theorem, we will show
1) the tightness of $(\pi^{kl,n})_n$ in $\mathcal{P}(\mathcal{P}(I\!D([0,T], I\!R^3)))$,
2) the identification of the limiting values of $(\pi^{kl,n})_{k,l,n}$ as solutions of the nonlinear martingale problem $(LMP)$.
One knows (cf. [22]) that the tightness of $(\pi^{kl,n})_{k,l,n}$ is equivalent to the tightness of the laws of the semimartingales $V^{kl,1n}$ belonging to $\mathcal{P}(I\!D([0,T], I\!R^3))$. This tightness is due to

$$\sup_{k,l,n} E(\sup_{t \leq T} |V_t^{kl,1n}|^4) < +\infty. \tag{4.4}$$

This moment condition is obtained by a good use of Burkholder-Davis-Gundy and Doob's inequalities for (4.3) and Lemma 1.1.
Let us now prove that all the limit values are solutions of the nonlinear martingale problem $(LMP)$. Consider $\pi^\infty \in \mathcal{P}(\mathcal{P}(I\!D([0,T], I\!R^3)))$ a limit value of $(\pi^{kl,n})$. It is the limit point of a subsequence we still denote by $(\pi^{kl,n})$.
We define for $\phi \in C_b^1(I\!R^3)$, $0 \leq s_1, ..., s_p \leq s < t$, $G \in C_b((I\!R^3)^p)$, $Q \in \mathcal{P}(I\!D_T)$ and for $X$ the canonical process on $I\!D([0,T], I\!R^3)$

$$F(Q) = \left\langle G(X_{s_1}, ..., X_{s_p}) \left( \phi(X_t) - \phi(X_s) - \int_s^t \langle L^\phi(X_u, v_*), Q_u(dv_*) \rangle dz du \right), Q \right\rangle. \tag{4.5}$$

Our aim is to prove that $< |F|, \pi^\infty > = 0$.
The mapping $F$ is not continuous since the projections are not continuous for the Skorohod topology. However, for any $Q \in \mathcal{P}(I\!D_T)$, the mapping $X \mapsto X_t$ is $Q$-almost surely continuous for all $t$ outside an at most countable set $D_Q$, and then $F$ is continuous at the point $Q$ if $s, t, s_1, ..., s_p$ are not in $D_Q$. Here we use the continuity and the boundedness of $\phi, G$ and also the continuity of $(q, v) \mapsto \int_{I\!R^3} L^\phi(v, w) q(dw)$ on $\mathcal{P}(I\!D([0,T], I\!R^3)) \times I\!R^3$. Thus, if $s, t, s_1, ..., s_p$ are not in $D_Q$, $F$ is $\pi^\infty$-a.s. continuous. Then,

$$\left\langle F^2, \pi^\infty \right\rangle = \lim_{k,l,n} \left\langle F^2, \pi^{kl,n} \right\rangle$$

But $\left\langle |F|, \pi^{kl,n} \right\rangle \leq \left\langle |F^{kl}|, \pi^{kl,n} \right\rangle + \left\langle |F - F^{kl}|, \pi^{kl,n} \right\rangle$ where

$$F^{kl}(Q) = \left\langle G(X_{s_1}, ..., X_{s_p}) \left( \phi(X_t) - \phi(X_s) - \int_s^t \langle K^\phi_{\beta^{\varepsilon l}, k}(X_u, v_*), Q_u(dv_*) \rangle du \right), Q \right\rangle. \tag{4.6}$$

in which $K^\phi_{\beta^{\varepsilon l}, k}$ is obtained as $K^\phi_{\beta^{\varepsilon l}, \gamma}$ but where $|z|^\gamma$ has been replaced by $|z|^\gamma \wedge k$. In this case and since $\int_0^\pi \beta^{\varepsilon l}(\theta) d\theta < +\infty$, $K^\phi_{\beta^{\varepsilon l}, k}$ also writes

$$
\begin{aligned}
K^\phi_{\beta^{\varepsilon l}, k}(v, v_*) &= \int_0^{2\pi} \int_0^\pi (\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v)) \psi_k(v - v_*) \beta^{\varepsilon l}(\theta) d\theta d\varphi \\
&= \int_0^{2\pi} \int_0^\pi \int_0^{kH} (\phi(v + a(v, v_*, \theta, \varphi) \mathbf{1}_{\{x \leq \psi_k(v - v_*)\}}) - \phi(v)) \beta^{\varepsilon l}(\theta) dx d\theta d\varphi.
\end{aligned}
$$

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Firstly,

$$\left\langle (F^{kl})^2, \pi^{kl,n} \right\rangle = E((F^{kl}(\mu^{kl,n}))^2)$$

$$= E\left( \left( \frac{1}{n} \sum_{i=1}^{n} (M_t^{kl,i\phi} - M_s^{kl,i\phi}) G(V_{s_1}^{kl,in}, ..., V_{s_p}^{kl,in}) \right)^2 \right)$$

$$= \frac{1}{n} E\left( \left( (M_t^{kl,1\phi} - M_s^{kl,1\phi}) G(V_{s_1}^{kl,1n}, ..., V_{s_p}^{kl,1n}) \right)^2 \right) \tag{4.7}$$

$$+ \frac{n-1}{n} E\left( (M_t^{l,1\phi} - M_s^{l,1\phi})(M_t^{l,2\phi} - M_s^{l,2\phi}) G(V_{s_1}^{kl,1n}, ..., V_{s_p}^{kl,1n}) G(V_{s_1}^{kl,2n}, ..., V_{s_p}^{kl,2n}) \right)$$

where $M^{kl,i\phi}$ is the martingale defined by

$$M_t^{kl,i\phi} = \phi(V_t^{kl,in}) - \phi(V_0^i)$$

$$- \frac{1}{n-1} \sum_{j=1}^{n} \int_0^t \int_0^{kH} \int_0^{2\pi} \int_0^{\pi} \left( \phi(V_s^{kl,in} + a(V_s^{kl,in}, V_s^{kl,jn}, \theta, \varphi) \mathbf{1}_{\{x \le \psi_k(V_s^{kl,in} - V_s^{kl,jn})\}}) \right.$$

$$\left. - \phi(V_s^{kl,in}) \right) \beta^{\varepsilon_l}(\theta) d\theta d\varphi dx ds$$

and with Doob-Meyer process given by

$$< M^{kl,i\phi} >_t$$

$$= \frac{1}{n-1} \sum_{j=1}^{n} \int_0^t \int_0^{kH} \int_0^{2\pi} \int_0^{\pi} \left( \phi(V_s^{kl,in} + a(V_s^{kl,in}, V_s^{kl,jn}, \theta, \varphi) \mathbf{1}_{\{x \le \psi_k(V_s^{kl,in} - V_s^{kl,jn})\}}) \right.$$

$$\left. - \phi(V_s^{kl,in}) \right)^2 \beta^{\varepsilon_l}(\theta) d\theta d\varphi dx ds$$

and for $i \ne j$,

$$< M^{kl,i\phi}, M^{kl,j\phi} >_t \tag{4.8}$$

$$= \frac{1}{n-1} \int_0^t \int_0^{kH} \int_0^{2\pi} \int_0^{\pi} \left( \phi(V_s^{kl,in} + a(V_s^{kl,in}, V_s^{kl,jn}, \theta, \varphi) \mathbf{1}_{\{x \le \psi_k(V_s^{kl,in} - V_s^{kl,jn})\}} - \phi(V_s^{kl,in}) \right)$$

$$\left( \phi(V_s^{kl,jn} + a(V_s^{kl,jn}, V_s^{kl,in}, \theta, \varphi) \mathbf{1}_{\{x \le \psi_k(V_s^{kl,in} - V_s^{kl,jn})\}} - \phi(V_s^{kl,jn}) \right) \beta^{\varepsilon_l}(\theta) d\theta d\varphi dx ds.$$

The right terms in (4.7) go to 0 thanks to the expression of the Doob-Meyer process, to the uniform integrability proved in (4.4) and to Lemma 1.1. Moreover the convergence is uniform on $k, l$. Hence

$$\lim_n \langle |F^{kl}|, \pi^{kl,n} \rangle = 0, \text{ uniformly in } k, l.$$

Otherwise, the quantity $\langle |F - F^{kl}|, \pi^{kl,n} \rangle = E(|F - F^{kl}|(\mu^{kl,n}))$ can be written in a form analogeous to the right term of (3.11) replacing $Q^\varepsilon$ by $\mu^{kl,n}$. Its study is thus controled in a similar way than the term $E_1$ in Section 3.3. Then it converges to 0 uniformly in $k$ and $n$ as $l$ tends to infinity.

Finally, we have proved that

$$\langle |F|, \pi^\infty \rangle = 0.$$

Thus, $F(Q)$ is $\pi^\infty$-a.s. equal to 0, for every $s, t, s_1, ..., s_p$ outside of the countable set $D_Q$. It is sufficient to assure that $\pi^\infty$-a.s., $Q$ is solution of the nonlinear martingale problem $(LMP)$. Let us remark to conclude that each solution $Q$ of the limiting martingale problem is in fact a probability measure on $\mathcal{C}_T$. This remark allows us to deduce immediately the following corollary. $\triangle$

**Corollary 4.2** *Assume $Q_0 \in \mathcal{P}_4(\mathbb{R}^3)$ and consider a sequence $\mu^{k_r l_r, n_r}$ which converges to $Q$. Then the probability measure-valued process $(\mu_t^{k_r l_r, n_r})_{t \geq 0}$ converges in probability to the flow $(Q_t)_{t \geq 0}$ in the space $\mathbb{D}([0, T], \mathcal{P}(\mathbb{R}^3))$ endowed with the uniform topology.*

# 5 The Monte-Carlo algorithm

We deduce from the above study an algorithm associated with the binary mean-field interacting particle system.

From now on, the quantities $h$, $\gamma$, $k$ and $\beta^\varepsilon$ defining the cross-section $B$, the initial distribution $Q_0$, the terminal time $T > 0$ and the size $n \geq 2$ of the particle system are fixed. We denote by $B_{k,\varepsilon}(z, \theta) = \psi_k(z)\beta^\varepsilon(\theta)$ the corresponding cross-section with cutoff. Because of Theorem 4.1 and Corollary 4.2, we simulate a particle system following (4.2), i.e. the whole path $(V_t^n)_{t \in [0,T]} \in \mathbb{D}([0, T], (\mathbb{R}^3)^n)$.

First of all, we assume that $V_0^n$ is simulated according to the initial distribution $Q_0^{\otimes n}$. Then, we denote by $0 < T_1 < ... < T_k$ the successive jump times until $T$ of a standard Poisson process with parameter $n\pi k H \|\beta^\varepsilon\|_1$.

Before the first collision, the velocities do not change, so that we set $V_s^n = V_0^n$ for all $s < T_1$. Let us describe the first collision. We choose at random a couple $(i, j)$ of particles according a uniform law over $\{(p, m) \in \{1, ..., n\}^2; \ m \neq p\}$. We choose $x$ uniformly on the interval $[0, kH]$, we choose the first angle of collision $\varphi$ uniformly on $[0, 2\pi]$ and we finally choose the collision angle $\theta$ following the law $\frac{\beta^\varepsilon(\theta)}{\|\beta^\varepsilon\|_1} d\theta$. Then we set

$$
\begin{aligned}
V_{T_1}^{n,i} &= V_0^{n,i} + a(V_0^{n,i}, V_0^{n,j}, \theta, \varphi) \mathbf{1}_{\{x \leq \psi_k(V_0^{n,i} - V_0^{n,j})\}} \\
V_{T_1}^{n,j} &= V_0^{n,j} + a(V_0^{n,j}, V_0^{n,i}, \theta, \varphi) \mathbf{1}_{\{x \leq \psi_k(V_0^{n,i} - V_0^{n,j})\}} \\
V_{T_1}^{n,p} &= V_0^{n,p} \quad \text{if } p \neq \{i, j\}
\end{aligned}
$$

Since nothing happens between $T_1$ and $T_2$, we set $V_s^n = V_{T_1}^n$ for all $s \in [T_1, T_2[$. Iterating this method, we simulate $V_{T_1}^n, V_{T_2}^n, ..., V_{T_k}^n$, i.e. the whole path $(V_t^n)_{t \in [0,T]}$, which was our aim.

Notice that this algorithm is very simple and takes a few lines of program and does not require to discretize time. It furthermore conserves momentum and kinetic energy. Let us remark that at least formally, this algorithm can be adapted in a similar way to the coulombian case, since the soft potential term is cutoffed for the simulations.

# 6 Numerical results

We use the previous Monte-Carlo algorithm to estimate the fourth-order moment of a solution of the Landau equation. By this method, one conserves momentum and kinetic energy, and one follows the asymptotics of grazing collisions.

We consider the cross-section $B_{k,\varepsilon}(z,\theta) = \psi_k(z)\beta^\varepsilon(\theta)$ with $\psi_k(z) = |z|^{-\gamma} \wedge k$ and $\beta^\varepsilon$ satisfying Assumptions (2.10), (3.5) and (3.6) of which the form will depend of the value of $\gamma$.

For each $\varepsilon$, $k$, we denote by $Q^{k,\varepsilon}$ the solution of the martingale problem with cross-section $B_{k,\varepsilon}$ obtained in Theorem 2.7. We know that for each $\varepsilon$, $k$, $(Q^{k,\varepsilon})$ is a cluster point as $n$ tends to infinity of the empirical measure $\mu^{k,\varepsilon,n}$ associated with a simulable particle system. We also know that $(Q^{k,\varepsilon})_{\varepsilon>0,k\geq0}$ is tight and that any limit point $P$ is a solution of the martingale problem $(LMP)$ associated with the Landau equation.

At last, we define :

$$m_\gamma^{k,\varepsilon,n}(t) = \int_{I\!R^3} |v|^4\, \mu_t^{k,\varepsilon,n}(dv) \quad ; \quad m_\gamma^{k,\varepsilon}(t) = \int_{I\!R^3} |v|^4\, Q_t^{k,\varepsilon}(dv)$$

$$\text{and } m_\gamma(t) = \int_{I\!R^3} |v|^4\, P_t(dv).$$

We mention that there is no explicit computation of the fourth-order moment $m_t$ for the Landau equation in our context.

## 6.1 The 'moderately soft' potentials case, $\gamma \in\, ] - 1, 0]$

We fix $\gamma = -0.8$ and we consider the following asymptotic

$$\beta^\varepsilon(\theta) = \frac{1}{2\pi\varepsilon^3 \sin\left(\frac{\theta}{2\varepsilon}\right)^2} \mathbf{1}_{\varepsilon \leq |\frac{\theta}{\varepsilon}| \leq \pi}$$

This function satisfies Assumptions (2.10) for any $\varepsilon > 0$ and (3.5), (3.6) when $\varepsilon$ tends to zero. We notice that $\|\beta^\varepsilon\|_1 = \frac{1}{\pi\varepsilon^2}\tan^{-1}(\varepsilon/2)$ and $\Lambda^\varepsilon = \pi\int\beta^\varepsilon(\theta)\sin^2\left(\frac{\theta}{2}\right)d\theta$ converges towards $\Lambda = \pi\ln 2$ as $\varepsilon$ tends to 0.

We also consider the initial distribution on $I\!R^3$, $Q_0(dv) = \mathbf{1}_{[-1/2;1/2]^3}(v)\,dv$.

We first estimate $m_{-0.8}(t)$ at time $t = \frac{1}{2\pi}$ . We consider $n = 50000$ particules.

First of all, when we consider the mean over 100 simulations of $m_{-0.8}^{k,0.1,50000}(\frac{1}{2\pi})$, we notice that it converges very fastly in $k$. Hence the error due to the *spatial* cutoff is small :

| $k$ | 1 | 4 | 6 | 10 | 50 |
|---|---|---|---|---|---|
| $m_{-0.8}^{k,0.1,50000}(\frac{1}{2\pi})$ | 0.09742 | 0.09873 | 0.09881 | 0.09878 | 0.09875 |

So we fix $k = 6$ in all what follows.

We now study the convergence of $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$ as $\varepsilon$ tends to zero. Taking each time the mean over 100 simulations, we observe in Figure 1 the convergence of the fourth-order moments for the Boltzmann equation to the one for the Landau equation when $\varepsilon$ becomes small.
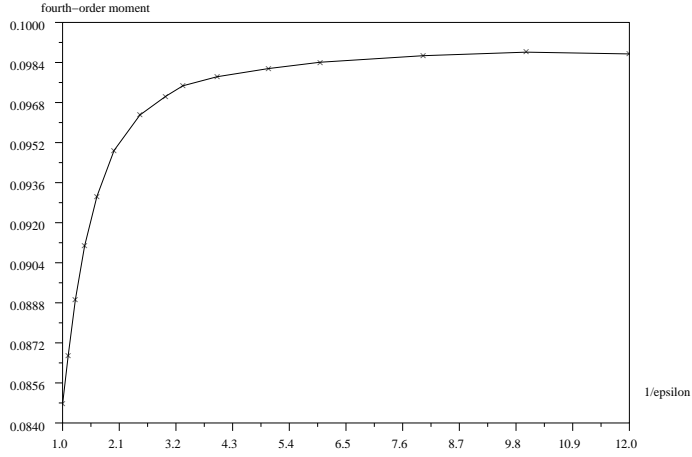
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Figure 1. Evolution in $1/\varepsilon$ of $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$.

One can notice that $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$ tends to 0.0988, with a speed of convergence in $\left|m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi}) - 0.0988\right| \simeq 0.015*\varepsilon^2$, when $\varepsilon$ tends to zero. Hence, the choice $\varepsilon = 0.1$ seems reasonable to describe the Landau behaviour.

Our algorithm describes precisely the convergence of the Boltzmann equation to the Landau equation. But we take into account all small jumps, then the duration of computation is not optimal. For example, when $\varepsilon = 0.1$ and $k = 6$, there is arround $25.10^6$ shocks of particles on the time interval $[0, 1]$.

Let us now study the speed of convergence of $m_{-0.8}^{6,0.1,n}(\frac{1}{2\pi})$ to $m_{-0.8}^{6,0.1}(\frac{1}{2\pi})$, when $n$ tends to infinity. We obtain the Figure 2.
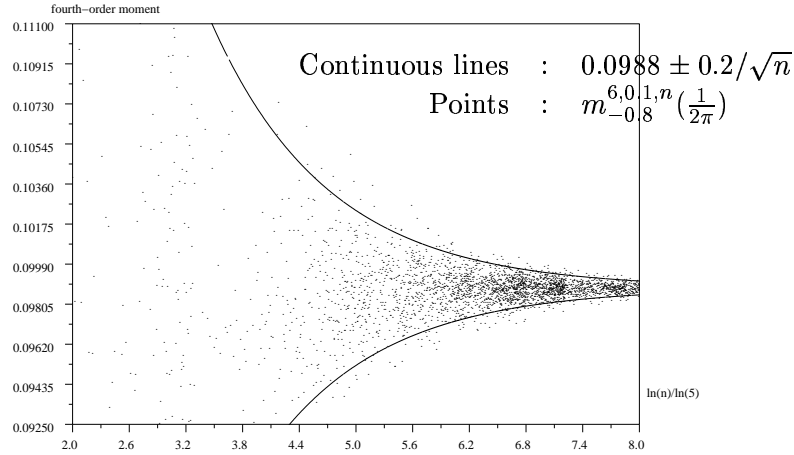


Figure 2. Evolution of $m_{-0.8}^{6,0.1,n}(\frac{1}{2\pi})$ as $n \to +\infty$.

The speed of convergence is in $1/\sqrt{n}$. It seems that a central limit theorem holds. (A proof of a similar central limit theorem has been obtained by Fournier-Méléard [10] from 2D Bolztmann equations without cutoff and for Maxwell molecules).

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At last, we observe the evolution in time of the fourth-order moment. (Our method conserves the energy, then the moment of order two is constant in time). We fix again $k = 6$ and $\varepsilon = 0.1$ and we observe the moments of order 4 for some values of $t \in [0,1]$ :
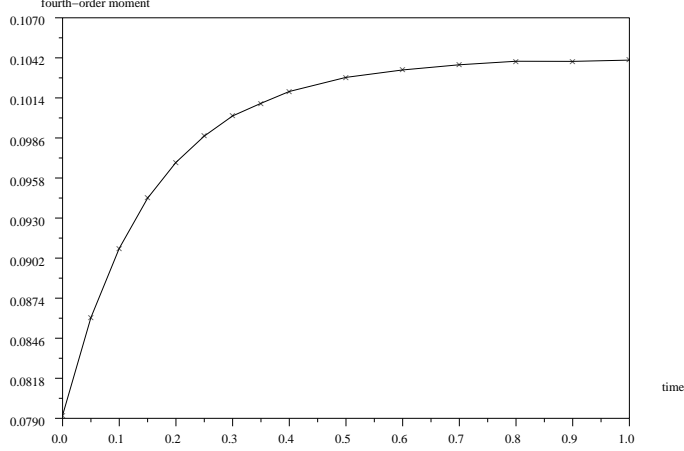


Figure 3. Evolution in time of $m_{-0.8}^{6,0.1,50000}(t)$.

## 6.2 The coulombian case

Our theorical results are satisfied for a potential $\gamma \in ]-1,0]$. But our numerical approach works in the interesting case for the Landau equation is the case of Coulomb molecules, $\gamma = -3$.

We consider now our algorithm with $\gamma = -3$ and with the same initial condition of Buet, Cordier, Degond and Lemou in [2]. We consider $n = 50000$ particles and each value is obtained taking the mean over 100 simulations. We take as initial condition $Q_0$ the mesure with the following density with respect the Lebesgue measure:

$$f(0,v) = \frac{1}{2}\left(M_{\mathcal{N},v_{01},v_{th}} + M_{\mathcal{N},v_{02},v_{th}}\right)$$

where $M_{\mathcal{N},u,v_{th}}$ is the Maxwellian function on $\mathbb{R}^3$

$$M_{\mathcal{N},u,v_{th}}(v) = \frac{\mathcal{N}}{(2\pi v_{th}^2)^{3/2}}exp\left(-\frac{|v-u|^2}{2v_{th}^2}\right)$$

with $\mathcal{N} = 5$, $v_{th} = 0.45$, $v_{01} = (2,3,3)$ and $v_{02} = (4,3,3)$.
Moreover we take the asymptotic defined in [4]:

$$\beta^\varepsilon(\theta) = \frac{1}{|\log \varepsilon|}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\mathbb{I}_{\theta \geq \varepsilon}$$

In this situation, $\Lambda^\varepsilon$ converges towards $\Lambda = \frac{1}{2}$ as $\varepsilon$ tends to 0. Thus, comparing our expression of the Landau equation (3.1) and the one of Buet, Cordier, Degond and Lemou in [2], we notice that we simulate the same quantity, there is no multiplicative factor.

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We notice that the initial data is not a measure of probability, its mass is equal to 5. Adapting the results obtained by Méléard in [21] for the Navier-Stokes equation, we just have to consider in our algorithm the following empirical measure

$$\mu^{k,\varepsilon,n} = \frac{5}{n} \sum_{i=1}^{n} \delta_{V^{k\varepsilon,in}}$$

and jump times on the interval $[0,T]$ of a standard Poisson process with parameter $5n\pi k\|\beta^\varepsilon\|_1$.

We first estimate the fourth-order moment $m_{-3}(t)$ at time $t = 0.06$.
As for the previous simulations, the algorithm converges very fastly in $k$. Then we fix from now $k = 6$.
We observe that the convergence in $\varepsilon$ of the fourth-order moment of the Boltzmann equation to the one of the Landau equation is very fast:

| $\varepsilon$ | 0.9 | 0.6 | 0.2 | 0.1 | 0.08 |
|---|---|---|---|---|---|
| $m_{-3}^{6,\varepsilon,50000}(0.06)$ | 4389.5 | 4389.1 | 4389.9 | 4388.9 | 4388.5 |

The choice of $\varepsilon = 0.2$ seems to be reasonable to describe the Landau moment.
At last, we fix $k = 6$ and $\varepsilon = 0.2$ and we observe the evolution in time of the fourth-order moment. We find the same evolution as described in [2].
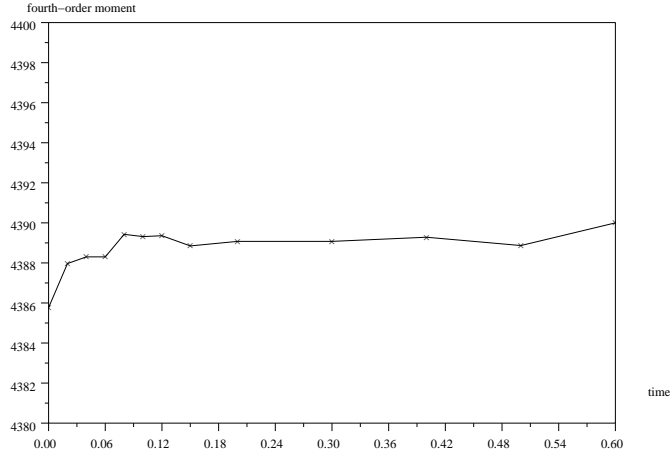


Figure 5. Evolution in time of $m_{-3}^{6,0.2,50000}(t)$.

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