MAXIMAL INEQUALITIES AND AN INVARIANCE PRINCIPLE FOR A CLASS OF WEAKLY DEPENDENT RANDOM VARIABLES

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Abstract

The aim of this paper is to investigate the properties of the maximum of partial sums for a class of weakly dependent random variables which includes the instantaneous filters of a Gaussian sequence having a positive continuous spectral density. The results are used to obtain an invariance principle for strongly mixing sequences of random variables in the absence of stationarity or strong mixing rates. An additional condition is imposed to the coefficients of interlaced mixing. The results are applied to linear processes of strongly mixing sequences.

Short Title: Maximal inequalities for sums of dependent random variables.

Key words and phrases: maximal inequalities, invariance principles, dependent random variables, Rosenthal inequality.

1 Introduction

Let (Ω, K, P) be a probability space and let $\mathcal{A}, \mathcal{B}$ be two sub σ-algebras of K. Define the strong mixing coefficient by

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(AB) - P(A)P(B)|$$

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and the maximal coefficient of correlation

$$\rho(\mathcal{A},\mathcal{B}) = \sup_{f\in L_2(\mathcal{A}),g\in L_2(\mathcal{B})} |\mathrm{corr}(f,g)|$$

A strictly stationary sequence $\{X_i\}_{i\in Z}$ is called strong mixing, or α-mixing, if $\alpha_n \to 0$ where

$$\alpha_n = \alpha(\sigma(X_i, i \le 0), \sigma(X_i, i \ge n)).$$

It should be noted that in order for the invariance principle to hold for a strictly stationary strong mixing sequence of random variables it is required the existence of moments of order strictly higher than two in combination with a polynomial mixing rate. (See Peligrad (1986a) for a survey and Doukhan, Massart and Rio (1994)).

In many situations the mixing rates are hard to estimate. Therefore it is interesting to replace the strong mixing rates by sufficient conditions imposed to some other dependence coefficients that might be in certain situations easier to verify. Papers by Bradley (1981) and Peligrad (1982), (1986b) are steps in this direction.

In this paper we shall consider the following mixing coefficients:

For a stationary sequence $\{X_k\}_{k\in Z}$ denote by $\mathcal{F}_T = \sigma(X_i; i \in T)$ where T is a family of integers. Define

$$\alpha_n^* = \sup \alpha(\mathcal{F}_T, \mathcal{F}_s)$$

$$\rho_n^* = \sup \rho(\mathcal{F}_T, \mathcal{F}_s)$$

where these sups are taken over all pairs of nonempty finite sets S, T of Z such that $\mathrm{dist}(S,T) \ge n$.

According to Bradley (1993) for every $n \ge 1$ (if $\alpha_k^* \to 0$)

$$\alpha_n^* \le \rho_n^* \le 2\pi\alpha_n^*$$

Bradley (1992) proved in the context of strictly stationary random fields that the condition $\alpha_n^* \to 0$ as $n \to \infty$ contains enough information to assure the CLT without any additional rate or moments higher than 2. Miller (1995) analyzed the fourth moment of partial sums of such a random field not necessarily stationary, and proved the CLT for some estimators of spectral density for strictly stationary random fields. Bryc and Smolenski (1993) found bounds for the moments of partial sums for sequences of random variables satisfying $\rho_1^* < 1$, or

$$\lim_{n\to\infty} \rho_n^* < 1 \tag{1.1}$$

For strongly mixing sequences satisfying either $\rho_1^* < 1$ or $\lim_{n\to\infty} \rho_n^* < 1$, by the Remark 3 in Bryc and Smolenski (1993) we know that these conditions do not necessarily imply $\lim_{n\to\infty} \rho_n^* = 0$. Moreover in some situations these coefficients, or closely related ones are easy to estimate. With the same notation as in Bradley (1992) we denote by

$$r_n^* = \sup |\mathrm{corr}(V,W)|$$

where the supremum is taken over all finite subsets S, T of Z such that $\mathrm{dist}(S,T) \ge n$ and over all the linear combinations $V = \sum_{i\in S} a_i X_i$ and $W = \sum_{i\in T} b_i X_i$.

According to the proof of Theorem 2 in Bradley (1993) and the Remark 3 in Bryc and Smolenski (1993) one can see that if $\{X_k\}_{k\in Z}$ has a bounded positive spectral density, i.e. $0 < m < f(t) < M$ for every t one has $r_1^* < 1 - m/M < 1$.

For stationary Gaussian sequences the coefficients ρ_n^* and r_n^* are identical (Kolmogorov and Rozanov (1960)). As a consequence our results are easily applicable to filters

$$\xi_{ni} = f_n(X_i, X_{i+1}, \ldots, X_{i+m_n})$$

where the underlying sequence $\{X_i\}$ is stationary, strongly mixing Gaussian sequences which has a bounded spectral density which stays away from 0. When $m_n = 0$ for every n such a sequence satisfies $\rho_1^* < 1$ and when $\sup_n m_n < \infty$ we have $\lim_{n\to\infty} \rho_n^* < 1$.

The strong mixing property for a Gaussian sequence can also be expressed in terms of the form of the spectral density (Ibragimov and Rozanov (1978), chapters 4,5).

Our results do not assume stationarity and they deal with triangular arrays of random variables, $\{\xi_{ni}, 1 \leq i \leq k_n\}$ where $k_n \to \infty$.

In this context we shall define

$$\overline{\alpha}_{nk} = \sup_{s\geq 1} \alpha(\sigma(\xi_{ni}, i \leq s), \sigma(\xi_{nj}, j \geq s+k)) \qquad \text{and} \qquad \overline{\alpha}_k = \sup_n \alpha_{nk}.$$

The triangular array will be called strongly mixing if $\lim_{k\to\infty} \overline{\alpha}_k = 0$. Similarly we define

$$\overline{\rho}_{nk}^* = \sup_k (\rho(\sigma(\xi_{ni}, i \in T), \sigma(\xi_{nj}, j \in S)) \qquad \text{and} \qquad \overline{\rho}_k^* = \sup_n \overline{\rho}_{nk}^*.$$

where $T, S \subset \{1, 2, \ldots, k_n\}$ are nonempty subsets such that $\text{dist}(T, S) \geq k$. In this paper we shall impose the condition

$$\lim_{n\to\infty} \overline{\rho}_n^* < 1 \tag{1.2}$$

Various moment inequalities for sums and maximum of partial sums are contained in papers by Bradley (1992), Bryc and Smolenski (1993), Peligrad (1998), Peligrad and Gut (1999), Bradley and Utev (1994). They are excellently surveyed in Bradley (1999). All these inequalities are important steps which allowed us to establish our general maximal inequality under the condition (1.1) (Theorem 2.1 in this paper).

In the nonstationary context Peligrad (1996) studied the importance of condition (1.2) in the CLT for strongly mixing sequences. She proved that a nonstationary strong mixing sequence satisfying Lindeberg condition and (1.2) satisfies the central limit theorem. Our Theorem 2.2 in this paper gives an invariance principle for random elements associated to sums of strongly mixing triangular arrays of random variables satisfying (1.2). What is notable is that only Lindeberg condition is assumed and no mixing rate is imposed. This invariance principle generalizes the corresponding results for independent random variables of Prohorov (1956). The strictly stationary case is studied in Peligrad (1998).

In the next text we shall denote by $[x]$ the integer part of $x, \Rightarrow$ denotes the weak convergence.

2 Results

Our first theorem is a Rosenthal type moment maximal inequality for sums of random variables in terms of the interlaced mixing coefficients $\{\rho_n^*\}$.

Theorem 2.1 *Let $\{X_i\}_{i\geq 1}$ be a sequence of random variables with $EX_i = 0$ and $E|X_i|^q < \infty$ for every $i \geq 1$, and a certain $q \geq 2$. Assume there is a N such that $\rho_N^* < 1$. Then there is a constant $D(q, N, \rho_N^*)$ such that*

$$E \max_{1\leq i\leq n} \left|\sum_{j=1}^{i} X_j\right|^q \leq D(q, N, \rho_N^*) \left(\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{\frac{q}{2}}\right)$$

for all n. $\square$

This theorem is a basic tool in establishing various almost sure results for sums of dependent random variables.

In this paper we shall use our maximal inequality to establish the functional form of the central limit theorem of Peligrad (1996).

For a triangular array $\{\xi_{ni}; 1 \leq i \leq k_n\}$ of square integrable ($E\xi_{ni}^2 < \infty$) centered ($E\xi_{ni} = 0$) random variables, we denote by $\sigma_{nm}^2 = \mathrm{Var}\left(\sum_{i=1}^m \xi_{ni}\right)$ for $m \leq k_n$, and $\sigma_n^2 = \sigma_{nk_n}^2$ and define, for $0 \leq t \leq 1$,

$$\nu_t = \inf\left\{m; 1 \leq m \leq k_n : \frac{\sigma_{nm}^2}{\sigma_n^2} \geq t\right\} \qquad \text{and} \qquad W_n(t) = \frac{\sum_{i=1}^{\nu_t} \xi_{ni}}{\sigma_n} \tag{2.1}$$

In the analysis of limit theorems for dependent random variables

one of the difficulties is the irregular behaviour of second order characteristics. To control it, we assume that there exists a constant C such that for all for all $0 \leq t < t + \delta \leq 1$,

$$\limsup_{n\to\infty} \frac{\sum_{i=\nu_t}^{\nu_{t+\delta}} \mathrm{Var}(\xi_{ni})}{\mathrm{Var}\left(\sum_{i=\nu_t}^{\nu_{t+\delta}} \xi_{ni}\right)} \leq C \tag{2.2}$$

Theorem 2.2 *Let $\{\xi_{ni}; 1 \leq i \leq k_n\}$ be a strongly mixing triangular array of square integrable centered random variables, which satisfies (1.2), (2.2) and in addition for every $\epsilon > 0$*

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{k_n} E\xi_{ni}^2 I(|\xi_{ni}| > \epsilon\sigma_n) \to 0 \quad as \quad n \to \infty. \tag{2.3}$$

Then

$$W_n(t) \overset{\mathcal{D}}{\Rightarrow} W(t) \tag{2.4}$$

where $W_n(t)$ is defined by (2.1) and $W(t)$ denotes the standard Brownian process on $[0, 1]$.

It follows from Lemma 3.1 below, that condition (2.2) is immediately satisfied when $\rho_1^* < 1$ and we derive

Corollary 2.1 *Let $\{\xi_{ni}; 1 \leq i \leq k_n\}$ be a strongly mixing triangular array of square integrable centered random variables which satisfies (2.3) and $\overline{\rho}_1^* < 1$. Then (2.4) holds.*

The next corollary is motivated by the asymptotic behavior of linear processes, where the main part of nonstationarity comes from nonrandom normalizers.

Corollary 2.2 *Suppose $\{X_k\}$ is a strongly mixing sequence of random variables which is centered and $\{X_k^2\}$ is a uniformly integrable family. Consider the triangular array of random variables $\{a_{nk}X_k, 1 \leq k \leq n\}$ where a_{nk} are numerical constants and denote $\sigma_n^2 = \mathrm{Var}(\sum_{i=1}^n a_{ni}X_i)$. Assume $\overline{\rho}_1^* < 1$ and $\max_{1\leq k\leq n} |a_{nk}|/\sigma_n \to 0$ as $n \to \infty$. Then (2.4) holds where $W_n(t)$ is defined by (2.1) with $\xi_{nk} = a_{nk}X_k; 1 \leq k \leq n$.*

Other class of triangular arrays when the condition (2.2) is simplified are weakly (second-order or covariance) stationary triangular arrays of centered variables that is

$$E\xi_{ni} = 0, 1 \leq i \leq n, \quad \text{and} \quad E[\xi_{ni}\xi_{n(i+j)}] = E[\xi_{n1}\xi_{n(1+j)}] : j = 0, \ldots, n-1, i = 1, \ldots, n-j$$

In this case, we introduce a weaker version of (2.2)

$$\limsup_{n\to\infty} \frac{n\mathrm{E}\xi_{n1}^2}{\sigma_n} \leq C \tag{2.5}$$

Corollary 2.3 *Let $\{\xi_{ni}; 1 \leq i \leq n\}$ be a strongly mixing weakly stationary triangular array of square integrable centered random variables which satisfies (1.2), (2.3) and (2.5). Then (2.4) holds.*

Finally, when $\xi_{kn} = X_k$, $n = 1, 2, \ldots$, $1 \leq k \leq n$, $\sigma_n^2 = \mathrm{Var}(X_1 + \ldots + X_n)$ and $\{X_k\}$ is a weakly stationary sequence, Theorem 2.2 gives as a corollary:

Corollary 2.4 *Suppose $\{X_k\}$ is a strongly mixing weakly stationary sequence of random variables which is centered and $\{X_k^2\}$ is a uniformly integrable family. Assume*

$$\lim_{n\to\infty} \rho_n^* < 1 \qquad \text{and} \qquad \lim_{n\to\infty} \sigma_n^2 = \infty$$

Then (2.4) holds.

Corollary 2.5 *Suppose $\{X_k\}$ is a strongly mixing weakly stationary sequence of nondegenerate random variables which is centered and $\{X_k^2\}$ is a uniformly integrable family. Assume $\rho_1^* < 1$. Then $\lim_{n\to\infty} \sigma_n^2 = \infty$, and (2.4) holds.*

3 Proofs

The proof of Theorem 2.1 uses the following two lemmas. The first lemma gives bounds for the variance of partial sums in terms of a coefficient based on the correlation of sums. It is Lemma 1 in Bradley (1992).

Lemma 3.1 *Suppose $0 < r < 1$. Suppose $\{X_1, X_2, \ldots, X_n\}$ is a family of square integrable centered random variables such that for any nonempty subset*

$$S \subset \{1, 2, \ldots, n\} \qquad S^* = \{1, 2, \ldots, n\} - S$$

$$corr\left(\sum_{k\in S} X_k\,,\ \sum_{k\in S^*} X_k\right) \le r$$

Then

$$\frac{1-r}{1+r}\sum_{k=1}^{n} EX_k^2 \le E\left(\sum_{k=1}^{n} X_k\right)^2 \le \frac{1+r}{1-r}\sum_{k=1}^{n} EX_k^2$$

The next lemma gives estimates of higher moments of partial sums. It is Theorem 1 in Peligrad and Gut (1999). Its first part (3.1), for $2 \le q \le 4$ is due to Bryc and Smolenski (1993).

Lemma 3.2 *Let $\{X_i\}_{i\ge 1}$ be a sequence of random variables and set $S_n = \sum_{k=1}^{n} X_k, n \ge 1$. Suppose that $EX_i = 0$ and that $E|X_i|^q < \infty$ for every $i \ge 1$ and for a certain $q \ge 2$. In addition suppose $\rho_1^* < 1$. Then there exists a constant $D(q,\rho_1^*)$, depending on q, and ρ_1^*, such that:*

$$E|S_n|^q \le D(q,\rho_1^*)\left(\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right), \quad \textit{for all } n. \tag{3.1}$$

For a certain constant $D'(q,\rho_1^*)$, we have

$$E\max_{1\le i\le n}|S_i|^q \le D'(q,\rho_1^*)\left[\left(E\max_{1\le i\le n}|S_i|\right)^q + \sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right] \quad \textit{for all } n. \tag{3.2}$$

The proof of Theorem 2.1 is based on the following new proposition:

Proposition 3.1 *Let $\{Y_i\}_{i\ge 1}$ be a sequence of square integrable centered random variables. Assume that $\rho_1^* < 1$. Then,*

$$E\max_{1\le i\le n}\left|\sum_{j=1}^{i} Y_j\right| \le 32\,K(\rho_1^*)\sqrt{\sum_{j=1}^{n} EY_j^2};$$

where $K = K(\rho_1^*) = D'(4,\rho_1^*) + \frac{1+\rho_1^*}{1-\rho_1^*}$.

Proof. For a positive integer n, define

$$a_n = \sup_Y\left(E\max_{1\le i\le n}\left|\sum_{j=1}^{i} Y_j\right| \Big/ \left[\sum_{j=1}^{n} EY_j^2\right]^{1/2}\right) \tag{3.3}$$

where the supremum is taken over all sequences $\{Y_i\}$ of square integrable centered random variables with $\rho_1^*(\{Y_i\}) \le \rho_1^*$.

Fix such a random sequence $\{Y_i\}$ and in addition without loss of generality assume that

$$\sum_{j=1}^{n} \mathrm{Var}(Y_j) = 1\,.$$

Let M be a positive integer that will be specified later. For $1 \leq j \leq n$ define:

$$\xi_j = Y_j I_{(|Y_j| \leq M^{-1/2})} - EY_j I_{(|Y_j| \leq M^{-1/2})}$$

and

$$\eta_j = Y_j I_{(|Y_j| > M^{-1/2})} - EY_j I_{(|Y_j| > M^{-1/2})}$$

so that

$$\sum_{j=1}^{i} Y_j = \sum_{j=1}^{i} \xi_j + \sum_{j=1}^{i} \eta_j.$$

Since

$$\sum_{j=1}^{n} E|\eta_j| \leq 2M^{1/2} \sum_{j=1}^{n} EY_j^2 = 2M^{1/2}$$

we get

$$E \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} Y_j \right| \leq E \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} \xi_j \right| + 2M^{1/2}$$

To estimate $E \max_{1\leq i\leq n} \left|\sum_{j=1}^{i} \xi_j\right|$, we shall use a blocking procedure.

Take $m_0 = 0$ and define the integers m_k recursively by

$$m_k = \min\left\{ m, m > m_{k-1} \sum_{j=m_{k-1}+1}^{m} E\xi_j^2 > \frac{1}{M} \right\}$$

Note that, if we denote by ℓ the number of integers produced by this procedure, i.e: $m_0, m_1, \ldots, m_{\ell-1}$, we have

$$1 \geq \sum_{k=1}^{\ell-1} \sum_{j=m_{k-1}+1}^{m_k} E\xi_j^2 > \frac{\ell-1}{M} \quad \text{so that} \quad \ell \leq M.$$

We partition the sum $\sum_{j=1}^{n} \xi_j$ in partial sums

$$\sum_{j=1}^{n} \xi_j = \sum_{k=1}^{\ell} X_k$$

where $X_k = \sum_{j=m_{k-1}+1}^{m_k} \xi_j$ for $1 \leq k \leq \ell - 1$ and for convenience, $m_\ell = b$ that is $X_\ell = \sum_{j=m_{\ell-1}+1}^{n} \xi_j$.

Obviously

$$\begin{aligned} E \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} \xi_j \right| &\leq E \max_{1\leq k\leq \ell} \left| \sum_{j=1}^{k} X_j \right| + E \max_{1\leq k\leq \ell} \left(\max_{m_{k-1}<j<m_k} \left| \sum_{t=m_{k-1}+1}^{j} \xi_t \right| \right) \\ &= I + II \end{aligned} \tag{3.4}$$

We evaluate the two terms in the right hand side of (3.4) separately:

By Cauchy-Schwartz inequality

$$I \leq \sum_{k=1}^{\ell} E|X_k| \leq \sum_{k=1}^{\ell} (EX_k^2)^{1/2}$$

By Lemma 3.1, since $K \geq (1+\rho_1^*)/(1-\rho_1^*)$, for all $1 \leq i \leq \ell$, we have:

$$E(X_k^2) \leq K \sum_{t=m_{k-1}+1}^{m_k} E\xi_t^2 \,.$$

By using now the Cauchy-Schwartz inequality for sequences we obtain

$$I \leq K^{1/2} \sum_{k=1}^{\ell} \left(\sum_{t=m_{k-1}+1}^{m_k} E\xi_t^2 \right)^{1/2} \leq (KM)^{1/2} \left[\sum_{j=1}^{n} E\xi_j^2 \right]^{1/2} \leq (KM)^{1/2} \,.$$

To estimate II we notice that

$$(II)^4 \leq \sum_{k=1}^{\ell} E \max_{m_{k-1}<j<m_k} \left| \sum_{t=m_{k-1}+1}^{j} \xi_t \right|^4$$

By Lemma 3.2, relation (3.2) and since $K \geq D'(4, \rho_1^*)$, we obtain:

$$E \max_{m_{k-1}<j<m_k} \left| \sum_{t=m_{k-1}+1}^{j} \xi_t \right|^4 \leq \tag{3.5}$$
$$K \left[E^4 \max_{m_{k-1}<j<m_k} \left| \sum_{t=m_{k-1}+1}^{j} \xi_t \right| + \sum_{t=m_{k-1}+1}^{m_k-1} E\xi_t^4 + \left(\sum_{t=m_{k-1}+1}^{m_k-1} E\xi_t^2 \right)^2 \right]$$

We estimate each term in the right hand side of (3.5) separately.
By the definition of m_k

$$\sum_{t=m_{k-1}+1}^{m_k-1} E\xi_t^2 \leq \frac{1}{M}$$

so that by using our notation (3.3) for a_n and the definition of ξ_k we obtain

$$\sum_{t=m_{k-1}+1}^{m_k-1} E\xi_t^4 \leq \frac{4}{M} \sum_{t=m_{k-1}+1}^{m_k-1} E\xi_t^2 \leq \frac{4}{M^2}$$

and

$$E^4 \max_{m_{k-1}<j<m_k} \left| \sum_{t=m_{k-1}+1}^{j} \xi_t \right| \leq a_n^4 \left(\sum_{t=m_{k-1}+1}^{m_k-1} E\xi_t^2 \right)^2 \leq \frac{a_n^4}{M^2}$$

Overall, by (3.5) and above considerations we obtain

$$(II)^4 \leq K \left[\frac{a_n^4}{M} + \frac{4}{M} + \frac{1}{M} \right]$$

Now by (3.4) and our estimates for I and II we get

$$\begin{aligned} E \max_{1 \leq i \leq n} \left| \sum_{j=1}^{i} Y_j \right| &\leq E \max_{1 \leq i \leq n} \left| \sum_{j=1}^{i} \xi_j \right| + 2M^{1/2} \\ &\leq \left[(KM)^{1/2} + (K/M)^{1/4} a_n + (5K/M)^{1/4} \right] + 2M^{1/2} \end{aligned}$$

Therefore, by the definition (3.3),

$$a_n \leq (K/M)^{1/4} a_n + (K^{1/2}+2)M^{1/2} + (5K/M)^{1/4}$$

Now we select $M = M(\rho_1^*)$ to be:

$$M = M(\rho_1^*) = [16K] + 1$$

and since $K \geq 1$, we obtain

$$a_n \leq \frac{a_n}{2} + K^{1/2}(1+2K^{-1/2})4K^{1/2}(1+K^{-1/2}/4) + 1 \leq \frac{a_n}{2} + 16\,K$$

which implies the desired result.

Proof of Theorem 2.1 The conclusion of Theorem 2.1 is an easy consequence of Proposition 3.1 after a standard reduction procedure. Let N be the integer mentioned in Theorem 2.1 such that $\rho_N^* < 1$. We consider now N sequences of random variable $\{Y_{ij}' : j \geq 1\}$, $0 \leq i < N$ defined by $Y_{ij} = X_{iN+j}$.

Notice that for each i, the first interlaced mixing coefficient for $\{Y_{ij} : j \geq 1\}$, $\tilde{\rho}_1^* \leq \rho_N^* < 1\}$ is smaller than ρ_n^* and therefore strictly smaller than 1. We can apply Lemma 3.3 to each of the N subsequences. The rest of the proof is standard and it is left to the reader.

3.1 Proof of Theorem 2.2: Normalization and truncation

It is clear that conditions of Theorem 3.2 are scale invariant so that without loss of generality we assume that

$$\sigma_n^2 = \mathrm{Var}(\sum_{i=1}^{k_n} \xi_{ni}) = 1\,.$$

Therefore, by condition (2.3), there exists a sequence of positive numbers $\epsilon_n \to 0$ such that

$$\sum_{i=1}^{k_n} E\xi_{ni}^2 I(|\xi_{ni}| > \epsilon_n) \to 0 \text{ as } n \to \infty. \tag{3.6}$$

We truncate now at the level ϵ_n.

Define

$$\eta_{ni} = \xi_{ni} I(|\xi_{ni}| \leq \epsilon_n) - E\xi_{ni} I(|\xi_{ni}| \leq \epsilon_n)$$

and

$$\varphi_{ni} = \xi_{ni} I(|\xi_{ni}| > \epsilon_n) - E\xi_{ni} I(|\xi_{ni}| > \epsilon_n).$$

Denote by $W_n'(t) = \sum_{i=1}^{\nu_t} \eta_{ni}$ and by $W_n''(t) = \sum_{i=1}^{\nu_t} \varphi_{ni}$ where $\nu_t = \inf\{m; 1 \leq m \leq k_n, \sigma_{nm}^2 \geq t\}$. Notice that,

$$W_n(t) = W_n'(t) + W_n''(t).$$

By Theorem 2.1 and (1.2) we can find a constant $K = K(p, \overline{\rho}_p^*)$, where $\overline{\rho}_p^* < 1$, which does not depend on n, such that

$$E[\sup_t |W''(t)|]^2 \leq E \max_{1\leq i\leq k_n} \left(\sum_{j=1}^{i} \varphi_{nj}\right)^2 \leq K \sum_{i=1}^{k_n} E\xi_{ni}^2 I(|\xi_{ni}| > \epsilon_n). \tag{3.7}$$

which converges to 0 by (3.6) as $n \to \infty$. Therefore $W_n''(t)$ is converging weakly to 0 and the limiting distribution of $W_n(t)$ is the same as the limiting distribution of $W_n'(t)$ if the last one exists: (i.e. $W_n(t) - W_n'(t) \overset{\mathcal{D}}{\Rightarrow} 0$ as $n \to \infty$.).

To study the limiting distribution of $W_n'(t)$ we shall apply Theorem 19.2 in Billingsley (1968).

It is very easy to prove that the strong mixing condition implies that $W_n'(t)$ has asymptotically independent increments. According to Theorem 19.2 and the Corollary from page 56, both in Billingsley (1968), we have only to show:

(i) $E[W_n'(t)]^2 \to t$ as $n \to \infty$,

and, for all t fixed for a certain $\delta < 1 - t$, we have:

(ii) The family

$$\left\{ \frac{1}{\delta} \max_{\nu_t \leq i \leq \nu_{t+\delta}} \left(\sum_{j=\nu_t}^{i} \eta_{nj}\right)^2 \right\}_n$$

is uniformly integrable in n.

To prove (i) we have to establish that $\lim_{n\to\infty} E\left(\sum_{i=1}^{\nu_t} \eta_{ni}\right)^2 = t$ which is equivalent to $\lim_{n\to\infty} E\left(\sum_{i=1}^{\nu_t} \xi_{ni}\right)^2 = t$ by (3.7). Let $\|X\| = (E[X]^2)^{1/2}$ denote the norm in L_2. Since

$$\left\|\sum_{i=1}^{\nu_t} \xi_{ni}\right\| \leq \left\|\sum_{i=1}^{\nu_t - 1} \xi_{ni}\right\| + \|\xi_{n\nu_t}\|$$

by the definition of ν_t, this relation gives:

$$\sqrt{t} \leq \left\|\sum_{i=1}^{\nu_t} \xi_{ni}\right\| \leq \sqrt{t} + \|\xi_{n\nu_t}\|$$

In order to prove (i) we have only to notice that Lindeberg condition (2.3) implies $\lim_{n\to\infty} \|\xi_{n\nu_t}\| = 0$.

To establish (ii) we shall use Theorem 3.1 with $q = 4$, and the definition of η_{ni}.

For a certain constant $D(p, \overline{\rho}_p^*)$ with $\overline{\rho}_p^* < 1$ we have

$$\begin{aligned} E\left(\max_{\nu_t \leq i \leq \nu_{t+\delta}} \left(\sum_{j=\nu_t}^{i} \eta_{ni}\right)^4\right) &\leq D(p, \rho_p^*) \left[\sum_{i=\nu_t}^{\nu_{t+\delta}} E(\eta_{ni})^4 + \left(\sum_{i=\nu_t}^{\nu_{t+\delta}} E\eta_{ni}^2\right)^2\right] \\ &\leq D(p, \rho_p^*) \left[\epsilon_n^2 \left(\sum_{i=\nu_t}^{\nu_{t+\delta}} E\eta_{ni}^2\right) + \left(\sum_{i=\nu_t}^{\nu_{t+\delta}} E\eta_{ni}^2\right)^2\right] \end{aligned}$$

By condition (2.2) and (3.7), for sufficiently large n,

$$\sum_{i=\nu_t}^{\nu_{t+\delta}} E\eta_{ni}^2 \le \sum_{i=1}^{k_n} E\eta_{ni}^2 \le 2\sum_{i=1}^{k_n} E\xi_{ni}^2 \le 2C \text{ Var}\left(\sum_{i=1}^{k_n} \xi_{ni}\right) = 2C\,.$$

Thus, random increments $\{W_n'(t) - W_n'(s) : t, s \in [0,1],\ n \ge 1\}$ have uniformly bounded moments of order 4. This fact together with the strong mixing condition, a uniform infinite smallness $|\eta_{ni}| \le 2\epsilon_n$ and a standard mixing inequality (the constant comes from Theorem 1.1 in Bradley and Bryc (1985))

$$|E(XY) - E(X)E(Y)| \le 2\pi\alpha^{1/2}(\sigma(X), \sigma(Y)E^{1/4}(X^4)E^{1/4}(Y^4)$$

implies that $\text{corr}[W_n'(t), W_n'(t+\delta) - W_n'(t)] \to 0$ as $n \to \infty$ uniformly in t, δ with $0 \le t \le t + \delta \le 1$ so that

$$\lim_{n\to\infty} E\left(\sum_{i=\nu_t}^{\nu_{t+\delta}} \eta_{ni}\right)^2 = \lim_{n\to\infty} E\left(\sum_{i=\nu_t}^{\nu_{t+\delta}} \xi_{ni}\right)^2 = \delta$$

uniformly in t, δ with $0 \le t \le t + \delta \le 1$ and thus by condition (2.2)

$$\limsup_{n\to\infty} \sum_{i=\nu_t}^{\nu_{t+\delta}} E\eta_{ni}^2 = \limsup_{n\to\infty} \sum_{i=\nu_t}^{\nu_{t+\delta}} E\xi_{ni}^2 \le C\delta.$$

As a consequence

$$\limsup_n E\left(\max_{\nu_t \le i \le \nu_{t+\delta}} \left(\sum_{j=\nu_t}^{i} \eta_{ni}\right)^4\right) \le D(p, \overline{\rho}_p^*)\delta^2$$

which proves (ii) and completes the proof of this theorem.

The proofs of Corollary 2.1-2.5 require only to verify the conditions of Theorem 2.2 which can be found in Peligrad (1996), and the arguments will not be repeated here.

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