ESCAPE FUNCTIONS CONDITIONS FOR THE OBSERVATION, CONTROL AND STABILIZATION OF THE WAVE EQUATION

LUC MILLER *

Abstract. For the time-independent wave equation on a connected compact Riemannian manifold (Ω, g) with C^3 boundary, the geodesics condition of C. Bardos, G. Lebeau, and J. Rauch (cf. *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), pp. 1024–1065) is characterized in terms of *escape functions*, which are some Lyapunov functions on the phase space $S^*\bar{\Omega}$ (the unit sphere cotangent bundle). The first order multipliers are shown to correspond to linear escape functions. This yields a straightforward geometric proof that the geodesics condition holds in the situations where the classical multiplier method applies. When the geodesics condition holds, necessary conditions for the existence of a linear escape function are given. They yield a class of simple situations (e.g. in $\mathbb{R}^2$ with constant coefficients), where the optimal control time or control regions are out of reach of first order multipliers techniques.

Key words. wave equation, exact controllability, stabilization, multiplier method, geometric optics, escape function.

AMS subject classifications. 35L05, 35L20, 37B25, 49K20, 53D25, 58J47, 93B05, 93B07, 93D20

1. Introduction. This paper is concerned with the so widely cited and scarcely used sharp sufficient bicharacteristics condition for the observation, control and stabilization of the wave equation from the interior or the boundary introduced by C. Bardos, G. Lebeau, and J. Rauch (cf. [1] and the appendix of [16]). Since we shall restrict to the time-independent case, this conditions can be stated in terms of generalized geodesics (i.e. the rays of geometrical optics) and we shall refer to it as the *geodesics condition*.

1.1. P.D.E. problems. Let us recall two simple examples of the partial differential equations problems under consideration.

Our first example is a problem of exact controllability from the boundary. Let Ω be a bounded open connected subset of $\mathbb{R}^n$, with C^1 boundary $\partial\Omega$, inside which waves propagate according to the wave equation $\Box u = 0$, where $\Box = \partial_t^2 - \Delta$ is the speed one d'Alembertian. Let $T > 0$ and $\theta \in C_c^0(]0, T[\times\partial\Omega)$ define the boundary region $\Sigma = \{(t, x) \in \mathbb{R}\times\partial\Omega |\ \theta(t, x) \neq 0\}$ where the Dirichlet boundary condition is controlled. The function θ is said to *control* Ω *exactly* if for all $(u_0, u_1) \in L^2(\Omega)\times H^{-1}(\Omega)$ there is a control function $v \in L^2(\mathbb{R} \times \partial\Omega)$ such that the solution of the mixed Dirichlet-Cauchy problem:

$$\Box u = 0 \quad \text{in }]0, T[\times\Omega, \quad u = \theta \times v \quad \text{on }]0, T[\times\partial\Omega, \tag{1.1}$$

with Cauchy data $(u, \partial_t u) = (u_0, u_1)$ at $t = 0$, satisfies $u = \partial_t u = 0$ at $t = T$.

Our second example is a problem of uniform internal stabilization. Let (M, g) be a smooth connected compact Riemannian manifold with boundary ∂M and let $\Box_g$ denote the associated d'Alembertian (cf. §2 for geometric definitions). Let Ω be the complementary set of the support of a nonnegative $a \in C^\infty(\bar{M})$ which defines the

* Centre de Mathématiques, U.M.R. 7640 du C.N.R.S., Ecole Polytechnique, 91128 Palaiseau Cedex, France (`miller@math.polytechnique.fr`) and Équipe Modal'X, Bâtiment G, Université de Paris X - Nanterre, 200, Avenue de la République, 92001 Nanterre Cedex, France.

damping region $\{x \in \bar{M} |\ a(x) > 0\} = \bar{M} \setminus \bar{\Omega}$. The function a is said to *stabilize* M *uniformly* if for all $(u_0, u_1) \in H_0^1(M) \times L^2(M)$ the energy of the solution to the the mixed Dirichlet-Cauchy problem :

$$\square_g u + 2a\partial_t u = 0 \quad \text{in } \mathbb{R}_+ \times M, \quad u = 0 \quad \text{on } \mathbb{R}_+ \times \partial M\,, \tag{1.2}$$

with Cauchy data $(u, \partial_t u) = (u_0, u_1)$ at $t = 0$, decays exponentially, i.e. there exists $\beta > 0$ and $\beta' \geqslant 1$ such that for all $t \geqslant 0$:

$$E(t) = \frac{1}{2}\int_M \left\{|\partial_t u(t,x)|^2 + |\nabla u(t,x)|_x^2\right\} d_g x \leqslant \beta' e^{\beta t} E(0). \tag{1.3}$$

1.2. Geodesics condition. The common geometric features of these examples are a compact Riemannian manifold (Ω, g) and an open subset Γ of its boundary, if we restrict the first example to a time-independent control region $\Sigma =]0, T[\times\Gamma$ where Γ is an open subset of $\partial\Omega$ and if, in the second example, we denote the part of the boundary of the damping region inside M by $\Gamma = \partial\Omega \cap M$ (cf. fig. 1.1).

In this context, the *generalized geodesics* (cf. fig. 1.2 and def. 2.1) are continuous trajectories $t \mapsto x(t) \in \bar{\Omega}$ which follow geodesic curves at unit speed in Ω (so that on these intervals $t \mapsto \dot{x}(t)$ is continuous); if they hit $\partial\Omega \setminus \Gamma$ transversally at time t_0, then they reflect as light rays or billiard balls (and $t \mapsto \dot{x}(t)$ is discontinuous at t_0); if they hit Γ transversally at time t_0, then for times $t > t_0$ they have "escaped" from $\bar{\Omega}$; if they hit $\partial\Omega$ tangentially at time t_0 then either there exists a geodesic in Ω which continues $t \mapsto (x(t), \dot{x}(t))$ continuously and they branch onto it, or there is no such geodesic curve in Ω and — depending on where $\partial\Omega$ was hit — if $x(t_0) \in \Gamma$ then they have "escaped" from $\bar{\Omega}$ at times $t > t_0$, if $x(t_0) \in \Omega \setminus \Gamma$ then they glide at unit speed along the geodesic of $\partial\Omega$ which continues $t \mapsto (x(t), \dot{x}(t))$ continuously until they may branch onto a geodesic in Ω, otherwise they reach $\partial\Gamma$ at a time t_1 and have "escaped" from $\bar{\Omega}$ at times $t > t_1$. Following [3], to ensure the unique continuation of these trajectories, we assume that :

(1.4) $\partial\Omega$ is at least C^3 and, for any $k \in \mathbb{N}$, $k \geqslant 3$, there are no contacts of order $k - 1$ between $\partial\Omega$ and its tangents in open subsets of $\partial\Omega$ in which it is only C^k.

(n.b. the hypothesis of theorem 3.8 in [1] corresponds to $k = \infty$, and unicity also holds for real analytic g and $\partial\Omega$).

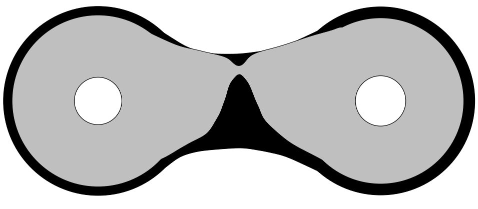

FIG. 1.1. *Ω is light, M is $\Omega\cup$dark, Γ is the frontier light/dark. The geodesics condition holds.*

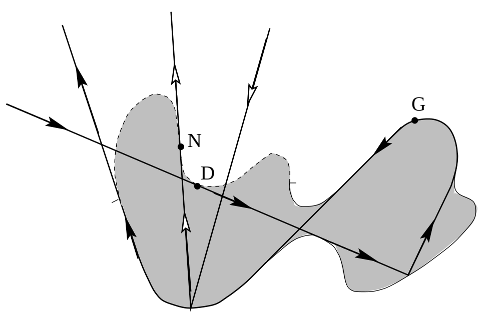

FIG. 1.2. *Two generalized geodesics. Γ is doted. N:nondiffractive, D:diffractive, G:gliding.*

The bicharacteristics condition of C. Bardos, G. Lebeau, and J. Rauch roughly says that every generalized bicharacteritics escapes $\mathbb{R} \times \bar{\Omega}$ through Σ. In our context of time-independent coefficients and region $\Sigma =]0, T[\times\Gamma$, we rephrase the bicharacteristics condition by saying that the time T and the boundary region Γ satisfy the *geodesics condition* if every generalized geodesic starting in $\bar{\Omega}$ has escaped $\bar{\Omega}$ through Γ at time $t = T$, in short : *every generalized geodesic of length T escapes $\bar{\Omega}$ through* Γ (e.g. in fig. 1.1, this condition is "easily seen" to hold for some T).

When $\partial\Omega$ is smooth and θ is the characteristic function $\mathbf{1}_\Sigma$ of Σ, it is proved in [1] using microlocal techniques (i.e. the results of [18] and [19] on the propagation of singularities at the boundary and a lifting lemma at nondiffractive points) that the bicharacterics condition is sufficient and almost necessary for the exact controllability of problem (1.1). Using microlocal measures techniques of P. Gérard, L. Tartar, P.-L. Lions and T. Paul (cf. [4] for a survey), N. Burq obtained the same result in [3] when $\Sigma =]0, T[\times\Gamma$ but $\partial\Omega$ is only C^k with $k \in \mathbb{N}$, $k \geqslant 3$ (cf. [9] for propagation results when Ω is convex with $C^{1,1}$ boundary and [5] for results about corners). Moreover, for any $\theta \in C^0_c(]0, T[\times\partial\Omega)$ it is proved with the same techniques in [6] that the bicharacteristics condition is both necessary and sufficient for θ to control Ω exactly in problem (1.1) (n.b. if $\mathbf{1}_\Sigma$ controls Ω exactly then θ does, and if θ does then so does $\mathbf{1}_{\Sigma'}$ for any open Σ' containing $\bar{\Sigma}$).

Another result of C. Bardos, G. Lebeau, and J. Rauch is that if there exists a time $T > 0$ such that the geodesics condition holds then a stabilizes M uniformly in problem (1.2) (n.b. the stabilization of a compact manifold M without boundary corresponds to $\Gamma = \partial\Omega$). No regularity of $\partial\Omega$ at points of $\bar{\Gamma}$ is required here because there is no boundary condition on Γ. This is the context in which microlocal measures techniques where first applied to control theory, namely by G. Lebeau in [15] to bound from above the best rate β of exponential decay in (1.3) by some means of a over generalized geodesics.

We refer to [13], [1], [14], [2] for more results allowing observation and stabilization from the boundary, time-dependent coefficients, lower-order terms, Neumann and mixed type boundary conditions, Schrödinger and plate equations, and more. We refer to [21], [20] for results on general boundary conditions and transmission problems, to [7] for general results on systems.

1.3. Motivation. While superseding in sharpness and scope earlier results obtained by the "multiplier method" (cf. §4), the results of [1] are often discarded for reasons already put forward in the introduction of [1]. In the first place, a lot of smoothness is required in [1], but we have recalled above the improvements brought by microlocal measures techniques in this respect. The second reason — more serious but less often noticed — is that the explicit computation of the constants appearing in the observation inequalities (which allow to predict how much energy is needed to control waves of given energy) are out of reach of the closed graph argument in [1] (or the argument by contradiction in the microlocal measures technique).

The third point — to which this article contributes — is that the conditions obtained may not be easy to verify for complicated operators and geometry as acknowledged in [1]. We may add that, even in applications in Euclidean geometry (where geodesics are straight lines) of dimension two or three, we are often in the awkward situation where it is intuitively clear whether the geodesics condition is satisfied but quite intricate to prove rigorously.

Therefore we felt the need to dwell on pages 1030 and 1031 of [1] which "illustrate the controllability criterion" and end with the following remark:

> "Our contention is not that we could not do any of these three [results] with sufficiently clever differential multiplier. Quite the contrary, the methods of Morawetz, Ralston and Strauss would surely suffice. However to create a general result, we would be led inevitably to the same geometric considerations, and avoiding pseudodifferential techniques would only make the task more complicated."

This quote refers to [24] where the decay of solutions of the wave equation outside obstacles is deduced from some "escape function" which is proved to exist in dimension three under the geodesics condition that the obstacle is "nontrapping" (R. Melrose later improved on this paper using the microlocal results of [18] and [19]). Increasingly clever first order differential multiplier had been applied earlier to this problem (radial in [22], gradient of a convex function in [23], "expansive" vector field in [26]) which all correspond to linear escape functions. As C. S. Morawetz, J. V. Ralston, and W. A. Strauss write in [24] :

> "The major point of the present work is that a linear escape function is too special."

In the context of resonances for Schrödinger operators with a potential, the same point was made by B. Helffer and J. Sjöstrand in [10] where they introduced nonlinear escape functions generalizing the radial case treated earlier by J. Aguilar and J. M. Combes.

This article makes the same point in the context of the observation, control and stabilization of waves from the interior or the boundary (cf. §5). We intend it as a track to find sufficient conditions that would be sharper than those obtained by the classical multiplier method and easier to verify — albeit less sharp — than the geodesics condition of [1] (cf. §6).

2. Generalized geodesics. Let (Ω, g) be a connected compact oriented n-dimensional Riemannian manifold with metric g of class C^2 and boundary $\partial\Omega$ satisfying (1.4). Let ν denote the exterior normal vector field and D the Levi-Civita connection of g.

Let $J : X \mapsto \xi$ denote the "flat" isomorphism between the tangent bundle $T\bar{\Omega}$ and cotangent bundle $T^*\bar{\Omega}$ defined by $\xi(Y) = g(X,Y)$, and let a denote the metric on $T^*\bar{\Omega}$ defined by $a(\xi,\eta) = g(X,Y)$ where $\xi = J(X)$ and $\eta = J(Y)$. In local coordinates $(x^1, x^2, \dots, x^n)$, we write a vector field : $X = \sum_i X^i \frac{\partial}{\partial x^i}$, a 1-form : $\xi = \sum_i \xi_i dx^i$, and for each $x \in \bar{\Omega}$ we write the scalar product on the tangent space : $\langle X, Y\rangle_x = \sum_{i,j} g_{i,j}(x) X^i Y^j$, so that $J(X)_j = \sum_i g_{i,j} X^i$ and $J^{-1}(\xi)^j = \sum_i a^{i,j}\xi_i$, where the matrix $A = (a^{i,j})$ is defined by $A^{-1} = (g_{i,j})$. We keep the same notation for the scalar product on the cotangent space $\langle \xi, \eta\rangle_x = {}^t\xi A \eta$ and denote the associated norms by $|\cdot|_x$.

Let $\nabla = J^{-1}d$ denote the gradient operator and Δ_g denote the Laplace-Beltrami operator in (Ω, g). The Sobolev spaces and the energy (1.3) are defined with respect to the measure dx_g. In local coordinates $(x^1, x^2, \dots, x^n)$: $dx_g = \sqrt{\det g}\, dx^1 \cdots dx^n$. and $\Delta_g f = (\sqrt{\det g})^{-1} \sum_{i,j} \partial_{x_j} \left(a^{i,j}\sqrt{\det g}\,\partial_{x_i} f\right)$. Let $p \in C^2(T^*(\mathbb{R}\times\bar{\Omega}))$ denote the principal symbol of the d'Alembertian $\Box_g = \partial_t^2 - \Delta_g$. In local coordinates, $p(t,x,\tau,\xi) = \sum_{i,j} a^{i,j}(x)\xi_i\xi_j - \tau^2$ so that p is also the principal symbol of the operator P defined in [1] by $P = \partial_t^2 - \sum_{i,j} a^{i,j}(x)\partial_{x_j}\partial_{x_i} + \text{lower order terms}$.

To link the bicharacteristics of p with the geodesics, it is convenient to consider the Hamiltonian function $h = p/2$ instead of p. Let H_h denote its Hamiltonian vector field, in local coordinates : $H_h = \partial_{\tau,\xi} h \partial_{t,x} - \partial_{t,x} h \partial_{\tau,\xi}$. The bicharacteristics are integral curves $s \mapsto (t(s), x(s), \tau(s), \xi(s)) = \exp(sH_h)(t(0), x(0), \tau(0), \xi(0))$ of H_h along which $h = 0$. Since p is time-independent, τ is constant along bicharacteristics.

By a linear change of parameter, we may restrict to the bicharacteristics defined on $S = \{(t, x, \tau, \xi) \in h^{-1}(0) | -\tau = |\xi|_x = 1\}$. They satisfy : $\dot{t}(s) = -\tau(s) = 1$ and $|\dot{x}(s)|_x = |J^{-1}(\xi(s))|_x = |\xi(s)|_x = 1$. Therefore, pushing them through the projections $\pi : T^*(\mathbb{R} \times \bar{\Omega}) \to \bar{\Omega}$ and $\Pi : T^*(\mathbb{R} \times \bar{\Omega}) \to T^*\bar{\Omega}$ (well defined since t is a global coordinate on $\mathbb{R} \times \bar{\Omega}$), we recover the geodesics curves parametrized at unit speed $t \mapsto x(t) = \pi \exp(tH_h)(0, x(0), 0, \xi(0))$ starting from $x(0)$ in the direction $\xi(0)$, which we will sometimes consider as strips $t \to (x(t), J(\dot{x}(t))) = \Pi \exp(tH_h)(0, x(0), 0, \xi(0))$ on the cosphere bundle $S^*\bar{\Omega} = \{(x, \xi) \in T^*\bar{\Omega} | \, |\xi|_x = 1\}$.

As proved by N. Burq in [3], (1.4) ensures that the generalized bicharacteristics introduced in [18] are uniquely defined through each point of $h^{-1}(0)$. Let $\mathcal{B}$ denote the second fundamental form, i.e. the symmetric bilinear form $\mathcal{B}_x(X, Y) = -\langle D_X \nu, Y\rangle_x$. Let $\partial\Omega = \phi^{-1}(0)$ be locally defined by a submersion ϕ such that $\Omega = \{\phi(x) > 0\}$. Then $\nabla\phi = -|\nabla\phi|\nu$, $\mathrm{Hess}\phi(X, Y) := Dd\phi(X, Y) = -|\nabla\phi|\mathcal{B}_x(X, Y)$ and $H_h^2\phi(x, J(X)) = -|\nabla\phi|_x\mathcal{B}_x(X, X)$. Therefore the *strictly gliding* points of the boundary are the points of $T^*\partial\Omega$ at which the second fundamental quadratic form is positive. The Hamiltonian field of h restricted to the symplectic space where $\phi = H_h\phi = 0$ is the gliding vector field $H_h^G = H_h + (H_h^2\phi/H_\phi^2 h)H_\phi$. The gliding bicharacteristics are the trajectories of H_h^G and their image through Π are the geodesic curves of the restriction of g to $\partial\Omega$ parametrized at unit speed. Recall that a generalized bicharacteristic pieces together trajectories of H_h in Ω and gliding bicharacteristics in the set $\mathcal{G}_g$ of strictly gliding points : in particular, if at time t it is at $\rho \in T^*\partial\Omega$, then in a one sided deleted neighborhood of t it coincides with either a bicharacteristic in Ω or a gliding bicharacteristic in $\mathcal{G}_g$. Recall that the *hyperbolic* points of the boundary are the transversal ones (i.e. not in $T^*\partial\Omega$). Recall that $\rho = (x, \xi) \in T^*\bar{\Omega}$ such that $x \in \partial\Omega$ is *nondiffractive* (cf. [1]) if the (nongeneralized) bicharacteristic through ρ at time t is out of $\bar{\Omega}$ at least in one of the one sided deleted neighborhoods of t, i.e. either $\rho \notin T^*\partial\Omega$ is hyperbolic or $\rho \in T^*\partial\Omega$ and the generalized bicharacteristic through ρ at time t is a gliding bicharacteristic in $\mathcal{G}_g$ at least in one of the one sided deleted neighborhoods of t.

Definition 2.1. *The* generalized geodesic strips *are the images of the generalized bicharacteristics of* $h = (|\xi|_x - \tau^2)/2$ *over* $S = \{-\tau = |\xi|_x = 1\}$ *through the bijection* $\Pi : S \to S^*\bar{\Omega}$. *The* generalized geodesic curves *are the projections of the generalized geodesic strips on* $\bar{\Omega}$.

The generalized geodesic curves are described in §1.2 for readers not familiar with generalized bicharacteristics. In §3, it will be convenient to think of generalized geodesics hitting Γ at a nondiffractive point at time $t = t_0$ as having escaped $\bar{\Omega}$ at times $t > t_0$. This interpretation is natural when Ω is an open subset of a larger manifold (M, g) : $\partial\Omega \setminus \Gamma$ is a border "obstacle" which confines geodesics inside $\bar{\Omega}$ and Γ is a border "hole" through which the geodesics may "escape" out of $\bar{\Omega}$.

Definition 2.2. *The* geodesics condition $G(T, \Gamma)$ *for the time* $T > 0$ *and the open region* $\Gamma \subset \partial\Omega$ *holds if every generalized geodesic of length greater than* T *passes through* Γ *at a nondiffractive point (i.e. every generalized geodesic of length greater than* T *escapes* $\bar{\Omega}$ *through* Γ*).*

As recalled in §1, $G(T, \Gamma)$ implies, for instance, that all $T' > T$ has the following control property : for all $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control function $v \in L^2(]0, T'[\times\Gamma)$ such that the solution of the mixed Dirichlet-Cauchy problem:

$$\square_g u = 0 \text{ in }]0, T'[\times\Omega, \quad u = v \text{ on }]0, T'[\times\Gamma, \quad u = 0 \text{ on }]0, T'[\times\partial\Omega \setminus \Gamma, \tag{2.1}$$

with Cauchy data $(u, \partial_t u) = (u_0, u_1)$ at $t = 0$, satisfies $u = \partial_t u = 0$ at $t = T'$.

Moreover, if T' satisfies this property, then $G(T',\Gamma)$ holds. When $k<\infty$, these results are implicit in [3].

Remark : If we also assume Ω to be convex then the second fundamental form $\mathcal{B}$ is nonnegative on $T^*\partial\Omega$, so that all generalized geodesics starting from $T^*\partial\Omega$ keep gliding on $\partial\Omega$ forever. This answers the question raised in remark 4.7 of [17] : there is never internal exact controllability for the wave equation in a convex open $\Omega\subset\mathbb{R}^n$ satisfying 1.4 with Dirichlet condition on $\partial\Omega$ from a control region G such that $\bar{G}\subset\Omega$.

3. Escape functions. In [24], a notion of "escape function" was introduced which characterizes (in dimension three) the "nontrapping" geodesics condition of P. D. Lax and R. S. Phillips for exterior problems (roughly: $\Gamma=\emptyset$ and there is no specific time, as for stabilization). We adapt the notion of escape function to the geodesics condition of C. Bardos, G. Lebeau, and J. Rauch by taking T into account.

DEFINITION 3.1. *An* escape function *adapted to $T>0$ and $\Gamma\subset\partial\Omega$ is a real function f defined on $S^*\bar{\Omega}$ such that :*

(i) *For all (x,ξ) and (y,η) in $S^*\bar{\Omega}$: $|f(x,\xi)-f(y,\eta)|\leqslant T$.*

(ii) *f increases at least as fast as the distance along the closure of any finite interval of geodesic strip in Ω.*

(iii) *For all $(x,\xi)\in S^*\bar{\Omega}$ such that $x\in\partial\Omega\setminus\Gamma$ and $\xi(\nu)>0$: $f(x,\xi')\geqslant f(x,\xi)$ for $\xi'=\xi-2\xi(\nu)J^{-1}(\nu)$.*

(iv) *f increases at least as fast as the distance along the closure of any finite interval of geodesic strip in $\partial\Omega\setminus\Gamma$ on which the second fundamental quadratic form is positive.*

The escape functions condition $E(T,\Gamma)$ *holds if there is an escape function adapted to T and Γ.*

Note that (i) says that f takes its values in an interval of length less or equal to T and that (iii) says that f is nondecreasing at reflexions on $\partial\Omega\setminus\Gamma$. When $f\in C^1(S^*\bar{\Omega})$ then (ii) says that for all $(x,\xi)\in S^*\Omega$: $H_hf(x,\xi)\geqslant 1$, and (iv) says that for all $(x,\xi)\in S^*\bar{\Omega}$ such that $x\in\partial\Omega\setminus\Gamma$, $\xi(\nu)=0$ and $\mathcal{B}_x(J^{-1}(\xi),J^{-1}(\xi))>0$: $H_h^Gf(x,\xi)\geqslant 1$. Moreover (ii) and (iii) imply (iv) by taking a limit as $\xi(\nu)$ tends to 0.

THEOREM 3.2. *$E(T,\Gamma)$ is equivalent to $G(T,\Gamma)$. Moreover f can be chosen continuous outside $\bar{\Gamma}$ and at hyperbolic and strictly gliding points of Γ.*

Proof. We first prove : $E(T,\Gamma)\Rightarrow G(T,\Gamma)$. Assume $E(T,\Gamma)$ holds. Let $x:[0,T']\to\bar{\Omega}$ be a generalized geodesic of length $T'>T$ which does not pass through Γ at a nondiffractive point. From (ii), (iii) and (iv) we deduce that $t\mapsto f(x(t),J(\dot{x}(t)))$ increases at least as fast as t. Hence :$f(x(T'),J(\dot{x}(T')))-f(x(0),J(\dot{x}(0)))\geqslant T'>T$, which contradicts (i). This proves that such an x does not exist, and therefore $G(T,\Gamma)$ also holds.

We now prove : $G(T,\Gamma)\Rightarrow E(T,\Gamma)$. Assume $G(T,\Gamma)$ holds. Let $T'>0$ and consider a generalized geodesic curve x which does not pass through Γ at a nondiffractive point for $t\in]0,T'[$, such that $x(t)\in\bar{\Gamma}$ for $t\in\{0,T'\}$ and there exists $\epsilon>0$ satisfying the following properties. If $x(0)\in\Gamma$, then we assume $x(t)\in\Gamma$ for $t\in]-\epsilon,0[$ (in particular $x(0)$ is nondiffractive) and $x(t)\in\Omega$ for $t\in]0,\epsilon[$. If $x(0)\in\partial\Gamma$, then we assume $x(t)\in\Gamma$ for $t\in]-\epsilon,0[$ and $x(t)\notin\Gamma$ for $t\in]0,\epsilon[$. If $x(T')\in\Gamma$, then we assume $x(t)\in\Gamma$ for $t\in]T',T'+\epsilon[$ (in particular $x(T')$ is nondiffractive) and $x(t)\in\Omega$ for $t\in]T'-\epsilon,T'[$. If $x(T')\in\partial\Gamma$, then we assume $x(t)\in\Gamma$ for $t\in]T',T'+\epsilon[$ and $x(t)\notin\Gamma$ for $t\in]T'-\epsilon,T'[$.

For each such x, we set $f(x(t),J(\dot{x}(t)))=t$ for $t\in[0,T']$, where the equality is understood as valid for derivatives from both sides at points of reflexion except when $t\in\{0,T'\}$. $G(T,\Gamma)$ ensures that $T'<T$ and therefore this f satisfies (i). By

definition, this f also satisfies (ii), (iii) and (iv) (with equalities instead of inequalities). The only points of $S^*\bar{\Omega}$ where f has not been yet defined are the points $\rho \in S^*\bar{\Gamma}$ such that the generalized bicharacteristic through ρ at time t is a gliding bicharacteristic in $\mathcal{G}_g$ in both one sided deleted neighborhoods of t. We set $f = 0$ at those points and recall that strictly gliding points of Γ have this property.

The continuity of compressed generalized bicharacteristics ensures that f is continuous outside $\bar{\Gamma}$ and at hyperbolic points of Γ. For any point $\rho \in \mathcal{G}_g \cap S^*\Gamma$ and $\delta > 0$ small enough, there is a neighborhood of ρ in Γ included in the union of $\mathcal{G}_g \cap S^*\Gamma$ and hyperbolic points of Γ which are endpoints of a generalized geodesic x of the preceding type with $T' < \delta$. Therefore f is also continuous at strictly gliding points. $\square$

4. Linear escape functions and the classical multiplier method. We discuss the geometrical relationship between the geodesics condition and the situations where first order multiplier techniques apply (cf. the books [16] and [11]).

Definition 4.1. *Consider a time $T > 0$ and an open region $\Gamma \subset \partial\Omega$.*
The escape vector field condition $EV(T, \Gamma)$ *holds if there is a C^1 section L of $T\bar{\Omega}$ such that :*

(i) *For all $x \in \bar{\Omega}$: $|L(x)|_x \leqslant T/2$.*
(ii) *For all $(x, X) \in S\bar{\Omega}$, $\langle D_X L, X\rangle_x \geqslant 1$.*
(iii) $\{x \in \partial\Omega \mid \langle L(x), \nu\rangle_x > 0\} \subset \Gamma$.

The escape potential condition $EP(T, \Gamma)$ *holds if there is a function $\varphi \in C^2(\bar{\Omega})$ such that :*

(i) *For all $x \in \bar{\Omega}$: $|d\varphi|_x \leqslant T/2$.*
(ii) *For all $(x, X) \in S\bar{\Omega}$,* $\mathrm{Hess}\varphi(X, X) := Dd\varphi(X, X) \geqslant 1$.
(iii) $\{x \in \partial\Omega \mid \frac{\partial\varphi}{\partial\nu}(x) := d_x\varphi(\nu) > 0\} \subset \Gamma$.

When Ω is a submanifold of $\mathbb{R}^n$ with the Euclidean metric, the radial condition $R(T, \Gamma)$ *holds if there is a point $x_0 \in \mathbb{R}^n$ such that :*

(i) $R(x_0) := \sup\{|x - x_0| \mid x \in \bar{\Omega}\} \leqslant T/2$.
(ii) $\{x \in \partial\Omega \mid \mid \langle x - x_0, \nu\rangle > 0\} \subset \Gamma$.

By taking $\phi(x) = |x - x_0|^2/2$, $L(x) = \nabla\phi$ and $f(x, \xi) = \xi(L(x))$, a straightforward computation yields (using, in the last step, that geodesic curves $t \mapsto x(t)$ are defined by $D_{\dot{x}}\dot{x} = 0$ and that D is defined by $D_X\langle L, X\rangle_x = \langle D_X L, X\rangle_x + \langle L, D_X X\rangle_x$):

Proposition 4.2. $R(T, \Gamma) \Rightarrow EP(T, \Gamma) \Rightarrow EV(T, \Gamma) \Rightarrow E(T, \Gamma)$.

The radial multiplier was introduced by C. S. Morawetz in [22] for exterior problems and condition $R(T, \Gamma)$ is a variation on her "star-shape" condition. Using this radial multiplier, sufficient conditions for exact controllability from the boundary were obtained by G. Chen (1979) and L. P. Ho (1986) and condition $R(T, \Gamma)$ is their sharper form due to J. L. Lions (cf. [16] and [11]). The condition $EP(T, \Gamma)$ is adapted from the convex function condition of C. S. Morawetz in [23] for exterior problems. Multiples of the square of the distance to a point (more generally to a convex set) are natural candidates for potential escape functions, at least when $\bar{\Omega}$ is included in the "injectivity domain" of the corresponding exponential function and the Hessian does not degenerate on $\bar{\Omega}$ (c.f. [27] for interesting remarks along this line). The condition $EV(T, \Gamma)$ is adapted from the condition of W. A. Strauss in [26] for exterior problems. The condition of W. A. Strauss was used in [8] for boundary stabilization with Euclidean metric, and later in [12] with less restrictive assumptions and hints for general metrics.

Remark 1 : The Riemann multiplier method developed by P. F. Yao in [27] is based on the condition $EV(T, \Gamma)$. As a byproduct we obtain the results stated in [27] under less restrictive assumptions (lower order terms and less regular $\partial\Omega$ and g are allowed). Although not stated in [27], Yao's proof (e.g of theorem 1.1) yields more in one respect (cf. the second reason in §1.3) since it allows to bound from above the constant appearing in the observation inequality in terms of a finite number of spectral data : one can conclude by an explicit computation from the high-frequencies inequality in lemma 2.3 of [27] (where this constant for waves which are orthogonal to the first m eigenmodes is an explicit function of the m-th eigenvalue), instead of using theorem 5.2 of [11] which loses track of the constants.

Remark 2 : If L_2 is a vector field satisfying $\langle D_X L_2, X\rangle_x = 0$ for all $(x, X) \in S\bar{\Omega}$, then adding L_2 to an escape vector field L_1 modifies the boundary condition (iii) without modifying the interior condition (ii). Hence the escape vector field $L = L_1 + L_2$ yields control regions which could not be obtained with L_1. The multipliers introduced by A. Osses (e.g. $L(x) = M(\theta)(x - x_0)$ where $M(\theta)$ is the rotation of angle θ) build on this remark: in [25], Ω is a submanifold of $\mathbb{R}^n$ with the euclidean metric, $L_1(x) = x - x_0$ is the radial vector field and $L_2(x) = A(x - x_0)$ is a "rotated" vector field, i.e. A is a skew-symmetric matrix.

Remark 3 : In [17], K. Liu introduces "piecewise" multipliers for internal exact controllability in a bounded connected open $M \in \mathbb{R}^n$ with Dirichlet condition on ∂M from a control region $G \subset M$ under the following geometric condition : there exists open sets $\Omega_j \subset M$ and points $x_j \in \mathbb{R}^n$ (for $j = 1, \ldots, J$) such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $G \supset M \cap \mathcal{N}_\epsilon [(\cup_j \Gamma_j) \cup (M \setminus \cup_j \Omega_j)]$ for some $\epsilon > 0$ where $\mathcal{N}_\epsilon[S]$ denotes an ϵ neighborhood of the set S and $\Gamma_j = \{x \in \partial\Omega_j \mid \langle x - x_j, \nu_j\rangle > 0\}$ where ν_j is the unit exterior normal to $\partial\Omega_j$. Remark 4.7 of [17] calls for a geometric argument proving that this condition implies the geodesics condition when ∂M is sufficiently smooth. Let Ω denote a connected component of $\bar{M} \setminus G$ and $\Gamma = \partial\Omega \cap \partial G \subset \partial G \cap M$. It is included in one of the Ω_j only and we fix this j henceforth. $(T_j, \partial\Omega_j \cap G)$ satisfies the radial condition in Ω_j with $T_j = 2R(x_j)$ since $\Gamma_j \subset G$, hence it also satisfies the geodesics condition. Every generalized geodesic of length T_j starting in $\bar{\Omega}$ and reflecting on $\partial\Omega \setminus \partial G \subset \partial\Omega_j \setminus (\partial\Omega_j \cap G) \subset \partial M$ reaches $\partial\Omega_j \cap G$ and a fortiori escapes $\bar{\Omega}$ through Γ. Therefore every generalized geodesics in $\bar{M}$ of length $T = \sup_j T_j$ reaches G, which proves that the condition of Liu implies the geodesics condition (no regularity of $\partial\Omega_j$ outside ∂M is needed since it carries no boundary condition).

5. Necessity of nonlinear escape functions. We illustrate the necessity of nonlinear escape functions by conditions which keep the optimal control time (cf. prop. 5.1 and fig. 5.1) or the optimal control region (cf. prop. 5.2 and fig. 5.2) out of reach of any linear escape function. This contrasts with the context of exterior problems where the geodesics condition implies the existence of a linear escape function in dimension $n = 2$ (cf. §4 of [24]).

Let $\operatorname{diam}_g(\bar{\Omega})$ denote the (finite) supremum of the lengths of the geodesics in Ω. A diameter of $\bar{\Omega}$ is a geodesic in Ω whose closure is of length $\operatorname{diam}_g(\bar{\Omega})$ (there is at least one since $\bar{\Omega}$ is compact).

PROPOSITION 5.1. *If there exists $y \in \bar{\Omega}$ and two distinct geodesics in Ω of length $T > 0$ issued from x, then $EV(T, \emptyset)$ does not hold. In particular, if $T = \operatorname{diam}_g(\bar{\Omega})$ and there are two diameters of $\bar{\Omega}$ issued from the same point $x \in \bar{\Omega}$, then $E(T, \partial\Omega)$ holds but $EV(T, \partial\Omega)$ does not.*

Proof. Assume L satisfies $EV(T, \partial\Omega)$. Denote by $[0, T] \ni t \mapsto x(t)$ the geodesic from $x(0) = z$ to $x(T) = y$. By (ii) : $\langle L(y), \dot{x}(T)\rangle_y - \langle L(z), \dot{x}(0)\rangle_z \geqslant T$, and by (i) we know that $\langle L(y), \dot{x}(T)\rangle_y \leqslant |L(y)|_y \leqslant T/2$ and $-\langle L(z), \dot{x}(0)\rangle_z \leqslant |L(z)|_z \leqslant T/2$, so that $\langle L(y), \dot{x}(T)\rangle_y = |L(y)|_y = T/2 = |L(z)|_z = -\langle L(z), \dot{x}(T)\rangle_z$ and therefore

$L(y) = \dot{x}(T)$. The same argument with the other geodesic proves both geodesics have the direction $L(y)$ at y which contradicts that they are distinct. □

PROPOSITION 5.2. *If there are J geodesics in Ω which are issued from points $x_j \in \partial\Omega \setminus \Gamma$ ($j = 1, \ldots, J$) with directions normal to the boundary and cross at $y \in \Omega$ with directions ξ_j such that the complementary set in $T_y\Omega$ of their polar cone $C = \{Y \in T_y\Omega \,|\, \forall j, \xi_j(Y) \leqslant 0\}$ is empty, then $EV(T, \Gamma)$ does not hold for any $T > 0$.*

Proof. Assume L satisfies $EV(T, \Gamma)$. Denote by $[0, T] \ni t \to x(t)$ the geodesic from $x(0) = x_j$ to $x(T) = y$. Since $\dot{x}(0) = -\nu(x_j)$, (iii) implies $\langle L(x_j), \dot{x}(0)\rangle_{x_j} \geqslant 0$. But by (ii), we know that $\langle L(y), \dot{x}(T)\rangle_y > \langle L(x_j), \dot{x}(0)\rangle_{x_j}$, so that $\langle L(y), \xi_j\rangle_y > 0$ since $\dot{x}(T) = \xi_j$. Repeating the argument with all j proves $L(y) \notin C$, which is a contradiction. □

In particular, this proves that exact controllability cannot be proved "with a sufficiently clever multiplier" of order one in the situation described in fig. 4 p. 1031 of [1]: a disk with some disconnected "minimal" boundary control region which we reproduce in fig. 5.2.

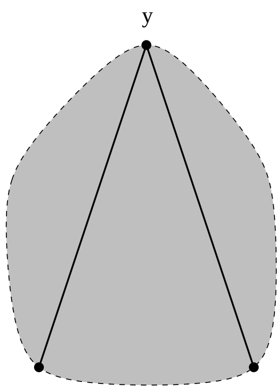

FIG. 5.1. *Segments are diameters of length T. By prop. 5.1: $G(T, \partial\Omega)$ holds but $EV(T, \partial\Omega)$ does not.*

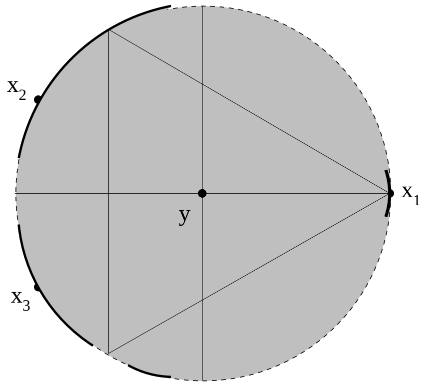

FIG. 5.2. *Segments explain why $G(T, \Gamma)$ holds for some T. $EV(T, \Gamma)$ does not by prop. 5.2 with $J = 3$.*

6. Open problem.

As a conclusion to this article we formulate a problem which is as relevant to the applications as finding more clever first order multipliers : find explicit classes of nonlinear escape functions (e.g. polynomial in ξ of fixed odd order) which apply to simple situations where linear escape functions do not, and which yield sufficient conditions easier to verify than the sharper geodesics condition of [1].

Acknowledgments. I am thankful to N. Burq, G. Lebeau and C. Margerin for stimulating discussions. N. Burq triggered this investigation by mentioning the function $f(x, \xi) = \xi.(x - x_0)$ as I was asking him whether he knew of a straightforward proof that the radial multiplier condition as found in [16] implies the bicharacteristics condition of [1].

REFERENCES

[1] C. BARDOS, G. LEBEAU, AND J. RAUCH, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), pp. 1024–1065.

[2] N. Burq, *Contrôle de l'équation des plaques en présence d'obstacles strictement convexes*, Mém. Soc. Math. France (N.S.), (1993), p. 126.

[3] ———, *Contrôlabilité exacte de l'équation des ondes dans des ouverts peu réguliers*, Asymptotic Anal., 14 (1997), pp. 157–191.

[4] ———, *Mesures semi-classiques et mesures de défaut*, Astérisque, (1997), pp. 167–195. Séminaire Bourbaki, Vol. 1996/97, Exp. No. 826.

[5] ———, *Contrôle de l'équation des ondes dans des ouverts comportant des coins*, Bull. Soc. Math. France, 126 (1998), pp. 601–637. With an appendix in collaboration with J.-M. Schlenker.

[6] N. Burq and P. Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes.*, C. R. Acad. Sci., Paris, Ser. I, Math., 25 (1997), pp. 749–752.

[7] N. Burq and G. Lebeau, *Mesures de défaut de compacité, application au sytème de Lamé.* Preprint, École Polytechnique, 1999, to appear in Ann. Sci. École Norm. Sup. (4).

[8] G. Chen, *A note on the boundary stabilization of the wave equation*, SIAM J. Control Optim., 19 (1981), pp. 106–113.

[9] P. Gérard and E. Leichtnam, *Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. J., 71 (1993), pp. 559–607.

[10] B. Helffer and J. Sjöstrand, *Résonances en limite semi-classique*, Mém. Soc. Math. France (N.S.), (1986), pp. iv+228.

[11] V. Komornik, *Exact controllability and stabilization: the multiplier method*, Masson, Paris, 1994.

[12] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Differential Equations, 50 (1983), pp. 163–182.

[13] G. Lebeau, *Control for hyperbolic equations*, in Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1992), École Polytech., Palaiseau, 1992, p. 24.

[14] ———, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. (9), 71 (1992), pp. 267–291.

[15] ———, *Équation des ondes amorties*, in Algebraic and Geometric Methods in Mathematical Physics, A. Boutet de Monvel and V. Marchenko, eds., Kluwer Academic, 1996, pp. 73–109.

[16] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, Masson, Paris, 1988. With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[17] K. Liu, *Locally distributed control and damping for the conservative systems*, SIAM J. Control Optim., 35 (1997), pp. 1574–1590.

[18] R. B. Melrose and J. Sjöstrand, *Singularities of boundary value problems. I*, Comm. Pure Appl. Math., 31 (1978), pp. 593–617.

[19] ———, *Singularities of boundary value problems. II*, Comm. Pure Appl. Math., 35 (1982), pp. 129–168.

[20] L. Miller, *Bicharacteristics conditions for exact controllability from the boundary in problems of transmission*. in preparation.

[21] ———, *Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary*, J. Math. Pures Appl. (9), 79 (2000), pp. 227–269.

[22] C. S. Morawetz, *The decay of solutions of the exterior initial-boundary value problem for the wave equation*, Comm. Pure Appl. Math., 14 (1961), pp. 561–568.

[23] ———, *Decay for solutions of the exterior problem for the wave equation*, Comm. Pure Appl. Math., 28 (1975), pp. 229–264.

[24] C. S. Morawetz, J. V. Ralston, and W. A. Strauss, *Decay of solutions of the wave equation outside nontrapping obstacles*, Comm. Pure Appl. Math., 30 (1977), pp. 447–508.

[25] A. Osses, *Une nouvelle famille de multiplicateurs et applications à la contrôlabilité exacte de l'équation d'ondes*, C. R. Acad. Sci. Paris Sér. I Math., 326 (1998), pp. 1099–1104.

[26] W. A. Strauss, *Dispersal of waves vanishing on the boundary of an exterior domain*, Comm. Pure Appl. Math., 28 (1975), pp. 265–278.

[27] P.-F. Yao, *On the observability inequalities for exact controllability of wave equations with variable coefficients*, SIAM J. Control Optim., 37 (1999), pp. 1568–1599.