Entropy, Universal coding, Approximation and Bases properties.

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Abstract

We shall present here results concerning the metric entropy of spaces of linear and non linear approximation under very general conditions. Our first result precises the metric entropy of the linear and non linear approximation spaces according to an unconditional basis verifying the Temlyakov property. This theorem shows that the second index r is not visible throughout the behavior of the metric entropy. However, metric entropy does discriminate between linear and non linear approximation.

Our second result extends and precises a result obtained in an hilbertian framework by Donoho. Since these theorems are given under the general context of Temlyakov property, they have a large spectrum of applications. For instance, it is proved in the last section, they can be applied, in the case of $_p$ norms for ^{d} for $1 < p < \infty$. We show that the lower bounds needed for this paper are in fact following from quite simple large deviation inequalities concerning hypergeometric or binomial distributions.

To prove the upper bounds, we provide a very simple universal coding based on a thresholding-quantizing procedure.

Key words and phrases : Entropy, coding, linear and non linear approximation, unconditonal bases, wavelet bases.

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1 Introduction

We are mainly interested in this paper, in the interplay between linear approximation, non linear approximation and metric entropy. Non linear approximation has been widely studied, at least these last 15 years (see for instance [4], [5], [9], [10], [11], [12]...). Especially the constructive methods, and particularly those using bases have been developped in the direction of important applications : image processing, statistical estimation, compression. (see for instance Donoho [15], Kerkyacharian Picard [25].)

On the other side, metric entropy has been introduced in the fifties by Kolmogorov (see [27] and also [31], [32]). In 1967 Birman and Solomyak [3] have computed the behavior of the metric entropy for balls of Besov spaces. More recently, in the framework of $_2-$approximation, D. Donoho in [15] has proved that there is an important link between the two topics. His proof uses the characterization of non linear approximation spaces as sequence spaces, and an interpretation of metric entropy via coding theory. The point of wiew of coding theory has also been recently studied by Birgé and Massart [2], and Cohen and al, [6].

We shall present here two principal results concerning the metric entropy of spaces of linear and non linear approximation under very general conditions. In section 2, we recall the main definitions of the objects that we are going to consider. More precisely, Schauder and unconditional bases, as well as the very useful Temlyakov property will be recalled. We also give the definitions of spaces of linear and non linear approximation according to a particular basis.

In section 3, we state our two principal results which are the following : The first theorem precises the metric entropy of the linear and non linear approximation spaces according to an unconditional basis verifying the Temlyakov property. This theorem shows that the second index r is not visible throughout the behavior of the metric entropy. However, metric entropy does discriminate between linear and non linear approximation.

The second theorem extends and precises the result obtained in an hilbertian framework by Donoho [15] : Under some geometrical condition for K, and some a priori compactness property subspaces of the non linear approximation space $A_\infty^s(\mathcal{E})$ are characterized by metric entropy behaviour.

Since these theorems are given under the general context of Temlyakov property, they have a large spectrum of applications. For instance, as it is proved in the last section, they can be applied, in the case of $_p$ norms for ^{d} for $1 < p < \infty$, when the basis is a compactly supported wavelet tensor-product basis. With some additional computations which can be found in [26], these results enable us to find back very simply the results of Birman and Solomiak, concerning the metric entropy of Besov spaces.

Section 4 and 5 are devoted to the proofs of the previous theorems. However, each of them has its own interest.

Section 4 concerns lower bounds. All the lower bounds needed for this paper are in fact following from an evaluation of large deviation type inequalities concerning hyper-

geometric or binomial distributions. It is then applied to special sets whose entropy is bounded below (see Proposition 4).

Section 5 is devoted to upper bounds. To prove the upper bounds, we provide a very simple universal coding (it is essentially a thresholding-quantizing procedure).

As was said before, the last section in concerning the application to wavelet in the context of ℓ_p spaces. We will first make a review of some embedded properties for a sequence $(e_i)_{i \in}$ of real valued functions defined on a measured space $(\mathcal{X}, \mu)$. The last one will be the p-Temlyakov property. The first one is the superconcentration property. We will prove that this property can be proved for wavelet bases. We will essentially need 2 steps: The first one studies the case of the Haar wavelet, where the properties is almost obvious; the second one uses the maximal functional to transfer results about Haar wavelets to general compactly supported wavelets.

2 Properties of Bases and Approximation.

Let us first recall some well known facts about bases in Banach spaces and approximation. We refer to [38] and [29].

2.1 Banach spaces and Schauder Bases.

All along the paper X will denote a Banach space, or a quasi-Banach space equipped with the norm $\| \ \|_X$. Let us recall (see [11]) that we can renormalized a quasi-Banach space in such way that it becomes a $\tau-$Banach with $0 < \tau \le 1$, i.e.

$$\forall f \in X, g \in X, \quad \|f + g\|_X^\tau \le \|f\|_X^\tau + \|g\|_X^\tau$$

To make things simpler, we will give the proofs for the Banach case. However, the case of a quasi-Banach case is straightforward. It is let to the reader. Typical examples of Banach spaces will be $X = \ell_p(\mathbb{Z}^d)$ $1 < p < \infty$ and $X = H_1$ the Hardy space, and of quasi-Banach spaces the Hardy spaces $X = H_p$, $0 < p < 1$.

We recall that a *topological basis (or Schauder basis)* is a sequence $\mathcal{E} = \{e_n, \ n \in \mathbb{N}\} \subset X$, such that :

$$\forall x \in X, \text{ there exists a unique sequence } \lambda_n \text{ such that } x = \sum \lambda_n e_n.$$

The sequence $\sum \lambda_n e_n$ is convergent in the X-norm.

This notion has been introduced by Schauder in 1927 see [37]. As an example, the Fourier basis is a topological basis of $\ell_p(\mathbb{T})$ for $1 < p < \infty$, if $\mathbb{T}$ is the torus.

2.1.1 Spaces of linear approximation according to a basis $\mathcal{E}$.

Let $\mathcal{E} = \{e_n, \ n \in \mathbb{N}\}$ be a Schauder basis of X. For $f = \sum_i \theta_i e_i$ in X, let us introduce

$$\rho_n(f, \mathcal{E}, X) = \|\sum_{i=n}^{\infty} \theta_i e_i\|_X.$$

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Of course, using the definition of a Schauder Basis, we get $\lim_{n\to\infty}\rho_n(f,\mathcal{E},X) = 0$. It is natural to precise the rate of convergence and define for all $0 < s < \infty$, and $0 < r < \infty$, the spaces :

$$B_r^s(\mathcal{E},X) = \{f = \sum_i \theta_i e_i \in X, \quad \{\sum_{n=1}^{\infty}[n^s\rho_{n-1}(f,\mathcal{E},X)]^r 1/n\}^{1/r} := \|f\|_{B_r^s(\mathcal{E},X)} < \infty\}$$

and for $r = \infty$,

$$B_\infty^s(\mathcal{E},X) = \{f = \sum \theta_i e_i \in X, \quad \sup_{1\leq n\leq\infty} n^s\rho_{n-1}(f,\mathcal{E},X) := \|f\|_{B_\infty^s(\mathcal{E},X)} < \infty\}$$

It can be shown that these spaces are (quasi-)Banach spaces, continuously embedded in X.

2.1.2 Spaces of non linear approximation.

Let $\mathcal{E} = \{e_n,\ n \in\ \}$ be a Schauder basis of X. For any integer $n \in\ $ and any function $f = \sum_0^\infty \theta_i e_i \in X$ we define the best rate of approximation of f by a linear combination of length n in $\mathcal{E}$, by:

$$\sigma_n(f,\mathcal{E},X) = \inf\{\|f - h\|_X, \quad h = \sum_{i\in\Lambda}\lambda_i e_i, \quad \text{card }(\Lambda) = n, \quad \lambda_i \in\ \} \tag{1}$$

Obviously, $\sigma_n \leq \rho_n$ since $\Lambda_n = \{1,\dots,n\}$ and $\lambda_i = \theta_i$ is a possible choice. Moreover, the introduction of all the other possible choices clearly leads to a non linear problem. This notion seems to go back at least to Birman and Solomiak [3], and has been impressively revisited in the last decade.

One also defines, for all $0 < s < \infty$, and $0 < r < \infty$, the space :

$$A_r^s(\mathcal{E},X) = \{f = \sum \theta_i e_i \in X, \quad \{\sum_1^{\infty}[n^s\sigma_{n-1}(f,\mathcal{E},X)]^r 1/n\}^{1/r} := \|f\|_{A_r^s(\mathcal{E},X)} < \infty\}$$

and for $r = \infty$,

$$A_\infty^s(\mathcal{E},X) = \{f = \sum \theta_i e_i \in X, \quad \sup_{1\leq n\leq\infty} n^s\sigma_{n-1}(f,\mathcal{E},X) := \|f\|_{A_\infty^s(\mathcal{E},X)} < \infty\}$$

It can be shown (see [9], [11], [12]) that these spaces are (quasi-) Banach spaces, continuously embedded in X. Obviously $B_r^s(\mathcal{E},X)$ is continously embedded in $A_r^s(\mathcal{E},X)$. Let us observe that $\sigma_n(f,\mathcal{E},X)$ is a non increasing function of n. Hence, we have an equivalent description of $A_r^s(\mathcal{E},X)$, by taking :

$$A_r^s(\mathcal{E},X) = \{f = \sum \theta_i e_i \in X, \quad \|f\|_X + \{\sum_{j=0}^{\infty}[2^{js}\sigma_{2^j}(f,\mathcal{E},X)]^r\}^{1/r} := \|f\|'_{A_r^s(\mathcal{E},X)} < \infty\}$$

$$\tag{2}$$

and for $r = \infty$,

$$A_\infty^s(\mathcal{E},X) = \{f = \sum \theta_i e_i \in X, \quad \|f\|_X + \sup_{0\leq j\leq\infty} 2^{js}\sigma_{2^j}(f,\mathcal{E},X) := \|f\|'_{A_\infty^s(\mathcal{E},X)} < \infty\}$$

$$\tag{3}$$

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2.2   Unconditional bases

We recall that an *unconditional basis* of X is a topological basis $\mathcal{E} = \{e_n, \}_{(n \in \ )}$ verifying the *Shrinkage property* : There exists an absolute constant C such that if $|\theta_i| \leq |\theta_i'|$ for all i, then

$$\| \sum_i \theta_i e_i \|_X \leq C \| \sum_i \theta_i' e_i \|_X. \tag{4}$$

The notion of unconditional basis also has been revisited in the recent year partly because its importance in statistical applications (see Donoho [14], [15] and Mallat [33]).

If $\mathcal{E} = \{e_n, \ n \in \ \}$ is a topological basis (resp. unconditional basis) then $\{\lambda_n e_n, \ n \in \ \}$ is also a topological basis (resp. unconditional basis) as soon as $\forall n \in \ , \ \lambda_n \neq 0$. Hence, we will naturally associate to an unconditional basis $\mathcal{E} = \{\psi_n, \ n \in \ \}$ the 'normalized' unconditional basis : $\{e_n = \dfrac{\psi_n}{\|\psi_n\|_X}, \ n \in \ \}$

2.2.1   Unconditional bases and linear approximation.

If $\mathcal{E} = \{e_n, \}_{(n \in \ )}$ is an unconditional basis, then it is possible to give a description of the elements of $B_r^s(\mathcal{E}, X)$ in terms of dyadics, as we did for $A_r^s(\mathcal{E}, X)$: let $f = \sum_{i=0}^{\infty} \theta_i e_i$ and $2^j \leq n < 2^{j+1}$. Then

$$\frac{1}{C} \rho_{2^{j+1}}(f) = \frac{1}{C} \| \sum_{i=2^{j+1}}^{\infty} \theta_i e_i \|_X \leq \rho_n(f) = \| \sum_{i=n}^{\infty} \theta_i e_i \|_X \leq C \| \sum_{i=2^j}^{\infty} \theta_i e_i \|_X = C \rho_{2^j}(f)$$

So :

$$B_r^s(\mathcal{E}, X) = \{f = \sum \theta_i e_i \in X, \ \ \|f\|_X + \{\sum_{j=0}^{\infty} [2^{js} \rho_{2^j}(f, \mathcal{E}, X)]^r\}^{1/r} := \|f\|'_{B_r^s(\mathcal{E}, X)} < \infty\}$$

with the usual modification for $r = \infty$.
We can also express the previous norms in even simpler forms, observing that:

$$\frac{1}{C} \| \sum_{2^j \leq i < 2^{j+1}} \theta_i e_i \|_X \leq \rho_{2^j}(f) = \| \sum_{i=2^j}^{\infty} \theta_i e_i \|_X \leq \sum_{k=j}^{\infty} \| \sum_{2^k \leq i < 2^{k+1}} \theta_i e_i \|_X$$

Using the following well known lemma, we obviously also obtain :

$$B_r^s(\mathcal{E}, X) = \{f = \sum \theta_i e_i \in X, \ \ \|f\|_X + \{\sum_{j=0}^{\infty} [2^{js} \| \sum_{2^j \leq i < 2^{j+1}} \theta_i e_i \|_X]^r 1\}^{1/r} := \|f\|''_{B_r^s(\mathcal{E}, X)} < \infty\} \tag{5}$$

(with the usual modification for $r = \infty$.)

Lemma 1. *Let* $0 < s < \infty$, $0 < r \le \infty$. *Let* (α_j) *be a sequence of real numbers and for all* j *in* , *let* $\eta_j = \sum_{k \ge j} |\alpha_k| 2^{-(k-j)s}$, *then*

$$\|(\alpha_j)\|_r \le \|(\eta_j)\|_r \le C_{r,s} \|(\alpha_j)\|_r.$$

$(\|(\alpha_j)\|_r^r = \sum_j |\alpha_j|^r.)$

2.2.2 Unconditional bases and non linear approximation.

Let us now observe that if $\mathcal{E} = \{e_i, \; i \in \;\}$ is an unconditional basis of X, for all $f \in X$, $\sigma_n(f, \mathcal{E}, X)$ can also be computed up to constants, in a much simpler way: In the case where X is a Hilbert space, in the definition (1) of σ_n we can obviously restrict to functions of the form $h = \sum_{i \in \Lambda} \theta_i e_i$ to approximate $f = \sum_i \theta_i e_i$. The following proposition proves that if we do so in the case where $\mathcal{E}$ is an unconditional basis, we only lose an absolute constant.

Proposition 1. *Let* $\mathcal{E} = \{e_i, \; i \in \;\}$ *an unconditional basis of* X. *Let* $f = \sum \theta_i e_i$ *in* X. *Then for any* $\Lambda \subset$ *and for any linear combination* $\sum_{i \in \Lambda} \lambda_i e_i$ *we have :*

$$\|f - \sum_{i \in \Lambda} \theta_i e_i\|_X \le (1 + C)\|f - \sum_{i \in \Lambda} \lambda_i e_i\|_X, \; and$$

$$\sigma_n(f, \mathcal{E}, X) \le \inf\{\|f - \sum_\Lambda \theta_i e_i\|_X, \quad card\,(\Lambda) = n\} \le (1 + C)\sigma_n(f, \mathcal{E}, X).$$

Proof: $\|f - \sum_\Lambda \theta_i e_i\|_X \le \|f - \sum_\Lambda \lambda_i e_i\|_X + \|\sum_\Lambda \theta_i e_i - \sum_\Lambda \lambda_i e_i\|_X$.
But $\|\sum_\Lambda \theta_i e_i - \sum_\Lambda \lambda_i e_i\|_X \le C\|f - \sum_\Lambda \lambda_i e_i\|_X$. $\qquad \square$

2.3 Unconditional bases with p-Temlyakov Property

Definition 1. *Let* $\mathcal{E} = \{e_i, \; i \in \;\}$ *be a normalized unconditional Basis of* X, $p > 0$. *This basis shares the 'p-Temlyakov' Property if there exists* $0 < C_1 < \infty$, *such that the following bounds are true for any finite subset* Λ *of* :

$$\frac{1}{C_1} \inf_{i \in \Lambda} |\theta_i| (\,card\,\Lambda)^{1/p} \le \|\sum_{i \in \Lambda} \theta_i e_i\|_X \le C_1 \sup_{i \in \Lambda} |\theta_i| (\,card\,\Lambda)^{1/p} \tag{6}$$

The pair of inequalities (6) has been introduced in DeVore (1998) [9] and Temlyakov (1999) [41] and is generally refered to as *Temlyakov's property*. We will prove in section 6 that fundamental examples of families verifying (6) are compactly supported wavelet bases, and associated wavelet-tensor product bases, when the space X is the $_p$ space on ^{d} or $[0, 1]^d$, for $1 < p < \infty$.

This property is a kind of quantified version of the following concept of democratic bases introduced in Konyagin, Temlyakov 1999 [28] : For all finite Λ, Λ' included in with the same cardinality, then $\|\sum_\Lambda e_i\| \sim \|\sum_{\Lambda'} e_i\|$.

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2.3.1 p-Temlyakov Property and non linear approximation.

Let us start with the following definition :

Definition 2. *For a countable set I, let us define the non increasing rearrangement of a family $(a_i)_{i \in I}$ of complex numbers : as the family $(|a|_{(n)})_{n \in I}$ where,*

$$\forall n \geq 1, \quad |a|_{(n)} = \inf[\lambda; \quad card \{i \in I/|a|_i > \lambda\} < n].$$

Obviously,

$$card \{i \in I/|a|_i > |a|_{(n)}\} < n \leq card \{i \in I/|a|_i \geq |a|_{(n)}\}. \tag{7}$$

We will have now the opportunity of precising the best non linear approximation of length n of an arbitrary function $f = \sum \theta_i e_i$. We have already proved in proposition (1) that if $\{e_i, \ i \in \ \}$ is an unconditional bases the search for the best non linear approximation can be restricted to only consider $\sum_{i \in \Lambda} \theta_i e_i$, and optimize the choice of Λ. If in addition, we have property (6), the following proposition precises an easy way to choose Λ.

This proposition is proved in [28] in a more general context. We give here a more direct proof for the reader's convenience :

For $f = \sum \theta_i e_i$, we say that the subset Λ_n of $\ $ is n-adjusted if

$$card (\Lambda_n) = n, \quad and \quad \inf_{i \in \Lambda_n} (|\theta_i|) = |\theta|_{(n)} \tag{8}$$

Without loss of generality, the sequence Λ_n can be chosen non decreasing.

Proposition 2. *(Konyagin and Temlyakov) Let $\mathcal{E} = \{e_i, \ i \in \ \}$ be a normalized unconditional basis verifying property (6). Let $f = \sum_{i=0}^{\infty} \theta_i e_i$ in X. For n arbitrary chosen in $\ $, let us suppose that Λ_n subset of $\ $ is n adjusted, then for any Λ such that $card (\Lambda) = n$, we have*

$$\| \sum_{i \notin \Lambda_n} \theta_i e_i \|_X \leq (1 + C + CC_1^2) \| \sum_{i \notin \Lambda} \theta_i e_i \|_X$$

Proof :

$$\sum_{i \notin \Lambda_n} \theta_i e_i \ = \ \sum_{i \notin \Lambda} \theta_i e_i + \sum_{i \in \Lambda \setminus \Lambda_n} \theta_i e_i - \sum_{i \in \Lambda_n \setminus \Lambda} \theta_i e_i \ \ hence,$$

$$\| \sum_{i \notin \Lambda_n} \theta_i e_i \|_X \ \leq \ \| \sum_{i \notin \Lambda} \theta_i e_i \|_X + \| \sum_{i \in \Lambda \setminus \Lambda_n} \theta_i e_i \|_X + \| \sum_{i \in \Lambda_n \setminus \Lambda} \theta_i e_i \|_X$$

But

$$\| \sum_{i \in \Lambda_n \setminus \Lambda} \theta_i e_i \|_X \leq C \| \sum_{i \notin \Lambda} \theta_i e_i \|_X$$

$$\| \sum_{i \in \Lambda \setminus \Lambda_n} \theta_i e_i \|_X \leq C_1 |\theta|_{(n)} \ card \ (\Lambda \setminus \Lambda_n)^{1/p} \ = \ C_1 |\theta|_{(n)} \ card \ (\Lambda_n \setminus \Lambda)^{1/p}$$

$$\leq \ C_1^2 \| \sum_{i \in \Lambda_n \setminus \Lambda} \theta_i e_i \|_X$$

$$\leq \ CC_1^2 \| \sum_{i \notin \Lambda} \theta_i e_i \|_X$$

Hence,

$$\|\sum_{i \notin \Lambda_n} \theta_i e_i\|_X \le (1 + C + CC_1^2)\|\sum_{i \notin \Lambda} \theta_i e_i\|_X$$

$\square$

2.3.2 Sequence spaces.

The property (6) of the unconditional basis $\mathcal{E} = \{e_i, i \in \ \}$, will allow us to define some particularly interesting sequence spaces.

Let us recall that for an arbitrary sequence (a_n) we have defined in definition 2 the non increasing rearrangement sequence $(|a|_{(n)})$. We also define the following norms.

For $0 < r < \infty$, $0 < q < \infty$,
$$\|(a_n)\|_{q,r} = [\sum_1^\infty (n^{1/q}|a|_{(n)})^r 1/n]^{1/r} \sim [\sum_0^\infty (2^{j/q}|a|_{(2^j)})^r]^{1/r}$$

$$\|(a_n)\|_{q,\infty} = \sup_{n \ge 1} n^{1/q}|a|_{(n)} \sim \sup_{j \ge 0} 2^{j/q}|a|_{(2^j)}$$

$$\|(a_n)\|_{\infty,\infty} = \sup_{n \ge 1} |a|_{(n)} = \|(a_n)\|_\infty.$$

We recall the following usual notation:

$$\|.\| \sim \|.\|' \iff \text{There exist } (A, B), \forall g, A\|g\| \le \|g\|' \le B\|g\| \tag{9}$$

The Lorentz space $l_{q,r}$, is defined as the space of sequences (a_n) verifying $\|(a_n)\|_{q,r} < \infty$. $\|(a_n)\|_{q,r}$ is a (quasi-)norm on $l_{q,r}$. It is well known that we have the following continuous embeddings :

$$l_{q,r} \subset l_{q,s}, \text{for } 0 < r < s \le \infty, \ 0 < q < \infty,.$$
$$l_{q',r} \subset l_{q,s}, \text{for } 0 < q' < q < \infty, \ \forall r, s,.$$
$$l_{q,q} = l_q, \text{ for } q, \ 0 < q \le \infty,.$$

We can also define the following space

$$l_{q,r}(\mathcal{E}, X) = \{f = \sum \theta_n e_n \in X, \text{ such that } \|f\|_{l_{q,r}(\mathcal{E},X)} := \|(\theta_n)\|_{q,r} < \infty.\}$$

2.3.3 p-Temlyakov property and non linear approximation.

The following theorem shows that the previous spaces $l_{q,r}(\mathcal{E}, X)$ for $0 < q < p < \infty$, can in fact be identified with the spaces of non linear approximation. Actually, this theorem is an extension in our framework of a result in DeVore, Jawerth and Popov [10]. It is worthwhile to notice that the proof given here is completely direct and will not use the interpolation theory as it is the case in [10]. [1]

Theorem 1. Let $\mathcal{E} = \{e_i, \ i \in \ \}$ be an unconditional basis.

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[1] While writing this manuscript we heard about the work of Gribonval, Nielsen [22] where the same result is proved using interpolation theory, together with other more general results on democratic bases.

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1. *If $\mathcal{E}$ verifies the p-Temlyakov property (6), and if $0 < s < \infty$*

$$\forall \, 0 < r \leq \infty, \quad and \quad q, \quad such \ that \ s = 1/q - 1/p, \ A_r^s(\mathcal{E}, X) = l_{q,r}(\mathcal{E}, X)$$

2. *If there exists $0 < s$ such that for $q : s = 1/q - 1/p$, we have*

$$A_q^s(\mathcal{E}, X) = l_{q,q}(\mathcal{E}, X)$$

then $\mathcal{E}$ verifies the p-Temlyakov property (6).

Proof :

1. The proof essentially relies on the following lemma which allows us to transform the norm of $A_r^s(\mathcal{E}, X)$.

   Lemma 2. *Let $f = \sum_{k \in} \theta_k e_k$, σ be a permutation of such that $|\theta_{\sigma(i)}| = |\theta|_{(i)}$. Then if*

   $$\|f\|_{A_r^s(\mathcal{E}, X)} \sim \left(\sum_{j=0}^{\infty} (2^{js} \| \sum_{2^j \leq i < 2^{j+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X)^r \right)^{1/r}$$

   and

   $$\|f\|_{A_\infty^s(\mathcal{E}, X)} \sim \sup_{j \geq 0} 2^{js} \| \sum_{2^j \leq i < 2^{j+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X.$$

   Proof of the lemma This lemma is a consequence of the proposition (2). We have the following equivalent quasi-norms for $A_r^s(\mathcal{E}, X)$:

   $$\|f\|_{A_r^s(\mathcal{E}, X)} \sim \left(\sum_{n=1}^{\infty} (n^s \| \sum_{n \leq i} \theta_{\sigma(i)} e_{\sigma(i)} \|_X)^r \frac{1}{n} \right)^{1/r} \quad ; \quad \|f\|_{A_\infty^s(\mathcal{E}, X)} \sim \sup_{n \geq 1} n^s \| \sum_{n \leq i} \theta_{\sigma(i)} e_{\sigma(i)} \|_X$$

   or

   $$\|f\|_{A_r^s(\mathcal{E}, X)} \sim \left(\sum_{j=0}^{\infty} (2^{js} \| \sum_{2^j \leq i} \theta_{\sigma(i)} e_{\sigma(i)} \|_X)^r \right)^{1/r} \quad ; \quad \|f\|_{A_\infty^s(\mathcal{E}, X)} \sim \sup_{n \geq 1} 2^{js} \| \sum_{2^j \leq i} \theta_{\sigma(i)} e_{\sigma(i)} \|_X$$

   Moreover, let us observe that, as the basis is unconditional,

   $$\| \sum_{2^j \leq i < 2^{j+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X \leq C \| \sum_{2^j \leq i} \theta_{\sigma(i)} e_{\sigma(i)} \|_X$$

   and

   $$\| \sum_{2^j \leq i} \theta_{\sigma(i)} e_{\sigma(i)} \|_X \leq \sum_{k \geq j} \| \sum_{2^k \leq i < 2^{k+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X$$

   Obviously, if

   $$\forall k \in \, , \, \| \sum_{2^k \leq i < 2^{k+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X = \alpha_k 2^{-ks}, \text{ with } (\alpha_i) \in l_q, \, 0 < q \leq \infty$$

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then
$$\| \sum_{2^j \le i} \theta_{\sigma(i)} e_{\sigma(i)} \|_X \le \sum_{k \ge j} \alpha_k 2^{-ks} = 2^{-js} \sum_{k \ge j} \alpha_k 2^{-(k-j)s} = 2^{-js} \eta_j$$

But using lemma 1, we have :

$$\|f\|_{A_r^s(\mathcal{E},X)} \sim \left(\sum_{j=0}^{\infty} (2^{js} \| \sum_{2^j \le i < 2^{j+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X)^r \right)^{1/r}$$

and

$$\|f\|_{A_\infty^s(\mathcal{E},X)} \sim \sup_{j \ge 0} 2^{js} \| \sum_{2^j \le i < 2^{j+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X .$$

To prove the theorem, we now only need to remark that, using (6):

$$\forall j \ge 0, \quad \frac{1}{C_1} 2^{j/p} |\theta|_{(2^{j+1}-1)} \le \| \sum_{2^j \le i < 2^{j+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X \le C_1 |\theta|_{(2^j)} 2^{j/p},$$

Hence, for $0 < q < p$,

$$\left(\sum_{j=0}^{\infty} (2^{j/q} |\theta|_{(2^j)})^r \right)^{1/r} \sim \left(\sum_{j=0}^{\infty} (2^{j(1/q-1/p)} \| \sum_{2^j \le i < 2^{j+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X)^r \right)^{1/r}$$

and

$$\sup_{j \ge 0} 2^{j/q} |\theta|_{(2^j)} \sim \sup_{j \ge 0} 2^{j(1/q-1/p)} \| \sum_{2^j \le i < 2^{j+1}} \theta_{\sigma(i)} e_{\sigma(i)} \|_X .$$

2. Let $0 < s = 1/q - 1/p, \quad A_q^s(\mathcal{E},X) = l_{q,q}(\mathcal{E},X)$. So

$$\forall f = \sum_{i \in} \theta_i e_i, \quad \{ \sum_1^{\infty} [k^{1/q-1/p} \sigma_{k-1}(f,\mathcal{E},X)]^q 1/k \}^{1/q} \sim (\sum_{i \in} |\theta_i|^q)^{1/q} \qquad (10)$$

Let us prove that p-Temlyakov property (6). Clearly it is equivalent to prove that

$$\forall i \in \quad . \quad \|e_i\|_X \sim 1, \text{which is obvious by (10)}$$

and

$$\forall \Lambda \subset \quad , \quad card(\Lambda) = n, \quad \| \sum_\Lambda e_i \|_X \sim n^{1/p}$$

So let $f = \sum_\Lambda e_i$. By (10) we have

$$\{ \sum_1^n k^{-q/p} [\sigma_{k-1}(f,\mathcal{E},X)]^q \}^{1/q} \sim n^{1/q}$$

but

$$\|f\|_X n^{1/q-1/p} \sim \|f\|_X \{ \sum_1^n k^{-q/p} \}^{1/q} \ge \{ \sum_1^n k^{-q/p} [\sigma_{k-1}(f,\mathcal{E},X)]^q \}^{1/q} \sim n^{1/q}$$

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This gives one half of the result. On the other hand, let $card(\Lambda) = 2n$, $f = \sum_\Lambda e_i$.

$$\sigma_n(f,\mathcal{E},X)n^{1/q-1/p} \sim \sigma_n(f,\mathcal{E},X)\{\sum_{n+1}^{2n} k^{-q/p}\}^{1/q} \leq \{\sum_1^{2n} k^{-q/p}[\sigma_{k-1}(f,\mathcal{E},X)]^q\}^{1/q} \sim (2n)^{1/q}$$

This means that there exists a constant A, such that for all subset $\Lambda \subset$, $card(\Lambda) = 2n$ there exists $\Lambda' \subset \Lambda$ $card(\Lambda') = n$, such that $\|\sum_{i\in\Lambda'} e_i\|_X \leq An$. Now, if $card(\Lambda) = 2^j$ we can split Λ in $j+1$ disjoint subsets $\Lambda_{-1}, \Lambda_0, ...\Lambda_{j-1}$, $card(\Lambda_{-1}) = 1$, and $card(\Lambda_l) = 2^l$ for $l = 0,1,...j-1$, such that $\|\sum_{i\in\Lambda_l} e_i\|_X \leq A2^{l/p}$. This , of course implies that $\|\sum_{i\in\Lambda} e_i\|_X \leq A'2^{j/p}$, with another constant A' depending only on A and X. This gives the second half of the result, at least for subset of of dyadic cardinality. but it is easy to conclude.

$\square$

We finish this section devoted to the properties of bases by their consequences on the approximation of functions.

2.3.4   Thresholding and Quantization.

Let $\mathcal{E} = \{e_i, i \in \}$ be a normalized unconditional basis verifying p-Temlyakov property (6). In this section we will derive from this assumption an upper bound for the two following well known types of approximation of an arbitrary function $f = \sum_i \theta_i e_i \in X$: Let $\lambda > 0$.

1. The λ-thresholding approximation of f : $\sum_i \theta_i e_i I\{|\theta_i| > \lambda\}$.

2. The λ-quantized approximation of f : $\sum_i Sign(\theta_i)k_i\lambda e_i$, where $k_i = [|\theta_i|/\lambda]$. ($[x]$ denotes the integer part of x.)

Proposition 3. Let $\mathcal{E} = \{e_i, i \in \}$ be a normalized unconditional basis verifying p-Temlyakov property (6). For $0 < q < p$ there exist constants C_q, D_q, only depending on q and the constant C_1 in (6), such that for any $f = \sum_i \theta_i e_i \in l_{q,\infty}(\mathcal{E},X)$ and $\lambda > 0$, $\epsilon > 0$,

$$\|f - \sum \theta_i e_i I\{|\theta_i| > \lambda\}\|_X = \quad \|\sum \theta_i e_i I\{|\theta_i| \leq \epsilon\}\|_X \quad \leq D_q \lambda^{1-\frac{q}{p}}\|\theta\|_{l_{q,\infty}}^{\frac{q}{p}}. \quad (11)$$

$$\|f - \sum_i Sign(\theta_i)k_i\lambda e_i\|_X = \quad \|\sum_i(\theta_i - Sign(\theta_i)k_i\lambda)e_i\|_X \quad \leq C_q\|\theta.\|_{l_{q,\infty}}^{\frac{q}{p}}\lambda^{1-\frac{q}{p}} \quad (12)$$

Where $k_i = [|\theta_i|/\lambda]$.

Proof :

$$\|f - \sum_{|\theta_i|>\lambda} \theta_i e_i\|_X = \|\sum_{|\theta_i|\leq\lambda} \theta_i e_i\|_p \leq \sum_{j\geq 0} \|\sum_{i\in\Lambda_j} \theta_i e_i\|_X;$$

Where $\Lambda_j = \{i/\lambda 2^{-j-1} < |\theta_i| \leq \lambda 2^{-j}\}$. By (6), we have : $\|\sum_{i\in\Lambda_j} \theta_i e_i\|_X \leq C_1 \lambda 2^{-j} (\ card\ (\Lambda_j))^{\frac{1}{p}}$. As $(\theta_n) \in l_{q,\infty}$, we have : $card\ (\Lambda_j) \leq \|(\theta_n)\|_{l_{q,\infty}}^q \lambda^{-q} 2^{(j+1)q}$, hence:

$$\|\sum_{|\theta_i|\leq\lambda} \theta_i e_i\|_X \leq C_1\|(\theta_n)\|_{l_{q,\infty}}^{\frac{q}{p}}\lambda^{1-\frac{q}{p}} \sum_{j\geq 0} 2^{-j}2^{(j+1)\frac{q}{p}}.$$

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This proves (11). For (12):

$$\|\sum_i (\theta_i - Sign(\theta_i)k_i\lambda)e_i\|_X \leq \|\sum_i \theta_i e_i I\{|\theta_i| < \lambda\}\|_X + \|\sum_i I\{|\theta_i| \geq \lambda\}(\theta_i - Sign(\theta_i)k_i\lambda)e_i\|_X$$

Using (11), : $\|\sum_i \theta_i e_i I\{|\theta_i| < \lambda\}\|_X \leq D_q \lambda^{1-q/p}\|\theta.\|_{l_{q,\infty}}^{\frac{q}{p}}$. It remains to see, using property (6), that

$$\|\sum_i I\{|\theta_i| \geq \lambda\}(\theta_i - Sign(\theta_i)k_i\lambda)e_i\|_X \leq C_1\lambda \ (\ \text{card} \ \{i, |\theta_i| \geq \lambda\})^{1/p} \leq C_1\|(\theta_i)\|_{l_{q,\infty}}^{\frac{q}{p}}\lambda^{1-q/p}.$$

$\square$

3 Metric Entropy.

Let us recall the following definitions.

- Let (K,d) a metric space. For every $\epsilon > 0$, we define $N(\epsilon, K, d)$ as the minimum number of balls of radius ϵ, covering K.

- We define the **metric Entropy** of K as $H(\epsilon, K, d) = \log_2(N(\epsilon, K, d))$

- Let (X,d) a metric space, and $K \subset X$. For every $\epsilon > 0$, we define $N(\epsilon, K, X, d)$ as the minimum number of balls of radius ϵ, centered in X, covering K.

- We define the **metric Entropy relative to** X as $H(\epsilon, K, X, d) = \log_2(N(\epsilon, K, X, d))$. If K is considered with the induced metric, it is obvious that :

$$H(\epsilon, K, d) \geq H(\epsilon, K, X, d) \geq H(2\epsilon, K, d).$$

3.1 Main Results

An important aspect of unconditional bases and spaces $l_{q,\infty}(\mathcal{E})$ has been pointed out in Donoho [15] expliciting their link with the metric entropy and the coding theory, in an hilbertian context. See also Cohen, Dahmen, Daubeuchies and DeVore (1999). We begin with the following theorem giving the metric entropy for balls of spaces of linear and non linear approximation.

Theorem 2. *Let* $\mathcal{E} = \{e_n, \ n \in \ \}$ *be an unconditional normalized basis of some space* X , *verifying property (6) with some* $0 < p < \infty$.

1. *The unit ball* $V_r^s(\mathcal{E}, X)$ *of* $A_r^s(\mathcal{E}, X)$ *is never compact, whatever* $0 < s < \infty$, $0 < r \leq \infty$ *are.*

2. *Let* $U_r^s(\mathcal{E}, X)$ *the unit ball of* $B_r^s(\mathcal{E}, X)$, *then if* $1 < p < \infty$,

$$H(\epsilon, U_r^s, X) \asymp \epsilon^{-1/s}$$

3. let $0 < \delta < s$ then

$$H(\epsilon, V_r^s(\mathcal{E}, X) \cap U_r^\delta(\mathcal{E}, X), X) \asymp \epsilon^{-1/s} \log(1/\epsilon)$$

Remarks :

- the notation $c(\epsilon) \asymp b(\epsilon)$ means that there exists two constant $c_1 > 0$, $c_2 > 0$, $c_1 c(\epsilon) \le b(\epsilon) \le c_2 c(\epsilon)$, $\forall \epsilon \le \epsilon_0 < 1$.

- As can be seen, the second index r is not visible throughout the behavior of the metric entropy. However, metric entropy does discriminate between linear and non linear approximation (at least for $1 < p < \infty$).

- Points 2 and 3 are proved in the forthcoming theorems 6 and theorem 7. Point 1 is proved in the remark at the beginning of section 4.4.

  $\star$

The following theorem extends and precises the result obtained in an hilbertian framework by Donoho [15]. Under some geometrical condition for K, and some a priori compactness property subspaces of $A_\infty^s(\mathcal{E})$ are characterized by metric entropy behaviour.

Theorem 3. *Let us consider K a subset of X. Let $\mathcal{E} = \{e_n, n \in\ \}$ be an unconditional, normalized basis of X verifying (6). Let us suppose :*

- *There exists $c > 0$ such that, if $f = \sum \theta_n e_n$ belongs to K then $\sum \omega_n \theta_n e_n$ also belongs to K for all $(\omega_n)_{n\in}\ \in \{0,1\}\ $. Then*

$$H(\epsilon, K, X, \| \ \|_X) \le C \epsilon^{-1/s} \implies K \subset A_\infty^s(\mathcal{E}, X).$$

- *K is contained in a ball of $B_\infty^\delta(\mathcal{E}, X)$ for some (small) $\delta > 0$. Then*

$$K \subset A_\infty^s(\mathcal{E}, X) \implies H(\epsilon, K, X, \| \ \|_X) \le C \epsilon^{-1/s} \log(1/\epsilon)$$

Hence, this theorem proves that if K has the property of orthosymmetry, then a polynomially bounded entropy implies an inclusion in a specific space of approximation. Reversely, if K is included in the previous space and polynomially tailed compact, then its entropy is polyniomially bounded. The proof of this theorem follows from the forthcoming theorem 4 and the previous one.

The two following sections are devoted to the proofs of these theorems. The first one is devoted to lower bounds. The bounds are in fact deriving from very simple concentration inequalities concerning binomial or hypergeometric distributions (see proposition 4). To prove the upper bounds, in section 5, we shall use coding theory.

4   Lower Bounds

This section will begin with a proposition evaluating lower bounds for the entropy of sets in the l_1 norm. This proposition will then be essential to establish the lower bounds for the entropy of the sets mentioned above.

4.1 Computations of the Metric Entropy for two model-sets.

Proposition 4. *Let us consider the following sets :*

$$\Omega_n = \{0,1\}^n, \ A_{k,n} = \{\omega \in \{0,1\}^n, \ \sum_{i=1}^n \omega_i = k\}, \ k \in$$

with the l_1 distance:

$$\|\omega - \omega'\| = \sum_{i=1}^n |\omega_i - \omega_i'|$$

The following bounds are true for $k \in \quad , \quad k \sim n^\alpha, \quad 0 < \alpha < 1$:

$$H(n/4, \Omega_n, l_1) \geq \frac{n}{8\log(2)} \tag{13}$$

$$H(k/2, A_{k,n}, l_1) \geq ((1-\alpha)\frac{3}{4}k\log k)(1 + o(1)) \tag{14}$$

Proof of the Proposition : Let us consider a covering of Ω_n by N balls of radius $n/4$:

$$\Omega_n = \cup_{k=1}^N B(\omega^k, n/4). \tag{15}$$

Let P the uniform probability measure on Ω_n (i.e. for all $\omega \in \Omega_n$, $P\{\omega\} = (\frac{1}{2})^n$). Because of (15), we have :

$$\sum_{k=1}^N P(B(\omega^k, n/4)) \geq 1.$$

Let us now compute $P(B(\omega^k, n/4))$.

$$B(\omega^k, n/4) = \{\omega \ / \ \sum_{i=1}^n |\omega_i - \omega_i^k| \leq n/4\} = \{\omega \ / \ \sum_{i=1}^n (1/2 - |\omega_i - \omega_i^k|) \geq n/4\}.$$

But, because of our choice for P, the random variables ω_i's are independent, so that the n random variables $Z_i(\omega) = 1/2 - |\omega_i - \omega_i^k|$, are also independent, centered Bernouilli variables. Using Hoeffding inequality (1963) [23], we get:

$$P(\{\omega \ / \ \sum_{i=1}^n Z_i(\omega) \geq n/4\}) \leq \exp -2\frac{(n/4)^2}{n} = \exp -n/8$$

So we have :

$$H(n/4, \Omega_n, l_1) \geq \frac{n}{8\log(2)}$$

This proves (13).

To prove (14) we again consider the uniform probability on the space $\Omega_n = \{0,1\}^n$ and a covering of $A_{k,n}$ by N balls $B(u^j, k/2)$, with $u^j \in A_{k,n}$. So

$$\frac{C_n^k}{2^n} = P(A_{k,n}) \leq \sum_{j=1}^N P(B(u^j, k/2) \cap A_{k,n}) \tag{16}$$

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Let us compute $P(B(u, k/2) \cap A_{k,n})$ with $u \in A_{k,n}$. As this quantity is obviously independent of $u \in A_{k,n}$, let us take $u = (1, \ldots, 1, 0, \ldots, 0)$.

$$B(u, k/2) \cap A_{k,n} = \{\omega \in \{0,1\}^n, \ (\sum_1^k (1 - X_i(\omega)) + \sum_{k+1}^n X_i(\omega)) \leq k/2\} \cap (\sum_1^n X_i(\omega) = k)\}$$

or again

$$B(u, k/2) \cap A_{k,n} = \bigcup_{0 \leq l \leq k/4} \{\omega \in \{0,1\}^n, \ (\sum_{k+1}^n X_i(\omega) = l) \cap (\sum_1^k X_i(\omega) = k - l)\}$$

So :

$$P(B(u, k/2) \cap A_{k,n}) = \sum_{0 \leq l \leq k/4} \frac{C_{n-k}^l}{2^{n-k}} \frac{C_k^{k-l}}{2^k} = \frac{1}{2^n} \sum_{0 \leq l \leq k/4} C_{n-k}^l C_k^l \leq \frac{1}{2^n} [\frac{k}{4}] C_{n-k}^{[k/4]} C_k^{[k/4]}$$

(We suppose here that $k << n$. In fact $\frac{k}{4} < \frac{n-k}{2}$ is enough.) This implies for covering number :

$$N(k/2, A_{k,n}, h) \geq \frac{C_n^k}{([\frac{k}{4}] + 1) C_{n-k}^{[k/4]} C_k^{[k/4]}}$$

But

$$\log\left(\frac{C_n^k}{([\frac{k}{4}] + 1) C_{n-k}^{[k/4]} C_k^{[k/4]}}\right) = \log(C_n^k) - \log(([\frac{k}{4}] + 1)) - \log(C_{n-k}^{[k/4]}) - \log(C_k^{[k/4]})$$

As $C_n^k = \prod_{j=1}^{j=k} \frac{n-k+j}{j}$, First we have,

$$\log(C_n^k) - \log(C_{n-k}^{[k/4]}) = \sum_{j=1}^{j=k} \log(1 + \frac{n-k}{j}) - \sum_{j=1}^{j=[k/4]} \log(1 + \frac{n-k-[k/4]}{j})$$

$$\geq \sum_{j=[k/4]}^{j=k} \log(1 + \frac{n-k}{j}) \geq \frac{3k}{4} \log(1 + \frac{n-k}{k}) = \frac{3k}{4} \log(\frac{n}{k})$$

Moreover,

$$\log(C_k^{[k/4]}) = \sum_{j=1}^{j=[k/4]} \log(1 + \frac{k - [k/4]}{j}) \leq \int_0^{[k/4]} \log(1 + \frac{k - [k/4]}{x}) dx$$

$$= k \log(k) - [k/4] \log([k/4]) - (k - [k/4]) \log(k - [k/4])$$

$$\leq k(\log(\frac{4}{3}) + \frac{\log(3)}{4}) = k \log(4.3^{-3/4}) \leq k \log(2)$$

So

$$\log\left(\frac{C_n^k}{\frac{k}{4} C_{n-k}^{[k/4]} C_k^{[k/4]}}\right) \geq \frac{3k}{4} \log(\frac{n}{k}) - k \log(2) - \log(\frac{k}{4} + 1)$$

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This gives, for $k \sim n^\alpha$,

$$H(k/2, A_{k,n}, l_1) \geq ((1-\alpha)\frac{3}{4}k \log k)(1 + o(1)).$$

$\square$

The two following subsections will use the inequalities proved in Proposition 4. First, we will consider the cases where we want to obtain lower bounds without a logarithmic factor, using inequality (13). The first proposition consider proves the lower bound inequalities of Theorem 2 for the spaces $B_r^s(\mathcal{E}, X)$. The second one proves the first part of Theorem 8.

4.2 $\epsilon^{1/s}$ lower bounds.

In this section we begin by proving the following proposition:

Proposition 5. *Let $U_r^s(\mathcal{E}, X)$ the unit ball of $B_r^s(\mathcal{E}, X)$ for $0 < s < \infty$, $0 < r \leq \infty$. Then there exists a constant $0 < c_s < \infty$ such that*

$$\forall \, 0 < \epsilon < 1, \quad H(\epsilon, U_r^s(\mathcal{E}, X), \| \ \|_X) \geq c_s \epsilon^{-1/s}.$$

This proposition is a consequence of the following lemma:

Lemma 3. *Let $K \subset X$ and $\mathcal{E} = \{e_i, i \in \ \}$ be a normalized unconditional basis verifying the p-Temlyakov property. Let $0 < s = \frac{1}{q} - \frac{1}{p}$, and let us suppose that there exists a non decreasing sequence of integers n_j, $\lim_{j \to \infty} n_j = \infty$, $c \leq \dfrac{n_j}{n_{j+1}}$. Let us suppose that, for each $j \in \ $, there exists a set $\Lambda_j \subset \ $, $\ \text{card}\,(\Lambda_j) = n_j$, verifying the following property :*

$$A_j = \{n_j^{-1/q} \sum_{i \in \Lambda_j} \omega_i e_i, \quad \omega_i \in \{0, 1\}\} \subset K.$$

Then there exists a constant $0 < c(K) < \infty$ such that

$$\forall 0 < \epsilon < 1, \quad H(\epsilon, K, \| \ \|_X) \geq c(K)\epsilon^{-1/s}.$$

Proof of the lemma : Clearly

$$\forall j \in \ , \ \forall \epsilon > 0, \quad H(\epsilon, K, \| \ \|_X) \geq H(\epsilon, A_j, \|\|_X)$$

Let us consider a covering of A_j by balls of radius ϵ in the X metric, with center in A_j. Let such a ball be $B(u, \epsilon)$, $u = n_j^{-1/q} \sum_{i \in \Lambda_j} \omega_i e_i$, for some $\omega \in \{0, 1\}^{\Lambda_j}$.

$$B(u, \epsilon) \cap A_j = \{n_j^{-1/q} \sum_{i \in \Lambda_j} \omega_i' e_i, \quad n_j^{-1/q} \| \sum_{i \in \Lambda_j} (\omega_i' - \omega_i) e_i \|_p \leq \epsilon, \}$$

Hence, using property 6,

$$C_1^p \sum_{i \in \Lambda_j} |\omega_i' - \omega_i| \leq \epsilon^p n_j^{p/q}.$$

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Let $\epsilon_j = \dfrac{C_1}{2^{1/p}} n_j^{-s}$, using inequality (13), proposition 4 :

$$H(\epsilon_j, A_j, \|\|_X) \geq H(n_j/2, \Omega_{n_j}, l_1) \geq \frac{n_j}{8\log(2)}.$$

But as $n_j = c\epsilon_j^{-1/s}$, the lemma is proved.

$\square$

Proof of the proposition. Let us use the quasi norms defined by (5) and :

$$\|f\|"_{B_r^s(\mathcal{E},X)} = \|f\|_X + (\sum_{j=0}^{\infty} (2^{js}\| \sum_{2^j \leq i < 2^{j+1}} \theta_i e_i \|_X)^r 1)^{1/r}$$

$$\|f\|"_{B_\infty^s(\mathcal{E},X)} = \|f\|_X + \sup_{0 \leq j \leq \infty} (2^{js}\| \sum_{2^j \leq i < 2^{j+1}} \theta_i e_i \|_X),$$

and apply the previous lemma with $n_j = 2^j$, $s = \frac{1}{q} - \frac{1}{p}$, $\Lambda_j = \{2^j \leq i < 2^{j+1}\}$ and

$$A_j = \{2^{-j/q} \sum_{2^j \leq i < 2^{j+1}} \omega_i e_i, \quad \omega_i \in \{0,1\}\}.$$

Let us again observe that we don't use here the r index.

4.3   Metric entropy obstruction.

In the next theorem, we see that, under some geometrical condition for K, a bounded subset of X, an entropy behavior of type $H(\epsilon, K, X, \| \|_X) \leq C_K \epsilon^{-1/s}$ implies $K \subset A_\infty^s(\mathcal{E}, X)$.

Theorem 4. *Let K be a bounded subset of X. Let $\mathcal{E} = \{e_n, n \in \ \}$ an unconditional, normalized basis of X verifying p-Temlyakov Property. Let us suppose that if $f = \sum \theta_n e_n$ belongs to K then $\sum \omega_n \theta_n e_n$ also belongs to K for any $(\omega_n)_{n \in} \in \{0,1\} \ $.*
    If K is not a bounded subset of $A_\infty^s(\mathcal{E}, X)$ then :

$$\limsup_{\epsilon \to 0} \epsilon^{1/s} H(\epsilon, K, X, \| \|_X) = \infty.$$

Proof of the Theorem
Let $\mathcal{K} \subset \ $ the set of all the sequences $(\theta_n)_{n \in}$ such that $f = \sum \theta_n e_n \in K$. By hypothesis

$$\text{if } (\theta_n)_{n \in} \in \mathcal{K}, \text{ then } (\omega_n \theta_n)_{n \in} \in \mathcal{K}, \ \forall \ (\omega_n)_{n \in} \in \{0,1\} \ . \tag{17}$$

Let us first observe that $\mathcal{K} \subset l_\infty(\ )$: As $\mathcal{E}$ is an unconditional, normalized basis, and K is a bounded subset of X, (say by $B < \infty$), we have by (4) for all $f = \sum \theta_n e_n \in K$,

$$\forall \ n \in \ , \ |\theta_n| = \|\theta_n e_n\|_X \leq C\|f\|_X \leq CB \tag{18}$$

Now by hypothesis, $\mathcal{K}$ is not a bounded subset of $l_{q,\infty}(\ )$ with $s = \frac{1}{q} - \frac{1}{p}$, by theorem 1.
So :

$$\sup_{(\theta_i) \in \mathcal{K}} (\sup_{n \in} n^{1/q} |\theta|_{(n)}) = \infty$$

So for all $N \in \mathbb{N}^*$ there exists $(\theta_j^N) \in \mathcal{K}$ and $n = n(N) \in \mathbb{N}$, such that $n^{1/q}|\theta^N|_{(n)} \geq N$.
Using (18), certainly $n^{-1/q}N \leq CB$. So $N^q \leq CBn$ and $n \to \infty$, with $N \to \infty$.
By (17), there exists $\Lambda \subset \mathbb{N}$, card $(\Lambda) = n$, and $(\lambda_i)_{i \in \Lambda}$ with $|\lambda_i| \geq n^{-1/q}N$, such that

$$K_n = \{\sum_{i \in \Lambda} \omega_i \lambda_i e_i, \quad \omega_i \in \{0,1\}\} \subset K.$$

Let us compute $H(\epsilon, K_n, X)$ for an appropriate ϵ :
Let $\phi = \sum_{i \in \Lambda} \omega_i \lambda_i e_i$ and $\phi' = \sum_{i \in \Lambda} \omega_i' \lambda_i e_i$, in K_n such that $\|\phi - \phi'\|_X \leq an^{-s}$. Using (6),

$$C_1^{-1} N n^{-1/q} (\sum_{i \in \Lambda} |\omega_i' - \omega_i|)^{1/p} \leq \|\sum_{i \in \Lambda} \omega_i' \lambda_i e_i - \sum_{i \in \Lambda} \omega_i \lambda_i e_i\|_X \leq an^{-s}$$

So

$$\sum_{i \in \Lambda} |\omega_i' - \omega_i| \leq na^p N^{-p} C_1^p$$

Let us take $a^p N^{-p} C_1^p = 1/4$.
For any covering of K_n by balls (in the $\| \ \|_X$ metric) of radius an^{-s}, we have a covering
of $\Omega_n = \{0,1\}^n$, (in the l_1 metric) of radius $n/4$. So

$$H(\frac{N}{C_1 4^{1/p}} n^{-s}, K, \| \ \|_X) \geq H(\frac{N}{C_1 4^{1/p}} n^{-s}, K_n, \| \ \|_X) \geq H(n/4, \Omega_n, l_1) \geq \frac{n}{8 \log(2)}$$

Let us observe that if for all $N \in \mathbb{N}$, $\frac{N}{C_1 4^{1/p}} n^{-s} > \alpha > 0$ as $n \to \infty$, then $H(\alpha, K, \| \ \|_X) = \infty$ and K is not relatively compact.
Otherwise if $\epsilon_N = \frac{N}{C_1 4^{1/p}} n^{-s}$, $\liminf_{N \to \infty} \epsilon_N = 0$ and

$$H(\epsilon_N, K, \| \ \|_X) \geq \frac{1}{8 \log(2)} (C_1 4^{1/p})^{-1/s} \epsilon_N^{-1/s} N^{1/s} = C' \epsilon_N^{-1/s} N^{1/s}$$

and we get the result.

4.4 $\epsilon^{1/s} \log(1/\epsilon)$ **lower bounds.**

We now want to compute the lower bound of the metric entropy of balls of the spaces $A_r^s(\mathcal{E}, X)$. First let us observe the following remark:

Remark : The balls of $A_r^s(\mathcal{E}, X)$ are not relatively compact subsets of X : The set $\mathcal{E}$ is in the unit ball of these spaces and moreover if $i \neq j$, then $\|e_i - e_j\|_X \geq \frac{1}{C}\|e_i\|_X = \frac{1}{C}$. Hence, to calculate the metric entropy, we will intersect $A_r^s(\mathcal{E}, X)$ with compact sets and especially with the following ones (also called polynomially tailed compact): balls of $B_\infty^\delta(\mathcal{E}, X)$, with small $0 < \delta$. $\star$

Theorem 5. *Let $V_r^s(\mathcal{E}, X)$ the unit ball of $A_r^s(\mathcal{E}, X)$ for $0 < s < \infty$, $0 < r \leq \infty$. Let $0 < \delta < s$. Then there exists a constant $0 < c_{s,\delta} < \infty$ such that*

$$\forall \ 0 < \epsilon < 1, \quad H(\epsilon, V_r^s(\mathcal{E}, X) \cap U_\infty^\delta(\mathcal{E}, X), \ _p) \geq c_{s,\delta} \epsilon^{-1/s} \log(1/\epsilon).$$

This theorem is a consequence of the following lemma:

Lemma 4. *Let $K \subset X$ and $\mathcal{E} = \{e_i, i \in \ \}$ be a normalized unconditional basis verifying p-Temlyakov property. Let $0 < s = \frac{1}{q} - \frac{1}{p}$, $0 < \alpha < 1$; $0 < c$. Let us suppose that there exists a non decreasing sequence of integers n_j, $\lim_{j \to \infty} n_j = \infty$, $c \leq \dfrac{n_j}{n_{j+1}}$. Let $k_j \sim n_j^\alpha$. Let us suppose now that for each $j \in \ $, there exists a set $\Lambda_j \subset \ $, $\mathrm{card}\,(\Lambda_j) \sim n_j$, verifying the following property :*

$$A_j = \{k_j^{-1/q} \sum_{i \in \Lambda_j} \omega_i e_i, \quad \sum_{i \in \Lambda_j} \omega_i = k_j\} \subset K.$$

Then there exists a constant $0 < c(K) < \infty$ such that

$$\forall 0 < \epsilon < 1, \quad H(\epsilon, K, X) \geq c(K) \epsilon^{-1/s} \log(1/\epsilon).$$

Proof of the lemma: Clearly

$$\forall j \in \ , n \ \forall \epsilon > 0, \quad H(\epsilon, \mathcal{K}, X) \geq H(\epsilon, A_j, X)$$

Let us consider a covering of A_j by balls of radius ϵ in the $X-$metric, with centers in A_j. Let such a ball be $B(u, \epsilon)$, $u = k_j^{-1/q} \sum_{i \in \Lambda_j} \omega_i e_i$, for some ω verifying $\sum_{i \in \Lambda_j} \omega_i = k_j$

$$B(u, \epsilon) \cap A_j = \{k_j^{-1/q} \sum_{i \in \Lambda_j} \omega_i' e_i, \quad \sum_{i \in \Lambda_j} \omega_i' = k_j, \quad k_j^{-1/q} \| \sum_{i \in \Lambda_j} (\omega_i' - \omega_i) e_i \|_p \leq \epsilon, \}$$

This implies, using p-Temlyakov property,

$$C_1^p \sum_{i \in \Lambda_j} |\omega_i' - \omega_i| \leq \epsilon^p k_j^{p/q}.$$

Let $\epsilon_j = \dfrac{C_1}{2^{1/p}} k_j^{-s}$, we have :

$$H(\epsilon_j, A_j, X) \geq H(k_j/2, A_{k_j, n_j}, l_1) \geq ((1 - \alpha)\frac{3}{4} k_j \log k_j)(1 + o(1)).$$

But

$$k_j \log k_j = c \ \epsilon_j^{-1/s} \log(1/\epsilon_j)$$

$\square$

Proof of the Theorem :
We apply Lemma 4 with $n_j = 2^j$, $s = \frac{1}{q} - \frac{1}{p}$, $\Lambda_j = \{2^j \leq i < 2^{j+1}\}$, $\alpha = \frac{\delta}{s}$ and $A_j = \{2^{-j\alpha/q} \sum_{2^j \leq i < 2^{j+1}} \omega_i e_i, \ \omega_i \in \{0, 1\}, \ \sum_{i \in \Lambda_j} \omega_i = 2^{-j\alpha}\}$. We again observe that we don't need to use the r index.

5 Upper bound. Universal coding.

Let $\{e_i, \ i \in \ \}$ be an unconditional basis of X verifying (6). We are going to consider the problem of evaluating upper bounds for the entropies of the sets considered above in a slightly more general framework involving their coding numbers. More precisely, we will investigate the length of a very simple universal coding.
Let us first recall the following definitions :

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5.1 Coding

- Let (X, d) a metric space and $K \subset X$. An ϵ- **coding of K of length** l is given by two functions :

  $C : K \longrightarrow \{0, 1\}^l$, (the "encoding" function) and

  $D : \{0, 1\}^l \longrightarrow X$, (the "decoding" function), such that

  $$d(DC(x), x) \leq \epsilon.$$

- Let us define $L(\epsilon, K, X, d)$ as the minimum length l of an ϵ- coding of K .

- It is obvious that :

  $$H(\epsilon, K, X, d) \leq L(\epsilon, K, X, d) \leq H(\epsilon, K, X, d) + 1.$$

Let us now give a description of the coding we shall consider:

5.2 Universal coding.

Our coding will have 2 tuning constants :

$$\lambda > 0 \text{ and } n \in \quad^*.$$

Of course, its length will highly depend on these 2 constants. Let us consider $f = \sum_i \theta_i e_i$ and the following steps :

1. Replace f by its λ quantized approximation

$$\widehat{f} = \sum_{i \leq n, |\theta_i| \geq \lambda} \left[\frac{\theta_i}{\lambda} \right] \lambda sign(\theta_i) e_i.$$

2. Consider the following subset of

$$\{1 \leq i \leq n, |\theta_i| \geq \lambda\} = \{i_1, i_2, \dots, i_k\}$$

3.

$$
\begin{aligned}
\forall x \in \quad^*, c'(sign(x)) &= 00, \text{ if } x > 0 \\
c'(sign(x)) &= 11, \text{ if } x < 0. \\
\forall m \in \quad, m \neq 0, \ c(m) &= a_j a_j \dots a_2 a_2 a_1 a_1 a_0 a_0 \\
\text{if } m &= a_0 + a_1 2 + a_2 2^2 \dots a_j 2^j.
\end{aligned}
$$

4. Encode f by the following formula :

$$c(i_1)01c'(sign(\lambda_{i_1})c([\frac{\lambda_{i_1}}{\lambda}])01c(i_2 - i_1)01$$

$$c'(sign(\lambda_{i_2})c([\frac{\lambda_{i_2}}{\lambda}]01 \dots$$

$$\dots c(i_l - i_{l-1})01c'(sign(\lambda_{i_l})c([\frac{\lambda_{i_l}}{\lambda}]01$$

$$\dots c(i_k - i_{k-1})01c'(sign(\lambda_{i_k})c([\frac{\lambda_{i_k}}{\lambda}]$$

Hence, we use 01 as separator, and we double the dyadic representation of the integers. So there is no ambiguity to reconstruct $\widehat{f}$. We have the following lemma :

Lemma 5. *The length of the code described above is :*

$$L(f,\lambda,n) = 10k(f,\lambda,n) + 2\sum_{1\leq j\leq k(f,\lambda,n)}\log_2(i_j - i_{j-1}) + 2\sum_{1\leq j\leq k}\log_2([\frac{\theta_{i_j}}{\lambda}]) \qquad (19)$$

We take for convention that $i_0 = 0$.

The proof is obvious just noticing that :

$$L(f,\lambda,n) = 6k(f,\lambda,n) + 2\sum_{1\leq j\leq k(f,\lambda,n)}(\log_2(i_j - i_{j-1}) + 1) + 2\sum_{1\leq j\leq k}(\log_2([\frac{|\theta_{i_j}|}{\lambda}]) + 1)$$

5.2.1 Entropy of balls of non-linear approximation spaces

Theorem 6. *Let $U_r^\delta(\mathcal{E},X)$ be the unit ball of $B_r^\delta(\mathcal{E},X)$ and $V_r^s(\mathcal{E},X)$ the unit ball of $A_r^s(\mathcal{E},X)$. For $0 < \delta < s < \infty,\quad 0 < r \leq \infty$, the previous coding with*

$$n^{-\delta} \sim \frac{\epsilon}{2} \qquad (20)$$

$$CC_q\lambda^{sq} = \frac{\epsilon}{2} \qquad (21)$$

is an ϵ-coding if q is such that $s = \dfrac{1}{q} - \dfrac{1}{p}$ and C_q is the constant given in inequality (12) of Proposition 3.
Moreover, there exists two constants $0 < c_{s,\delta} \leq C_{s,\delta} < \infty$ such that

$$\forall\, 0 < \epsilon,\quad c_{s,\delta}\epsilon^{-1/s}\log(1/\epsilon) \leq H(\epsilon, V_r^s(\mathcal{E},X)\cap U_\infty^\delta(\mathcal{E},X),\ _p) \leq C_{s,\delta}\epsilon^{-1/s}\log(1/\epsilon).$$

Proof of the Theorem :
Using inequality (12) and Theorem 1, it is clear that the constants are exactly chosen in such a way that the previous coding is an ϵ-coding.

The left hand side of the inequality is given by theorem 5.

We will prove the righthand side for $V_\infty^s(\mathcal{E},X)$. This will be enough, due to the inclusion $V_r^s(\mathcal{E},X) \subset V_\infty^s(\mathcal{E},X)$.
Hence, we have to find an upper bound to (19) for $f \in V_\infty^s(\mathcal{E},X)\cap U_\infty^\delta(\mathcal{E},X)$.
Let us observe that, $\|(\theta_i)\|_{l_{q,\infty}} \leq 1$, implies $|\theta_i| \leq 1$ for all i.
Because $f \in V_\infty^s(\mathcal{E},X)$, we have

$$k(f,\lambda,n) \leq (\frac{1}{\lambda})^q = \mathcal{O}(\epsilon^{-1/s}).$$

Hence,

$$\sum_{1\leq j\leq k(f,\lambda,n)}\log_2([\frac{\theta_{i_l}}{\lambda}]) \leq k(f,\lambda,n)\log_2(\frac{1}{\lambda}) \leq (\frac{1}{\lambda})^q\log_2(\frac{1}{\lambda}) = \mathcal{O}(\epsilon^{-1/s}\log(\frac{1}{\epsilon}))$$

Moreover, using Jensen inequality :

$$\sum_{1\leq j\leq k(f,\lambda,n)} \log_2(i_j - i_{j-1}) = k(f,\lambda,n)\frac{1}{k(f,\lambda,n)}\sum_{1\leq j\leq k(f,\lambda,n)} \log_2(i_j - i_{j-1})$$

$$\leq k\log_2(\frac{1}{k}\sum_{1\leq j\leq k}(i_j - i_{j-1}))$$

$$\leq k\log_2\frac{n}{k}$$

But :

$$\sup_{0\leq x\leq K} x\log_2\frac{K}{x} = \frac{K}{e\log 2} \;;\; \text{and} \quad \sup_{0\leq x\leq a\leq\frac{K}{e}} x\log_2\frac{K}{x} = a\log_2\frac{K}{a} \tag{22}$$

$$\text{as } \epsilon^{-1/s} << n \sim \epsilon^{-1/\delta}, \quad k\log_2\frac{n}{k} \leq \epsilon^{-1/s}\log_2\frac{\epsilon^{-1/\delta}}{\epsilon^{-1/s}} = \mathcal{O}(\epsilon^{-1/s}\log(\frac{1}{\epsilon}))$$

$\square$

5.2.2 Entropy of balls of linear approximation spaces

Theorem 7. *Let $\{e_i, \; i\in\;\}$ an unconditional basis of X verifying (6) with $1 < p < \infty$. Let $U_r^s(\mathcal{E}, X)$ be the unit ball of $B_r^s(\mathcal{E}, X)$ for $0 < s < \infty$, $0 < r \leq \infty$. The previous coding with*

$$n \quad \sim \quad (\frac{\epsilon}{2})^{-\frac{1}{s}} \tag{23}$$

$$CC_q\lambda^{sq} = \frac{\epsilon}{2} \tag{24}$$

is an ϵ-coding if q is such that $s = \dfrac{1}{q} - \dfrac{1}{p}$ and C_q is the constant given in inequality (12) of Proposition 3.
Moreover, there exists a constant $0 < c_s \leq C_s < \infty$ such that

$$\forall 0 < \epsilon, \quad c_s\epsilon^{-1/s} \leq H(\epsilon, U_r^s(\mathcal{E}, X), X) \leq C_s\epsilon^{-1/s}.$$

Proof of the Theorem:
As above, it is clear using inequality (12) that the coding is an ϵ-coding.

The left hand side of the entropy inequality is a corollary (5). Again, we will prove the righthand side for $U_\infty^s(\mathcal{E}, X)$. This will be enough, due to the inclusion $U_r^s(\mathcal{E}, X) \subset U_\infty^s(\mathcal{E}, X)$. We have to find an upper bound to (19) for $f \in U_\infty^s(\mathcal{E}, X)$.
Let us introduce J such that :
$$n = 2^J \sim \epsilon^{-1/s}.$$

$$L(f,\lambda,n) = \mathcal{O}(\epsilon^{-1/s}) + \sum_{1\leq j\leq k(f,\lambda,n)}\log_2(i_j - i_{j-1}) + \sum_{1\leq j\leq k(f,\lambda,n)}\log_2([\frac{|\theta_{i_j}|}{\lambda}])$$

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Again, using (22),

$$\begin{aligned}
\sum_{1 \le j \le k} \log_2(i_j - i_{j-1}) &= k\frac{1}{k} \sum_{1 \le j \le k} \log_2(i_j - i_{j-1}) \\
&\le k \log_2(\frac{1}{k} \sum_{1 \le j \le k}(i_j - i_{j-1})) \\
&\le k \log_2(\frac{n}{k}) \le \frac{n}{e \log 2} \sim \frac{\epsilon^{-1/s}}{e \log 2}
\end{aligned}$$

Moreover, let $A_j = \{2^j \le i \le 2^{j+1}, \ |\theta_i| \ge \lambda\}$. We have:

$$\| \sum_{i \in A_j} \theta_i e_i \|_X \le C\| \sum_{i=2^j}^{\infty} \theta_i e_i \|_X \le C2^{-js}$$

As $\epsilon \sim 2^{-Js}$ and $\lambda \sim 2^{-J/q}$

$$\sum_{1 \le j \le k(f,\lambda,n)} \log_2[\frac{|\theta_{i_j}|}{\lambda}] = \sum_{j=0}^{J} \sum_{i \in A_j} \log_2[\frac{|\theta_i|}{\lambda}]$$

Using Jensen inequality and denoting $|A_j| = \text{card } A_j$,

$$\begin{aligned}
\sum_{i \in A_j} \log_2[\frac{|\theta_i|}{\lambda}] &= |A_j|\frac{1}{|A_j|} \sum_{i \in A_j} \log_2[\frac{|\theta_i|}{\lambda}] \\
&\le |A_j| \log_2 \left(\frac{1}{|A_j|} \sum_{A_j} [\frac{|\theta_i|}{\lambda}] \right) \\
&= |A_j| \log_2 \left(\frac{1}{|A_j|} \sum_{1 \le l} l \text{ card } \{l \le \frac{|\theta_i|}{\lambda} < l+1, \ i \in A_j\} \right) \\
&= |A_j| \log_2 \left(\frac{1}{|A_j|} \sum_{1 \le l} \text{ card } \{l \le \frac{|\theta_i|}{\lambda}, \ i \in A_j\} \right)
\end{aligned}$$

Using (6), we have

$$(l\lambda)^p \text{ card } \{l \le \frac{|\theta_i|}{\lambda}, \ i \in A_j\} \le C_1\| \sum_{i \in A_j} \theta_i e_i \|_X^p \le C_1 C 2^{-jsp}$$

So

$$\sum_{i \in A_j} \log_2([\frac{|\theta_i|}{\lambda}]) \le |A_j| \log_2 \left(\frac{1}{|A_j|} \sum_{1 \le l} \frac{C_1 C 2^{-jsp}}{(l2^{-J/q})^p} \right) = |A_j| \log_2 \left(\frac{1}{|A_j|} \frac{C_1 C 2^{-jsp}}{2^{-Jp/q}} \sum_{1 \le l} \frac{1}{l^p} \right)$$

But $p/q = sp + 1$, **and as** $p > 1$, $\quad \sum_{1 \le l} \frac{1}{l^p} = C_p < \infty$.

$$\sum_{i \in A_j} \log_2([\frac{|\theta_i|}{\lambda}]) \le |A_j| \log_2 \left(\frac{1}{|A_j|} C_1 C C_p 2^{-jsp} 2^{Jp/q} \right)$$

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By (22)), we have, as $|A_j| \leq 2^j << CC_p 2^{-jsp} 2^{Jp/q}$.

$$|A_j| \log_2 \left(\frac{1}{|A_j|} C_1 C C_p 2^{-jsp} 2^{Jp/q} \right) \leq 2^j \log_2 \left(\frac{1}{2^j} C_1 C C_p 2^{-jsp} 2^{Jp/q} \right) = 2^j \log_2 \left(C_p' 2^{(J-j)p/q} \right)$$

So :

$$
\begin{aligned}
\sum_{1 \leq j \leq k} \log_2([\frac{\theta_{i_j}}{\lambda}]) &= \sum_{j=0}^{J} \sum_{i \in A_j} \log_2([\frac{|\theta_i|}{\lambda}]) \\
&\leq \sum_{j=0}^{J} 2^j \left((J-j)p/q + \log_2(C_p') \right) \\
&\leq 2^J \left(p/q \sum_{j=0}^{J} 2^{-(J-j)}(J-j) + \log_2(C_p') \sum_{j=0}^{J} 2^{-(J-j)} \right) = \mathcal{O}(\epsilon^{-1/s})
\end{aligned}
$$

$\square$

6 $\quad_p-$ spaces and wavelet bases : Unconditional bases and p-Temlyakov property.

6.1 Unconditional bases in $\quad_p$ spaces.

In this paragraph we give a characterization of unconditional bases in $\quad^p$ spaces for $1 < p < \infty$. These restriction is due to the fact that in general there is no unconditional basis if $p \notin (1, \infty)$ (see Lindenstrauss and Tzafriri 1977, Proposition 1.d.1).

Theorem 8. *Let $(\mathcal{X}, \tau)$ be a measure space. Let $\{\psi_i, \ i \in \quad\}$ a sequence in $\quad^p(\mathcal{X}, \tau)$, with , $1 < p < \infty$. The following facts are equivalent:*

1. *$\{\psi_i, \ i \in \quad\}$ is an unconditional basis of $\quad^p(\mathcal{X}, \tau)$.*

2. *$\{\psi_i, \ i \in \quad\}$ is a total system in $\quad^p(\mathcal{X}, dx)$, and there exists $K > 0$, such that, for any set F included in , and any choice of coefficients c_i 's,*

$$K^{-1} \| \sum_{i \in F} c_i \psi_i \|_p \leq \| (\sum_{i \in F} |c_i \psi_i|^2)^{1/2} \|_p \leq K \| \sum_{i \in F} c_i \psi_i \|_p.$$

This result is stated in [13]. The proof relies on Khinchine inequality. The major striking fact about Theorem 8 is that if $\{\psi_i, \ i \in I\}$ is an unconditional basis of $\quad^p$, then for

$$f = \sum_{i \in} \theta_i \psi_i, \quad \|f\|_p \sim \|(\sum_{i \in} |\theta_i \psi_i|^2)^{\frac{1}{2}}\|_p. \tag{25}$$

Notably the Fourier basis is not an unconditional basis of $\quad_p$ for $p \neq 2$ (see Kahane et al. 1977 [24]).

However, it is a classical result, that **wavelet bases, with compact support, are unconditional bases for** $\quad_p$ **spaces on** or $[0, 1]$. This beautiful results is a consequence of Calderon-Zygmund theory (see Meyer [34]). We can immediately deduce as a corollary of Theorem 8 an unconditional basis for $\quad_p$ spaces on ^{d} or $[0, 1]^d$, by just taking the usual tensor product of the wavelet basis. However, this usual tensor product is no longer linked with the multiresolution analysis. It is no longer a wavelet basis, in the sense that it is not obtained by dilating by a single scale and translating.

In this framework, it is often more convenient to consider the so called wavelet-tensor product which preserves the property of being a wavelet basis.

6.1.1 Wavelet-tensor products.

Let, ϕ, ψ a system, where ψ is the wavelet and ϕ the scaling function. $\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k)$, $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, the associated translated-dilated functions. Let us consider the triples $(j, \mathbf{k}, A)$ where $j \in$, $\mathbf{k} = (k_1, \ldots, k_d) \in$ ^{d} and $A \in S_d$ the set of all the non void subsets of $\{1, \ldots, d\}$. For such a triplet we consider the functions $\phi^{\mathbf{k}}$, $\psi_{(j,\mathbf{k},A)}$: $^d \to$,

$$\phi^{\mathbf{k}}(x_1, \ldots, x_d) = \prod_{i=1}^{d} \phi_{0k_i}(x_i),$$

$$\psi_{(j,\mathbf{k},A)}(x_1, \ldots, x_d) = \prod_{i \in A} \psi_{jk_i}(x_i) \prod_{i \notin A} \phi_{jk_i}(x_i).$$

The system $\{\phi^{\mathbf{k}}, \mathbf{k} \in$ ^{d}, $\psi_{(j,\mathbf{k},A)}, j \in$, $\mathbf{k} \in$ ^{d}, $A \in S_d\}$ is an orthonormal family of ^{d} called *the wavelet-tensor product*, see Meyer [34]. One can also prove (see Meyer [34], ch 6) that these wavelet-tensor products constructed on compactly supported wavelets are *unconditional bases* for $\quad_p$ spaces on ^{d} for $p \in (1, \infty)$ or $[0, 1]^d$ as well, with the usual modifications (see Cohen, Daubechies and Vial 1993). In this setting, the index $(j, \mathbf{k}, A)$ is composed of a pair (l, A) of a dyadic cube $l = (j, \mathbf{k})$ in ^{d}, and a non-void subset A of $\{1, 2, \ldots, d\}$. Let us denote by $\tilde{I}$ the set of such pairs. We will also us denote by $(\psi_{(l,A)})_{(l,A) \in \tilde{I}}$, this wavelet-tensor product basis.

6.1.2 The Hardy space H_p for $0 < p \leq 1$.

For $0 < p \leq 1$, formula 25 no more characterizes $\quad_p$ spaces. These spaces, usually, don't have inconditional bases. However, if the family e_i is a suitable wavelet basis (see [34] for precise conditions) then formula 25 characterizes the Hardy spaces $H_p(\ ^d)$. So the previous results are also valid for these spaces if e_i verifies the condition 6 which will be established in the following lines.

6.2 Superconcentration Inequality

We will first make a review of some embedded properties for a sequence $(e_i)_{i \in}$ of real valued functions defined on a measured space $(\mathcal{X}, \mu)$. The last one will be the p-Temlyakov property. The first one is the superconcentration property. We will prove that this property can be proved for wavelet bases.

- **Superconcentration property (for p.)** Let $\{e_i,\ i \in\ \}$ be a sequence of real valued functions defined on some measured space $(\mathcal{X}, \mu)$. We say that this sequence of functions satisfies a superconcentration inequality (for p) if :

  For any arbitrary $0 < r < \infty$ there exists a constant C (only depending on p, r) such that :
  $$\forall\ \Lambda \subset\ ,\ \ \|[\sum_{i \in \Lambda} |e_i|^r]^{\frac{1}{r}}\|_p \leq C(p,r)\|\sup_{i \in \Lambda} |e_i|\|_p.$$

- **H_α property (for p)** The sequence $\{e_i,\ i \in\ \}$ satisfies the H_α property (for p) if there exists $0 < K_\alpha < \infty$, such that any arbitrary
  $$\forall\ \ \Lambda \subset\ ,\ \int_\Lambda (\sum |e_i|^{p\alpha})^{1/\alpha} \leq K_\alpha \int_\Lambda \sum |e_i|^p$$

  Of course only $0 < \alpha < 1$ is of interest.

- **p-Temlyakov property (for p)** The sequence $(e_i)_{i \in}$ satisfies p-Temlyakov property (for p) if there exists $0 < c_0 \leq C_0 < \infty$, such
  $$\forall\ \ \Lambda \subset\ ,\ c_0 \int \sum_{i \in \Lambda} |e_i|^p \leq \int (\sum_{i \in \Lambda} |e_i|^2)^{\frac{p}{2}} \leq C_0 \int \sum_{i \in \Lambda} |e_i|^p.$$

  It is an elementary consequence of Theorem 8, that for p, the p- Temlyakov property takes this form (if of course $\forall i \in\ ,\ \ \|e_i\|_p \asymp 1$).

The following proposition is establishing the implications between the properties mentionned above.

Proposition 6. *1. The superconcentration property implies the H_α property, for any $0 < \alpha < 1$,*

2. If a sequence $\{e_i,\ i \in\ \}$ verifies the H_α property for any $0 < \alpha < 1$, then it verifies the p-Temlyakov property.

Proof of the proposition

1.
$$\int_\Lambda (\sum |e_i|^{p\alpha})^{1/\alpha} = \int (\sum_\Lambda |e_i|^{p\alpha})^{p/p\alpha}$$

Let us put $r = p\alpha$, using the superconcentration property we have

$$\int [\sum_{i \in \Lambda} |e_i|^r]^{\frac{p}{r}} \leq C(p,r)^p \int \sup_{i \in \Lambda} |e_i|^p \leq C(p,r)^p \int \sum_{i \in \Lambda} |e_i|^p.$$

2. (a) Let $2 \leq p$, then $\sum_\Lambda |e_i|^p \leq (\sum_\Lambda |e_i|^2)^{p/2}$. Hence, $\int \sum_\Lambda |e_i|^p \leq \int(\sum_\Lambda |e_i|^2)^{p/2}$. On the other hand if the H_α property is verified for $\alpha = 2/p \leq 1$ then

$$\int (\sum_{i \in \Lambda} |e_i|^2)^{\frac{p}{2}} = \int (\sum_\Lambda |e_i|^{p\alpha})^{1/\alpha} \leq K_\alpha \int_\Lambda \sum |e_i|^p$$

(b) Let $2 > p$, then $(\sum_\Lambda |e_i|^2)^{p/2} \le \sum_\Lambda |e_i|^p$. So $\int(\sum_\Lambda |e_i|^2)^{p/2} \le \int \sum_\Lambda |e_i|^p$. But using the following lemma with $\psi_i = |e_i|^p$, $r = \alpha < 1 = s$, $1 < t = 2/p$ then there exists a constant K independant of Λ such that

$$\int \sum_\Lambda |e_i|^p \le K \int (\sum_\Lambda |e_i|^2)^{p/2}.$$

Lemma 6. *Let $(\psi_i \ i \in \ )$, a family of measurable functions, defined on some measured space $(\mathcal{X}, d\tau)$. Let $0 < r < s$. If :*

$$\int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^r)^{\frac{1}{r}} d\tau \le C \int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^s)^{\frac{1}{s}} d\tau < \infty.$$

Then $\forall \ s < t < \infty$,

$$\int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^s)^{\frac{1}{s}} d\tau \le C^{\frac{r(t-s)}{t(s-r)}} \int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^t)^{\frac{1}{t}} d\tau.$$

Proof of the lemma : As $0 < r < s < t < \infty$, there exists $0 < \beta < 1$, such that :

$$s = \beta r + (1-\beta)t = \frac{t-s}{t-r} r + \frac{s-r}{t-r} t.$$

So, using Hölder inequality :

$$(\sum_{i\in} |\psi_i|^s)^{\frac{1}{s}} = (\sum_{i\in} |\psi_i|^{\beta r}|\psi_i|^{(1-\beta)t})^{\frac{1}{s}} \le \left(\sum_{i\in} |\psi_i|^r\right)^{\frac{\beta}{s}} \left(\sum_{i\in} |\psi_i|^t\right)^{\frac{1-\beta}{s}}$$

As $1 = \frac{\beta r}{s} + \frac{(1-\beta)t}{s}$, using again Hölder inequality and the assumption:

$$\int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^s)^{\frac{1}{s}} d\tau \ \le \ \left(\int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^r)^{\frac{1}{r}} d\tau\right)^{\frac{r\beta}{s}} \left(\int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^t)^{\frac{1}{t}} d\tau\right)^{\frac{t(1-\beta)}{s}}$$

$$\le \ \left(C \int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^s)^{\frac{1}{s}} d\tau\right)^{\frac{r\beta}{s}} \left(\int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^t)^{\frac{1}{t}} d\tau\right)^{\frac{t(1-\beta)}{s}}$$

So,

$$\int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^s)^{\frac{1}{s}} d\tau \le C^{\frac{r\beta}{t(1-\beta)}} \int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^t)^{\frac{1}{t}} d\tau = C^{\frac{r(t-s)}{t(s-r)}} \int_{\mathcal{X}} (\sum_{i\in} |\psi_i|^t)^{\frac{1}{t}} d\tau.$$

$\square$

Remark : The previous proposition can be slightly weakened: For a fixed $0 < p < \infty$, to verify p-Temlyakov property, we only need the property H_α for $\alpha = 2/p$ if $2 \le p$; and if $p < 2$, it is enough to have the property H_α for some $0 < \alpha < 1$.

$\star$

6.3   Superconcentration Inequality for wavelet bases.

Theorem 9. *Let $0 < \rho < \infty$. Let $(\psi_i)_{i \in \tilde{I}}$ a compactly supported wavelet tensor product basis in ^{d} (or $[0,1]^d$.), normalized in such way that $\|\psi_i\|_\rho = 1$. Let $0 < p < \infty, \ 0 < r < \infty$ be arbitrary. Then there exists a constant C, only depending on p, r, the dimension d and the wavelet system, such that : for all $\Lambda \subset \tilde{I}$:*

$$\|[\sum_{i \in \Lambda} |\psi_i|^r]^{\frac{1}{r}}\|_p \leq C\|\sup_{i \in \Lambda} |\psi_i|\|_p.$$

The end of this section is devoted to the proof of Theorem 9. We will essentially need 2 steps: The first one will study the case of the Haar wavelet, the second one will use the maximal functional to transfer results about Haar wavelets to general compactly supported wavelets.

6.3.1   The Haar wavelet case.

Let, for $x \in$ the classical Haar function $h(x) = 1_{[0,1]}(x)$ and $g(x) = 1_{[0,1/2]}(x) - 1_{[1/2,1]}(x)$ is such that the family

$$\{h_k, \ k \in \ , \ g_{j,k'} \ j \in \ , k' \in \ \}, \ h_k(x) = h(x-k), \ g_{j,k'}(x) = 2^{j/\rho} g(2^j x - k')$$

is an orthogonal family of , normalized in such way that $\|g_{j,k'}\|_\rho = 1$. Let us call $(\theta_{(l,\epsilon)})_{(l,\epsilon) \in \tilde{I}}$, the wavelet-tensor product associated to the Haar basis. The following proposition is our first step :

Proposition 7. *For any $0 < q < \infty$, there exists $C_{q,d}$ such that, for all $\Lambda \subset \tilde{I}$,*

$$[\sum_{(l,\epsilon) \in \Lambda} |\theta_{l,\epsilon}|^q]^{1/q} \leq C_{q,d} \sup_{(l,\epsilon) \in \Lambda} |\theta_{l,\epsilon}|$$

Proof.

For the proof of the proposition, we only need to consider the case of finite Λ and we observe that for $x \in \ ^d$, we only need to consider those among the $\theta_{l,\epsilon}$'s such that $\theta_{l,\epsilon}(x) \neq 0$, and we certainly have

$$\sum_{(l,\epsilon) \in \Lambda} |\theta_{l,\epsilon}(x)|^q \leq (2^d - 1)(\sup_{(l,\epsilon) \in \Lambda} |\theta_{l,\epsilon}(x)|)^q \sum_{j \geq 0} 2^{-j\frac{dq}{\rho}}.$$

This inequality is obvious. The important point is only to remark that we consider $\theta_{(l,\epsilon)}$ which are supported by the dyadic *cube l* containing x, and any cube sharing this property contains the smallest one which is the support of the function with maximal absolute value. The term $(2^d - 1)$ only comes from the fact that because of the index ϵ, it may happen that in Λ, exactly $(2^d - 1)$ have the same smallest support. $\square$

In the sequel, we will forget the double subscrit (i, ϵ), and just for convenience, only keep i.

6.3.2 The maximal function and the Fefferman-Stein lemma.

Let us recall the definition of the Hardy-Littlewood maximal function (actually a somewhat generalized version.)

Definition 3. *Let $\mathcal{X} = {}^d$ (or $[0,1]^d$) and $\mathcal{B}$ the set of the cubes of $\mathcal{X}$. For each B in $\mathcal{B}$, let $|B|$ its Lebesgue measure. Let $0 < s < \infty$. For any measurable fonction f, we define $M_s^*(f)$:*

$$\forall x \in \mathcal{X}, \quad M_s^*(f)(x) = \sup_{B \in \mathcal{B}, \; x \in B} \left(\frac{1}{|B|} \int_B |f|^s\right)^{\frac{1}{s}}.$$

Remark : It is clear that

$$M_s^*(f)(x) = [M_1^*(|f|^s)(x)]^{1/s}$$

$\star$

Then we have :

Lemma 7. *Let θ_i the wavelet tensor product associated with the Haar system in ^{d}, and ψ_i another compactly supported wavelet tensor product in the same space, with $\|\theta_i\|_\rho = \|\theta_i\|_\rho = 1$, $\forall i$.*
Then for all $s > 0$, there exists $C(s, d, \theta, \psi)$, $C'(s, d, \theta, \psi)$ such that for any x

$$|\psi_i|(x) \leq C(s, d, \theta, \psi) M_s^*(\theta_i)(x) \tag{26}$$

$$|\theta_i|(x) \leq C'(s, d, \theta, \psi) M_s^*(\psi_i)(x) \tag{27}$$

The proof is simple, and it is omitted. It just relies on the same localization and size of the two system of functions, for the same index.

The previous lemma will be used in association with the following important result :

Lemma 8. *(Fefferman-Stein [21])*
Let f_n be a sequence of measurable functions defined on ^{d} (or $[0,1]^d$). Let $0 < s < p < \infty$, $s < q \leq \infty$. Then, there exists $C(p, q, s, d)$ such that

$$\| \left[\sum_n (M_s^*(f_n))^q\right]^{1/q} \|_p \leq C(p, q, s, d) \| \left[\sum_n |f_n|^q\right]^{1/q} \|_p.$$

(with the usual modification for $q = \infty$.)

Actually in Fefferman-Stein [21] the previous statement is given, and proved for $s = 1$, and $1 < p, q < \infty$. But it is not difficult to derive lemma 8, using the remark 6.3.2. Furthermore, for $q = \infty$, it is enough to observe that :

$$\sup_n M_1^*(f_n) \leq M_1^*(\sup_n |f_n|)$$

and to apply the classical maximal theorem :

$$\text{for } 1 < p < \infty \quad : \|M_1^*(\sup_n |f_n|)\|_p \leq C_p \|\sup_n |f_n|\|_p.$$

6.3.3   Proof of Theorem 9: second step.

We can now begin the proof of our theorem. Let $0 < s < r$, $0 < s < p$.
Let θ_i denote the Haar wavelet-tensor product basis on the same space and with the same index, as the ψ_i's. Let $\Lambda \subset I$. Using (26),

$$(\sum_{i \in \Lambda} |\psi_i|^r)^{\frac{1}{r}} \leq C(s,d,\theta,\psi)[\sum_{i \in \Lambda} |M_s^*(\theta_i)|^r]^{\frac{1}{r}}$$

Using Lemma 8 :

$$\|[\sum_{i \in \Lambda} |M_s^*(\theta_i)|^r]^{\frac{1}{r}}\|_p \leq C(p,r,s,d)\|[\sum_{i \in \Lambda} |\theta_i|^r]^{\frac{1}{r}}\|_p$$

Using Proposition 7

$$[\sum_{i \in \Lambda} |\theta_i|^r]^{\frac{1}{r}} \leq C_{r,d} \sup_{i \in \Lambda} |\theta_i|$$

Using (27),

$$\sup_{i \in \Lambda} |\theta_i| \leq C'(s,d,\theta,\psi) \sup_{i \in \Lambda} |M_s^*(\psi_i)|.$$

Using Lemma 8 :

$$\|\sup_{i \in \Lambda} |M_s^*(\psi_i)|\|_p \leq C(p,\infty,s,d)\|\sup_{i \in \Lambda} |\psi_i|\|_p.$$

This proves :

$$\|(\sum_{i \in \Lambda} |\psi_i|^r)^{\frac{1}{r}}\|_p \leq C(s,d,\theta,\psi)C(p,r,s,d)C_{r,d}C'(s,d,\theta,\psi)C(p,\infty,s,d)\|\sup_{i \in \Lambda} |\psi_i|)\|_p$$

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