Existence results for 2D homogeneous Boltzmann equations without cutoff and for non Maxwell molecules by use of Malliavin calculus

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Abstract

Tanaka showed in [21] a way to relate the measure-solution $\{Q_t\}_t$ of a spatially homogeneous Boltzmann equation of Maxwell molecules without angular cutoff to the solution V_t of a Poisson-driven stochastic differential equation: for each t, Q_t is the law of V_t.

Using a typically probabilistic substitution in the Boltzmann equation,we extend Tanaka's probabilistic interpretation to much more general spatially homogeneous Boltzmann equations.

Then we introduce an adapted stochastic calculus of variations on the Poisson space, to prove that for each $t > 0$, the law of V_t admits a density $f(t, .)$. The function $f(t, v)$ is solution of the Boltzmann equation, and this existence result improves the existing analytical results.

Since the "Malliavin derivative" of V_t does not belong to $L^2(\Omega)$, and thus cannot be a "L^2-derivative", we introduce a criterion of absolute continuity based on the use of a.s. derivatives.

Key words : Boltzmann equations without cutoff, Nonlinear stochastic differential equations, Jump measures, Stochastic calculus of variations.

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1 Introduction and general setting.

The Boltzmann equation we consider describes the evolution of the density $f(t, v)$ of particles with velocity $v \in \mathbb{R}^2$ at time t in a rarefied homogeneous gas:

$$\frac{\partial f}{\partial t} = Q(f, f) \tag{1.1}$$

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where Q is a quadratic collision kernel preserving momentum and kinetic energy, of the form

$$Q(f,f)(t,v) = \int_{v_* \in I\!R^2} \int_{\theta=-\pi}^{\pi} \Big(f(t,v')f(t,v'_*) - f(t,v)f(t,v_*) \Big) B(|v-v_*|,\theta) d\theta dv_* \quad (1.2)$$

with

$$v' = v + A(\theta)(v - v_*) \ ; \ v'_* = v_* - A(\theta)(v - v_*) \quad (1.3)$$

and

$$A(\theta) = \frac{1}{2} \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix} \quad (1.4)$$

Notice that for each $\theta \in [-\pi, \pi] \setminus \{0\}$,

$$|A(\theta)| \leq K|\theta| \quad (1.5)$$

The cross-section B is a positive function, even in the θ-variable. If the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s \geq 2$, then $B(z,\theta) = z^{\frac{s-5}{s-1}} d(|\theta|)$ where $d \in L^\infty_{loc}(]0,\pi])$ and $d(\theta) \sim K(s)\theta^{-\frac{s+1}{s-1}}$ when θ goes to zero, for some $K(s) > 0$. Physically, this explosion comes from the accumulation of grazing collisions.

In this general (spatially homogeneous) setting, the Boltzmann equation is very difficult to study. A large literature deals with the non physical equation with angular cutoff, namely under the assumption $\int_0^\pi B(z,\theta)d\theta < \infty$. More recently, the case of Maxwell molecules, for which the cross section $B(z,\theta) = \beta(\theta)$ only depends on θ, has been much studied without the cutoff assumption. In the Maxwell context, Tanaka, [21] was considering the case where $\int_0^\pi \theta\beta(\theta)d\theta < \infty$, and Desvillettes, [5], Desvillettes, Graham, Méléard, [6] and Fournier, [8] have worked under the general physical assumption $\int_0^\pi \theta^2\beta(\theta)d\theta < +\infty$.

The case in which B depends on z is really harder and there is just a few results on it. We can just mention the paper of Alexandre-Desvillettes-Villani-Wennberg [1]. In [10], a natural probabilistic approach is proposed to study the case one of non Maxwell molecules under the condition $\int_0^\pi \theta B(z,\theta)d\theta < \infty$, when $B(z,\theta) = \psi(z)\beta(\theta)$ where ψ is positive and bounded and locally Lipschitz continuous. One proves in this case the existence of a measure-solution of the equation for any initial probability data with a second order moment. But this approach makes appear an indicator function in the model, which makes very hard the study of the regularity of the obtained measure-solution. In particular, we are not able, for the moment, to prove the existence of a function-solution with such a method.

In the present paper, we extend Tanaka's probabilistic interpretation, [21], who was dealing with Maxwell molecules, to much more general spatially homogeneous Boltzmann equations, under the condition $\int_0^\pi \theta^2 B(z,\theta)d\theta < \infty$: using a tricky transformation of the cross-section, we relate a solution of the equation to the solution V_t of a Poisson-driven stochastic differential equation. We obtain thus measure-valued solutions for the nonlinear equation. Then, we develop, in the case where $\int_0^\pi \theta B(z,\theta)d\theta < \infty$, an adapted stochastic calculus of variations on the Poisson space to prove that for each $t > 0$, the law of V_t has a density. As an immediate corollary, we obtain the existence of a function-solution to the Boltzmann equation. This result improves the existing analytical results. We will see that since the Malliavin derivative of V_t is not in L^2, we need a very weak criterion of absolute continuity based on the use of almost sure derivatives.

The reasons why we have to assume the condition $\int_0^\pi \theta B(z,\theta)d\theta < \infty$ to obtain the existence of a function-solution are technical. This condition yields that our Poisson driven process V_t has almost surely finite variations, which makes easy the computations. A priori, the method should extend to the general case where $\int_0^\pi \theta^2 B(z,\theta)d\theta < \infty$, but we are not able, for the moment, to get rid of the stronger L^1 condition.

Notation 1.1 *The terminal time $T > 0$ is arbitrarily fixed.*
$I\!D_T$ will denote the Skorohod space $I\!D([0,T], I\!R^2)$ of càdlàg functions from $[0,T]$ into $I\!R^2$. The space $I\!D_T$ endowed with the Skorohod topology is a Polish space.
$\mathcal{P}(I\!D_T)$ will denote the space of probability measures on $I\!D_T$ and $\mathcal{P}_2(I\!D_T)$ will be the subset of probability measures with a second order moment : Q belongs to $\mathcal{P}_2(I\!D_T)$ if

$$\int_{x\in I\!D_T} \sup_{[0,T]} |x(t)|^2 Q(dx) < \infty \tag{1.6}$$

K will denote a real positive constant of which the value may change from line to line.

In order to prove the existence of measure-solutions, we will assume that

Assumption (S) : for each $x \in I\!R_+$, $B(x,\theta)$ is an even strictly positive function on $[-\pi,\pi]/\{0\}$ satisfying

$$\text{for all } x \in I\!R_+, \quad \int_{-\pi}^{\pi} B(x,\theta)d\theta = \infty \tag{1.7}$$

and

$$\sup_{x\in I\!R_+} \int_{-\pi}^{\pi} \theta^2 B(x,\theta)d\theta < \infty \tag{1.8}$$

For $X \in I\!R^2$, we will denote by $B(X,\theta)$ the quantity $B(|X|,\theta)$.

To prove the existence of function-solutions, we will suppose the stronger

Assumption (S') : the same as (S) but with

$$\sup_{x\in I\!R_+} \int_{-\pi}^{\pi} |\theta| B(x,\theta)d\theta < \infty \tag{1.9}$$

Equation (1.1) has to be understood in a weak sense, i.e. f is a solution of the equation if for each bounded test function ϕ,

$$\frac{\partial}{\partial t} < f,\phi > = < Q(f,f),\phi > \tag{1.10}$$

where $< .,. >$ denotes the duality bracket between L^1 and L^∞ functions. A standard integration by parts would give that f satisfies for each bounded ϕ

$$\frac{\partial}{\partial t}\int_{I\!R^2} f(t,v)\phi(v)dv = \int_{I\!R^2\times I\!R^2}\int_{-\pi}^{\pi}(\phi(v')-\phi(v))B(v-v_*,\theta)d\theta f(t,v)dvf(t,v_*)dv_* \tag{1.11}$$

But under (S), the form of v' and the fact that $\int |\theta| B(x,\theta)$ might be infinite necessitates to consider a compensated form of the collision term, which may explode in the previous form. This remark leads us to the following definition of solutions of (1.1).

Assume (S). First of all, we define, for $q \in \mathcal{P}_2(I\!R^2)$, each $\phi \in C_b^2(I\!R^2)$,

$$\begin{aligned} L_q\phi(v) &= \int_{I\!R^2}\int_{-\pi}^{\pi} \Big(\phi(v + A(\theta)(v-v^*)) - \phi(v) - A(\theta)(v-v^*).\nabla\phi(v)\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad B(v-v^*,\theta)d\theta q(dv^*) \\ &\quad - \int_{I\!R^2}(v-v^*).\nabla\phi(v)b(v-v^*)q(dv^*) \end{aligned} \tag{1.12}$$

with for each $X \in I\!R^2$,

$$b(X) = \frac{1}{2}\int_{-\pi}^{\pi} B(X,\theta)(1-\cos\theta)d\theta. \tag{1.13}$$

This kernel is well defined thanks to (1.5) and (1.8). Notice that if (S') holds, then this kernel can be written in the simpler form :

$$L_q\phi(v) = \int_{I\!R^2}\int_{-\pi}^{\pi} \Big(\phi(v + A(\theta)(v-v^*)) - \phi(v)\Big) B(v-v^*,\theta)d\theta q(dv^*) \tag{1.14}$$

Definition 1.2 *Assume (S) (or (S')). Consider Q_0 a probability measure on $I\!R^2$. We say that a probability measure family $\{Q_t\}_{t\in[0,T]}$ is a measure-solution of the Boltzmann equation (1.1) with initial data Q_0 if for each $\phi \in C_b^2(I\!R^2)$ (or $\phi \in C_b^1(I\!R^2)$), all $t \in [0,T]$,*

$$\langle \phi, Q_t\rangle = \langle \phi, Q_0\rangle + \int_0^t \langle L_{Q_s}\phi(v), Q_s(dv)\rangle\, ds, \tag{1.15}$$

If furthermore for all $t \in]0,T]$, the probability measure Q_t admits a density $f(t,.)$ with respect to the Lebesgue measure on $I\!R^2$, the obtained function $f(t,v) :]0,T] \times I\!R^2 \mapsto I\!R_+$ is said to be a function-solution of the Boltzmann equation (1.1).

The probabilitistic approach consists in considering (1.15) as the evolution equation of the flow of time-marginals of a Markov process, solution of the following nonlinear martingale problem.

Definition 1.3 *Let B be a cross section satisfying (S) (or (S')) and let Q_0 in $\mathcal{P}_2(I\!R^2)$. We say that $Q \in \mathcal{P}_2(I\!D_T)$ solves the nonlinear martingale problem (MP) starting at Q_0 if for X the canonical process under Q, the law of X_0 is Q_0 and for any $\phi \in C_b^2(I\!R^2)$, any $t \in [0,T]$,*

$$\phi(X_t) - \phi(X_0) - \int_0^t L_{Q_s}\phi(X_s)ds \tag{1.16}$$

is a square-integrable martingale. Here, the nonlinearity appears through Q_s which denotes the law of X_s under Q.

Remark 1.4 *Taking expectations in (1.16), we observe that if Q is a solution of (MP), then its marginal flow $(Q_t)_{t\in[0,T]}$ is a measure-solution of the Boltzmann equation, in the sense of Definition 1.2.*

2 Transformation of the Boltzmann equation and main results.

This whole work is based on the following substitution in L_q.

Notation 2.1 *For each $X \in I\!R^2$, we consider the function h_X defined on $[-\pi,\pi]/\{0\}$ by*

$$h_X(\theta) = \int_\theta^\pi B(X,\varphi)d\varphi \text{ if } \theta > 0 \; ; \; h_X(\theta) = -\int_{-\pi}^\theta B(X,\varphi)d\varphi \text{ if } \theta < 0 \tag{2.1}$$

Thanks to (S), it is clear that for each X, $h_X(\theta)$ is strictly decreasing from 0 to $-\infty$ between $\theta = -\pi$ and $\theta = 0^-$, and from $+\infty$ to 0 between $\theta = 0^+$ and $\theta = \pi$. We thus can set, for each $X \in I\!R^2$ and each $z \in I\!R^*$,

$$g(X,z) = h_X^{-1}(z), \quad \text{i.e.} \quad h_X(g(X,z)) = z \tag{2.2}$$

Notice that for each X, z, the derivative $\frac{\partial}{\partial z} g(X,z) = -1/B(X, g(X,z)) < 0$, thanks to (S). The function $g(X,z)$ is thus strictly decreasing from 0 to $-\pi$ between $-\infty$ and 0^-, and from π to 0 between 0^+ and $+\infty$.

Notice also that $g(X,.)$ is odd and depends only on $|X|$.

Finally remark that (1.13) can be written as

$$b(X) = \int_{I\!R^*} (1 - \cos g(X,z))dz \tag{2.3}$$

that (1.8) becomes

$$\sup_{X \in I\!R^2} \int_{I\!R^*} g^2(X,z)dz < +\infty \tag{2.4}$$

while (1.9) can be written as

$$\sup_{X \in I\!R^2} \int_{I\!R^*} |g(X,z)|dz < +\infty \tag{2.5}$$

We introduce again some notations.

Notation 2.2 *For $X \in I\!R^2$ and $z \in I\!R^*$, we set*

$$\gamma(X,z) = A(g(X,z)).X \; : \; I\!R^2 \times I\!R^* \mapsto I\!R^2 \tag{2.6}$$

$$\delta(X) = b(X)X \; : \; I\!R^2 \mapsto I\!R^2. \tag{2.7}$$

Finally,

Proposition 2.3 *Assume (S). Then for each $q \in \mathcal{P}_2(I\!R^2)$, each $\phi \in C_b^2(I\!R^2)$,*

$$\begin{aligned} L_q\phi(v) = & \int_{I\!R^2} \int_{z \in I\!R^*} \Big(\phi(v + \gamma(v - v^*, z)) - \phi(v) \\ & \qquad - \gamma(v - v^*, z).\nabla\phi(v) \Big) dz q(dv^*) \\ & - \int_{I\!R^2} \delta(v - v^*).\nabla\phi(v) q(dv^*) \end{aligned} \tag{2.8}$$

If furthermore (S') holds, then

$$L_q\phi(v) \quad = \quad \int_{I\!R^2}\int_{z\in I\!R^*}\Big(\phi(v+\gamma(v-v^*,z))-\phi(v)\Big)dzq(dv^*) \tag{2.9}$$

Proof. It suffices to use the substitution

$$\theta = g(v-v^*,z) \quad ; \quad z = h_{v-v^*}(\theta) \quad ; \quad dz = -B(v-v^*,\theta)d\theta \tag{2.10}$$

in (1.12) and (1.13) △

Remark 2.4 *We now give an idea of the probabilistic approach we will use, following the main ideas of Tanaka, [21], who was dealing with the much more simple case of Maxwell molecules (i.e. $B(X,\theta)=\beta(\theta)$). In this case, the jump measure appearing in the analogous of (2.8) is $\beta(\theta)d\theta q(dv^*)$ independent of v. The main interest of the transformation described above is to transform the jump measure $B(v-v^*,\theta))d\theta q(dv^*)$ in a measure $dzq(dv^*)$ independent of v. That will allow us to have a probabilistic interpretation in terms of Poisson measure.*

Let us consider two probability spaces : the first one is the abstract space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},P)$ and the second one is the auxiliary space $([0,1],\mathcal{B}([0,1]),d\alpha)$ introduced to model the nonlinearity by the Skorohod representation theorem. In order to avoid any confusion, the processes on $([0,1],\mathcal{B}([0,1]),d\alpha)$ will be called α-processes, the expectation under $d\alpha$ will be denoted by E_α, and the laws $\mathcal{L}_\alpha$.

Notation 2.5 *We will denote by L^2_T the space of $I\!D_T$-valued processes Y such that*

$$E\left(\sup_{t\in[0,T]}|Y_t|^2\right) < +\infty \tag{2.11}$$

and by L^2_T-α the space of α-processes W such that

$$E_\alpha\left(\sup_{t\in[0,T]}|W_t|^2\right) < +\infty \tag{2.12}$$

Definition 2.6 *Assume (S) (or (S')). We will say that (V,W,N,V_0) is a solution of (SDE) if*
(i) (V_t) is an adapted L^2_T-process on Ω,
(ii) (W_t) is a L^2_T-α-process on $[0,1]$,
(iii) $N(\omega,dt,d\alpha,dz)$ is a Poisson measure on $[0,T]\times[0,1]\times I\!R^*$ with intensity measure

$$m(dt,d\alpha,dz) = dtd\alpha dz \tag{2.13}$$

(iv) V_0 is a square integrable variable independent of N,
(v) The laws of V and W on their respective probability spaces are the same, i.e. $\mathcal{L}(V)=\mathcal{L}_\alpha(W)$,
(vi) The following S.D.E. is satisfied :

$$V_t = V_0+\int_0^t\int_0^1\int_{I\!R^*}\gamma(V_{s-}-W_{s-}(\alpha),z)\tilde{N}(ds,d\alpha,dz)-\int_0^t\int_0^1\delta(V_{s-}-W_{s-}(\alpha))d\alpha ds \tag{2.14}$$

where $\tilde{N}$ denotes the compensated Poisson point process associated with N.

Notice that under (S'), equation (2.14) can be written in the simpler form:

$$V_t = V_0 + \int_0^t \int_0^1 \int_{I\!R^*} \gamma(V_{s-} - W_{s-}(\alpha), z) N(ds, d\alpha, dz) \tag{2.15}$$

The following remark shows the connection between (SDE) and the Boltzmann equation (1.1).

Remark 2.7 *If (V, W, N, V_0) is a solution of (SDE), one easily proves by using the Itô formula, that $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$ is a solution of the martingale problem* (1.16) *with initial law $Q_0 = \mathcal{L}(V_0)$, and thus $\{\mathcal{L}(V_s)\}_{s\in[0,T]}$ is a measure-solution of (1.15) with initial data Q_0.*

Let us now state an hypothesis, which, combined with (S), will be sufficient for proving the existence of a solution to (SDE) (and thus a solution to (MP), and hence a measure-solution to (1.1)).

Assumption (MS) : (i) There exists a constant $K \in I\!R_+$ such that for all $X \in I\!R^2$,

$$\int_{I\!R^*} \gamma^4(X, z) dz \leq K(1 + |X|^4) \tag{2.16}$$

(ii) There exists a function S from $I\!R^2 \times I\!R^2 \to I\!R^+$, locally bounded, such that for each $X, Y \in I\!R^2$,

$$|\delta(X) - \delta(Y)|^2 + \int_{I\!R^*} (\gamma(X, z) - \gamma(Y, z))^2 \, dz \leq |X - Y|^2 S^2(X, Y) \tag{2.17}$$

(iii) The initial data Q_0 admits a moment of order 4.

Remark 2.8 *Thanks to (2.16), (2.6), (2.7), (2.3) and (2.4); there exists a constant $K \in I\!R_+$ such that for all $X \in I\!R^2$,*

$$\delta^4(X) + \left(\int_{I\!R^*} \gamma^2(X, z) dz \right)^2 + \int_{I\!R^*} \gamma^4(X, z) dz \leq K(1 + |X|^4) \tag{2.18}$$

Then the following result will hold.

Theorem 2.9 *Assume hypotheses (S) and (MS). Then*
1) The martingale problem (MP) with initial data Q_0 admits a solution $Q \in \mathcal{P}_2(I\!D_T)$.
2) Let Q be any solution of (MP). Let W be any α-process with law Q. On an enlarged probability space from the canonical space $(I\!D_T, \mathcal{D}_T, Q)$ there exist a Poisson measure N with intensity m and an independent square integrable variable V_0 with law Q_0 such that (X, W, N, V_0) is solution of (SDE), where X is the canonical process. (That means that there exists a weak solution to (SDE)).

Remark 2.10 *Let us remark that there is no assumption on Q_0, except to have a forth order moment, and that allows us to consider degenerate initial data, as Dirac measures. Theorem 2.9 exhibits in particular a measure-solution to the Boltzmann equation (1.1) for every initial data $Q_0 \in \mathcal{P}_4(I\!R^2)$.*

We now introduce an hypothesis more stringent than (MS).

Assumption (FS) : (i) The map $\gamma(X,z) : \mathbb{R}^2 \times \mathbb{R}^* \mapsto \mathbb{R}^2$ is of class C^2. There exist $p \in \mathbb{N}$, $K \in \mathbb{R}^+$, and a bounded positive function $\eta : \mathbb{R}^* \mapsto \mathbb{R}^+$, satisfying the integrability condition

$$\bar{\eta}(z) = \sup_{|u-z|\leq |z|/2 \wedge 1/|z|} \eta(u) \ \in \ L^1(\mathbb{R}^*, dz) \tag{2.19}$$

and such that

$$|\gamma(X,z)| \leq (1+|X|)\,\eta(z) \tag{2.20}$$

$$|\gamma'_X(X,z)| + |\gamma''_{XX}(X,z)| \leq (1+|X|^p)\,\eta(z) \tag{2.21}$$

$$|\gamma'_z(X,z)| + |\gamma''_{zz}(X,z)| + |\gamma''_{Xz}(X,z)| \leq K\,(1+|X|^p) \tag{2.22}$$

(ii) The initial distribution Q_0 admits moments of all orders, and is not a Dirac mass.

Notice that the integrability condition (2.19) is not much more stringent than the simple condition $\eta \in L^1(\mathbb{R}^*, dz)$.

Then the following result holds :

Theorem 2.11 *Assume (S') and (FS). Consider a solution (V, W, N, V_0) of (SDE), as built in Theorem 2.9. Then for all $t > 0$ the law of V_t admits a density $f(t,.)$ with respect to the Lebesgue measure on $\mathbb{R}^2$.*

The next corollary, immediately deduced from Theorem 2.11, states our main result.

Corollary 2.12 *Assume (S') and (FS). Then there exists a function-solution*

$$f \in L^\infty(]0,T], L^1((1+|v|^2)dv)) \tag{2.23}$$

to the Boltzmann equation without cutoff, for non Maxwell molecules, with initial data Q_0, and $f(t,.)$ is for each $t > 0$ a probability density function.

We now give examples of application.

Remark 2.13 *Assume that the cross section is of the form $B(X,\theta) = \psi(X)/|\theta|^\alpha$, with ψ positive and $\alpha \in [1,3[$.*
1) Then (S) and (MS) are satisfied if ψ is strictly positive, bounded, and locally lipschitz continuous on $\mathbb{R}^2$.
2) (S') and (FS) are satisfied if $\alpha \in [1,2[$, if ψ is of class C^2 on $\mathbb{R}^2$, if ψ' and ψ'' have at most a polynomial growth, and if there exist $0 < \epsilon < M < \infty$ such that for all $X \in \mathbb{R}^2$, $\epsilon \leq \psi(X) \leq M$.

Proof. Observing that when $\alpha = 1$, $g(X,z) = sign(z)e^{-|z|/\psi(X)}$, and when $\alpha > 1$, $g(X,z) = sign(z)\left(\frac{\pi^{\alpha-1}\psi(X)}{(\alpha-1)|z|\pi^{\alpha-1}+\psi(X)}\right)^{\frac{1}{\alpha-1}}$, the remark can be proved by using simple computations. △

3 Existence of a solution to (SDE).

The aim of this section is to prove Theorem 2.9. This will be done in many steps. We first introduce the following notations.

Notation 3.1 *Let us consider for $n \in I\!N$ the functions γ_n and δ_n defined respectively from $I\!R^2 \times I\!R^2 \times I\!R^*$ and from $I\!R^2 \times I\!R^2$ into $I\!R^2$ by*

$$\begin{aligned} i)\ \gamma_n(v,w,z) &= \gamma(v \wedge n \vee (-n) - w \wedge n \vee (-n), z) &(3.1)\\ ii)\quad \delta_n(v,w) &= \delta(v \wedge n \vee (-n) - w \wedge n \vee (-n)) &(3.2)\end{aligned}$$

where $v \wedge n$ (resp. $v \vee (-n)$), denotes the vector $(v_1 \wedge n, v_2 \wedge n)$ (resp. $(v_1 \vee (-n), v_2 \vee (-n))$, if $v = (v_1, v_2)$.

Similarly to Definition 2.6, a solution (V^n, W^n, N, V_0) of $(SDE)_n$ is defined exactly as a solution of (SDE), but with

$$V_t^n = V_0 + \int_0^t \int_0^1 \int_{I\!R_*} \gamma_n(V_{s-}^n, W_{s-}^n(\alpha), z)\tilde{N}(ds, d\alpha, dz) - \int_0^t \int_0^1 \delta_n(V_{s-}^n, W_{s-}^n(\alpha))d\alpha ds \quad (3.3)$$

instead of (2.14). We also denote by $(MP)_n$ the nonlinear martingale problem associated with the operator L_q^n defined as L^q (by (1.12)) but with γ_n and δ_n instead of γ and δ.

Proposition 3.2 *Assume (S) and (MS), and let $n \in I\!N$ be fixed. For each pair (V_0, N), V_0 being with law Q_0 and N a Poisson measure with intensity m, the equation $(SDE)_n$ admits a solution (V^n, W^n, N, V_0) and*

$$\sup_{n \geq 1} E(\sup_{t \in [0,T]} |V_t^n|^4) < +\infty. \quad (3.4)$$

Moreover, $Q^n = \mathcal{L}(V^n) = \mathcal{L}_\alpha(W^n)$ is the unique solution of the nonlinear martingale problem $(MP)_n$.

Proof. Following Tanaka [21], Desvillettes-Graham-Méléard [6] or Fournier [8], we construct a specific iteration of Picard which allows us to obtain the existence of a pair (V^n, W^n) of identically distributed processes, such that (V^n, W^n, N) is a solution of $(SDE)_n$.
Since n is fixed, we drop the superscript. We first consider the process X^0 identically equal to V_0, then consider Y^0 defined on $[0,1]$ such that $\mathcal{L}_\alpha(Y^0) = \mathcal{L}(X^0)$. By induction, assuming that $X^0, X^1, ..., X^k$ and $Y^0, Y^1, ..., Y^k$ are constructed, one defines X^{k+1} by

$$X_t^{k+1} = V_0 + \int_0^t \int_0^1 \int_{I\!R^*} \gamma_n(X_{s-}^k, Y_{s-}^k(\alpha), z)\tilde{N}(ds, d\alpha, dz) - \int_0^t \int_0^1 \delta_n(X_{s-}^k, Y_{s-}^k(\alpha))d\alpha ds \quad (3.5)$$

and one considers on $[0,1]$ a process Y^{k+1} such that

$$\mathcal{L}_\alpha(Y^0, Y^1, ..., Y^{k+1}) = \mathcal{L}(X^0, X^1, ..., X^{k+1}) \quad (3.6)$$

and so on. One proves easily, by Doob's and Cauchy-Schwarz's inequality, the existence of a constant K_n such that

$$E(\sup_{s \in [0,t]} |X_s^{k+1} - X_s^k|^2)$$

$$\leq K_n \int_0^t \int_0^1 E(|X^k_{s-} - X^{k-1}_{s-}|^2 + |Y^k_{s-}(\alpha) - Y^{k-1}_{s-}(\alpha)|^2) d\alpha ds$$

$$\leq K_n \int_0^t E(\sup_{u\leq s} |X^k_u - X^{k-1}_u|^2) ds. \tag{3.7}$$

We deduce easily that there exist an adapted process X with $E(\sup_{t\in[0,T]} |X_t|^2) < \infty$ and a α-process Y with $\mathcal{L}_\alpha(Y) = \mathcal{L}(X)$ such that

$$E(\sup_{t\in[0,T]} |X^k_t - X_t|^2) = E_\alpha(\sup_{t\in[0,T]} |Y^k_t - Y_t|^2) \tag{3.8}$$

tends to zero as k tends to infinity. Then (X, Y, N, V_0) is solution of $(SDE)_n$.

Now, let us rename $X = V^n$, $Y = W^n$, and denote by Q^n the law of V^n.

The proof of the uniqueness in law of a solution of $(SDE)_n$ is obtained by a coupling argument, exactly as in [6]. One proves that if $(U, W, \hat{N}, \hat{V_0})$ is a solution of $(SDE)_n$, then $\mathcal{L}(U) = \mathcal{L}(V^n) = Q^n$, where $(V^n, W^n, \hat{N}, \hat{V_0})$ is the Picard iteration constructed on the probability space associated with $\hat{V_0}$ and $\hat{N}$.

Then we deduce the uniqueness for the martingale problem $(MP)_n$. If R is another solution of the martingale problem, one proves by using a comparision between the Itô formula and the martingale problem that the canonical process X is under R the sum of the drift $-\int_0^t \int_0^1 \delta_n(X_{s-}, W_{s-}(\alpha)) d\alpha ds$ and of a pure jump process of which Lévy's measure is the image measure of the measure $m(ds, d\alpha, dz) = ds d\alpha dz$ by the mapping $(\alpha, z, s) \mapsto \gamma_n(X_{s-}, W_{s-}(\alpha), z)$, where $W_s(\alpha)$ is any process on the probability space $[0, 1]$ with law R. Then, by using the representation theorem proved in Grigelionis [13] and El Karoui-Lepeltier [7] (see also [20]), we know that there exist on an enlarged probability space a square integrable variable V_0 and an independent point Poisson measure N with intensity m such that (X, W, N, V_0) is a solution of $(SDE)_n$. Then by the uniqueness in law for $(SDE)_n$ (proved above), R is equal to Q^n and the martingale problem $(MP)_n$ has a unique solution.

Now, it remains to prove (3.4), where we have denoted for each n by (V^n, W^n) a solution of $(SDE)_n$. Remark 2.8 implies that there exists a constant K independent of n such that

$$\delta_n^4(x, y) + \left(\int_{I\!R^*} \gamma_n^2(x, y, z) dz\right)^2 + \int_{I\!R^*} \gamma_n^4(x, y, z) dz \leq K(1 + |x|^4 + |y|^4) \tag{3.9}$$

and we obtain, using twice the Burkholder inequality,

$$\begin{aligned} E(\sup_{s\leq t} |V^n_s|^4) \quad &\leq \quad KE(|V_0|^4) + KE\left(\left|\int_0^t \int_0^1 \int_{I\!R^*} \gamma_n^2(V^n_{s-}, W^n_{s-}(\alpha), z) N(ds, d\alpha, dz)\right|^2\right) \\ &\quad + KE\left(\int_0^t \int_0^1 \delta_n^4(V^n_{s-}, W^n_{s-}(\alpha)) d\alpha ds\right) \\ &\leq \quad KE(|V_0|^4) + KE\Bigg(\left|\int_0^t \int_0^1 \int_{I\!R^*} \gamma_n^2(V^n_{s-}, W^n_{s-}(\alpha), z) dz d\alpha ds\right|^2 \end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^t\int_0^1\int_{\mathbb{R}^*}\gamma_n^2(V^n_{s-},W^n_{s-}(\alpha),z)\tilde N(ds,d\alpha,dz)\right|^2\Bigg)\\
&+KE\left(\int_0^t\int_0^1\delta_n^4(V^n_{s-},W^n_{s-}(\alpha))d\alpha ds\right)\\
\leq\;& KE(|V_0|^4)++\int_0^t\int_0^1\left(\int_{\mathbb{R}^*}\gamma_n^2(V^n_{s-},W^n_{s-}(\alpha),z)dz\right)^2 d\alpha ds\\
&+KE\left(\int_0^t\int_0^1\int_{\mathbb{R}^*}\gamma_n^4(V^n_{s-},W^n_{s-}(\alpha),z)dzd\alpha ds\right)\\
&+KE\left(\int_0^t\int_0^1\delta_n^4(V^n_{s-},W^n_{s-}(\alpha))d\alpha ds\right)\\
\leq\;& KE(|V_0|^4)+K\int_0^t\int_0^1\left[E(|V^n_s|^4)+|W^n_s(\alpha)|^4\right]ds\\
\leq\;& KE(|V_0|^4)+K\int_0^t E(|V^n_s|^4)ds \qquad (3.10)
\end{aligned}
$$

where K does not depend on n. The last inequality comes from the equality $\mathcal{L}(V^n)=\mathcal{L}_\alpha(W^n)$. Gronwall's lemma allows us to conclude that (3.4) holds. △

Proposition 3.3 *Assume* (S) *and* (MS). *The sequence of probability measures* $(Q^n)_n$ *on* $\mathbb{D}_T$ *is tight. Any limiting point* Q *of this sequence satisfies the martingale problem* (MP).

Proof. We need first to prove that the sequence Q^n is tight. Thanks to (3.4), we just need to verify the Aldous criterion. We have, for any stopping times τ and τ' satisfying a.s. $0\leq\tau\leq\tau'\leq(\tau+\delta)\wedge T$,

$$
\begin{aligned}
E(|V^n_{\tau'}-V^n_\tau|^2)\;\leq\;& KE\left(\int_\tau^{\tau'}\int_0^1\int_{\mathbb{R}^*}\gamma_n^2(V^n_{s-},W^n_{s-}(\alpha),z)dzd\alpha ds\right)\\
&+KE\left(\int_\tau^{\tau'}\int_0^1\delta_n^2(V^n_{s-},W^n_{s-}(\alpha))d\alpha ds\right) \qquad (3.11)\\
\leq\;& KE\left(\int_\tau^{\tau'}\int_0^1\left(|V^n_{s-}|^2+E_\alpha(|W^n_{s-}(\alpha)|^2)\right)d\alpha ds\right)\\
\leq\;& KE\left((\tau'-\tau)\sup_{t\leq T}|V^n_t|^2\right)+KE(\tau'-\tau)E_\alpha(\sup_{t\leq T}|W^n_t|^2)\leq K\delta
\end{aligned}
$$

by (3.4), where K is independent of n, τ, and τ'. Then we deduce that for each $\eta>0$,

$$
\sup_n\sup_{\{\tau,\tau'\;;\;0\leq\tau\leq\tau'\leq(\tau+\delta)\wedge T\}}P(|V^n_{\tau'}-V^n_\tau|\geq\eta) \qquad (3.12)
$$

tends to 0 as δ tends to 0, and the Aldous criterion is satisfied. Hence the sequence (Q^n) is tight.

We have now to identify each limit point of (Q^n). Let Q be a limit value of this sequence. We consider the canonical process X on $I\!D_T$ and for $\phi \in C_b^2(I\!R^2)$, $t > 0$, we set

$$H_t^\phi = \phi(X_t) - \phi(X_0) + \int_0^t \int_{w\in I\!R^2} \nabla\phi(X_u)\delta(X_u, w)Q_u(dw)du \tag{3.13}$$

$$- \int_0^t \int_{I\!R^*} \int_{w\in I\!R^2} (\phi(X_u + \gamma(X_u, w, z)) - \phi(X_u) - \gamma(X_u, w, z).\nabla\phi(X_u))\, Q_u(dw)dzdu$$

and $H_t^{n,\phi}$ denotes a similar quantity with γ_n, δ_n instead of γ, δ and Q_u^n instead of Q_u. The probability measure Q will be a solution of the nonlinear martingale problem (MP) with initial law Q_0 if it satisfies for each $0 \leq s_1 < ... < s_k < s < t \leq T$, each $g_1, ...g_k \in C_b(I\!R^2)$,

$$< (H_t^\phi - H_s^\phi)g_1(X_{s_1})...g_k(X_{s_k}), Q >= 0 \tag{3.14}$$

Since Q^n is solution of $(PM)_n$, we already know that

$$< (H_t^{n,\phi} - H_s^{n,\phi})g_1(X_{s_1})...g_k(X_{s_k}), Q^n >= 0 \tag{3.15}$$

Since the sequence (Q^n) satisfies the Aldous criterion, the law Q is the law of a quasi-càg process (cf. [15] p. 321). Then the mapping

$$F : x \mapsto (\phi(x_t) - \phi(x_s))g_1(x_{s_1})...g_k(x_{s_k}) \tag{3.16}$$

is Q-almost everywhere continuous and bounded from $I\!D_T$ to $I\!R$. Thus $< F, Q^n >$ tends to $< F, Q >$ as n tends to infinity.
Next, let us prove that α_n defined by

$$< \Big(g_1(X_{s_1})...g_k(X_{s_k}) \int_s^t \int_{I\!R^*} \int_{I\!R^2} \Big(\phi(X_u + \gamma(X_u, w, z)) - \phi(X_u + \gamma_n(X_u, w, z))$$

$$-\nabla\phi(X_u).(\gamma((X_u, w, z)) - \gamma_n(X_u, w, z))\Big) dzQ_u^n(dw)du\Big), Q^n > \tag{3.17}$$

tends to 0 as n tends to infinity. We have :

$$\int_{I\!R^*} |\phi(X_u + \gamma(X_u, w, z)) - \phi(X_u + \gamma_n(X_u, w, z)) - \nabla\phi(X_u).(\gamma((X_u, w, z)) - \gamma_n(X_u, w, z))|$$

$$\leq K \int_{I\!R^*} |\gamma(X_u, w, z) - \gamma_n(X_u, w, z)|^2 dz$$

$$\leq K(|w|^2 + |X_u|^2)(\mathbf{1}_{\{|X_u|\geq n\}} + \mathbf{1}_{\{|w|\geq n\}}) \tag{3.18}$$

Then

$$|\alpha_n| \leq K\Pi_{i=1,...,k}\|g_i\|_\infty < \int_s^t \int_{I\!R^2} (|w|^2 + |X_u|^2)(\mathbf{1}_{\{|X_u|\geq n\}} + \mathbf{1}_{\{|w|\geq n\}})Q_u^n(dw)du, Q^n >$$

$$\leq K \int_{x\in I\!D_T} \int_{y\in I\!D_T} \left(\sup_{t\leq T} |x(t)|^2 + \sup_{t\leq T} |y(t)|^2\right)$$

$$\left(\mathbf{1}_{\{\sup_{t\leq T} |x_t|\geq n\}} + \mathbf{1}_{\{\sup_{t\leq T} |y_t|\geq n\}}\right) Q^n(dx)Q^n(dy)$$

$$\leq K\Big(\Big(\int_{x\in \mathbb{D}_T}(\sup_{t\leq T}|x(t)|^2)Q^n(dx)\Big)\times\Big(\int_{x\in \mathbb{D}_T}\left(\mathbf{1}_{\{\sup_{t\leq T}|x_t|\geq n\}}\right)Q^n(dx)\Big)$$

$$+\int_{x\in \mathbb{D}_T}(\sup_{t\leq T}|x(t)|^2)\left(\mathbf{1}_{\{\sup_{t\leq T}|x_t|\geq n\}}\right)Q^n(dx)\Big) \qquad (3.19)$$

By (3.4), we know that $\int_{x\in \mathbb{D}_T}(\sup_{t\leq T}|x(t)|^4)Q^n(dx)$ is bounded uniformly in n, that $\int_{x\in \mathbb{D}_T}\left(\mathbf{1}_{\{\sup_{t\leq T}|x_t|\geq n\}}\right)Q^n(dx)$ tends to 0 as n tends to infinity, and by Cauchy-Schwarz inequality that $\int_{x\in \mathbb{D}_T}(\sup_{t\leq T}|x(t)|^2)\left(\mathbf{1}_{\{\sup_{t\leq T}|x_t|\geq n\}}\right)Q^n(dx)$ tends to 0 as n tends to infinity. Then α_n tends to 0 as n tends to infinity.

The same arguments yield that

$$\alpha'_n =< \int_s^t\int_{\mathbb{R}^2}\nabla\phi(X_u).(\delta(X_u,w)-\delta_n(X_u,w))Q^n_u(dw)du\times g_1(X_{s_1})...g_k(X_{s_k}),Q^n > \qquad (3.20)$$

tends to 0 as n tends to infinity.

It remains to prove that $< G(x,y),Q^n(dx)\otimes Q^n(dy) >$ tends to $< G(x,y),Q(dx)\otimes Q(dy) >$, where

$$G(x,y)=\Big(\int_s^t\int_{\mathbb{R}^*}(\phi(x_u+\gamma(x_u,y_u,z))-\phi(x_u)-\gamma(x_y,y_u,z).\nabla\phi(x_u))\,dzdu\Big)\,g_1(x_{s_1})...g_k(x_{s_k}). \qquad (3.21)$$

The measure $Q^n\otimes Q^n$ converges obviously to $Q\otimes Q$. The function G is $Q\otimes Q$ continuous a.s. by a similar argument as before but not bounded. We thus only know that for each fixed real positive number C, the quantities $< G\wedge C, Q^n\otimes Q^n >$ converge to $< G\wedge C, Q\otimes Q >$. But one has moreover that

$$|G(x,y)|\leq K\left(\sup_{t\leq T}|x(t)|^2+\sup_{t\leq T}|y(t)|^2\right) \qquad (3.22)$$

Then,

$$\begin{aligned}|G(x,y)|\mathbf{1}_{\{|G(x,y)|\geq C\}} &\leq K\left(\sup_{t\leq T}|x(t)|^2+\sup_{t\leq T}|y(t)|^2\right)\mathbf{1}_{\{\sup_{t\leq T}|x(t)|+\sup_{t\leq T}|y(t)|\geq C/K\}}\\ &\leq K\left(\sup_{t\leq T}|x(t)|^2+\sup_{t\leq T}|y(t)|^2\right)\\ &\qquad\times\left(\mathbf{1}_{\{\sup_{t\leq T}|x(t)|\geq C/2K\}}+\mathbf{1}_{\{\sup_{t\leq T}|y(t)|\geq C/2K\}}\right) \end{aligned} \qquad (3.23)$$

We have already seen that

$$\sup_n <\left(\sup_{t\leq T}|x(t)|^2+\sup_{t\leq T}|y(t)|^2\right)\left(\mathbf{1}_{\{\sup_{t\leq T}|x(t)|\geq C/2K\}}+\mathbf{1}_{\{\sup_{t\leq T}|y(t)|\geq C/2K\}}\right),Q^n\otimes Q^n > \qquad (3.24)$$

tends to 0 as C tends to infinity, thanks to (3.4), and thus, $< G,Q^n\otimes Q^n >$ goes to $< G,Q\otimes Q >$.

Finally, we use the same arguments to prove that $< \bar{G}(x,y), Q^n \otimes Q^n >$ converges to $< \bar{G}(x,y), Q \otimes Q >$ where

$$\bar{G}(x,y) = \int_s^t \nabla(x_u).\delta(x_u, y_u)du \times g_1(x_{s_1})...g_k(x_{s_k}).$$

Now the conclusion is obvious and the proposition is proved. △

Remark 3.4 *Proposition 3.3 proves the first point of Theorem 2.9.*

Let us now deduce the point (2) of Theorem 2.9.

Proposition 3.5 *Assume (S) and (MS). Consider the canonical space $\mathbb{D}_T$, X the canonical process and Q the solution of (MP) obtained in Proposition 3.3. Consider a α-process W such that $\mathcal{L}_\alpha(W) = Q$, then there exist a Poisson measure N with intensity m on an enlarged probability space and an independent square integrable variable V_0 such that (X, W, N, V_0) is a solution of (SDE).*

Proof. The proof is exactly similar to the end of the one of Proposition 3.2. Since Q is solution of a martingale problem, the canonical process X is a semimartingale under Q. Then a comparison between the Itô formula and the martingale problem proves that X is a pure jump process with drift $-\int_0^t \int_0^1 \delta(X_{s-} - W_{s-}(\alpha))d\alpha ds$, and that its Lévy measure is the image measure of the measure m on $[0,T] \times [0,1] \times \mathbb{R}^*$ by the mapping $(z, \alpha, s) \mapsto \gamma(X_{s-}, W_{s-}(\alpha), z)$. Then always by the representation theorem for point measures [7], there exist on an enlarged space a square integrable variable V_0 and a point Poisson measure N with intensity m such that (X, W, N, V_0) is a solution of (SDE). △

4 Existence of a function-solution by use of Malliavin calculus

In this section we will prove Theorem 2.11. We assume from now (S') and (FS). We thus consider a fixed solution (V, W, N, V_0) of (SDE). In this case, (SDE) has the simpler form (2.15). Our aim is to prove that for any $t > 0$, the law of V_t admits a density with respect to the Lebesgue measure on $\mathbb{R}^2$. In fact we will only study the case of V_T, for $T > 0$ the fixed terminal time, which of course suffices since $T > 0$ is arbitrarily fixed.

We begin with two lemmas.

Lemma 4.1 *Under (S') and (FS), for any $q \in \mathbb{N}$,*

$$E\left(\sup_{[0,T]} |V_t|^q\right) = E_\alpha\left(\sup_{[0,T]} |W_t|^q\right) < \infty \tag{4.1}$$

The proof is standard, consists in the application of Burkholder's inequality and in the use of assumption (FS). It is left to the reader.

Lemma 4.2 *Assume* (S') *and* (FS). *Then*
1) the conservation of momentum holds, i.e. for all $t \in [0,T]$, $E(V_t) = E(V_0)$.
2) the conservation of kinetic energy holds, i.e. for all $t \in [0,T]$, $E(|V_t|^2) = E(|V_0|^2)$.
3) for all $t \in]0,T]$, *the law of* V_t *(and thus that of* W_t*) is not a Dirac mass.*

Proof. 1) Since V is a solution of (SDE), we now that the family $\{Q_t\}_{t\in[0,T]} = \{\mathcal{L}(V_t)\}_{t\in[0,T]}$ is a solution of (1.1) in the sense of Definition 1.2. Let $\phi(v) = v$. We deduce from equations (1.14) and (1.15) that

$$\langle \phi, Q_t \rangle = \langle \phi, Q_0 \rangle + \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} K(v, v_*) Q_s(dv_*) Q_s(dv) ds \tag{4.2}$$

where

$$K(v, v_*) = \int_{-\pi}^{\pi} (v + A(\theta)(v - v_*) - v)\, B(v - v_*, \theta) d\theta = -(v - v_*) b(v - v_*) \tag{4.3}$$

Since $b(v - v_*)$ depends only on $|v - v_*|$, we obtain, using symetry arguments, that for any $q \in \mathcal{P}_2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} K(v, v_*) q(dv_*) q(dv) = 0 \tag{4.4}$$

Hence $\langle \phi, Q_t \rangle = \langle \phi, Q_0 \rangle$ for any t, and 1) is checked.

2) The same arguments, with $\phi(v) = |v|^2$, yield that

$$\langle \phi, Q_t \rangle = \langle \phi, Q_0 \rangle + \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} b(v - v_*) \left(|v_*|^2 - |v|^2 \right) Q_s(dv_*) Q_s(dv) ds \tag{4.5}$$

and we conclude as in 1).

3) Assume that for some $t \in [0,T]$, some $a \in \mathbb{R}^2$, $V_t = a$ a.s. Then $E(|V_t - a|^2) = 0$. But we deduce from 1) and 2) that

$$\begin{aligned} E(|V_0 - a|^2) &= E(|V_0|^2) + |a|^2 - 2a.E(V_0) \\ &= E(|V_t|^2) + |a|^2 - 2a.E(V_t) \\ &= E(|V_t - a|^2) = 0 \end{aligned} \tag{4.6}$$

Thus $V_0 = a$ a.s., which contradicts (ii) in (FS). △

We now divide the proof of Theorem 2.11 in several steps. The main idea is the following. For $\Lambda \subset \mathbb{R}^2$ a neighborhood of 0, we will build a family of "perturbed" processes $\{V^\lambda\}$, such that :

(i) the map $\lambda \mapsto V_T^\lambda$ is a.s. C^1 at 0, and $V_T^0 = V_T$,
(ii) for each λ, the law of V^λ is absolutely continuous with respect to that of V,
(iii) the derivative $\frac{\partial}{\partial\lambda} V_T^\lambda \Big|_{\lambda=0}$ is a.s. invertible.

We will prove in Subsection 4.1 that these conditions will imply the existence of a density for the law of V_T. In Subsection 4.2, we will build some absolutely continuous changes of measures, on our Poisson space, which will allow to define the perturbed processes V^λ. In fact, we will define a "class" of changes of measure, depending on the "direction" in which we want to perturbe our process. The a.s. differentiability of V^λ with respect to λ is studied in Subsection 4.3. In Subsection 4.4, we choose a "direction", and we prove that the associated $\frac{\partial}{\partial\lambda} V_T^\lambda\Big|_{\lambda=0}$ is a.s. invertible. We finally conclude in Subsection 4.5.

4.1 A general criterion of absolute continuity using a.s. derivatives.

The following criterion is a very weak form of the usual Malliavin calculus criterion of absolute continuity, see Nualart [18] (for the Wiener case), and Bichteler, Jacod, [3] (for the Poisson case). We however really need this robust criterion, because we will see in the sequel that our a.s. derivatives are not some L^2 derivatives, since they do not seem to belong to $L^2(\Omega)$.

Theorem 4.3 *Let $d \in \mathbb{N}^*$, and let X be a $\mathbb{R}^d$-valued random variable on a probability space $(\Omega, \mathcal{F}, P)$. Let Λ be a neighborhood of 0 in $\mathbb{R}^d$. Assume that there exists a family $\{X^\lambda\}_{\lambda\in\Lambda}$ of $\mathbb{R}^d$-valued random variables such that*
(i) For each $\lambda \in \Lambda$, the law of X^λ is absolutely continuous with respect to that of X. We denote by $G^\lambda = \dfrac{dX}{dX^\lambda}$ the associated Radon-Nykodym density. The family G^λ satisfies the integrability condition

$$\sup_\lambda E\left(|G^\lambda|^2\right) < \infty \tag{4.7}$$

(ii) For almost all ω, there exists a neighborhood $\mathcal{V}(\omega)$ of 0 in $\mathbb{R}^d$ on which the map $\lambda \mapsto X^\lambda(\omega)$ is of class C^1.
(iii) For almost all ω, the derivative $\dfrac{\partial}{\partial\lambda} X^\lambda\Big|_{\lambda=0}$ is invertible.
Then the law of X is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Proof. Let A be a negligible subset of $\mathbb{R}^d$. We have to prove that $P(X \in A) = 0$.

Step 1 : applying the inverse local theorem, we deduce from (ii) and (iii) that for almost all ω, there exists a neighborhood $\bar{\mathcal{V}}(\omega)$ of 0 in $\mathbb{R}^d$ on which the map $\lambda \mapsto X^\lambda(\omega)$ is a C^1 diffeomorphism. We now set, for $n \in \mathbb{N}_*$,

$$\Omega_n = \left\{\omega \in \Omega \;\Big/ [-1/n, 1/n]^d \subset \bar{\mathcal{V}}(\omega)\right\} \tag{4.8}$$

Then it is clear that Ω_n grows to some $\tilde{\Omega}$, with $P(\tilde{\Omega}) = 1$.

Step 2 : the aim of this step is to check that

$$P(X \in A) = \lim_{n\to\infty} E\left[\left(\frac{n}{2}\right)^d \int_{[-1/n,1/n]^d} 1_A(X^\lambda) G^\lambda d\lambda \times 1_{\Omega_n}\right] \tag{4.9}$$

For each $\lambda \in \Lambda$, we deduce from (i) that

$$P(X \in A) = E\left(1_A(X^\lambda) G^\lambda\right) \tag{4.10}$$

and hence, for each n,

$$P(X \in A) = E\left[\left(\frac{n}{2}\right)^d \int_{[-1/n,1/n]^d} 1_A(X^\lambda) G^\lambda d\lambda\right] \tag{4.11}$$

Hence

$$\left| P(X \in A) - E\left[\left(\frac{n}{2}\right)^d \int_{[-1/n,1/n]^d} 1_A(X^\lambda) G^\lambda d\lambda 1_{\Omega_n}\right]\right|$$

$$\leq E\left[\left(\frac{n}{2}\right)^d \int_{[-1/n,1/n]^d} 1_A(X^\lambda) G^\lambda (1 - 1_{\Omega_n})\right]$$

$$\leq \left(\frac{n}{2}\right)^d \int_{[-1/n,1/n]^d} E\left[G^\lambda d\lambda (1 - 1_{\Omega_n})\right] d\lambda$$

$$\leq \sup_\lambda E\left[G^\lambda (1 - 1_{\Omega_n})\right]$$

$$\leq \sup_\lambda E\left[|G^\lambda|^2\right]^{1/2} P(\Omega/\Omega_n)^{1/2} \tag{4.12}$$

which goes to 0 thanks to (i).

Step 3 : we conclude proving that for each n, each $\omega \in \Omega_n$,

$$\int_{[-1/n,1/n]^d} 1_A(X^\lambda) G^\lambda d\lambda = 0 \tag{4.13}$$

It of course suffices that

$$I_n(\omega) = \int_{[-1/n,1/n]^d} 1_A(X^\lambda) d\lambda = 0 \tag{4.14}$$

But ω belongs to Ω_n, thus $\lambda \mapsto X^\lambda(\omega)$ is a C^1 diffeomorphism from $[-1/n, 1/n]^d$ into some set $D_n(\omega)$. Substituting $y = X^\lambda(\omega)$ in (4.14), denoting by $J_n(\omega, y)$ the associated Jacobian, we obtain

$$I_n(\omega) = \int_{D_n(\omega)} 1_A(y) J_n(\omega, y) dy \tag{4.15}$$

which of course vanishes since A is Lebesgue-negligible. This concludes the proof. △

4.2 The case of Poisson functionnals.

We now consider the Poisson case. We will build, following the ideas of Bichteler, Jacod, [3], a family of shifts S^λ on Ω, such that the family $V_t^\lambda = V_t \circ S^\lambda$ satisfies the assumptions of the criterion given in Theorem 4.3. We begin with a definition, which describes in which "directions" we are authorized to perturbe our process.

Definition 4.4 *We say that a predictable function $v(\omega, s, \alpha, z) : \Omega \times [0, T] \times [0, 1] \times \mathbb{R}^* \mapsto \mathbb{R}^2$ is a* **direction** *if it is of class C^1 in z, and if there exists a deterministic positive function $\rho(z) : \mathbb{R}^* \mapsto \mathbb{R}^+$ such that*

$$|v(\omega, s, \alpha, z)| + |v'(\omega, s, \alpha, z)| \leq \rho(z) \tag{4.16}$$

(where $v' = \frac{\partial}{\partial z} v$), and

$$\rho \in L^1(I\!R^*, dz) \tag{4.17}$$

$$\rho(z) \leq |z|/2 \wedge 1/|z| \tag{4.18}$$

$$\forall\, z \in I\!R^*, \;\; \rho(z) \leq 1/2 \tag{4.19}$$

Let now v be a fixed direction. We associate with v many objects.

We consider a neighborhood Λ of 0 in $B(0,1) \subset I\!R^2$. For $\lambda \in \Lambda$, we define the following perturbation:

$$\Gamma^\lambda(\omega, t, z, \alpha) = z + \lambda.v(\omega, t, z, \alpha) = z + \lambda_x v_x(\omega, t, z, \alpha) + \lambda_y v_y(\omega, t, z, \alpha) \tag{4.20}$$

One can check that for every $\lambda \in \Lambda$, for every ω, t, α, the map $z \mapsto \Gamma^\lambda(\omega, t, z, \alpha)$ is an increasing bijection from $I\!R^*$ into itself.
For $\lambda \in \Lambda$, we set $N^\lambda = \Gamma^\lambda(N)$: if A is a Borel set of $[0,T] \times [0,1] \times I\!R^*$,

$$N^\lambda(A) = \int_0^T \int_0^1 \int_{I\!R^*} \mathbf{1}_A(s, \Gamma^\lambda(\omega, s, z, \alpha), \alpha) N(\omega, dz, d\alpha, ds) \tag{4.21}$$

We consider the shift S^λ defined by

$$V_0 \circ S^\lambda(\omega) = V_0(\omega) \;,\; N \circ S^\lambda(\omega) = N^\lambda(\omega) \tag{4.22}$$

We now look for a family of probability measures P^λ on Ω satisfying $P^\lambda \circ (S^\lambda)^{-1} = P$. To this end, we consider the following predictable real valued function on $\Omega \times [0,T] \times I\!R^* \times [0,1]$

$$Y^\lambda(\omega, s, z, \alpha) = 1 + \lambda_x v'_x(\omega, s, z, \alpha) + \lambda_y v'_y(\omega, s, z, \alpha). \tag{4.23}$$

We have

$$|Y^\lambda(\omega, s, z, \alpha) - 1| \leq |\lambda| \rho(z). \tag{4.24}$$

Then we consider the following square integrable Doléans-Dade martingale:

$$G_t^\lambda = 1 + \int_0^t \int_0^1 \int_{I\!R^*} G_{s-}^\lambda (Y^\lambda(s, z, \alpha) - 1) \tilde{N}(dz, d\alpha, ds). \tag{4.25}$$

Proposition 4.5 *G_t^λ is strictly positive for every $t \in [0,t]$. If P^λ is the probability measure defined by $P^\lambda = G_T^\lambda.P$, then $P^\lambda \circ (S^\lambda)^{-1} = P$. Furthermore,*

$$\sup_\lambda E\left[(G_T^\lambda)^2\right] < \infty \tag{4.26}$$

Proof. Recall that if

$$M_t^\lambda = \int_0^t \int_0^1 \int_{I\!R^*} (Y^\lambda(\omega, s, z, \alpha) - 1) \tilde{N}(dz, d\alpha, ds) \tag{4.27}$$

then (see Jacod-Shiryaev [15] p.59),

$$G_t^\lambda = e^{M_t^\lambda} \Pi_{s \leq t} (1 + \Delta M_s^\lambda) e^{-\Delta M \lambda_s} \tag{4.28}$$

Since by construction, $|Y^\lambda(\omega,s,z,\alpha)-1|\le \frac{1}{2}$ for $z\in I\!R^*{}_+$, the jumps of M^λ are greater than $-\frac{1}{2}$, and thus G^λ_t is strictly positive. Now, using the definition of the shift S^λ and the Girsanov theorem (see Jacod-Shiryaev [15] p.157), we see that the compensator of N under P^λ is $\Gamma^\lambda(Y^\lambda.m)$. But Y^λ has been chosen such that $\Gamma^\lambda(Y^\lambda.m)=m$. Indeed, considering a Borel set A of $[0,T]\times I\!R^*\times[0,1]$, we have

$$\Gamma^\lambda(Y^\lambda.m)(A)=\int_0^t\int_0^1\int_{I\!R^*}\mathbf{1}_A(s,\Gamma^\lambda(s,z,\alpha),\alpha)Y^\lambda(s,z,\alpha)dzd\alpha ds. \tag{4.29}$$

The substitution $z'=\Gamma^\lambda(s,z,\alpha)$ implies that $\Gamma^\lambda(Y^\lambda.m)(A)=m(A)$. Hence since the law of a Poisson point measure is characterized by its intensity, we deduce that $\mathcal{L}(N^\lambda|P^\lambda)=\mathcal{L}(N|P)$. Finally, since V_0 is independent of G^λ, it is clear that $\mathcal{L}(V_0|P^\lambda)=\mathcal{L}(V_0|P)$.
It remains to prove (4.26). Let λ be fixed. We deduce from (4.28) that

$$E[(G^\lambda_t)^2]\le 2+2\int_0^t\int_0^1\int_{I\!R^*}E\left[(G^\lambda_s)^2(Y^\lambda(s,\alpha,z)-1)^2\right]dzd\alpha ds \tag{4.30}$$

But we deduce from (4.24) and the fact that $\rho\in L^1\cap L^\infty(I\!R^*,dz)$ that

$$E[(G^\lambda_t)^2]\le 2+K\int_0^t E\left[(G^\lambda_s)^2\right]ds \tag{4.31}$$

where $K=2\int_{I\!R^*}\rho^2(z)dz$. Gronwall's lemma allows us to conclude. △

4.3 Perturbation and derivation of V_t.

In this subsection, we consider a fixed direction v, we use the notations of the previous subsection, and we study the smoothness of the map $\lambda\mapsto V^\lambda_t=V_t\circ S^\lambda$. Here the α-process W is fixed, deterministic (from the point of view of the probability space Ω), and thus behaves as a parameter.

Proposition 4.6 *Let $\lambda\in\Lambda$ be fixed. The perturbed process V^λ, defined by $V^\lambda_t=V_t\circ S^\lambda$, satisfies the following equation :*

$$V^\lambda_t=V_0+\int_0^t\int_0^1\int_{I\!R^*}\gamma(V^\lambda_{s-}-W_{s-}(\alpha),\Gamma^\lambda(s,\alpha,z))N(dz,d\alpha,ds) \tag{4.32}$$

Proof. It suffices to replace everywhere ω by $S^\lambda(\omega)$ in equation (2.15). △

We will need the following Lemma.

Lemma 4.7 *Assume (S') and (FS). For each λ, equation (4.32) admits a unique solution belonging a.s. to $I\!D([0,T],I\!R^2)$.*
We furthermore have a.s.

$$\sup_{\lambda,0\le t\le T}|V^\lambda_t|<\infty \tag{4.33}$$

We omit the proof of this lemma, because it can be done in the same way as that of the next one.

The following lemma deals with the possible derivative of V^λ_t, which should satisfy the equation obtained by differentiating formally equation (4.32).

Lemma 4.8 *Assume* (S') *and* (FS). *For each* λ, *the equation*

$$D_t^\lambda = \int_0^t \int_0^1 \int_{I\!R^*} \gamma'_X(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s,\alpha,z)) D_{s-}^\lambda N(ds,d\alpha,dz) \tag{4.34}$$

$$+ \int_0^t \int_0^1 \int_{I\!R^*} \gamma'_z(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s,\alpha,z)) v(s,\alpha,z) N(ds,d\alpha,dz)$$

admits a unique solution belonging a.s. to $I\!D([0,T], \mathcal{M}_{2\times 2}(I\!R))$.
We have furthermore almost surely

$$\sup_{\lambda, 0\le t\le T} |D_t^\lambda| < \infty \tag{4.35}$$

Remark that there is no reason why that for some λ fixed, say for $\lambda = 0$, D_T^0 belongs to L^2. The only assumption that makes D_T^0 belonging to L^2 easily is the Maxwell assumption $B(X,\theta) = \beta(\theta)$, which yields that $\gamma(X,z) = A(g(z)).X$, with g no more depending on X, and thus $\gamma'_X(X,z) = A(g(z))$. In any other case, D_T^0 behaves almost as the Doléans-Dade exponential of a pure jump process with finite variations, belonging to all the L^qs, but this does not imply that D_T^0 belongs to L^2. (One easily builds semimartingales which belong to all the L^qs, and of which the Doléans-Dade exponential is not in L^1). This is the reason why we have to use the a.s. derivatives and the weak criterion given by Theorem 4.1.

Proof. 1) We first prove the uniqueness. We will use Lemma 5.1 of the Appendix, for λ and ω fixed. Let thus λ be fixed, and let D and E be two càdlàg solutions of (4.34). A simple computation shows that

$$|D_t - E_t| \le \int_0^t \int_0^1 \int_{I\!R^*} |D_{s-} - E_{s-}| \times \left| \gamma'_X \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s,\alpha,z) \right) \right| N(ds,d\alpha,dz) \tag{4.36}$$

Since $\Gamma^\lambda(s,\alpha,z) = z + < \lambda, v(s,\alpha,z) >$, we deduce from (4.16) and (4.19) that $|\Gamma^\lambda(s,\alpha,z) - z| \le |z|/2 \wedge 1/|z|$. Hence, using (2.19) and (2.21) in (FS), we obtain the existence of a constant C such that

$$\left| \gamma'_X \left(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s,\alpha,z) \right) \right| \le C \left(1 + |V_{s-}^\lambda|^p + |W_{s-}(\alpha)|^p \right) \bar{\eta}(z) \tag{4.37}$$

We set $\tilde{\eta}(s,\alpha,z) = (1 + |W_{s-}(\alpha)|^p) \bar{\eta}(z)$. Then $\tilde{\eta}$ belongs to $L^1(ds,d\alpha,dz)$, thanks to (FS) and Lemma (4.1), and hence $\tilde{\eta}$ belongs a.s. to $L^1(N(ds,d\alpha,dz))$. We also set $a = 1 + \sup_{\lambda, s\in[0,T]} |V_{s-}^\lambda|^p$, which is a.s. finite thanks to Lemma (4.7). We finally obtain

$$|D_t - E_t| \le Ka \int_0^t \int_0^1 \int_{I\!R^*} |D_{s-} - E_{s-}| \times \tilde{\eta}(s,\alpha,z) N(ds,d\alpha,dz) \tag{4.38}$$

Applying Lemma (5.1), we finally deduce that

$$\sup_{[0,T]} |D_t - E_t| = 0 \quad a.s. \tag{4.39}$$

which was our aim.

2) We now prove the existence. We still fix λ. We first consider the simpler equation, for $\epsilon > 0$ fixed,

$$\begin{aligned}\bar{D}_t^\epsilon &= \int_0^t \int_0^1 \int_{|z|\leq 1/\epsilon} \gamma_X'(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s,\alpha,z))\bar{D}_{s-}^\epsilon N(ds,d\alpha,dz) \\ &+ \int_0^t \int_0^1 \int_{I\!R^*} \gamma_z'(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s,\alpha,z))v(s,\alpha,z)N(ds,d\alpha,dz) \qquad (4.40)\end{aligned}$$

We denote by U_t the last term of this equation. Notice that thanks to (2.19) and (2.22) in (FS), and thanks to (4.16), a.s., $\sup_{[0,T]} |U_t| \leq A$, where

$$A = \left(1 + \sup_{\lambda,u} |V_u^\lambda|^p\right) \int_0^t \int_0^1 \int_{I\!R^*} (1 + |W_{s-}(\alpha)|^p)\, \rho(z) N(ds,d\alpha,dz) \qquad (4.41)$$

is a.s. finite thanks to (4.17) and Lemma 4.7.
Since $N|_{[0,T]\times[0,1]\times\{|z|\leq 1/\epsilon\}}$ is a finite counting measure, it can be written (for each ω) as a (finite) sum of n Dirac measures at some points (T_i, α_i, z_i), and one may assume that $0 < T_1 < T_2 < ... < T_n < T$. Thus equation (4.40) can be solved by working recursively on the time intervals $[T_i, T_{i+1}[$:
for $t \in [0, T_1[$, we set $\bar{D}_t^\epsilon = U_t$
for $t \in [T_1, T_2[$, we set $\bar{D}_t^\epsilon = \gamma_X'(V_{T_1-}^\lambda - W_{T_1-}(\alpha_1), \Gamma^\lambda(T_1 s, \alpha_1, z_1)) + U_t$
and so on...
Then we have to prove that for (almost) all ω,

$$\sup_{\epsilon, t\in[0,T]} |\bar{D}_t^\epsilon| < \infty \qquad (4.42)$$

Using the same arguments and notations as in the proof of uniqueness, we obtain :

$$|\bar{D}_t^\epsilon| \leq A + Ka \int_0^t \int_0^1 \int_{I\!R^*} |\bar{D}_{s-}^\epsilon| \tilde{\eta}(s,\alpha,z) N(ds,d\alpha,dz) \qquad (4.43)$$

Lemma 5.1 allows us to conclude that

$$\sup_{[0,T]} |\bar{D}_t^\epsilon| \leq A \exp\left(\int_0^T \int_0^1 \int_{I\!R^*} \ln(1 + Ka\tilde{\eta}(s,\alpha,z))\, N(ds,d\alpha,dz)\right) \qquad (4.44)$$

and (4.42) is proved. We finally check that the family $\bar{D}^\epsilon$ is Cauchy for the supremum norm (for almost all ω fixed). Let $\epsilon < \epsilon'$ be fixed. Then

$$\begin{aligned}|\bar{D}_t^\epsilon - \bar{D}_t^{\epsilon'}| &\leq \int_0^t \int_0^1 \int_{I\!R^*} |\gamma_X'(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s,\alpha,z))||\bar{D}_{s-}^\epsilon - \bar{D}_{s-}^{\epsilon'}| N(ds,d\alpha,dz) \\ + \sup_{\epsilon, u\in[0,T]} |\bar{D}_u^\epsilon| &\times \int_0^T \int_0^1 \int_{1/\epsilon' < |z| < 1/\epsilon} |\gamma_X'(V_{s-}^\lambda - W_{s-}(\alpha), \Gamma^\lambda(s,\alpha,z))| N(ds,d\alpha,dz) \qquad (4.45)\end{aligned}$$

Still using the same notations, we obtain

$$|\bar{D}_t^\epsilon - \bar{D}_t^{\epsilon'}| \leq Ka \int_0^t \int_0^1 \int_{I\!R^*} \tilde{\eta}(s,\alpha,z) |\bar{D}_{s-}^\epsilon - \bar{D}_{s-}^{\epsilon'}| N(ds,d\alpha,dz) + Z^{\epsilon,\epsilon'} \qquad (4.46)$$

where

$$Z^{\epsilon,\epsilon'} = \sup_{\epsilon,u\in[0,T]} |\bar{D}^\epsilon_u| \times aK \int_0^T \int_0^1 \int_{1/\epsilon'<|z|<1/\epsilon} \tilde{\eta}(s,\alpha,z)N(ds,d\alpha,dz) \qquad (4.47)$$

Since $\tilde{\eta}$ belongs (a.s.) to $L^1(N)$, it is clear that when ϵ,ϵ' go to 0, $Z^{\epsilon,\epsilon'}$ goes to 0. Lemma 5.1 yields immediately that

$$\sup_{[0,T]} |\bar{D}^\epsilon_t - \bar{D}^{\epsilon'}_t| \leq B \times Z^{\epsilon,\epsilon'} \qquad (4.48)$$

where B is an a.s. finite random variable. The family $\bar{D}^{\epsilon'}_t$ is thus a.s. Cauchy for the supremum norm on $[0,T]$, and this concludes the proof of the existence.

3) We finally check (4.35). Still using the same arguments and notations, we obtain

$$|D^\lambda_t| \leq Ka \int_0^t \int_0^1 \int_{I\!R^*} \tilde{\eta}(s,\alpha,z)|D^\lambda_{s-}|N(ds,d\alpha,dz), \qquad (4.49)$$

and Lemma 5.1 allows to conclude as usual. △

Lemma 4.9 *Assume* (S') *and* (FS)*. For almost all* ω*, there exists* $A(\omega) < \infty$ *such that for all* $0 \leq t \leq T$*, all* $\lambda,\mu \in \Lambda$*,*

$$|V^\lambda_t - V^\mu_t| \leq A|\lambda-\mu| \qquad (4.50)$$

Proof. Let λ,μ be fixed. Notice that thanks to (FS), to the definition of Γ^λ, and to the properties of the direction v, there exists a constant C such that

$$\begin{aligned} &|\gamma(V^\lambda_{s-} - W_{s-}(\alpha),\Gamma^\lambda(s,\alpha,z)) - \gamma(V^\mu_{s-} - W_{s-}(\alpha),\Gamma^\mu(s,\alpha,z))| \\ &\leq C\left(1+|V^\lambda_{s-}|^p + |V^\mu_{s-}|^p + |W_{s-}(\alpha)|^p\right)\bar{\eta}(z)|V^\lambda_{s-} - V^\mu_{s-}| \\ &+C\left(1+|V^\mu_{s-}|^p + |W_{s-}(\alpha)|^p\right)|\Gamma^\lambda(s,\alpha,z) - \Gamma^\mu(s,\alpha,z)| \\ &\leq C\sup_{\lambda',u}\left(1+|V^{\lambda'}_u|^p\right) \times \left(\bar{\eta}(z)(1+|W_{s-}(\alpha)|^p) \times |V^\lambda_{s-} - V^\mu_{s-}| + |\lambda-\mu|\rho(z)\right) \\ &= Ca\left(\tilde{\eta}(s,\alpha,z)|V^\lambda_{s-} - V^\mu_{s-}| + |\lambda-\mu|\rho(z)\right) \end{aligned} \qquad (4.51)$$

where the last inequality defines some notations. As in the previous proofs, $\tilde{\eta} \in L^1(N)$ almost surely, and a is a.s. finite.
We thus deduce that

$$\begin{aligned} |V^\lambda_t - V^\mu_t| &\leq Ca\int_0^t \int_0^1 \int_{I\!R^*} |V^\lambda_{s-} - V^\mu_{s-}|\tilde{\eta}(s,\alpha,z)N(ds,d\alpha,dz) \\ &+Ca|\lambda-\mu|\int_0^T \int_0^1 \int_{I\!R^*} \rho(z)N(ds,d\alpha,dz) \end{aligned} \qquad (4.52)$$

Lemma 5.1 allows one more time to conclude. △

Proposition 4.10 *For almost all ω, the map $\lambda \mapsto V_T^\lambda$ is differentiable on Λ, and $\frac{\partial}{\partial \lambda} V_T^\lambda = D_T^\lambda$.*

Proof. the proof is very similar to that of the previous proposition. One can check the existence of an a.s. finite random variable B such that a.s., for all $0 \leq s \leq T$, all $\lambda, \mu \in \Lambda$,

$$|V_s^\lambda - V_s^\mu - D_s^\lambda(\lambda - \mu)| \leq B|\lambda - \mu|^2 \tag{4.53}$$

△

4.4 Choice of v and inversiblity of D_T^0.

We still have to check that for a good choice of v, D_T^0 is a.s. invertible. (That will provide the condition (iii) in Theorem 4.3). Recall that

$$D_t^0 = \int_0^t dX_s.D_{s-}^0 + H_t \tag{4.54}$$

where

$$X_t = \int_0^t \gamma'_X\left(V_{s-} - W_{s-}(\alpha), z\right) N(ds, d\alpha, dz) \tag{4.55}$$

and

$$H_t = \int_0^t \gamma'_z\left(V_{s-} - W_{s-}(\alpha), z\right) v(s, \alpha, z) N(ds, d\alpha, dz) \tag{4.56}$$

Using Jacod [14], we compute explicitely D_T^0. First, we denote by K_t the Doléans-Dade exponential of X : for I the unit matrix of $\mathcal{M}_{2\times 2}(I\!R)$,

$$K_t = \mathcal{E}(X)_t = I + \int_0^t dX_s.K_{s-} = \prod_{s\leq t}(I + \Delta X_s) \tag{4.57}$$

Then we consider the following sequence of stopping times :

$$S_0 = 0 \quad ; \quad S_{n+1} = \inf\left\{t \in]S_n, T] \ / \ \det(I + \Delta X_t) = 0\right\} \tag{4.58}$$

with the convention $\inf \emptyset = \infty$. Then the sequence S_n is totally inaccessible, and we have, a.s., for all n, $T \neq S_n$. Furthermore, it is clear that for all n, all $t \in]S_n, S_{n+1}[$

$$\mathcal{E}(X - X^{S_n})_t = \prod_{S_n < s \leq t} (I + \Delta X_s) \tag{4.59}$$

is invertible.
We thus know, still from [14], that if ω satisfies $S_n < T < S_{n+1} = \infty$, then

$$D_T^0 = \mathcal{E}(X - X^{S_n})_T. \left[\Delta H_{S_n} + \int_{]S_n,T]} \mathcal{E}(X - X^{S_n})_{s-}^{-1} \left(I + \Delta X_s\right)^{-1} dH_s\right] \tag{4.60}$$

We finally rewrite (4.60) explicitely :

Proposition 4.11 *For almost all ω, there exists n such that $S_n < T < S_{n+1}$, and*

$$D_T^0 = \mathcal{E}(X - X^{S_n})_T.\Big[\Delta H_{S_n} +$$

$$\int_{]S_n,T]} \int_0^1 \int_{I\!R^*} \mathcal{E}(X - X^{S_n})_{s-}^{-1} \left(I + \gamma_X' \left(V_{s-} - W_{s-}(\alpha), z\right)\right)^{-1} \gamma_z' \left(V_{s-} - W_{s-}(\alpha), z\right)$$

$$v(s, \alpha, z) N(ds, d\alpha, dz)\Big] \tag{4.61}$$

We now choose v. First of all, we denote by k a function from $\mathcal{M}_{2\times 2}(I\!R)$ into $[0,1]$ such that

$$k(M) = 0 \quad \Longleftrightarrow \quad \det M = 0 \tag{4.62}$$

and such that the map

$$M \mapsto \begin{cases} k(M)(M^{-1})^T & \text{if} \quad \det M \neq 0 \\ 0 & \text{if} \quad \det M = 0 \end{cases} \tag{4.63}$$

is of class C_b^∞ from $\mathcal{M}_{2\times 2}(I\!R)$ into itself.
We also consider a C^1 function f from $I\!R^*$ into $]0,1]$ such that for some $c \in]0,1]$

$$|f| + |f'| \leq c \quad ; \quad |f(z)| + |f'(z)| \leq |z|/2 \wedge 1/|z| \quad ; \quad |f| + |f'| \in L^1(I\!R^*, dz) \tag{4.64}$$

Definition 4.12 *We set*

$$v(s, \alpha, z) = \frac{\gamma_z' \left(V_{s-} - W_{s-}(\alpha), z\right)^T}{1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p}.$$

$$\frac{\left(I + \gamma_X' \left(V_{s-} - W_{s-}(\alpha), z\right)\right)^{-1,T} \times k\left(I + \gamma_X' \left(V_{s-} - W_{s-}(\alpha), z\right)\right)}{1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p}$$

$$.\mathcal{E}(X - X^{S_n})_{s-}^{-1,T} k(\mathcal{E}(X - X^{S_n})_{s-}) \times f(z) \tag{4.65}$$

If c is small enough, which we assume, then v is a direction in the sense of Definition 4.4.

Lemma 4.13 *Almost surely, $\Delta H_{S_n} = 0$ for all n such that $S_n < T$.*

Proof. The stopping time S_n is a time of jump of the Poisson measure. Let us denote by (α_{S_n}, z_{S_n}) the associated jump. We know, from the definition of S_n, that $\det(I + \Delta X_{S_n}) = 0$, which implies that $\det(I + \gamma_X'(V_{S_n} - W_{S_n}(\alpha_{S_n}), z_{S_n})) = 0$. Hence, thanks to the definition of v and k, we deduce that $v(S_n, \alpha_{S_n}, z_{S_n}) = 0$, which clearly implies the result. △

Remark 4.14 *(i) We deduce from the lemma above that in order to prove that D_T^0 is a.s. invertible, it suffices to check that for any n, for all ω satisfying $S_n < T < S_{n+1}$, Δ_t^n is a.s. invertible, where*

$$\Delta_T^n = \int_{]S_n,T]} \int_0^1 \int_{I\!R^*} \mathcal{E}(X - X^{S_n})_{s-}^{-1} \left(I + \gamma_X' \left(V_{s-} - W_{s-}(\alpha), z\right)\right)^{-1}$$

$$\gamma_z' \left(V_{s-} - W_{s-}(\alpha), z\right) v(s, \alpha, z) N(ds, d\alpha, dz) \tag{4.66}$$

(ii) We can also write, using the explicit expression of v,

$$\Delta_T^n = \int_{]S_n,T]} \mathcal{E}(X - X^{S_n})_{s-}^{-1}.dR_s.\mathcal{E}(X - X^{S_n})_{s-}^{-1,T} \tag{4.67}$$

where

$$R_t = \int_{]S_n,T]} \int_0^1 \int_{I\!R^*} J(V_{s-} - W_{s-}(\alpha), z) \times h(s, \alpha, z) \times f(z) N(ds, d\alpha, dz) \tag{4.68}$$

with, for $X \in I\!R^2$,

$$J(X, z) = (I + \gamma'_X(X, z))^{-1} \gamma'_z(X, z)\gamma'_z(X, z)^T (I + \gamma'_X(X, z))^{-1,T} \tag{4.69}$$

and

$$h(s, \alpha, z) = \frac{1}{(1 + |V_{s-}|^p + |W_{s-}(\alpha)|^p)^2} \times k\left(I + \gamma'_X\left(V_{s-} - W_{s-}(\alpha), z\right)\right) k(\mathcal{E}(X - X^{S_n})_{s-}) \tag{4.70}$$

For all X, z, $J(X, z)$ is a symmetric nonnegative matrix. The function h is always nonnegative. Hence R_t is nonnegative, symmetric, and increasing for the strong order. Since h does never vanish, and since $\mathcal{E}(X - X^{S_n})_{s-}^{-1}$ is invertible for all $s \in]S_n, T]$, it suffices to prove that a.s., for all $0 \le s < t \le T$, $\bar{R}_t - \bar{R}_s$ is invertible, where

$$\bar{R}_t = \int_{]S_n,T]} \int_0^1 \int_{I\!R^*} J(V_{s-} - W_{s-}(\alpha), z) \times f(z) N(ds, d\alpha, dz) \tag{4.71}$$

We finally prove

Proposition 4.15 *With our choice of v, D_T^0 is a.s. invertible.*

Proof. We of course use the previous remark. The proof necessitates several steps.

Step 1 : Let X and Y be two non zero vectors of $I\!R^2$. Then

$$\int_{I\!R^*} 1_{\{Y^T\gamma'_z(X,z)\gamma'_z(X,z)^TY \neq 0\}} dz = \infty \tag{4.72}$$

To prove this, we first set $I(X, z) = \gamma'_z(X, z)\gamma'_z(X, z)^T$. Notice that, by definition of γ,

$$I(X, z) = (g'_z(X, z))^2 A'(g(X, z))XX^T A'(g(X, z))^T \tag{4.73}$$

But it is clear, see Section 2, that g'_z does never vanish. Hence, thanks to the substitution $\theta = g(X, z)$, we obtain (see Section 2 again)

$$\int_{I\!R^*} 1_{\{Y^T I(X,z)Y \neq 0\}} dz = \int_{-\pi}^{\pi} 1_{\{Y^T A'(\theta)XX^T A'(\theta)^T Y \neq 0\}} B(X, \theta) d\theta \tag{4.74}$$

But a simple computation shows that $Y^T A'(\theta)XX^T A'(\theta)^T Y$ does $1_{[-\pi,\pi]}(\theta)d\theta$-almost never vanish (for $X \neq 0$ and $Y \neq 0$ fixed). Since $\int_{-\pi}^{\pi} B(X, \theta)d\theta = \infty$, the proof of Step 1 is finished.

Step 2 : For all $s \in [0,T]$, for almost all ω,

$$\int_0^1 1_{\{V_{s-}-W_{s-}(\alpha)\neq 0\}} d\alpha > 0 \tag{4.75}$$

Indeed, we know from assumption (FS) (ii) and Lemma 4.2 that $\mathcal{L}(V_s)$, and thus $\mathcal{L}_\alpha(W_s)$ is not a Dirac mass. Hence, for any deterministic $X \in I\!R^2$,

$$\int_0^1 1_{\{X-W_{s-}(\alpha)\neq 0\}} d\alpha = P_\alpha(W_s \neq X) > 0 \tag{4.76}$$

Since ω is fixed, $V_{s-}(\omega)$ is "α-deterministic", and hence (4.76) holds for $X = V_{s-}(\omega)$, which drives immediately to (4.75).

Step 3 : Associating Steps 1 and 2, we finally deduce : for all non-zero vector $Y \in I\!R^2$, all $s \in [0,T]$, a.s.,

$$\int_0^1 \int_{I\!R^*} 1_{\{Y^T \gamma'_z(V_{s-}-W_{s-}(\alpha)X,z)\gamma'_z(V_{s-}-W_{s-}(\alpha),z)^T Y \neq 0\}} d\alpha dz = \infty \tag{4.77}$$

Step 4 : Let $s > 0$ and $Y \in I\!R^2/\{0\}$ be fixed. We now prove that on the set $S_n < T < S_{n+1} = \infty$, for all $s > S_n$ a.s. for all $t \in]s,T]$,

$$Y^T(\bar{R}_t - \bar{R}_s)Y > 0 \tag{4.78}$$

To this end, we introduce the following stopping time.

$$\tau(Y) = \inf\left\{ u > s \;\Big/\; \int_s^u \int_{I\!R^*} \int_0^1 1_{\{Y^T J(V_{s-}-W_{s-}(\alpha)X)Y > 0\}} N(ds,d\alpha,dz) > 0 \right\} \tag{4.79}$$

We just have to check that a.s., $\tau(Y) = 0$. We have, by construction, a.s.,

$$\int_s^{\tau(Y)} \int_{I\!R^*} \int_0^1 1_{\{Y^T J(V_{s-}-W_{s-}(\alpha)X)Y > 0\}} N(ds,d\alpha,dz) \leq 1 \tag{4.80}$$

Taking the expectation in this expression, we obtain

$$E\left(\int_s^{\tau(Y)} \int_{I\!R^*} \int_0^1 1_{\{Y^T J(V_{s-}-W_{s-}(\alpha)X)Y > 0\}} ds d\alpha dz \right) \leq 1 \tag{4.81}$$

and, we deduce that a.s.,

$$\int_s^{\tau(Y)} \int_{I\!R^*} \int_0^1 1_{\{Y^T J(V_{s-}-W_{s-}(\alpha)X)Y > 0\}} ds d\alpha dz < \infty \tag{4.82}$$

Due to (4.77), this is not possible, except if $\tau(Y) = 0$ a.s.

Step 5 : The previous step shows that on the set $S_n < T < S_{n+1} = \infty$, for all $s \in]S_n,T]$, a.s., for all $u \in]s,T]$, $\bar{R}_u - \bar{R}_s$ is invertible.
What we have to prove is that on the set $S_n < T < S_{n+1} = \infty$, for all a.s., $s \in]S_n,T]$, for all $u \in]s,T]$, $\bar{R}_u - \bar{R}_s$ is invertible.
This extension is not hard, by using the fact that $\bar{R}$ is increasing.

The proof is complete

$\triangle$

4.5 Conclusion.

We finally are able to conclude the

Proof of Theorem 2.11. Since $T > 0$ is arbitrarily fixed, it of course suffices to prove that the law of V_T admits a density. We thus apply Theorem 4.3 with $X = V_T$. The family X^λ is defined by $V_T^\lambda = V_T \circ S^\lambda$, the shift S^λ being defined by (4.22), relatively to the direction v chosen in Definition 4.12. Condition (i) of Theorem 4.3 is satisfied thanks to Proposition 4.5. Condition (ii) holds thanks to Proposition 4.10. Finally, Proposition 4.15 shows that condition (iii) is met. Hence the law of V_T admits a density, which was our aim. $\triangle$

5 Appendix.

Our purpose is to prove the following Gronwall type lemma.

Lemma 5.1 *Let $\mathcal{X}$ be a measurable space. We consider a counting σ-finite measure $\mu(dt, dx)$ on $[0, T] \times \mathcal{X}$. Let $\eta(s, x)$ be a positive function belonging to $L^1(\mu)$. Then every bounded positive function φ_t on $[0, T]$, such that for all $t > 0$,*

$$\varphi_t \leq a + \int_0^t \int_{\mathcal{X}} \varphi_{s-} \eta(s, x) \mu(ds, dx) \tag{5.1}$$

is bounded by

$$\sup_{[0,T]} \varphi_t \leq a \exp\left(\int_0^T \int_{\mathcal{X}} \ln(1 + \eta(s, x)) \mu(ds, dx) \right) \tag{5.2}$$

Proof. We divide the proof in two steps.

<u>Step 1</u> : We begin with the case where $\mu(\eta \neq 0) < \infty$. In this case, we can consider that the support of μ is finite, and thus that μ is of the form $\sum_{i=1}^n \delta_{(T_i, X_i)}$, with $0 < T_1 < T_2 < ... < T_n < T$. Then we use (5.1). First, for all $t < T_1$,

$$\varphi_t \leq a \tag{5.3}$$

from which we deduce, for all $t \in [T_1, T_2[$,

$$\varphi_t \leq a + a\eta(T_1, X_1) \leq a(1 + \eta(T_1, X_1)) \tag{5.4}$$

which clearly also holds for all $t \in [0, T_2[$. And so on... We finally obtain that for all $t \in [0, T]$,

$$\begin{aligned} \varphi_t &\leq a(1 + \eta(T_1, X_1)) \times ... \times (1 + \eta(T_n, X_n)) \\ &\leq a \exp\left(\sum_{i=1}^n \ln(1 + \eta(T_i, X_i)) \right) \\ &\leq a \exp\left(\int_0^T \int_{\mathcal{X}} \ln(1 + \eta(s, x)) \mu(ds, dx) \right) \end{aligned} \tag{5.5}$$

which was our aim.

Step 2 : If $\mu(\eta \neq 0) < \infty$, then we split the space $\mathcal{X} = \mathcal{X}_\epsilon \cup \mathcal{X}_\epsilon^c$, in such a way that for all ϵ, $\mu([0,T] \times \mathcal{X}_\epsilon) < \infty$, and such that $\mathcal{X}_\epsilon$ grows to $\mathcal{X}$ when ϵ goes to 0. Then we rewrite (5.1) as

$$\varphi_t \leq (a + u_\epsilon) + \int_0^t \int_{\mathcal{X}_\epsilon} \varphi_{s-} \eta(s,x) \mu(ds,dx) \tag{5.6}$$

where

$$u_\epsilon) = \| \varphi \|_\infty \int_0^t \int_{\mathcal{X}_\epsilon^c} \eta(s,x) \mu(ds,dx) \tag{5.7}$$

clearly goes to 0 since $\eta \in L^1(\mu)$. Applying Step 1, we obtain for each ϵ

$$\sup_{[0,T]} \varphi_t \leq (a + u_\epsilon) \exp\left(\int_0^T \int_{\mathcal{X}_\epsilon} \ln(1 + \eta(s,x)) \mu(ds,dx) \right) \tag{5.8}$$

Making ϵ tend to 0 drives immediately to the conclusion. △

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