Solving Landau equation for some soft potentials through a probabilistic approach

Hélène Guérin*

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Abstract

This article deals with a way to solve the spatially homogeneous Landau equation using probabilistic tools. Thanks to the study of a nonlinear stochastic differential equation driven by a space-time white noise, we state the existence of a measure solution of the Landau equation with a probability measure initial data, for a generalization of the Maxwellian molecules case. Then, by approximation of the Landau coefficients, the first result helps us to state the existence of a measure solution for some soft potentials ($\gamma \in (-1,0)$). This second part is based on the use of nonlinear stochastic differential equations and some martingale problems.

Key Words : Landau Equation, White Noise, Nonlinear Stochastic Differential Equation, Nonlinear Martingale Problems.
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1 Introduction

The Landau equation is obtained as a limit of the Boltzmann equation, when all the collisions become grazing. In the spatially homogeneous case, it writes:

$$\frac{\partial f}{\partial t}(v,t) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d} dv_* a_{ij}(v-v_*)\left[f(v_*,t)\frac{\partial f}{\partial v_j}(v,t) - f(v,t)\frac{\partial f}{\partial v_{*j}}(v_*,t)\right]\right\} \tag{1}$$

where $f(v,t) \geq 0$ is the density of particles having the velocity $v \in \mathbb{R}^d$ at time $t \in \mathbb{R}^+$, and $(a_{ij}(z))_{1\leq i,j\leq d}$ is a nonnegative symmetric matrix depending on the interaction between the particles.

This equation is also called the Fokker-Planck-Landau equation. Arsen'ev and Buryak (see [1]) have shown that the solutions of the Boltzmann equation converge toward the solutions of the Landau equation when grazing collisions prevail. On that topic, one can read the paper of Villani ([10]), which gives a lot of references.

If we assume, for example, that any two particles at distance r interact with a force proportional to $\frac{1}{r^s}$, the matrix a has the following expression, up to a multiplicative constant,

$$a_{ij}(z) = |z|^{\gamma+2}\,\Pi_{ij}(z)$$

where

- $|z|$ is the euclidean norm of z in $\mathbb{R}^d$,

*Université Paris 10, MODALX, UFR SEGMI, 200 avenue de la République, 92000 Nanterre, hguerin@ccr.jussieu.fr

- $\Pi(z)$ is the orthogonal projection on $z^{\perp}$ $(z \neq 0)$, i.e. $\Pi_{ij}(z) = \delta_{ij} - \dfrac{z_i z_j}{|z|^2}$,

- $\gamma = \dfrac{s-(2d-1)}{s-1}$.

The Landau equation has a physical sense when $d=3$. However, we will prove some results in more general cases ($d \geq 1$). Moreover, in this paper, we will consider a matrix a of the form:

$$a_{ij}(z) = |z|^{\gamma+2} \Pi_{ij}(z) h\left(|z|^2\right)$$

where h is a bounded nonnegative continuous function. We define

$$b_i(z) = \sum_{j=1}^{d} \partial_j a_{ij}(z)$$

So by integration by parts, for any test function φ, we can write a weak formulation of the Landau equation, at least formally (see [10]),

$$\begin{aligned} \frac{d}{dt}\int \varphi(v) f(v,t)\, dv &= \frac{1}{4}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(v,t) f(v_*,t)\, a_{ij}(v-v_*)\left(\partial_{ij}\varphi(v) + \partial_{ij}\varphi(v_*)\right) \\ &+ \frac{1}{2}\sum_{i=1}^{d}\int_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(v,t) f(v_*,t)\, b_i(v-v_*)\left(\partial_i\varphi(v) - \partial_i\varphi(v_*)\right) \quad (2) \end{aligned}$$

where $\partial_i \varphi = \dfrac{\partial \varphi}{\partial v_i}$ and $\partial_{ij}\varphi = \dfrac{\partial^2 \varphi}{\partial v_i \partial v_j}$.

The properties of the equation depend heavily on γ :

- $\gamma > 0$, one speaks of hard potentials,
- $\gamma = 0$ corresponds to the case of Maxwellian molecules,
- $\gamma < 0$, one speaks of soft potentials,
- $\gamma = -3$ corresponds to the Coulomb interaction.

C. Villani studies carefully the Landau equation for Maxwellian molecules in [11]. L. Desvillettes and C. Villani prove in [3] the existence of solution, in a weak sense, for hard potentials under some conditions on the initial data. Little is known about soft potentials, we can mention the work of C. Villani, [10], and the one of T. Goudon, [6]. Those two independent articles prove the existence of a weak function solution of the Landau equation when $\gamma \in (-2,0)$ and when the initial data is a nonnegative function with finite mass, energy and entropy, using the convergence of the solutions of the Boltzmann equation toward the solutions of the Landau equation.

Our paper deals with an original probabilistic way to solve the spatially homogeneous Landau equation for $\gamma \in (-1,0]$. Thanks to this method, we can assume weaker conditions on the initial data than in the previous articles.

Remark 1 *Choosing $\varphi(v) = 1, v_i$, or $\dfrac{|v|^2}{2}$, we can easily check that the mass, the momentum and the kinetic energy are conserved.*

So, if we suppose that $\int_{\mathbb{R}^d} f(v,0)\,dv = 1$, we can define the probability flow $(P_t)_{t\geq 0}$ by

$$P_t(dv) = f(v,t)\,dv$$

Since the functions $z \longmapsto a_{ij}(z)$ and $z \longmapsto b_i(z)$ are respectively even and odd for any i,j, we obtain a new expression of the Landau weak formulation, which will be the base of our study,

$$\begin{aligned}\frac{d}{dt}\int \varphi(v)\,P_t(dv) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\,a_{ij}(v-v_*)\right)\partial_{ij}\varphi(v) \\ &\quad+\sum_{i=1}^{d}\int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\,b_i(v-v_*)\right)\partial_i\varphi(v)\end{aligned} \tag{3}$$

Definition 2 *Let P_0 be a probability measure on $\mathbb{R}^d$ with a finite two-order moment (i.e. $\int_{\mathbb{R}^d}|v|^2 P_0(dv) < \infty$). A measure solution of the Landau equation (3) with initial data P_0 is a probability flow $(P_t)_{t\geq 0}$ on $\mathbb{R}^d$ satisfying (3) for any function $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d,\mathbb{R})$, where $\mathcal{C}_b^2(\mathbb{R}^d,\mathbb{R})$ is the space of bounded functions of class $\mathcal{C}^2$ on $\mathbb{R}^d$ with bounded derivatives.*

Remark 3 *With an abuse of notation, we will still say that a probability measure P on $\mathcal{C}([0,T],\mathbb{R}^d)$ is a measure solution of the Landau equation when its time-marginals flow is a measure solution in the sense of Definition 2.*

There are two ways to solve the equation (3) in a probabilistic sense. The first consists in finding a probability measure P which satisfies a nonlinear martingale problem. Funaki solves in [5] this martingale problem when the matrix a is a nondegenerate matrix. But, the coefficients of the Landau equation are degenerated. The second way consists in associating with the Landau equation (3) a nonlinear stochastic differential equation driven by a space-time white noise. Those two methods are in relation. Indeed, a solution of the differential equation is a solution of the martingale problem and a solution of the martingale problem is a weak solution of the differential equation (see the article of N. El Karoui and S. Méléard, [4]).

The benefit of the second method is that one can develop a Malliavin calculus to state the existence of a density and then to show the existence of a weak function-solution of the Landau equation (2). If, for any $t > 0$, P_t has a density with respect the Lebesgue measure on $\mathbb{R}^d$, i.e. there exists a nonnegative function $f(.,t)$ such that $P_t(dv) = f(v,t)\,dv$, then f is a weak function-solution of the Landau equation (2). This question is studied in [7].

In this paper, we are firstly interested in solving the Landau equation with regular coefficients (for example, $\gamma = 0$ and $h =$ constant). In this case, we solve a nonlinear differential stochastic equation driven by a white noise to find a measure solution of the Landau equation.

Secondly, using the results obtained in the first part, we study the Landau equation with $\gamma \in (-1,0]$ and h some bounded continuous function. We approximate the coefficients by some coefficients having the same regularity as in the first part. Then, thanks to the study of martingale problems and of nonlinear stochastic differential equations, we state the existence of a measure solution of the Landau equation with $\gamma \in (-1,0]$. Moreover, we obtain a weak solution for the associated nonlinear stochastic differential equation.

Notations

- $\mathcal{C}([0,T],\mathbb{R}^d)$ is the space of continuous functions from $[0,T]$ to $\mathbb{R}^d$, and for $k \in \mathbb{N}$, $\mathcal{C}_b^k([0,T],\mathbb{R}^d)$ is the space of functions of class $\mathcal{C}^k$ with all its derivatives bounded up to order k.
- $\mathcal{M}_{d,d'}(\mathbb{R})$ is the set of $d \times d'$ matrix on $\mathbb{R}$.

- If $(P^n)_{n\geq 0}$ and P are probability measures, we denote by $P^n \Longrightarrow P$ the convergence in distribution of the sequence (P^n) toward P.

- K is an arbitrary notation for a constant (K can change from line to line).

We consider, as it was mentioned above, a matrix a which has the following form:

$$a_{ij}(z) = |z|^{\gamma+2} h\left(|z|^2\right)\left(\delta_{ij} - \frac{z_i z_j}{|z|^2}\right) \tag{4}$$

with h some bounded nonnegative continuous function on $\mathbb{R}_+$ and $\gamma \in (-1,0]$. Then, the vector b has the following expression

$$\begin{aligned} b_i(z) &= \sum_{j=1}^{d} \partial_j a_{ij}(z) \\ &= -(d-1)\, h\left(|z|^2\right)|z|^\gamma z_i \end{aligned} \tag{5}$$

For example, in dimension 2, a and b are given by

$$\begin{aligned} a(z) &= |z|^\gamma h\left(|z|^2\right)\begin{bmatrix} {z_2}^2 & -z_1 z_2 \\ -z_1 z_2 & {z_1}^2 \end{bmatrix} \\ b(z) &= -|z|^\gamma h\left(|z|^2\right)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned}$$

and in dimension 3, they are given by

$$\begin{aligned} a(z) &= |z|^\gamma h\left(|z|^2\right)\begin{bmatrix} {z_2}^2 + {z_3}^2 & -z_1 z_2 & -z_1 z_3 \\ -z_1 z_2 & {z_1}^2 + {z_3}^2 & -z_2 z_3 \\ -z_1 z_3 & -z_2 z_3 & {z_1}^2 + {z_2}^2 \end{bmatrix} \\ b(z) &= -2\,|z|^\gamma h\left(|z|^2\right)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \end{aligned}$$

As a is a symmetric nonnegative matrix, there exists a matrix σ in $\mathcal{M}_{d,d'}(\mathbb{R})$ such that

$$a = \sigma.\sigma^* \tag{6}$$

where σ^* is the adjoint matrix of σ and d' is an integer≥ 1. There is not uniqueness of σ, one can take for example

$$\sigma(z) = \frac{1}{|z|^{\frac{\gamma}{2}+1}\sqrt{h\left(|z|^2\right)}}\, a(z) \tag{7}$$

($\Pi(z)$ *is a projection, then* $a(z).a(z) = |z|^{\gamma+2} h\left(|z|^2\right) a(z)$), or in dimension two

$$\sigma(z) = |z|^{\frac{\gamma}{2}}\sqrt{h\left(|z|^2\right)}\begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} \tag{8}$$

and in dimension three

$$\sigma(z) = |z|^{\frac{\gamma}{2}}\sqrt{h\left(|z|^2\right)}\begin{bmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{bmatrix} \tag{9}$$

If we denote by c a constant > 0 such that

$$\forall z \in \mathbb{R}^d \; h\left(|z|^2\right) \leq c$$

one can notice that

$$\begin{array}{rcl} |a(z)| & \leq & c\,|z|^{\gamma+2} \\ |b(z)| & \leq & (d-1)\,c\,|z|^{\gamma+1} \end{array}$$

and in the previous examples

$$|\sigma(z)| \leq \sqrt{c}\,|z|^{\frac{\gamma}{2}+1}$$

2 The Landau equation with regular coefficients

2.1 A nonlinear stochastic differential equation associated with the Landau equation

We associate with the Landau equation a nonlinear stochastic differential equation driven by a space-time white noise which gives a probabilistic interpretation of the Landau equation (3). We highlight the nonlinearity using two probability spaces.

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space and $([0,1], \mathcal{B}([0,1]), d\alpha)$ be an auxiliary probability space, where $d\alpha$ is the Lebesgue measure on $[0,1]$.

The Skorohod Theorem (see [9]) links up those two spaces : it states that for any probability measure P on the polish space $\mathcal{C}\left([0,T], \mathbb{R}^d\right)$, with the topology of the uniform convergence, there exists a random variable $Y : ([0,1], \mathcal{B}([0,1]), d\alpha) \to \mathcal{C}\left([0,T], \mathbb{R}^d\right)$ which has the distribution P.

For the clarity of the exposition, we will denote by E the expectation and $\mathcal{L}$ the distribution of a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and E_α, $\mathcal{L}_\alpha$ for a random variable on $([0,1], \mathcal{B}([0,1]), d\alpha)$.

For $k \geq 2$, we define $\mathcal{P}_k$ the space of continuous adapted processes $X = (X_t)_{t\geq 0}$ from $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ to $\mathbb{R}^d$, such that $\forall T > 0$

$$E\left[\sup_{0\leq t\leq T} |X_t|^k\right] < \infty$$

and $\mathcal{P}_{k,\alpha}$ the space of continuous processes $Y = (Y_t)_{t\geq 0}$ from $([0,1], \mathcal{B}([0,1]), d\alpha)$ to $\mathbb{R}^d$, such that $\forall T > 0$

$$E_\alpha\left[\sup_{0\leq t\leq T} |Y_t|^k\right] < \infty$$

Let $T > 0$ be arbitrary fixed. ρ_T denotes the Vaserstein metric on the space of probability measure on $\mathcal{C}\left([0,T], \mathbb{R}^d\right)$ defined by

$$\rho_T^2(P,Q) = \inf\left\{E\left(\sup_{0\leq t\leq T} |A_t - B_t|^2\right) : \begin{array}{c} A \text{ and } B \text{ processes on } \mathcal{C}\left([0,T], \mathbb{R}^d\right) \\ \text{with distribution } P \text{ and } Q \text{ respectively} \end{array}\right\}$$

We define the d'-dimensional process $W^{d'}$

$$W^{d'} = \begin{pmatrix} W_1 \\ \vdots \\ W_{d'} \end{pmatrix}$$

where the W_i are independent space-time white noises with covariance measure $d\alpha dt$ on $[0,1] \times [0,\infty)$ (according to Walsh's Definition, [12]).

Let X_0 be a random vector on $\mathbb{R}^d$ independent of $W^{d'}$ with a finite moment of order 2.

Let σ and b be the functions defined by (6) and (5) respectively.

We consider the following nonlinear stochastic differential equation

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds \qquad (NSDE(\sigma, b))$$

with $\mathcal{L}(X_t) = \mathcal{L}_\alpha(Y_t)\ \forall t \geq 0$.

Proposition 4 *Let P_0 be a probability measure with a finite moment of order 2 . Let X_0, Y_0 be random variables such that $\mathcal{L}(X_0) = \mathcal{L}_\alpha(Y_0) = P_0$. If we assume that there exists a solution (X, Y) of $(NSDE(\sigma, b))$, in $\mathcal{P}_2 \times \mathcal{P}_{2,\alpha}$, with initial data (X_0, Y_0), such that $\forall t\ \ \mathcal{L}(X_t) = \mathcal{L}_\alpha(Y_t)$. Then the common flow $(P_t)_{t\geq 0}$ is a measure solution of the Landau equation with initial data P_0.*

Proof. Let $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$.

Using Itô's formula, we obtain

$$\begin{aligned}
\varphi(X_t) &= \varphi(X_0) + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \int_0^1 \partial_{ij}\varphi(X_s)\, a_{ij}(X_s - y)\, P_s(dy)\, ds \\
&\quad + \sum_{i=1}^d \int_0^t \int_0^1 \partial_i\varphi(X_s)\, b_i(X_s - y)\, P_s(dy)\, ds \\
&\quad + \sum_{i=1}^d \sum_{k=1}^{d'} \int_0^t \int_0^1 \sigma_{i,k}(X_s - Y_s(\alpha))\, \partial_i\varphi(X_s)\, W_k(d\alpha, ds)
\end{aligned}$$

According to Theorem 2.5 in [12], $\forall i, k\ \ \int_0^t \int_0^1 \sigma_{i,k}(X_s - Y_s(\alpha))\, \partial_i\varphi(X_s)\, W_k(d\alpha, ds)$ is a martingale. So the expectation of $\varphi(X_t)$ satisfies

$$\begin{aligned}
E[\varphi(X_t)] &= E[\varphi(X_0)] + \frac{1}{2}\sum_{i,j=1}^d \int_0^t E\left[\partial_{ij}\varphi(X_s)\left(\int_{\mathbb{R}^d} a_{ij}(X_s - y)\, P_s(dy)\right)\right] ds \\
&\quad + \sum_{i=1}^d \int_0^t E\left[\partial_i\varphi(X_s)\left(\int_{\mathbb{R}^d} b_i(X_s - y)\, P_s(dy)\right)\right] ds
\end{aligned}$$

Since $\mathcal{L}(X_t) = P_t\ \forall t \geq 0$, the proposition is proved. ■

Consequently, it is enough to solve the nonlinear stochastic differential equation to find a measure solution of the Landau equation.

2.2 Solving a nonlinear stochastic differential equation driven by a white noise

We use the same notations as in part 2.1.

Definition 5 *Let η and f be two continuous functions. Let $W^{d'}$ be a process on $\mathbb{R}^{d'}$ having independent white noises components on $[0,1] \times [0,+\infty)$ with covariance measure $d\alpha dt$ and X_0 be a random variable with finite moment of order 2. We consider Y_0 a random variable on $([0,1], \mathcal{B}([0,1]), d\alpha)$ such that $\mathcal{L}_\alpha(Y_0) = \mathcal{L}(X_0)$. We will say that a couple (X, Y) is solution of the nonlinear stochastic differential $(NSDE(\eta, f))$ if for any $t \geq 0$*

$$X_t = X_0 + \int_0^t \int_0^1 \eta\left(X_s - Y_s(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s - Y_s(\alpha)\right) d\alpha ds$$

and $\mathcal{L}(X) = \mathcal{L}_\alpha(Y)$.

We state the existence of a solution of $(NSDE\,(\eta, f))$ under some conditions on the regularity of the functions η and f:

Assumption (H): η and f are globally lipschitz continuous functions from $\mathbb{R}^d$ respectively to $\mathcal{M}_{d,d'}(\mathbb{R})$ and to $\mathbb{R}^d$, where d and d' are integers ≥ 1.

To simplify the expressions, we consider in this part $\mathbf{d} = \mathbf{d'} = \mathbf{1}$. Nevertheless the same arguments can be applied when the dimensions are higher.

The following method, based on a stochastic calculus for a white noise, is a variation of the method built by L. Desvillettes, C. Graham and S. Méléard in [2] in the different case of Poisson measure.

Definition 6 *Let W be a space-time white noise with covariance measure $d\alpha dt$ on $[0,1]\times[0,+\infty)$, X_0 an independent random variable with finite 2-order moment, Z a $\mathcal{P}_2$-process and Y a $\mathcal{P}_{2,\alpha}$-process. The following equation*

$$X_t = X_0 + \int_0^t \int_0^1 \eta\left(Z_s - Y_s(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(Z_s - Y_s(\alpha)\right) d\alpha ds \tag{10}$$

defines an application Φ

$$Z, Y, X_0, W \longmapsto X = \Phi\left(Z, Y, X_0, W\right).$$

We first state a technical lemma:

Lemma 7 *If X_0 and W are such as in Definition 6. For $i = 1, 2$, we consider the processes $Z^i \in \mathcal{P}_2$ and $Y^i \in \mathcal{P}_{2,\alpha}$. We define $X^i = \Phi\left(Z^i, Y^i, X_0, W\right)$, $i = 1, 2$. Then $X^i \in \mathcal{P}_2$. Moreover, for any $T > 0$, there exists a constant $K > 0$ such that*

$$E\left[\sup_{0\leq t\leq T}\left|X_t^1 - X_t^2\right|^2\right] \leq K\left\{\int_0^T E\left[\left|Z_s^1 - Z_s^2\right|^2\right] ds + \int_0^T E_\alpha\left[\left|Y_s^1 - Y_s^2\right|^2\right] ds\right\}$$

Proof. It is clear that the processes $\left(X_t^i\right)_{t\geq 0}$ are continuous.

Let $T > 0$. Using the Burkholder-Davis-Gundy and the Hölder inequalities, we obtain that

$$\begin{aligned} E\left[\sup_{0\leq t\leq T}\left|X_t^1 - X_t^2\right|^2\right] \leq\ & 2\left\{CE\left(\int_0^T \int_0^1 \left[\eta\left(Z_s^1 - Y_s^1(\alpha)\right) - \eta\left(Z_s^2 - Y_s^2(\alpha)\right)\right]^2 d\alpha ds\right)\right. \\ & \left. + TE\left(\int_0^T \int_0^1 \left[f\left(Z_s^1 - Y_s^1(\alpha)\right) - f\left(Z_s^2 - Y_s^2(\alpha)\right)\right]^2 d\alpha ds\right)\right\} \end{aligned}$$

Since η and f are lipschitz continuous, if we denote by K_η and K_f their lipschitz constant respectively, we have

$$E\left[\sup_{0\leq t\leq T}\left|X_t^1 - X_t^2\right|^2\right] \leq 4\left(CK_\eta^2 + TK_f^2\right)\left\{\int_0^T E\left(\left|Z_s^1 - Z_s^2\right|^2\right) ds + \int_0^T E_\alpha\left(\left|Y_s^1 - Y_s^2\right|^2\right) ds\right\}$$

The lemma is proved. ■

We give now a solving method for a linear stochastic differential equation.

Theorem 8 *Assume that W is a space-time white noise with covariance measure $d\alpha dt$ on $[0,1]\times[0,\infty)$, X_0 is an independent random variable with finite 2-order moment and Y a $\mathcal{P}_{2,\alpha}$-process. If η and f satisfy the Assumption (H), the equation*

$$X_t = X_0 + \int_0^t \int_0^1 \eta\left(X_s - Y_s(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s - Y_s(\alpha)\right) d\alpha ds \tag{11}$$

has a unique strong solution X belonging to $\mathcal{P}_2$.

Proof. We prove the existence of a solution of (11) which belongs to $\mathcal{P}_2$ using a standard method of approximation of the solution by the following Picard sequence, for any $t \geq 0$

$$
\begin{aligned}
X_t^0 &= X_0 \\
X_t^{n+1} &= X_0 + \int_0^t \int_0^1 \eta\left(X_s^n - Y_s(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s^n - Y_s(\alpha)\right) d\alpha ds
\end{aligned}
$$

(The proof is easy and can be adapted from the proof of theorem 10.)

Moreover, using Gronwall's Lemma, we state the strong uniqueness on $[0, T]$ for any $T > 0$. ■

Remark 9 *Let X be a solution of the linear stochastic differential equation*

$$
X_t = X_0 + \int_0^t \int_0^1 \eta\left(X_s - Y_s(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s - Y_s(\alpha)\right) d\alpha ds \tag{12}
$$

The strong uniqueness of X implies as usual the uniqueness in law. Moreover, the distribution of X depends only on the distribution of Y through its flow $\left(\mathcal{L}_\alpha(Y_t)\right)_{t\geq 0}$.

Proof. We define, for any $t \geq 0$, the flow $P_t = \mathcal{L}_\alpha(Y_t)$ and a martingale measure W^P on $[0,1] \times [0, +\infty)$ such that $\forall A \in \mathcal{B}([0,1]), \forall t \geq 0$,

$$
W_t^P(A) = \int_0^t \int_0^1 \mathbb{I}_A\left(Y_s(\alpha)\right) W(d\alpha, ds)
$$

We notice that

$$
\begin{aligned}
\mathcal{L}\left(W_t^P(A)\right) &= \mathcal{N}\left(0, \int_0^t \int_0^1 \mathbb{I}_A\left(Y_s(\alpha)\right) d\alpha ds\right) \\
&= \mathcal{N}\left(0, \int_0^t \int_{\mathbb{R}} \mathbb{I}_A(v) P_s(dv) ds\right)
\end{aligned}
$$

where $\mathcal{N}(\lambda, k)$ is the Normal distribution with expectation λ and variance k. Moreover, if $A \cap B = \emptyset$, we have $W_t^P(A \cup B) = W_t^P(A) + W_t^P(B)$.

So W^P is a white noise with covariance measure $P_s(dv)\,ds$ (according to Walsh's Definition, [12]). Then, we can rewrite (12) in the following way

$$
X_t = X_0 + \int_0^t \int_{\mathbb{R}} \eta(X_s - y) W^P(dy, ds) + \int_0^t \int_{\mathbb{R}} f(X_s - y) P_s(dy) ds
$$

A white noise is entirely defined by its covariance measure, and the one of W^P is $\nu(dy, ds) = P_s(dy)\,ds$. Consequently, the distribution of X depends only on the distribution of Y through its flow $\left(\mathcal{L}_\alpha(Y_t)\right)_{t\geq 0}$. ■

We now study the nonlinear stochastic differential equation $(NSDE(\eta, f))$.

Theorem 10 *Assume that $W^{d'}$ is a process on $\mathbb{R}^{d'}$ having independent white noises components on $[0,1] \times [0, +\infty)$ with covariance measure $d\alpha dt$, and assume that X_0 is an independent random vector on $\mathbb{R}^d$ with finite moment of order 2. Then, under the Assumption (H), there exists a couple (X, Y) solution of the nonlinear equation $(NSDE(\eta, f))$. Moreover, $(X, Y) \in \mathcal{P}_2 \times \mathcal{P}_{2,\alpha}$.*

We notice that the distribution of X depends only on the distribution $P_0 = \mathcal{L}(X_0)$ and not on the specific choice of the white noise and of X_0.

Proof. We prove this theorem in dimension $d = d' = 1$. The proof is almost the same in higher dimension if we work with each component, but the expressions are more complex.

We now use a generalization of the Picard iteration method. We construct two recursive sequences:

- Let X^0 such that $\forall s \geq 0$ $X_s^0 = X_0$ and Y^0 such that $\forall s \geq 0$ $Y_s^0 = Y_0$, where Y_0 is a random variable on $([0,1], \mathcal{B}([0,1]), d\alpha)$ such that $\mathcal{L}_\alpha(Y_0) = \mathcal{L}(X_0)$ (obtained by Skorohod's Theorem).

- We define

$$X_t^{n+1} = X_0 + \int_0^t \int_0^1 \eta\left(X_s^n - Y_s^n(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s^n - Y_s^n(\alpha)\right) d\alpha ds$$

  On the probability space $([0,1], \mathcal{B}([0,1]), d\alpha)$, we construct a continuous process Y^{n+1} such that

$$\mathcal{L}_\alpha\left(Y^{n+1} \mid Y^0, .., Y^n\right) = \mathcal{L}\left(X^{n+1} \mid X^0, .., X^n\right)$$

  In particular, we have for any $n \geq 0$

$$\mathcal{L}_\alpha\left(Y^0, .., Y^n\right) = \mathcal{L}\left(X^0, .., X^n\right)$$

We define

$$g_n(t) = E\left[\sup_{0 \leq s \leq t} \left(X_s^{n+1} - X_s^n\right)^2\right]$$

Lemma 7 implies

$$\begin{aligned}
g_n(t) &\leq K\left\{\int_0^T E\left[\left|X_s^n - X_s^{n-1}\right|^2\right] ds + \int_0^T E_\alpha\left[\left|Y_s^n - Y_s^{n-1}\right|^2\right] ds\right\} \\
&= 2K \int_0^T E\left[\left|X_s^n - X_s^{n-1}\right|^2\right] ds \leq 2K \int_0^T g_{n-1}(s) ds \\
&\vdots \\
&\leq (2K)^n \int_0^t dt_1 \int_0^{t_1} \ldots \int_0^{t_{n-1}} g_0(t_n) dt_n
\end{aligned}$$

For a fixed $T > 0$, it is easy to state that g_0 is bounded on $[0,T]$. If we define $C = \sup_{0 \leq t \leq T} g_0(t)$, we have

$$g_n(t) \leq C \frac{(2K)^n T^n}{n!}$$

Then, for any $T > 0$, the sequence $(X^n)_{n \geq 0}$ converges for the norm $\|U\| = \left\|\sup_{0 \leq s \leq T} U_s\right\|_{\mathbb{L}^2}$ and, using Borel-Cantelli's Lemma, (X^n) converges almost surely uniformly on $[0,T]$ toward a continuous process X. Consequently, $(Y^n)_{n \geq 0}$ converges also in $\mathbb{L}^2$ and a.s.. We denote by Y its limit. Since $\mathcal{L}_\alpha\left(Y^0, .., Y^n\right) = \mathcal{L}\left(X^0, .., X^n\right)$ $\forall n$, we have $\mathcal{L}_\alpha(Y) = \mathcal{L}(X)$.

In particular, for any $T > 0$,

$$\sup_{n \geq 0} E\left(\sup_{0 \leq t \leq T} \left|X_t^n\right|^2\right) = \sup_{n \geq 0} E_\alpha\left(\sup_{0 \leq t \leq T} \left|Y_t^n\right|^2\right) < \infty$$

Using dominated convergence theorem, we easily check that (X, Y) is effectively a solution of the nonlinear stochastic differential equation

$$X_t \overset{a.s.}{=} X_0 + \int_0^t \int_0^1 \eta\left(X_s - Y_s(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s - Y_s(\alpha)\right) d\alpha ds$$

Moreover, thanks to the strong uniqueness proved in Theorem 8 and consequently to the uniqueness in law, the distribution of X depends only on $P_0 = \mathcal{L}(X_0)$. ■

Theorem 11 *Under Assumption (H), uniqueness in law holds for a solution of $((NSDE\,(\eta,f))$.*

Proof. Assume that (U,V) is a solution on $\mathcal{C}\left([0,T],\mathbb{R}^d\right)$ with initial data X_0 of

$$U=\Phi\left(U,V,X_0,W\right) \quad \text{with } \mathcal{L}_\alpha\left(V\right)=\mathcal{L}\left(U\right)=Q$$

Assume that (X,Y) is the solution given by Theorem 10 of

$$X=\Phi\left(X,Y,X_0,W\right) \text{ with } \mathcal{L}_\alpha\left(Y\right)=\mathcal{L}\left(X\right)=P$$

We want to state that $P=Q$.

Let $T>0$.

Let $\tau\in\,]0,T]$, let ρ_τ be the Vaserstein metric on the space of probability measures on $\mathcal{C}\left([0,\tau],\mathbb{R}^d\right)$ defined by

$$\rho_\tau\left(P,Q\right)^2=\inf\left\{E_\alpha\left(\sup_{0\le t\le\tau}\left|A_t-B_t\right|^2\right):\mathcal{L}_\alpha\left(A\right)=P,\mathcal{L}_\alpha\left(B\right)=Q\right\}$$

We prove that there exists at least one $\tau>0$ such that $\rho_\tau\left(P,Q\right)=0$.

Let $\varepsilon>0$, there exists A^ε and B^ε, two $\mathcal{P}_{2,\alpha}$-processes, such that $\mathcal{L}_\alpha\left(A^\varepsilon\right)=P,\mathcal{L}_\alpha\left(B^\varepsilon\right)=Q$ and

$$\rho_\tau\left(P,Q\right)^2\le E_\alpha\left(\sup_{0\le t\le\tau}\left|A_t^\varepsilon-B_t^\varepsilon\right|^2\right)\le\rho_\tau\left(P,Q\right)^2+\varepsilon$$

Let X^ε be the solution of $X^\varepsilon=\Phi\left(X^\varepsilon,A^\varepsilon,X_0,W\right)$ given by Theorem 8. Since $\mathcal{L}_\alpha\left(A^\varepsilon\right)=\mathcal{L}_\alpha\left(Y\right)=P$ and following Remark 9, we have $\mathcal{L}\left(X^\varepsilon\right)=\mathcal{L}\left(X\right)$.

If U^ε is the solution of $U^\varepsilon=\Phi\left(U^\varepsilon,B^\varepsilon,X_0,W\right)$ obtained in Theorem 8, we have also $\mathcal{L}\left(U^\varepsilon\right)=\mathcal{L}\left(U\right)$.

Lemma 7 implies

$$\begin{aligned}
E\left[\sup_{0\le t\le\tau}\left|X_t^\varepsilon-U_t^\varepsilon\right|^2\right] &\le 4\left(CK_\eta^2+\tau K_f^2\right)\left\{\int_0^\tau E\left[\left|X_s^\varepsilon-U_s^\varepsilon\right|^2\right]ds+\int_0^\tau E_\alpha\left[\left|A_s^\varepsilon-B_s^\varepsilon\right|^2\right]ds\right\}\\
&\le 4\left(CK_\eta^2+\tau K_f^2\right)\left\{\int_0^\tau E\left[\sup_{0\le u\le s}\left|X_u^\varepsilon-U_u^\varepsilon\right|^2\right]ds+\tau\left(\rho_\tau\left(P,Q\right)^2+\varepsilon\right)\right\}
\end{aligned}$$

and by Gronwall's Lemma, we have

$$E\left[\sup_{0\le t\le\tau}\left|X_t^\varepsilon-U_t^\varepsilon\right|^2\right]\le 4\tau\left(CK_\eta^2+\tau K_f^2\right)\left(\rho_\tau\left(P,Q\right)^2+\varepsilon\right)\exp\left(4\tau\left(CK_\eta^2+\tau K_f^2\right)\right)$$

Thus, for any $\varepsilon>0$,

$$\rho_\tau\left(P,Q\right)^2\le 4\tau\left(CK_\eta^2+\tau K_f^2\right)\exp\left(4\tau\left(CK_\eta^2+\tau K_f^2\right)\right)\left(\rho_\tau\left(P,Q\right)^2+\varepsilon\right)$$

If we choose $\tau>0$ such that

$$4\tau\left(CK_\eta^2+\tau K_f^2\right)\exp\left(4\tau\left(CK_\eta^2+\tau K_f^2\right)\right)<1$$

then $\rho_\tau\left(P,Q\right)=0$.

We have uniqueness in law on $[0,\tau]$, but we would like to obtain uniqueness in law on $[0,T]$. We will extend the property by iteration.

For $n\ge1$, we define $X^n=\left(X_{n\tau+t}\right)_{t\ge0}$, and we define similarly Y^n,U^n,V^n ....

Let us assume that we have uniqueness in law on $[0, n\tau]$. Then, in particular, $\mathcal{L}(X_{n\tau}) = \mathcal{L}(U_{n\tau})$. We consider the process $\tilde{U}$ solution of

$$\tilde{U}_{t+n\tau} = X_{n\tau} + \int_{n\tau}^{t+n\tau} \int_0^1 \eta\left(\tilde{U}_s - V_s(\alpha)\right) W(d\alpha, ds) + \int_{n\tau}^{t+n\tau} \int_0^1 f\left(\tilde{U}_s - V_s(\alpha)\right) d\alpha ds \quad (13)$$

with initial data $X_{n\tau}$.

We can rewrite (13) in the following way

$$\tilde{U}_t^n = X_{n\tau} + \int_0^t \int_0^1 \eta\left(\tilde{U}_s^n - V_s^n(\alpha)\right) \tilde{W}(d\alpha, ds) + \int_0^t \int_0^1 f\left(\tilde{U}_s^n - V_s^n(\alpha)\right) d\alpha ds$$

where $\tilde{W}$ is a white noise with covariance $d\alpha dt$ on $[0,1] \times [0,\infty)$ defined by $\forall A \in \mathcal{B}([0,1])$

$$\begin{aligned} \tilde{W}(A \times [0,t]) &= W(A \times [0, n\tau + t]) - W(A \times [0, n\tau]) \\ &= W(A \times [n\tau, n\tau + t]) \end{aligned}$$

(if $A \in \mathcal{B}([0,1])$ is fixed, $(W(A \times [0,t]))_{t \geq 0}$ is an independent increment process).

According to the uniqueness in law, obtained in Remark 9, $\mathcal{L}\left(\tilde{U}^n\right) = \mathcal{L}(U^n)$ on $[0,\tau]$ and thus $\mathcal{L}\left(\tilde{U}^n\right) = \mathcal{L}_\alpha(V^n)$ on $[0,\tau]$.

Therefore, we have $\mathcal{L}\left(\tilde{U}^n\right) = \mathcal{L}(X^n)$ on $[0,\tau]$. We deduce from the recurrent hypothesis that the flows $(\mathcal{L}_\alpha(V_t))_{0 \leq t \leq \tau + n\tau}$ and $(\mathcal{L}_\alpha(Y_t))_{0 \leq t \leq \tau + n\tau}$ are the same. According to Remark 9, we have $\mathcal{L}(X) = \mathcal{L}(U)$ on $[0, (n+1)\tau]$. Hence, by iteration, we conclude $\mathcal{L}(X) = \mathcal{L}(U)$ on $[0,T]$ for any $T > 0$. ■

2.3 Existence of a measure solution of the Landau equation with regular coefficients

In the previous part, we have proved the existence and uniqueness in law of a solution of the nonlinear stochastic differential equation $(NSDE(\eta, f))$ when η and f satisfy Assumption (H).

According to Proposition 4, we have finally stated the following theorem

Theorem 12 *Assume that P_0 is a probability measure with a finite moment of order 2. There is a measure solution $(P_t)_{t \geq 0}$ with initial data P_0 to the Landau equation*

$$\begin{aligned} \frac{d}{dt} \int \varphi(v) P_t(dv) &= \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} P_t(dv) \left(\int_{\mathbb{R}^d} P_t(dv_*) a_{ij}(v - v_*) \right) \partial_{ij} \varphi(v) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}^d} P_t(dv) \left(\int_{\mathbb{R}^d} P_t(dv_*) b_i(v - v_*) \right) \partial_i \varphi(v) \end{aligned}$$

where $(a_{ij})_{0 \leq i,j \leq d}$ is a matrix of the form $a = \sigma.\sigma^*$, with σ and b satisfying Assumption (H).

Remark 13 *If we assume that $\gamma = 0$, choosing for σ the expression (8) in dimension 2, or (9) in dimension 3, we can notice that if h is a bounded nonnegative function of class C^1 such that there exists a constant $K > 0$ with $h'(x) \leq \frac{K}{x^2}$ when $x \to +\infty$, σ and b satisfy Assumption (H). In particular, if h is a constant function (the Maxwellian case), σ and b satisfy Assumption (H). We can generalize those properties in dimension $d \geq 3$.*

When the initial data is a probability measure with a finite moment of order 2, we have thus proved the existence of a measure solution of the Landau equation (3) under some conditions on the function h. Nevertheless, with this approach, we cannot state the uniqueness of a measure solution.

3 Study of the Landau equation for some soft potential $(\gamma \in (-1, 0])$

We use the same notations as in chapter 2.

The case $\gamma \in (-1, 0]$ with h some bounded continuous nonnegative function is more difficult than the previous case, because the continuous coefficients b and σ are no more lipschitz continuous on $\mathbb{R}^d$. We will use the results obtained in the chapter 2 approaching the coefficients σ and b by two sequences (σ^n) and (b^n) of lipschitz continuous functions. Then, for any $n \geq 0$, we build a sequence of random couples (X^n, Y^n) solution of the nonlinear differential equation:

$$X_t^n = X_0 + \int_0^t \int_0^1 \sigma^n \left(X_s^n - Y_s^n(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b^n \left(X_s^n - Y_s^n(\alpha)\right) d\alpha ds \tag{$NSDE(\sigma^n, b^n)$}$$

Our aim is to show that the sequence (X^n) converges, in a certain sense, toward a process X, and, if we denote by P the distribution of X, to state that P satisfies a nonlinear martingale problem. We will see that this last property has two mains consequences: the existence of a measure solution of the Landau equation when $\gamma \in (-1, 0]$ and h some bounded continuous function, and the existence of a weak solution of the nonlinear stochastic differential equation:

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds \tag{$NSDE(\sigma, b)$}$$

where σ and b are defined by (6), (4) and (5).

For the last stage (theorem 20), we use results obtained by N. El Karoui and S. Méléard in [4], and thereby we need a symmetric condition on σ *(consequently, $d' = d$)*. So we choose in this section the expression (7) given in the introduction, i.e.

$$\sigma_{ij}(z) = |z|^{\frac{\gamma}{2}+1} \sqrt{h\left(|z|^2\right)} \left(\delta_{ij} - \frac{z_i z_j}{|z|^2}\right)$$

As $\gamma \in (-1, 0]$, we can notice that σ and b have linear growth: if we denote by $c = \sup_{z \in \mathbb{R}^d} h\left(|z|^2\right)$, we have (differentiating the case the case $|z| \geq 1$ from the case $|z| < 1$)

$$\begin{aligned} |b(z)| &\leq c\,|z|^{\gamma+1} \\ &\leq c\,(d-1)\left(|z|+1\right) \end{aligned} \tag{14}$$

$$\begin{aligned} |\sigma(z)| &\leq \sqrt{c}\,|z|^{\frac{\gamma}{2}+1} \\ &\leq \sqrt{c}\left(|z|+1\right) \end{aligned} \tag{15}$$

Those inequalities will be very helpful below.

We give first a technical lemma.

Lemma 14 *If $k \geq 2$, we assume that $X_0 \in \mathbb{L}^k$.*

If $Z \in \mathcal{P}^k$ and $Y \in \mathcal{P}_\alpha^k$, then the process X defined by

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(Z_s - Y_s(\alpha)\right).W^d(d\alpha, ds) + \int_0^t \int_0^1 b\left(Z_s - Y_s(\alpha)\right) d\alpha ds$$

belongs to $\mathcal{P}^k$.

Proof. The i^{th} component of X is given by

$$X_{i,t}=X_{i,0}+\sum_{j=1}^{d}\int_0^t\int_0^1\sigma_{i,j}\left(Z_s-Y_s\left(\alpha\right)\right)W_j\left(d\alpha,ds\right)+\int_0^t\int_0^1 b_i\left(Z_s-Y_s\left(\alpha\right)\right)d\alpha ds$$

For some $T>0$, by the Burkholder-Davis-Gundy and the Hölder inequalities, we have

$$\begin{aligned}E\left[\sup_{0\leq t\leq T}\left|X_{i,t}\right|^k\right] &\leq 3^{k-1}\left\{E\left[\left|X_{i,0}\right|^k\right]+C_k d^{\frac{k-2}{2}}T^{\frac{k-2}{2}}\sum_{j=1}^{d}\int_0^T\int_0^1 E\left(\left|\sigma_{i,j}\left(Z_s-Y_s\left(\alpha\right)\right)\right|^k\right)d\alpha ds\right.\\ &\left.+T^{k-1}\int_0^T\int_0^1 E\left(\left|b_i\left(Z_s-Y_s\left(\alpha\right)\right)\right|^k\right)d\alpha ds\right\}\end{aligned}$$

Since $\gamma\in(-1,0]$, using (14) and (15), we obtain

$$\begin{aligned}E\left[\sup_{0\leq t\leq T}\left|X_{i,t}\right|^k\right] &\leq 3^{k-1}\left\{E\left[\left|X_{i,0}\right|^k\right]+C_k d^{\frac{k}{2}}T^{\frac{k-2}{2}}c^{\frac{k}{2}}\int_0^T\int_0^1 E\left(\left(\left|Z_s-Y_s\left(\alpha\right)\right|+1\right)^k\right)d\alpha ds\right.\\ &\left.+T^{k-1}c^k\left(d-1\right)^k\int_0^T\int_0^1 E\left(\left(\left|Z_s-Y_s\left(\alpha\right)\right|+1\right)^k\right)d\alpha ds\right\}\end{aligned}$$

So, there exists $K>0$ such that

$$E\left[\sup_{0\leq t\leq T}\left|X_t\right|^k\right]\leq K\left\{E\left[\left|X_0\right|^k\right]+\int_0^T E\left(\left|Z_s\right|^k\right)ds+\int_0^T E_\alpha\left(\left|Y_s\right|^k\right)ds\right\}$$

The lemma is proved. ■

3.1 Approximation of the solution

3.1.1 Construction of the approximation

Let χ be the following even smooth function

$$\chi\left(z\right)=\begin{cases}1 \text{ if } \left|z\right|\geq 2\\ 0 \text{ if } \left|z\right|\leq 1\end{cases}$$

such that for any $z\in\mathbb{R}^d$, $0\leq\chi\left(z\right)\leq 1$.

We define

$$\begin{aligned}a^n\left(z\right) &= \chi\left(nz\right)a\left(z\right)\\ b^n\left(z\right) &= \chi\left(nz\right)b\left(z\right)\\ \sigma^n\left(z\right) &= \sqrt{\chi\left(nz\right)}\sigma\left(z\right)\end{aligned}$$

Then, σ^n and b^n satisfy the Assumption (H) of chapter 2. Moreover, we can notice that

$$\left|a^n\right|\leq\left|a\right|,\ \ \left|b^n\right|\leq\left|b\right| \text{ and } \left|\sigma^n\right|\leq\left|\sigma\right|.$$

We consider the following approximation of the Landau equation: for any $\varphi\in\mathcal{C}_b^2\left(\mathbb{R}^d,\mathbb{R}\right)$

$$\begin{aligned}\frac{d}{dt}\int\varphi\left(v\right)P_t^n\left(dv\right) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}P_t^n\left(dv\right)\left(\int_{\mathbb{R}^d}P_t^n\left(dv_*\right)a_{ij}^n\left(v-v_*\right)\right)\partial_{ij}\varphi\left(v\right) \qquad (16)\\ &+\sum_{i=1}^{d}\int_{\mathbb{R}^d}P_t^n\left(dv\right)\left(\int_{\mathbb{R}^d}P_t^n\left(dv_*\right)b_i^n\left(v-v_*\right)\right)\partial_i\varphi\left(v\right)\end{aligned}$$

(We have chosen χ even to keep the conservation of the mass, of the momentum and of the energy in the approximation of the Landau equation.)

For any arbitrary $T > 0$, we define as follows the martingale problem (MP^n) associated with this equation: let X be the canonical process on $\mathcal{C}\left([0,T],\mathbb{R}^d\right)$ (i.e., for $w \in \mathcal{C}\left([0,T],\mathbb{R}^d\right)$ $X_t(\omega) = w(t)$), and let us define the second order differential operator

$$L^n(Q)\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{d}\int a_{ij}^n(x-y)\,Q(dy)\,\partial_{ij}^2\varphi(x) + \sum_{i=1}^{d}\int b_i^n(x-y)\,Q(dy)\,\partial_i\varphi(x)$$

where Q is a probability measure and $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d,\mathbb{R}\right)$.

We will say that a probability measure Q on $\mathcal{C}\left([0,T],\mathbb{R}^d\right)$ is a solution of the nonlinear martingale problem (MP^n) if

$$M_t^n = \varphi(X_t) - \varphi(X_0) - \int_0^t L^n(Q_s)\varphi(X_s)\,ds$$

is a Q-martingale for any $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d,\mathbb{R}\right)$, where $Q_s = Q \circ X_s^{-1}$. Taking the expectation of M_t^n, we notice that a solution of the martingale problem is a measure solution of (16).

If we assume that $X_0 \in \mathbb{L}^k$, adapting the proofs of Section 2.2, we show the existence of a solution $(X^n, Y^n) \in \mathcal{P}^k \times \mathcal{P}_\alpha^k$, unique in law, of the nonlinear stochastic differential equation

$$X_t^n = X_0 + \int_0^t\int_0^1 \sigma^n\left(X_s^n - Y_s^n(\alpha)\right).W^d(d\alpha,ds) + \int_0^t\int_0^1 b^n\left(X_s^n - Y_s^n(\alpha)\right)d\alpha ds$$

Moreover, if we denote by $P^n = \mathcal{L}(X^n) = \mathcal{L}_\alpha(Y^n)$ the common distribution, P^n satisfies the martingale problem (MP^n), for any n.

3.1.2 Tightness of the sequence (P^n)

Proposition 15 *Assume that X_0 is a square integrable random vector of $\mathbb{R}^d$, the sequence of probability distributions (P^n) built in Section 3.1.1 is tight.*

Proof. According to Aldous's criterium and Rebolledo's criterium (see [8]), it is enough to prove the following lemma to state the theorem.

Lemma 16 *If $X_0 \in \mathbb{L}^k$, with $k \geq 2$, there is a constant $C > 0$ such that for any $T > 0$*

$$\sup_{n\geq 0} E\left[\sup_{0\leq t\leq T}\left|X_t^n\right|^k\right] < C$$

We suppose Lemma 16 proved. X_0 is random vector with a finite 2-order moment. If we denote by $M^n + A^n$ the Doob-Meyer decomposition of X^n, i.e.

$$\begin{aligned}
M_t^n &= \int_0^t\int_0^1 \sigma^n\left(X_s^n - Y_s^n(\alpha)\right).W^d(d\alpha,ds)\\
A_t^n &= X_0 + \int_0^t\int_0^1 b^n\left(X_s^n - Y_s^n(\alpha)\right)d\alpha ds
\end{aligned}$$

then for any $T > 0$ there exists a constant $K > 0$ such that for any $\eta > 0$, $\delta > 0$, $t \in [0,T]$ and $n \geq 0$

$$\sup_{\theta\leq\delta}\mathbb{P}\left(\left|A_{t+\theta}^n - A_t^n\right| > \eta\right) \leq KE\left[\sup_{0\leq s\leq T}\left|X_s^n\right|^2\right]\frac{\delta^2}{\eta^2}$$

and

$$\sup_{\theta\leq\delta}\mathbb{P}\left(\left|\langle M^n\rangle_{t+\theta}-\langle M^n\rangle_t\right|>\eta\right)\leq KE\left[\sup_{0\leq s\leq T}|X_s^n|^2\right]\frac{\delta}{\eta}$$

where $\langle M\rangle$ is the bracket of M. According to Lemma 16, the two sequences (A^n) and (M^n) satisfy the hypothesis of Aldous's criterium. Then, for any $T>0$, according to Rebolledo's criterium, the sequence (P^n), where P^n is the distribution of (X^n) is tight in the space of probability measures on $\mathcal{C}\left([0,T],\mathbb{R}^d\right)$.

Proof. (Lemma 16)

We notice that,

$$\begin{aligned}|\sigma^n(z)| &\leq \sqrt{c}\,|z|^{\frac{\gamma}{2}+1}\\ |b^n(z)| &\leq c\,(d-1)\,|z|^{\gamma+1}\end{aligned}$$

where $c=\sup_{z\in\mathbb{R}^d}h\left(|z|^2\right)$. Since $\mathcal{L}(X^n)=\mathcal{L}_\alpha(Y^n)$, according to the proof of Lemma 14, we have

$$\begin{aligned}E\left[\sup_{0\leq u\leq t}|X_u^n|^k\right] &\leq 3^{k-1}\left\{E\left[|X_0|^k\right]+K\int_0^t E\left(|X_s^n|^{k\left(\frac{\gamma}{2}+1\right)}+|X_s^n|^{k(\gamma+1)}\right)ds\right\}\\ &\leq K_1+K_2\int_0^t E\left(\sup_{0\leq u\leq s}|X_u^n|^k\right)ds\end{aligned}$$

with K_1 and K_2 independent of n. Using Gronwall's Lemma, we have

$$E\left[\sup_{0\leq u\leq T}|X_u^n|^k\right]\leq K_1e^{K_2T}$$

The lemma is proved. ■ ■

Consequently, there is a subsequence of (P^n) which converges toward a probability distribution P. Let us now identify this distribution.

3.2 The nonlinear martingale problem associated with the probability measure P

For a probability measure Q, we define the elliptic operator

$$L(Q)\varphi(x)=\frac{1}{2}\sum_{i,j=1}^d\int a_{ij}(x-y)\,Q(dy)\,\partial_{ij}^2\varphi(x)+\sum_{i=1}^d\int b_i(x-y)\,Q(dy)\,\partial_i\varphi(x)$$

where $\varphi\in\mathcal{C}_b^2\left(\mathbb{R}^d,\mathbb{R}\right)$.

For any arbitrary $T>0$, we define the nonlinear martingale problem (MP) : a probability measure Q on $\mathcal{C}\left([0,T],\mathbb{R}^d\right)$ is a solution of (MP) if

$$M_t=\varphi(X_t)-\varphi(X_0)-\int_0^t L(Q_s)\,\varphi(X_s)\,ds$$

is a Q-martingale, where $Q_s=Q\circ X_s^{-1}$.

Theorem 17 *Assume that P_0 has a finite moment of order 4. Let P^n be a solution of (MP^n) with initial data P_0 for any $n\geq 0$ and P be a cluster point of the sequence (P^n). Then P satisfies the martingale problem (MP).*

Remark 18 *Assume that P_0 has a finite k-order moment. Let P be a cluster point of (P^n). Thanks to Lemma 16, there exists a constant $C > 0$ such that*

$$E_P \left[\sup_{0 \leq t \leq T} |X_t|^k \right] < C$$

where E_P is the expectation under the distribution P.

Proof. (Remark 18)

Up to a subsequence, (P^n) converges toward the distribution P. According to the Skorohod Theorem (see [9]), there exists a sequence of random processes $(Y^n)_{n \geq 0}$ and a process Y defined on $([0,1], \mathcal{B}([0,1]), d\alpha)$ such that

$$\begin{aligned} \mathcal{L}_\alpha (Y^n) &= P^n \quad \forall n \geq 0 \\ \mathcal{L}_\alpha (Y) &= P \end{aligned}$$

and $Y^n \underset{n \to \infty}{\longrightarrow} Y$ a.s.. Using Fatou's Lemma and Lemma 16, we notice that

$$E_P \left[\sup_{0 \leq t \leq T} |X_t|^k \right] \leq \liminf_{n \to \infty} E_{P^n} \left[\sup_{0 \leq t \leq T} |X_t|^k \right] \leq C$$

■

Proof. (Theorem 17)

Let M the process define by

$$M_t = \varphi(X_t) - \varphi(X_0) - \int_0^t L(P_s) \varphi(X_s) \, ds$$

To prove that P satisfies the martingale problem (MP), we have to state that M is a P-martingale.

Let (g_i) be a sequence of bounded functions. M is a P-martingale if and only if for any $0 \leq s \leq t$, $k \geq 1$ and $0 \leq s_1 \leq .. \leq s_k \leq s$, M satisfies

$$E_P \left[(M_t - M_s) \, g_1 (X_{s_1}) ... g_k (X_{s_k}) \right] = 0$$

We choose $0 \leq s \leq t$, $k \geq 1$ and $0 \leq s_1 \leq .. \leq s_k \leq s$.

We know that, for any n, P^n is a solution of the martingale problem (MP^n). We will still denote by (P^n) a subsequence of (P^n) which converges toward $P : P^n \Longrightarrow P$.

As M^n is a P^n-martingale, we have

$$E_{P^n} \left[(M_t^n - M_s^n) \, g_1 (X_{s_1}) ... g_k (X_{s_k}) \right] = 0$$

Let us prove in the following that

$$E_{P^n} \left[(M_t^n - M_s^n) \, g_1 (X_{s_1}) ... g_k (X_{s_k}) \right] \underset{n \to \infty}{\longrightarrow} E_P \left[(M_t - M_s) \, g_1 (X_{s_1}) ... g_k (X_{s_k}) \right] \tag{17}$$

Since φ, g_1,..., g_k are bounded continuous functions and $x \to x_t$ is a continuous function, $x \to \varphi(x_t) \, g_1(x_t) ... g_k(x_t)$ is a bounded continuous function, and then $\forall t \geq 0$

$$E_{P^n} \left[\varphi(X_t) \, g_1 (X_t) ... g_k (X_t) \right] \underset{n \to \infty}{\longrightarrow} E_P \left[\varphi(X_t) \, g_1 (X_t) ... g_k (X_t) \right] \tag{18}$$

Knowing convergence (18), we just have to check the following convergence

$$\begin{aligned} & E_{P^n} \left[\left(\int_s^t L^n (P_u^n) \varphi(X_u) \, du \right) g_1 (X_{s_1}) ... g_k (X_{s_k}) \right] \\ & \underset{n \to \infty}{\longrightarrow} E_P \left[\left(\int_s^t L(P_u) \varphi(X_u) \, du \right) g_1 (X_{s_1}) ... g_k (X_{s_k}) \right] \end{aligned} \tag{19}$$

We can write

$$
\begin{aligned}
&E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]-E_P\left[\left(\int_s^t L\left(P_u\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\\
&=E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]-E_{P^n}\left[\left(\int_s^t L\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\\
&+E_{P^n}\left[\left(\int_s^t L\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]-E_P\left[\left(\int_s^t L\left(P_u\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]
\end{aligned}
\tag{20}
$$

We will use a product-space to simplify those expressions. As $P^n \Longrightarrow P$, $P^n \bigotimes P^n \Longrightarrow P \bigotimes P$. If we denote by (X,Y) the canonical process on $\mathcal{C}\left([0,T],\mathbb{R}^d\right)\times\mathcal{C}\left([0,T],\mathbb{R}^d\right)$, we notice that

$$
\begin{aligned}
&E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\\
&=\frac{1}{2}\sum_{i,j=1}^d\int_s^t E_{P^n\bigotimes P^n}\left[a_{ij}^n\left(X_u-Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]du\\
&+\sum_{i=1}^d\int_s^t E_{P^n\bigotimes P^n}\left[\int b_{ij}^n\left(X_u-Y_u\right)P_u^n\left(dy\right)\partial_i\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]du
\end{aligned}
$$

We make the same transformation for the others expectations of the second term of (20), and we divide in two parts the convergence study of (19).

As $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d,\mathbb{R}\right)$, and $g_1,...,g_k$ are bounded functions, there exists a finite constant $m>0$ such that

$$
m=\sup\left(\left\|\partial\varphi\right\|_\infty,\left\|\varphi\right\|_\infty,\left\|g_i\right\|_\infty,i=1,..,k\right)
$$

Part I: We state that

$$
\begin{aligned}
&\left|E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\right.\\
&\left.-E_{P^n}\left[\left(\int_s^t L\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\right|\underset{n\to\infty}{\longrightarrow}0
\end{aligned}
$$

We first study the convergence of the term with the coefficients a_{ij}^n and a_{ij}.

$$
\begin{aligned}
E_1 &= \left|E_{P^n\bigotimes P^n}\left[\int_s^t a_{ij}^n\left(X_u-Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\right.\\
&\quad\left.-E_{P^n\bigotimes P^n}\left[\int_s^t a_{ij}\left(X_u-Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\right|\\
&\leq m^{k+1}E_{P^n\bigotimes P^n}\left[\int_s^t\left|a_{ij}^n\left(X_u-Y_u\right)-a_{ij}\left(X_u-Y_u\right)\right|du\right]
\end{aligned}
$$

Since $\left|a_{ij}^n\left(z\right)\right|\leq\left|a_{ij}\left(z\right)\right|\leq c\left|z\right|^{\gamma+2}$ and $a_{ij}^n\left(z\right)=a_{ij}\left(z\right)$ on $\left|z\right|\geq\frac{2}{n}$,

$$
E_1\leq 2m^{k+1}c\int_s^t E_{P^n\bigotimes P^n}\left[\left|X_u-Y_u\right|^{\gamma+2}\mathbb{I}_{\left|X_u-Y_u\right|\leq\frac{2}{n}}\right]du
$$

As $\gamma+2>0$, there finally exists a constant $K>0$ such that

$$
\begin{aligned}
&\left|E_{P^n\bigotimes P^n}\left[\int_s^t a_{ij}^n\left(X_u-Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\right.\\
&\left.-E_{P^n\bigotimes P^n}\left[\int_s^t a_{ij}\left(X_u-Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\right|\leq\frac{K}{n^{\gamma+2}}
\end{aligned}
$$

We can use the same arguments for the term with the coefficients b_i^n and b_i. Hence, we obtain

$$\begin{aligned}&\left|E_{P^n \bigotimes P^n}\left[\int_s^t b_i^n\left(X_u - Y_u\right)\partial_i\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\right.\\ &\left.-E_{P^n \bigotimes P^n}\left[\int_s^t b_i\left(X_u - Y_u\right)\partial_i\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\right| \leq \frac{K}{n^{\gamma+1}}\end{aligned}$$

Consequently, since $\gamma \in (-1, 0]$, we have proved,

$$\begin{aligned}&\left|E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\right.\\ &\left.-E_{P^n}\left[\left(\int_s^t L\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}^n\right)...g_k\left(X_{s_k}^n\right)\right]\right| \underset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

<u>Part II:</u> We now state that,

$$\begin{aligned}&\left|E_{P^n}\left[\left(\int_s^t L\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\right.\\ &\left.-E_{P}\left[\left(\int_s^t L\left(P_u\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\right| \underset{n\to\infty}{\longrightarrow} 0 \qquad (21)\end{aligned}$$

Like in the part I, we first study the term with the coefficients a_{ij}, i.e. the convergence

$$\begin{aligned}&\left|E_{P^n \bigotimes P^n}\left[\int_s^t a_{ij}\left(X_u - Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\right.\\ &\left.-E_{P \bigotimes P}\left[\int_s^t a_{ij}\left(X_u - Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\right| \underset{n\to\infty}{\overset{?}{\longrightarrow}} 0 \qquad (22)\end{aligned}$$

Let $f : \mathcal{C}\left([0,t], \mathbb{R}^d\right) \times \mathcal{C}\left([0,t], \mathbb{R}^d\right) \longrightarrow \mathbb{R}$ be the function defined by

$$(x, y) \longmapsto f(x, y) = \int_s^t a_{ij}\left(x_u - y_u\right)\partial_{ij}\varphi\left(x_u\right)du.g_1\left(x_{s_1}\right)...g_k\left(x_{s_k}\right)$$

So, we can rewrite (22):

$$\left|E_{P^n \bigotimes P^n}\left[f(X,Y)\right] - E_{P \bigotimes P}\left[f(X,Y)\right]\right| \underset{n\to\infty}{\overset{?}{\longrightarrow}} 0$$

f is not a bounded function, hence we cannot just use the convergence in distribution to conclude. We cut off the function f.

Let Ψ_R be a bounded continuous function such that

$$\begin{aligned} i)\quad \Psi_R(x,y) &= 1 \text{ if } |(x,y)| \leq R\\ &= 0 \text{ if } |(x,y)| \geq 2R\\ ii)\quad |\Psi_R| &\leq 1\\ \text{where } |(x,y)| &= \sup\left(\sup_{0\leq u\leq t}|x_u|, \sup_{0\leq u\leq t}|y_u|\right)\end{aligned}$$

and $f_R : (x,y) \longmapsto f(x,y)\Psi_R(x,y)$. Hence, f_R is a continuous bounded function, and then, using the convergence in distribution, we have

$$\left|E_{P^n \bigotimes P^n}\left[f_R(X,Y)\right] - E_{P \bigotimes P}\left[f_R(X,Y)\right]\right| \underset{n\to\infty}{\longrightarrow} 0 \qquad (23)$$

We now compare the expectations $E_{P^n \bigotimes P^n}\left[f(X,Y)\right]$ and $E_{P^n \bigotimes P^n}\left[f_R(X,Y)\right]$, as well as $E_{P \bigotimes P}\left[f(X,Y)\right]$ and $E_{P \bigotimes P}\left[f_R(X,Y)\right]$. We notice that:

1. $|f(x,y) - f_R(x,y)| \leq 2|f(x,y)| \mathbb{I}_{|(x,y)|>R}$
2. $|f(x,y)| \leq m^{k+1} \int_s^t |a_{ij}(x_u - y_u)| du$
   $\leq m^{k+1} c \int_s^t |x_u - y_u|^{\gamma+2} du$
   $\leq K \left(\sup_{0\leq u\leq t} |x_u|^{\gamma+2} + \sup_{0\leq u\leq t} |y_u|^{\gamma+2} \right)$

So,

$$\begin{aligned}
& \left| E_{P^n \bigotimes P^n}[f(X,Y)] - E_{P^n \bigotimes P^n}[f_R(X,Y)] \right| \\
\leq \ & 2E_{P^n \bigotimes P^n}\left[|f(X,Y)| \mathbb{I}_{|(X,Y)|>R} \right] \\
\leq \ & 2K \left\{ E_{P^n \bigotimes P^n}\left[\sup_{0\leq u\leq t} |X_u|^{\gamma+2} \mathbb{I}_{\sup_{0\leq u\leq t}|X_u|>R} \right] + E_{P^n \bigotimes P^n}\left[\sup_{0\leq u\leq t} |X_u|^{\gamma+2} \mathbb{I}_{\sup_{0\leq u\leq T}|Y_u|>R} \right] \right. \\
& \left. + E_{P^n \bigotimes P^n}\left[\sup_{0\leq u\leq t} |Y_u|^{\gamma+2} \mathbb{I}_{\sup_{0\leq u\leq t}|X_u|>R} \right] + E_{P^n \bigotimes P^n}\left[\sup_{0\leq u\leq t} |Y_u|^{\gamma+2} \mathbb{I}_{\sup_{0\leq u\leq t}|Y_u|>R} \right] \right\} \\
\leq \ & 4K \left\{ E_{P^n}\left[\sup_{0\leq u\leq t} |X_u|^{2\gamma+4} \right]^{\frac{1}{2}} P^n\left(\sup_{0\leq u\leq t} |X_u| > R \right)^{\frac{1}{2}} + E_{P^n}\left[\sup_{0\leq u\leq t} |X_u|^{\gamma+2} \right] P^n\left(\sup_{0\leq u\leq t} |X_u| > R \right) \right\}
\end{aligned}$$

Using Lemma 16 with $k = 4$, there exists a finite constant K, independent of n and of R, such that

$$\left| E_{P^n \bigotimes P^n}[f(X,Y)] - E_{P^n \bigotimes P^n}[f_R(X,Y)] \right| \leq \frac{K}{R} \tag{24}$$

As well, according to Remark 18, there exists a constant K' such that

$$\left| E_{P \bigotimes P}[f(X,Y)] - E_{P \bigotimes P}[f_R(X,Y)] \right| \leq \frac{K'}{R} \tag{25}$$

Then, using convergence (23) and inequalities (24) and (25), we have finally proved

$$\left| E_{P^n \bigotimes P^n}[f(X,Y)] - E_{P \bigotimes P}[f(X,Y)] \right| \underset{n\to\infty}{\longrightarrow} 0$$

To finish the proof of (21), we still have to check

$$\begin{aligned}
& \left| E_{P^n \bigotimes P^n}\left[\int_s^t b_i(X_u - Y_u) \partial_i \varphi(X_u) g_1(X_{s_1}) ... g_k(X_{s_k}) du \right] \right. \\
& \left. - E_{P \bigotimes P}\left[\int_s^t b_i(X_u - Y_u) \partial_i \varphi(X_u) g_1(X_{s_1}) ... g_k(X_{s_k}) du \right] \right| \underset{n\to\infty}{\overset{?}{\longrightarrow}} 0
\end{aligned}$$

We define the function $\tilde{f} : \mathcal{C}\left([0,t], \mathbb{R}^d\right) \times \mathcal{C}\left([0,t], \mathbb{R}^d\right) \longrightarrow \mathbb{R}$ by

$$(x,y) \longmapsto \tilde{f}(x,y) = \int_s^t b_i(x_u - y_u) \partial_i \varphi(x_u) du . g_1(x_{s_1}) ... g_k(x_{s_k})$$

As for the function f, we state that

$$\left| E_{P^n \bigotimes P^n}\left[\tilde{f}(X,Y) \right] - E_{P \bigotimes P}\left[\tilde{f}(X,Y) \right] \right| \underset{n\to\infty}{\longrightarrow} 0$$

Conclusion: according to Parts I and II, we have proved the convergence (19). Then, thanks to (18), we have

$$E_{P^n}\left[(M_t^n - M_s^n) g_1(X_{s_1}) ... g_k(X_{s_k}) \right] \underset{n\to\infty}{\longrightarrow} E_P\left[(M_t - M_s) g_1(X_{s_1}) ... g_k(X_{s_k}) \right]$$

Hence, since $E_{P^n}\left[\left(M_t^n - M_s^n\right) g_1\left(X_{s_1}\right) ... g_k\left(X_{s_k}\right)\right] = 0$, we have, for any $0 \leq s < t, 0 \leq s_1 \leq ... \leq s_k \leq s$,

$$E_P\left[\left(M_t - M_s\right) g_1\left(X_{s_1}\right) ... g_k\left(X_{s_k}\right)\right] = 0$$

So, P satisfies the martingale problem (MP). ■

There are two main consequences of this theorem. The first one concerns the existence of a solution to the Landau equation when $\gamma \in (-1, 0]$:

Theorem 19 *Let a and b be defined by (4) and (5) respectively. Let P_0 be a probability measure with a finite moment of order 4. There exists a measure solution $(P_t)_{t\geq 0}$ with initial data P_0 of the Landau equation*

$$\begin{aligned} \frac{d}{dt}\int \varphi(v) P_t(dv) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*) a_{ij}(v - v_*)\right)\partial_{ij}\varphi(v) \\ &\quad + \sum_{i=1}^{d}\int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*) b_i(v - v_*)\right)\partial_i\varphi(v) \end{aligned}$$

when $\gamma \in (-1, 0]$ and h is a bounded continuous nonnegative function.

The second one states that the distribution P can be also interpreted as the distribution of a weak solution of a nonlinear stochastic differential equation:

Theorem 20 *Let the matrix a and the vector b be defined by (4) and (5) respectively. Let X_0 be a random variable with a finite moment of order 4. Then, there exists a weak solution X of the nonlinear stochastic differential equation:*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right).W^d(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds \qquad (NSDE(\sigma, b))$$

where σ is a symmetric matrix such that $\sigma^*\sigma = a$.

Proof. Let P be a cluster point of (P^n). Let X be a process with distribution P.

We firstly state the following lemma:

Lemma 21 *The process*

$$M_t = X_t - X_0 - \int_0^t b\left(X_s, P_s\right) ds$$

where $b(X_s, P_s) = \int b(X_s - y) P_s(dy)$, is a continuous local P-martingale and its bracket is given by

$$\langle M_i, M_j\rangle_t = \int_0^t a_{ij}\left(X_s, P_s\right) ds$$

where $a_{ij}(X_s, P_s) = \int a_{ij}(X_s - y) P_s(dy)$.

Proof. We denote by $a \wedge b = \min(a, b)$ and $B_R = \left\{x \in \mathbb{R}^d : |x| \leq R\right\}$.

Using the functions $\varphi_i \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$ such that $\varphi_i(x) = x_i$ on B_R, for $i = 1, .., d$, it is easy to check that M is a continuous local P-martingale.

Using the functions $\varphi_{ij} \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$ such that $\varphi_{ij}(x) = x_i x_j$ on B_R, we state that the processes

$$\begin{aligned} N_{ij,t} &= X_{i,t}X_{j,t} - X_{i,0}X_{j,0} - \int_0^t a_{ij}\left(X_s, P_s\right) ds \\ &\quad - \int_0^t b_i\left(X_s, P_s\right) X_{j,s} ds - \int_0^t b_j\left(X_s, P_s\right) X_{i,s} ds \end{aligned}$$

are continuous local P-martingales, $i,j \in \{1,..,d\}$.

Moreover,

$$M_{i,t}M_{j,t} - \int_0^t a_{ij}(X_s, P_s)\, ds = N_{ij,t} - X_{i,0}M_{j,t} - X_{j,0}M_{i,t}$$

Then,

$$\langle M_i, M_j \rangle_t = \int_0^t a_{ij}(X_s, P_s)\, ds$$

■

According to Theorem III-10 in [4] *(σ is a symmetric matrix)*, we conclude that there are on an extension of the probability space d continuous orthogonal martingale measures $\left(W_k^P\right)_{k=1..d}$ with intensity $P_s(dy)\, ds$ on $\mathbb{R}^d \times [0,\infty)$ such that, for any $k = 1...d$

$$M_{i,t} = \sum_{k=1}^{d} \int_0^t \int_{\mathbb{R}^d} \sigma_{ik}(X_s - y)\, W_k^P(ds, dy)$$

As the measure $P_s(dy)\, ds$ is deterministic, using Theorem III-3 in [4], we deduce that the W_k^P are white noises and

$$X_{i,t} = X_0 + \sum_{k=1}^{d} \int_0^t \int \sigma_{ik}(X_s - y)\, W_k^P(ds, dy) + \int_0^t \int b_i(X_s - y)\, P_s(dy)\, ds \qquad (26)$$

We can easily rewrite the equation (26) under the expression $(NSDE(\sigma, b))$ (see the proof of Remark 9). Consequently, we have proved the theorem. ■

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