Harnack Type Estimates for Linear Elliptic Systems and Applications

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Part I : The Basic Estimates

1 Introduction

During the last twenty years there have been many attempts to extend to weakly coupled cooperative elliptic systems of second order the theory of scalar elliptic equations, and to describe the particularities due to the "coupling" in the system. We believe that this work will bring us closer to a definitive answer to that question.

In Part I of this paper we establish counterparts, for cooperative linear elliptic systems, of the fundamental Alexandrov-Bakelman-Pucci and Harnack-Krylov-Safonov estimates for scalar equations. These estimates have been essential in many areas of PDE's, such as existence and regularity of solutions of nonlinear elliptic equations, Liouville type theorems, qualitative properties of solutions. In Part II we prove the existence of a principal eigenvalue and a principal eigenfunction for an arbitrary cooperative system and present a number of applications to higher order elliptic equations.

We study the system

$$\left\{\begin{array}{rcl} L_i u_i + \sum_{j=1}^{n} c_{ij}(x) u_j & = & f_i(x) \\ i & = & 1, \ldots, n, \end{array}\right. \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$; $n, N \geq 1$.

In order to simplify the notations we write (1) in the form

$$LU + \mathcal{C}U = F,$$

where $L = \operatorname{diag}(L_1, \ldots, L_n)$, $\mathcal{C}(x) = (c_{ij}(x))_{i,j=1}^n$, $U = (u_1, \ldots, u_n)$, and $F = (f_1, \ldots, f_n)$.

The second-order elliptic operators $L_1, \ldots, L_n$ are supposed to be in general non-divergence form

$$L_k = \sum_{i,j=1}^N a_{ij}^k(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i^k(x) \frac{\partial}{\partial x_i},$$

to be uniformly elliptic :

(H1) there exists $c_0 \in (0,1)$ such that for all $\xi \in \mathbb{R}^N$, all $k = 1, \ldots, n$, and almost every $x \in \Omega$ we have

$$c_0 |\xi|^2 \leq \sum_{i,j=1}^N a_{ij}^k(x) \xi_i \xi_j \leq c_0^{-1} |\xi|^2.$$

We assume that the operators $L_1, \ldots, L_n$ and the matrix $\mathcal{C}$ have bounded measurable coefficients, with

(H2) $$\max_{1 \leq k \leq n} \sum_{i=1}^N \|b_i^k\|_{L^\infty(\Omega)}^2 = b^2 \leq \nu, \qquad \max_{1 \leq k \leq n} \sum_{i=1}^n \|c_{ki}\|_{L^\infty(\Omega)} \leq \nu.$$

We assume that the system (1) is cooperative, that is, for all indices $i, j \in \{1, \ldots, n\}$, with $i \neq j$,

(H3) $$c_{ij} \geq 0 \quad \text{a.e. in} \quad \Omega.$$

This is a widely used assumption in the study of elliptic systems. It is important to note that there can be no hope to obtain Harnack type estimates for general non-cooperative systems (see Remark 1 at the end of this section).

We study (1) in the framework of viscosity solutions, i.e. functions

(H4) $$U \in C(\Omega, \mathbb{R}^n)$$

satisfying (1) in a sense that we make precise in Section 2 (see Definition 2.2). We point out that, in particular, any strong solution of (1) (that is, any $U \in W_{\mathrm{loc}}^{2,N}(\Omega, \mathbb{R}^n)$, satisfying (1) a.e. in Ω) is a viscosity solution – see Section 2, Proposition 2.2. We make the convention that, throughout the paper, *all differential equations and inequations are assumed to hold in the viscosity sense, unless otherwise stated. Besides, all relations between vectors are understood to hold component-wise.*

We suppose that the right-hand side of (1) satisfies

(H5) $$F \in L^N(\Omega, \mathbb{R}^n).$$

Our first result is an Alexandrov-Bakelman-Pucci (ABP) type estimate for (1). To our knowledge, this is the first result of this kind for systems.

We use the following notation

$$\begin{array}{rclrcl} u \vee v(x) & = & \max(u(x), v(x)), & u \wedge v(x) & = & \min(u(x), v(x)), \\ u^+(x) & = & \max(u(x), 0), & u^-(x) & = & \max(-u(x), 0). \end{array}$$

Theorem 1.1 *(a) We suppose that*

(Ψ) *there exists a function* $\Psi = (\psi_1, \ldots, \psi_n) \in W^{2,N}_{\text{loc}}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ *such that*

$$\begin{cases} L\Psi + \mathcal{C}\Psi \le 0 & \textit{a.e. in} \quad \Omega \\ \Psi > 0 & \textit{in} \quad \overline{\Omega}. \end{cases} \tag{2}$$

If (H1)–(H5) hold and $U \in C(\overline{\Omega}, \mathbb{R}^n)$ *satisfies*

$$LU + \mathcal{C}U \ge -F \qquad \textit{in} \quad \Omega,$$

then

$$\sup_{\Omega} (u_1 \vee \ldots \vee u_n) \le C \left(\sup_{\partial\Omega} (u_1^+ \vee \ldots \vee u_n^+) + \|f^+\|_{L^N(\Omega)} \right), \tag{3}$$

where $f = f_1 \vee \ldots \vee f_n$. *The constant* C *depends only on* N, c_0, ν, Ψ, *and* diam Ω.

(b) If instead of (Ψ) *one assumes the stronger (take* $\Psi = (1, \ldots, 1)$*) condition*

$$\sum_{j=1}^n c_{ij}(x) \le 0 \qquad \textit{a.e. in} \quad \Omega, \quad \textit{for every } i = 1, \ldots, n, \tag{4}$$

then the following stronger conclusion holds true

$$\sup_{\Omega} u_1 \vee \cdots \vee u_n \le \sup_{\partial\Omega} u_1^+ \vee \cdots \vee u_n^+ + C_{\text{ABP}} ||f^+||_{L^N(\Omega)}, \tag{5}$$

where the constant C_{ABP} *depends only on* N, c_0, ν, *and* diam Ω.

Remark 1. Taking $n = 1$, $U \in W^{2,N}_{\text{loc}}(\Omega, \mathbb{R})$ in (5), we obtain Theorem 9.1 in [GT] (compare also with [BNV], Theorem 1.3). This means our results contain the classical ABP estimate for scalar equations.

Remark 2. Hypothesis (Ψ) implies that the matrix operator $L + \mathcal{C}$ satisfies the maximum principle in Ω (see [FM2]).

Remark 3. In the case when the second order coefficients of the elliptic operators $L_1, \ldots, L_n$ are continuous functions and $\partial\Omega$ has some regularity, we can weaken hypothesis (Ψ). More precisely, instead of $\Psi > 0$ in $\overline{\Omega}$ we could suppose that $\Psi > 0$ in Ω, with either $\Psi \not\equiv 0$ on $\partial\Omega$ or $L\Psi + \mathcal{C}\Psi \not\equiv 0$ in Ω. This weaker condition will be shown to be equivalent to (Ψ) (see in particular Lemma 9.1 in Part II).

Remark 4. The dependence in Ψ of the constant in Theorem 1.1 is expressed in terms of upper bounds on $\left|\dfrac{\nabla\Psi}{\psi_1 \wedge \ldots \wedge \psi_n}\right|_{L^\infty(\Omega)}$ and $\left|\dfrac{\psi_1 \vee \ldots \vee \psi_n}{\psi_1 \wedge \ldots \wedge \psi_n}\right|_{L^\infty(\Omega)}$.

Further, we establish a Harnack inequality for non-negative solutions of system (1).

Since we deal with a system, it is natural to expect that some of the functions $u_1, \ldots, u_n$ depend on each other. In order to describe this dependence we give two definitions.

The first definition introduces the frequently used notion of full coupling.

Definition 1.1 *A matrix $\mathcal{C}(x) = (c_{ij}(x))_{i,j=1}^n$, which satisfies (H3), is called irreducible in Ω, and the system $LU + \mathcal{C}U = F$ is called fully coupled in Ω, provided for any non-empty sets $I, J \subset \{1, \ldots, n\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \ldots, n\}$, $I \neq \emptyset$, $J \neq \emptyset$, there exist $i_0 \in I$ and $j_0 \in J$ for which*

$$\text{meas}\{x \in \Omega \mid c_{i_0 j_0}(x) > 0\} > 0. \tag{6}$$

For simplicity, when (6) holds we write $c_{i_0 j_0} \not\equiv 0$ in Ω.

In other words, a system is fully coupled provided it cannot be split into two subsystems, one of which does not depend on the other. Note that any 1×1 matrix is irreducible, that is, any scalar equation is a fully coupled system.

Next we give a notion of partial coupling for a non-fully coupled system. It is easy to see, by renumbering lines and columns (i.e. by permuting the indices of $u_1, \ldots, u_n$), that any matrix $\mathcal{C}$ can be written in the block triangular form

$$\mathcal{C} = (\mathcal{C}_{kl})_{k,l=1}^m, \tag{7}$$

so that the matrix which we obtain still satisfies (H3) ; here $1 \leq m \leq n$, $\mathcal{C}_{kl}$ are $t_k \times t_l$ matrices, $\sum_{k=1}^m t_k = n$, $\mathcal{C}_{kk}$ is an *irreducible* matrix for all $k = 1, \ldots, m$, and $\mathcal{C}_{kl} \equiv 0$ in Ω, for all $k, l \in \{1, \ldots, m\}$ with $k < l$. Note that $m = 1$ means $\mathcal{C}$ is irreducible, while $m = n$ means $\mathcal{C}$ is in triangular form.

From now on, we suppose that $\mathcal{C}(x)$ is written in the form (7). We set $s_0 = 0, s_k = \sum_{i=1}^k t_i$, $S_k = \{s_{k-1} + 1, \ldots, s_k\}$ and

$$\overline{u_k} = u_{s_{k-1}+1} \vee \ldots \vee u_{s_k}, \qquad \underline{u_k} = u_{s_{k-1}+1} \wedge \ldots \wedge u_{s_k},$$

for all $k \in \{1, \dots, m\}$.

Definition 1.2 *Let (1) be a non-fully coupled system and let $k > l$, for some $k, l \in \{1, \dots, m\}$. We call the system (1) (kl)-partially coupled, provided there exist indices $(i,j) \in S_k \times S_l$ such that $c_{ij} \not\equiv 0$ in Ω. For simplicity, in this case we write $\mathcal{C}_{kl} \not\equiv 0$ in Ω.*

We fix a point $x_0 \in \Omega$ and a ball $B_{3R} := B(x_0, 3R) \subset \Omega$. We suppose the matrix $\mathcal{C}(x)$ is written in the form (7) in B_{3R} (i.e. $\mathcal{C}_{kl} \equiv 0$ in B_{3R} for $k < l$).

We set, for all $k, l \in \{1, \dots, m\}$,

$$\Lambda_{kl}(B_{3R}) = \{\, (i,j) \mid (i,j) \in S_k \times S_l,\ i \neq j \text{ and } c_{ij} \not\equiv 0 \text{ in } B_{3R}\},\ \Lambda = \bigcup_{k,l=1}^{m} \Lambda_{kl},$$

(Λ can be empty) and fix $\alpha \in (0,3)$ such that $\Lambda_{kl}(B_{\alpha R}) = \Lambda_{kl}(B_{3R})$, for all $k, l \in \{1, \dots, m\}$.

We shall prove a Harnack inequality in the ball $B_{\alpha R}$. To avoid heavy notations, we take $\alpha = 1$. We set, for all $(i,j) \in \Lambda$,

$$\omega_{ij} = \{x \in B_R \mid c_{ij}(x) \geq \rho\},$$

where $\rho > 0$ is taken so that $\text{meas}(\omega_{ij}) > 0$.

Theorem 1.2 *Suppose (H1)–(H5) are satisfied and let $U \geq 0$ be a solution of (1) in B_{3R}. Then*

$$\sup_{B_R} \overline{u_k} \leq C \left(\inf_{B_R} \underline{u_k} + R\|f\|_{L^N(B_{3R})} \right), \tag{8}$$

for all $k \in \{1, \dots, m\}$.

If, in addition, (1) is (kl)-partially coupled, then

$$\sup_{B_R} \overline{u_k} \vee \overline{u_l} \leq C \left(\inf_{B_R} \underline{u_k} + R\|f\|_{L^N(B_{3R})} \right). \tag{9}$$

The constants C in (8) and (9) depend only on $n, N, c_0, \nu R^2, \rho$, and a positive lower bound for meas$(\omega_{ij}), (i,j) \in \Lambda \cap \{1, \dots, s_k\}^2$.

A particular case of (8) is the following

Corollary 1.1 *Under the hypotheses of Theorem 1.2 we have*

$$\sup_{B_R} u_i \leq C \left(\inf_{B_R} u_i + R\|f\|_{L^N(B_{3R})} \right), \tag{10}$$

for all $i = 1, \dots, n$.

If, in addition, the system (1) is fully coupled, then

$$\sup_{B_R} u_1 \vee \dots \vee u_n \leq C \left(\inf_{B_R} u_1 \wedge \dots \wedge u_n + R\|f\|_{L^N(B_{3R})} \right). \tag{11}$$

In the literature there have been some results on Harnack inequalities for elliptic systems. In particular, extending earlier results by Mandras [M], Chen and Zhao ([CZ]) obtained Corollary 1.1 for strong solutions of (1), in the case $F \equiv 0$, under Hölder regularity assumptions on the coefficients of the elliptic operators $L_1, \ldots, L_n$. Their proof is rather involved and makes use of estimates on the Green functions of $L_1, \ldots, L_n$. The result of Chen and Zhao was shown to hold for first and zero order coefficients in L^∞ by Arapostatis, Ghosh and Marcus ([AGM]). Finally, using a Nash-Moser type iteration technique, Muscalu ([Mu]) obtained a weak form of the Harnack inequality for a class of elliptic systems in divergence form.

The leading idea of our work is to use the properties of viscosity solutions of PDE's, which permit us to relate system (1) to a set of fully nonlinear elliptic equations and then apply the Alexandrov-Bakelman-Pucci and Harnack estimates for viscosity solutions of such equations. Our approach is nonlinear in nature and could apply to systems of nonlinear equations.

In the course of the proof of the Harnack estimate we obtain two estimates of independent interest, for subsolutions and supersolutions of system (1).

We set, for any $p > 0$,

$$|u|_{p,R} = \left(\frac{1}{\text{meas}(B_R)} \int_{B_R} |u|^p \right)^{1/p} .$$

Proposition 1.1 (local maximum principle) *Suppose (H1)–(H5) hold. Let*

$$LU + \mathcal{C}U \geq -F \tag{12}$$

in B_{2R}. Then for all $p > 0$ we have

$$\sup_{B_R} u_1 \vee \ldots \vee u_n \leq C \left(|u_1^+ \vee \ldots \vee u_n^+|_{p,2R} + R||f_1^+ \vee \ldots \vee f_n^+||_{L^N(B_{2R})} \right),$$

where $C = C(N, c_0, p, \nu R^2)$.

Proposition 1.2 (weak Harnack inequality) *Suppose (H1)–(H5) hold. Let $U \geq 0$ satisfy*

$$LU + \mathcal{C}U \leq F \tag{13}$$

in B_{3R}. Then there exists a number $p = p(N, c_0, \nu R^2) > 0$ such that for any $k \in \{1, \ldots, m\}$,

$$|\overline{u_k}|_{p,2R} \leq C \left(\inf_{B_R} \underline{u_k} + R||f_{s_{k-1}+1}^+ \vee \ldots \vee f_{s_k}^+||_{L^N(B_{3R})} \right), \tag{14}$$

and, in case (1) is (kl)-partially coupled,

$$|\overline{u_k} \vee \overline{u_l}|_{p,2R} \leq C \left(\inf_{B_R} \underline{u_k} + R||f_1^+ \vee \ldots \vee f_k^+||_{L^N(B_{3R})} \right), \tag{15}$$

where C is as in Theorem 1.2.

Remark. All our results fail for general non-cooperative systems. A simple example is provided by the system

$$\left\{ \begin{array}{lcl} \Delta u_1 - u_2 & = & 0 \\ \Delta u_2 & = & 0 \qquad \text{in } \ B_1. \end{array} \right. \tag{16}$$

This system satisfies all hypotheses of Theorems 1.1 and 1.2, except for (H3). By taking $u_1 = 1 - |x|^2, u_2 = -2N$, we see that Theorem 1.1 is false for (16). A counterexample for Proposition 1.2 (and Theorem 1.2) is obtained by setting $u_1 = |x|^2, u_2 = 2N$.

The rest of Part I (Sections 2 to 5) is devoted to the proofs of Theorems 1.1 and 1.2.

2 Definition and properties of viscosity solutions

In this section we define the class of viscosity solutions to which our results apply. The concept of a viscosity solution has found many applications in PDE's (starting with the fundamental work [CIL]).

A basic tool in the proof of the ABP estimate and the local maximum principle in our work is the ABP estimate for viscosity solutions of fully nonlinear elliptic equations, obtained by Caffarelli et al. in [C], Crandall et al. in [Cr], and Wang in [W]. We shall state, for the reader's convenience, the results from [W] and [C] that we need. We refer to [Caf] and [CC] for results on another class of viscosity solutions of fully nonlinear elliptic equations. Actually, it was remarked in [CC] (see notes at the end of Chapter 2 in [CC]) that their results can be extended to a degree which would also suit our needs.

Consider a measurable function $G : \mathcal{S}_N(\mathbb{R}) \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ ($\mathcal{S}_N(\mathbb{R})$ denotes the space of real symmetric $N \times N$ matrices) such that $G(\cdot, \cdot, \cdot, x)$ is locally uniformly continuous, uniformly for $x \in \Omega \backslash \mathcal{N}$, where $\mathcal{N}$ is a Lebesgue null set. Take a measurable function f defined on Ω. We shall give a precise meaning to the equation

$$G(D^2u, Du, u, x) = f \qquad \text{in } \ \Omega, \tag{17}$$

provided G is uniformly elliptic, that is, there exists $c_0 > 0$ such that

$$c_0|M'| \leq G(M+M',P,u,x) - G(M,P,u,x) \leq c_0^{-1}|M'|,$$

for any matrix $M \in \mathcal{S}_N(\mathbb{R})$, any positive definite matrix $M' \in \mathcal{S}_N(\mathbb{R})$, any $P \in \mathbb{R}^N$, $u \in \mathbb{R}$ and a.e. $x \in \Omega$.

Definition 2.1 *We say that $u \in C(\Omega)$ is a viscosity subsolution (supersolution) of (17), provided for any $\varepsilon > 0$, any open subset $\mathcal{O} \subset \Omega$, and any $\varphi \in W^{2,N}(\mathcal{O})$ (we call φ a test function), such that*

$$\begin{array}{rcl} G(D^2\varphi(x), D\varphi(x), u(x), x) & \leq & f(x) - \varepsilon \\ (G(D^2\varphi(x), D\varphi(x), u(x), x) & \geq & f(x) + \varepsilon) \qquad a.e.\ in\ \mathcal{O}, \end{array}$$

the function $u - \varphi$ cannot achieve a local maximum (minimum) equal to zero in $\mathcal{O}$. In this case we say that the function u satisfies the inequation

$$G(D^2u, Du, u, x) \geq (\leq) f$$

in the viscosity sense in Ω.

We say that u is a solution of (17) if u is at the same time a subsolution and a supersolution of (17).

Remark. This definition is equivalent to Definition 2.1 in [C], for $p = N$. It is easy to see, under our hypotheses on G, that the class of viscosity solutions the above definition introduces is smaller than the class considered by Wang in [W].

Definition 2.2 *The system of (in)equations*

$$LU + \mathcal{C}U \leq F \ (resp. \geq, =)$$

is said to be satisfied in the viscosity sense by $U \in C(\Omega, \mathbb{R}^n)$, provided the scalar (in)equations

$$\left\{ \begin{array}{rcl} L_i u_i & \leq & -\sum_{j=1}^{n} c_{ij}(x) u_j + f_i(x) \quad (resp. \geq, =) \\ i & = & 1, \ldots, n, \end{array} \right. \tag{18}$$

are satisfied in the viscosity sense, in the terms of Definition 2.1.

Remark. It is easily shown that the summation term in (18) can be put equivalently on the left-hand side of the inequality.

Proposition 2.1 *The maximum of two viscosity subsolutions is a viscosity subsolution. The minimum of two viscosity supersolutions is a viscosity supersolution.*

Proof. This is obvious from Definition 2.1. $\square$

Proposition 2.2 *Let $f \in L^N(\Omega)$ and let L_0 be a linear uniformly elliptic second order operator with bounded measurable coefficients. Suppose that $u \in W^{2,N}_{\mathrm{loc}}(\Omega)$ is such that $L_0 u \geq f$ a.e. in Ω. Then $L_0 u \geq f$ in the viscosity sense.*

Proof. Suppose for contradiction that there exists an open set $\mathcal{O} \subset \Omega$ (we can assume $\mathcal{O} \subset\subset \Omega$), $\varepsilon > 0$, $x_0 \in \mathcal{O}$ and a function $\varphi \in W^{2,N}(\mathcal{O})$ such that $\varphi - u$ vanishes at x_0, $\varphi - u \geq 0$ and

$$L_0\varphi \leq f - \varepsilon$$

a.e. in $\mathcal{O}$. Hence

$$L_0(\varphi - u) \leq -\varepsilon < 0 \tag{19}$$

a.e. in $\mathcal{O}$. We obtain a contradiction by applying the strong maximum principle (Theorem 9.22 in [GT]) to (19). $\square$

Next, we define the Pucci extremal operators

$$\begin{aligned} \mathcal{M}^+(M) &= c_0^{-1}\sum_{e_i>0} e_i + c_0 \sum_{e_i<0} e_i, \\ \mathcal{M}^-(M) &= c_0\sum_{e_i>0} e_i + c_0^{-1} \sum_{e_i<0} e_i, \end{aligned}$$

for $M \in \mathcal{S}_N(\mathbb{R})$, where $e_1, \dots, e_N$ denote the eigenvalues of M. Then (see [CC])

$$\mathcal{M}^+(M) = \sup_{A\in\mathcal{A}} \mathrm{tr}(AM)\ ,\ \mathcal{M}^-(M) = \inf_{A\in\mathcal{A}} \mathrm{tr}(AM), \tag{20}$$

where $\mathcal{A}$ denotes the set of all symmetric matrices whose eigenvalues lie in the interval $[c_0, c_0^{-1}]$. To relate our notations to that of [C] one has to note that $\mathcal{M}^+ = -\mathcal{P}^-$ and $\mathcal{M}^- = -\mathcal{P}^+$, with $\mathcal{P}^+, \mathcal{P}^-$ defined in [C]. It is not difficult to check that (see [CC])

$$-\mathcal{M}^+(-M) = \mathcal{M}^-(M)\ ,\ \mathcal{M}^+(\eta M) = \eta\mathcal{M}^+(M) \tag{21}$$

and

$$\begin{aligned} \mathcal{M}^+(M) + \mathcal{M}^-(N) \leq\ & \mathcal{M}^+(M+N) &\leq \mathcal{M}^+(M) + \mathcal{M}^+(N), \\ \mathcal{M}^-(M) + \mathcal{M}^-(N) \leq\ & \mathcal{M}^-(M+N) &\leq \mathcal{M}^-(M) + \mathcal{M}^+(N), \end{aligned} \tag{22}$$

for every two symmetric matrices M, N, and every $\eta \geq 0$.

We define

$$\mathcal{L}^+(D^2u, Du) = \mathcal{M}^+(D^2u) + b|Du|,$$
$$\mathcal{L}^-(D^2u, Du) = \mathcal{M}^-(D^2u) - b|Du|$$

(b is defined in (H2), $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$), so that

$$\mathcal{L}^+(D^2u, Du) = -\mathcal{L}^-(-D^2u, -Du).$$

The following lemma contains several easy, but important properties of $\mathcal{L}^+$ and $\mathcal{L}^-$.

Lemma 2.1 *(a) Suppose L_k satisfies (H1) and (H2), for all $k = 1, \ldots, n$. Then $L_k u \leq f$ (resp. $\geq$) implies*

$$\mathcal{L}^-(D^2u, Du) \leq f \qquad (\textit{resp. } \mathcal{L}^+(D^2u, Du) \geq f).$$

Conversely, $\mathcal{L}^+(D^2u, Du) \leq f$ implies $L_k u \leq f$, and $\mathcal{L}^-(D^2u, Du) \geq f$ implies $L_k u \geq f$, for all $k = 1, \ldots, n$.

(b) Let $u \in W^{2,N}_{\mathrm{loc}}(\Omega)$. There exist linear uniformly elliptic second order operators L_0^+, L_0^-, with bounded measurable coefficients, such that

$$\mathcal{L}^+(D^2u, Du) = L_0^+ u, \quad \mathcal{L}^-(D^2u, Du) = L_0^- u.$$

Furthermore, c_0 is an ellipticity constant for the operators L_0^+, L_0^-, and ν is an upper bound for the L^∞-norms of their first order coefficients.

Proof. Part (a) is a direct consequence of (20) and Definition 2.1. Part (b) follows from the fact that the supremum and the infimum in (20) are attained. For instance, we take

$$L_0^+ u(x) = \mathrm{tr}\left(A_0^+(x) D^2(x)\right) + \vec{b}(x).Du(x),$$

where $x \longrightarrow A_0^+(x)$ is a measurable selection of elements of $\mathcal{A}$ at which

$$\sup_{A \in \mathcal{A}} \mathrm{tr}\left(A D^2 u(x)\right)$$

is attained, and

$$\vec{b}(x) = \begin{cases} \dfrac{bDu(x)}{|Du(x)|}, & \text{if} \quad Du(x) \neq 0 \\ 0\,, & \text{if} \quad Du(x) = 0. \end{cases}$$

We shall use the following Alexandrov-Bakelman-Pucci inequality for fully nonlinear scalar equations.

Proposition 2.3 *(a) Let $u \in C(\overline{\Omega})$ be twice differentiable a.e. on its upper contact set, defined by*

$$\Gamma^+ = \{x \in \Omega \mid u(x) = \gamma(u)(x)\}, \tag{23}$$

where $\gamma(u)$ denotes the concave envelope of u, i.e.

$$\gamma(u) = \inf\{v \mid v \geq u \text{ and } v \text{ is concave}\}. \tag{24}$$

Suppose that $(-\mathcal{L}^+(D^2u, Du))^+ \in L^N(\Gamma^+)$. Then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + C\|(-\mathcal{L}^+(D^2u, Du))^+\|_{L^N(\Gamma^+)}, \tag{25}$$

where the constant C depends only on N, c_0, ν and $\operatorname{diam}\Omega$.
(b) Let $u \in C(\overline{\Omega})$ and $f \in L^N(\Omega)$ satisfy

$$-\mathcal{L}^+(D^2u, Du) \leq f \qquad \text{in } \ \Omega \cap \{u > 0\}.$$

Then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C\|f\|_{L^N(\Gamma^+)},$$

where C depends on the same quantities as in (a).

Part (a) and a particular case of part (b) of this proposition are due to Wang (see Theorem 3.14 and Lemma 3.19 in [W]). Statement (b) in its full generality is proved in [C] (Proposition 3.3 in [C]).

We shall make use of the following weak Harnack inequality for scalar equations (Corollary 4.14 in [W]; see Section 4.6 in [W] for more general equations).

Proposition 2.4 *Let $u \in C(\Omega)$ be a non-negative solution of*

$$\mathcal{L}^-(D^2u, Du) - cu \leq f \qquad \text{in } \ B_2 \subset \Omega,$$

where $c \in L^\infty(B_2)$, with $0 \leq c(x) \leq \nu$ a.e. in B_2, and $f \in L^N(B_2)$. Then there exists $p = p(N, c_0, \nu) > 0$ such that

$$|u|_{p,1} \leq C\left(\inf_{B_1} u + \|f\|_{L^N(B_2)}\right),$$

where $C = C(N, c_0, \nu)$.

In fact, this proposition was proved in [W] in the particular case $f \in L^\infty$, but extension to L^N is straightforward, since the proof in [W] relies only on the ABP inequality, which holds true for right-hand sides in L^N (see Proposition 2.3 above).

The following existence result for maximal operators will be useful in the sequel.

Proposition 2.5 *Let $c, f \in L^\infty(B_2)$, and $0 \le c \le \nu$ a.e. in B_2. Then there exists a unique solution $u \in W^{2,N}_{\mathrm{loc}}(B_2) \cap C(\overline{B_2})$ of the following problem*

$$\begin{cases} -\mathcal{L}^-(D^2u, Du) + cu &= f \quad \text{a.e. in } B_2 \\ u &= 0 \quad \text{on } \partial B_2 \end{cases} \tag{26}$$

Moreover, one has the interior estimate

$$\|u\|_{W^{2,N}(B_1)} \le C\left(\|u\|_{L^\infty(B_2)} + \|f\|_{L^N(B_2)}\right), \tag{27}$$

where $C = C(N, c_0, \nu)$.

Proof. When $c \equiv 0$ this result was proved in [C] (Corollary 3.10 in that paper). Exactly the same proof works for $c \ge 0$, since the authors use Theorem 17.17 in [GT] and the ABP estimate, which both hold when $c \ge 0$.

Finally, we prove a lemma concerning sums and products of viscosity solutions and test functions.

Lemma 2.2 *Suppose the linear operator*

$$L_0 = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} \tag{28}$$

is uniformly elliptic in Ω, and suppose L_0 has bounded measurable coefficients.

(a) Let $u \in C(\Omega)$ and $\psi \in W^{2,N}_{\mathrm{loc}}(\Omega)$ satisfy $L_0 u \ge f$ in Ω and $L_0\psi \ge g$ a.e. in Ω. Then

$$L_0(u + \psi) \ge f + g \quad \text{in } \Omega. \tag{29}$$

(b) Let $u \in C(\Omega)$ satisfy $L_0 u \ge f$ in Ω and let $\psi \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be strictly positive in $\overline{\Omega}$. Define $\tilde{u} = \dfrac{u}{\psi}$ and $\tilde{f} = \dfrac{f}{\psi}$. Then we have

$$\tilde{L}\tilde{u} \ge \tilde{f} \quad \text{in } \Omega,$$

where $\tilde{L}$ is defined by

$$\tilde{L} = L_0 + 2\sum_{i,j=1}^N a_{ij}(x)\frac{\partial_i \psi}{\psi}\frac{\partial}{\partial x_j} + \frac{L_0\psi}{\psi}.$$

Proof. Suppose (29) does not hold. Then, by Definition 2.1, we can find an open set $\mathcal{O} \subset \Omega$ (we can assume $\mathcal{O} \subset\subset \Omega$) , $\varepsilon > 0$, $x_0 \in \mathcal{O}$ and a function $\varphi \in W^{2,N}(\mathcal{O})$ such that

$$\begin{aligned} L_0\varphi &\leq f + g - \varepsilon \\ &\leq f + L_0\psi - \varepsilon \quad \text{a.e. in } \mathcal{O}, \end{aligned} \tag{30}$$

$\varphi \geq u+\psi$ in $\mathcal{O}$ and $\varphi(x_0) = u(x_0)+\psi(x_0)$. Since $\psi \in W^{2,N}(\mathcal{O})$, the function $\varphi - \psi$ is a legitimate test function for the equation $L_0 u \geq f$. This leads to a contradiction.

Next, suppose (b) is false. Then we can find an open set $\mathcal{O} \subset\subset \Omega$, $\varepsilon > 0$, $x_0 \in \mathcal{O}$, $\varphi \in W^{2,N}(\mathcal{O})$ such that $\varphi \geq \tilde{u}$ in $\mathcal{O}$, $\varphi(x_0) = \tilde{u}(x_0)$, and

$$\tilde{L}\varphi \leq \tilde{f} - \varepsilon \quad \text{a.e. in } \mathcal{O}. \tag{31}$$

A simple computation transforms (31) into

$$\begin{aligned} L_0(\varphi\psi) &\leq f - \varepsilon\psi \\ &\leq f - \varepsilon \min_{\mathcal{O}} \ \psi \qquad \text{a.e. in } \mathcal{O}. \end{aligned} \tag{32}$$

Since $\varphi\psi \in W^{2,N}(\Omega)$ ($W^{2,N}$ is an algebra), $\varphi\psi \geq u$ in $\mathcal{O}$, $\varphi\psi(x_0) = u(x_0)$, we obtain a contradiction with $L_0 u \geq f$. □

3 Proof of the ABP Estimate

This section is concerned with the proof of Theorem 1.1. First, let us prove part (b). We suppose that (1) satisfies the condition

$$\sum_{j=1}^{n} c_{ij}(x) \leq 0 \quad \text{a.e. in} \quad \Omega, \tag{33}$$

for all $i = 1, \ldots, n$. We set

$$v = u_1^+ \vee \cdots \vee u_n^+ - \sup_{\partial\Omega} u_1^+ \vee \cdots \vee u_n^+,$$

so that $v \leq 0$ on $\partial\Omega$.

Lemma 3.1 *We have*

$$-\mathcal{L}^+(D^2 v, Dv) \leq f_1^+ \vee \cdots \vee f_n^+ \quad \text{in } \Omega \tag{34}$$

in the viscosity sense.

Proof. Since (34) does not change when we add a constant to v, we can suppose in the proof of this lemma that $v = u_1^+ \vee \cdots \vee u_n^+$.

Suppose for contradiction that there exist an open set $\mathcal{O} \subset\subset \Omega$, $x_0 \in \mathcal{O}$, $\varepsilon > 0$, and a function $\varphi \in W^{2,N}(\mathcal{O})$ such that $v \leq \varphi$ in $\mathcal{O}$, $v(x_0) = \varphi(x_0)$, and

$$-\mathcal{L}^+(D^2\varphi, D\varphi) \geq f_1^+ \vee \cdots \vee f_n^+ + \varepsilon \quad \text{a.e. in } \mathcal{O}. \tag{35}$$

We have to show that (35) is impossible. Fix $k \in \{1, \ldots, n\}$ such that $u_k^+(x_0) = v(x_0)$. Then

$$\varphi \geq u_k^+ \geq 0 \text{ in } \mathcal{O} \quad \text{and} \quad \varphi(x_0) = u_k^+(x_0). \tag{36}$$

On the other hand, by (35) and Lemma 2.1 (a),

$$-L_k\varphi \geq f_1^+ \vee \ldots \vee f_k^+ + \varepsilon > 0 \quad \text{a.e. in} \quad \mathcal{O}. \tag{37}$$

We distinguish two cases. First, if $\varphi(x_0) = 0$, we see that φ attains a local minimum at x_0. Then we apply the strong maximum principle (Theorem 9.22 in [GT]) to (37) and obtain $\varphi \equiv 0$ in $\mathcal{O}$. This contradicts (37).

Second, if $\varphi(x_0) > 0$, we have, by (33), (37) and (H3),

$$\begin{aligned} -L_k\varphi &\geq \left(\sum_{j=1}^n c_{kj}\right) u_k + f_k^+ + \varepsilon & (38)\\ &\geq \sum_{j=1}^n c_{kj}u_j + f_k^+ + \frac{\varepsilon}{2} \qquad \text{a.e. in } \mathcal{O}_1, & (39)\end{aligned}$$

where $\mathcal{O}_1 \subset \mathcal{O}$ is an open neighbourhood of x_0 in which

$$u_k > 0 \quad \text{and} \quad u_k \geq u_j - \varepsilon/2\nu,$$

for all $j \in \{1, \ldots, n\}$ ($\mathcal{O}_1$ exists, since $u_k(x_0) = u_1 \vee \ldots \vee u_n(x_0) > 0$).

Since u_k is a (viscosity) solution of

$$-L_k u_k \leq \sum_{j=1}^n c_{kj}u_j + f_k^+,$$

we obtain a contradiction with (36) and (39). Lemma 3.1 is proved. □

Now we can apply Proposition 2.3 to (34). We obtain

$$\sup_\Omega v \leq C||f_1^+ \vee \cdots \vee f_n^+||_{L^N(\Omega)},$$

which gives part (b) of Theorem 1.1. Here, as in the rest of the paper, C denotes a constant which may vary from line to line.

In case the system (1) satisfies condition (Ψ), we make the following change of functions

$$\overline{u}_k = \frac{u_k}{\psi_k}, \quad \overline{f}_k = \frac{f_k}{\psi_k}. \tag{40}$$

A simple computation yields

$$\frac{1}{\psi_k} L_k u_k = \overline{L}_k \overline{u}_k + \overline{u}_k \left(\frac{L_k \psi_k}{\psi_k} \right),$$

for all $k = 1, \dots, n$, where, as in Lemma 2.2,

$$\overline{L}_k = L_k + 2 \sum_{i,j=1}^{N} a_{ij}^k(x) \frac{\partial_i \psi_k}{\psi_k} \frac{\partial}{\partial x_j}. \tag{41}$$

We have then, by Lemma 2.2,

$$-\overline{L}_k \overline{u}_k \leq \sum_{j=1}^{n} \overline{c}_{kj} \overline{u}_j + \overline{f}_k, \tag{42}$$

where

$$\overline{c}_{kj}(x) = \frac{c_{kj} \psi_j}{\psi_k} + \delta_{kj} \frac{L_k \psi_k}{\psi_k}.$$

We see that (Ψ) implies

$$\sum_{j=1}^{n} \overline{c}_{kj}(x) \leq 0 \quad \text{in} \quad \Omega, \tag{43}$$

for all $k = 1, \dots, n$. Thus, by making the change of functions (40), we obtain a new cooperative system which satisfies (33). By applying the ABP estimate for such systems, which we already proved, we obtain

$$\sup_{\Omega} u_1 \vee \cdots \vee u_n \leq C \frac{\sup_{\Omega} \psi_1 \vee \cdots \vee \psi_n}{\inf_{\Omega} \psi_1 \wedge \cdots \wedge \psi_n} \left(\sup_{\partial\Omega} u_1^+ \vee \cdots \vee u_n^+ + ||f^+||_{L^N(\Omega)} \right).$$

Theorem 1.1 is proved. □

4 Proof of the Local Maximum Principle

This section is devoted to the proof of Proposition 1.1. From now on, we suppose that $R = 1$ and $x_0 = 0$, the general case being obtained by means of the coordinate transformation $x \longrightarrow \dfrac{x - x_0}{R}$.

First, we claim that the function u_k^+ satisfies the inequation

$$-\mathcal{L}^+(D^2u_k^+, Du_k^+) \le \nu\, u_1^+ \vee \ldots \vee u_n^+ + f^+ \qquad \text{in } B_2 \tag{44}$$

(in the viscosity sense), for all $k = 1, \ldots, n$. Recall that $f = f_1 \vee \ldots \vee f_n$.

Proof of (44). Set $v = u_1^+ \vee \ldots \vee u_n^+$. Suppose (44) is false. Then, by Definition 2.1, there exist an open set $\mathcal{O} \subset\subset \Omega$, $x_0 \in \mathcal{O}$, $\varphi \in W^{2,N}(\mathcal{O})$, and a real number $\varepsilon > 0$ such that $v \le \varphi$ in $\mathcal{O}$, $v(x_0) = \varphi(x_0)$ and

$$-\mathcal{L}^+(D^2\varphi, D\varphi) \ge \nu v + f^+ + \varepsilon \quad \text{a.e. in } \mathcal{O}. \tag{45}$$

Observe that the hypotheses of Proposition 1.1 imply

$$-L_k u_k \le \nu v + c_{kk}^- u_k^- + f^+ \quad \text{in } \Omega \tag{46}$$

for all $k = 1, \ldots, n$. We then proceed as in the proof of the ABP estimate. If $v(x_0) = \varphi(x_0) > 0$, we take k such that $v(x_0) = u_k(x_0)$ and obtain a contradiction with (45) and (46) in some open set $\mathcal{O}_1 \subset \mathcal{O}$. If $\varphi(x_0) = 0$, φ attains a local minimum at x_0 and we obtain a contradiction with the strong maximum principle, as in the proof of the ABP estimate. $\square$

Using the fact that the maximum of subsolutions is a subsolution (Proposition 2.1), we obtain from (44)

$$-\mathcal{L}^+(D^2v, Dv) \le \nu v + f^+ \qquad \text{in } B_2 \tag{47}$$

in the viscosity sense.

In order to obtain the conclusion of Proposition 1.1 we use Proposition 2.3, combined with the localization argument in the proof of Theorem 9.20 in [GT]. We are going to show that this argument can be adapted to our situation.

We set $w = \eta v$, where, as in [GT],

$$\eta(x) = 4^{-\beta}(4 - |x|^2)^\beta,$$

for some $\beta \ge 1$.

Lemma 4.1 *For any $\delta > 0$ there exists a positive constant K, depending on δ, N, c_0, ν and β, such that*

$$-\mathcal{M}^+(D^2w) - K|Dw| \le K(v + f) \qquad \text{in } B_{2-\delta}. \tag{48}$$

Before proving Lemma 4.1, let us show how Proposition 1.1 follows from this lemma. The sense of Lemma 4.1 is that w is a (viscosity) subsolution of a fully nonlinear elliptic PDE with right-hand side in L^N.

By Proposition 3.5 in [C] w is a twice superdifferentiable function a.e. in $B_{2-\delta}$ (for a definition of twice sub- and super-differentiability, see page 381 in [C]). Since w is concave on its upper contact set Γ^+, is it clearly twice subdifferentiable a.e. on Γ^+. The function w, being at the same time twice subdifferentiable and superdifferentiable a.e. on $\Gamma^+ \cap B_{2-\delta}$, is twice differentiable a.e. on this set (by the generalized Rademacher-Stepanov theorem, used as in [C], page 384). Hence v is a twice differentiable function almost everywhere on $\Gamma^+ \cap B_{2-\delta}$. Letting $\delta \to 0$ we see that w and v are twice differentiable a.e. on Γ^+.

Remark. The use of the Rademacher-Stepanov theorem can be avoided in the case $f \in L^\infty$. In this case Wang gave a simple proof of the fact that w is $C^{1,1}$ (hence twice differentiable a.e.) on Γ^+ – see Corollary 3.17 in [W].

We continue the proof of Proposition 1.1. By (20), (21) and (22), we have

$$\begin{aligned}
-\mathcal{L}^+(D^2w, Dw) \;&\leq\; -\mathcal{M}^+(\eta D^2 v) - 2\mathcal{M}^-(D\eta \otimes Dv) \\
&\qquad -\mathcal{M}^-(vD^2\eta) - b\,|\eta Dv + vD\eta| \\
&\leq\; \eta\left(-\mathcal{M}^+(D^2 v) - b|Dv|\right) \\
&\qquad +2\mathcal{M}^+((-D\eta)\otimes Dv) + \mathcal{M}^+(-vD^2\eta) + bv|D\eta| \\
&\leq\; \nu w + f^+ + 2Nc_0^{-1}\left(|D\eta||Dv| + |D^2\eta|v\right) + bv|D\eta|
\end{aligned}$$

a.e. on Γ^+.

By a concavity argument we get, as in the proof of Theorem 9.20 in [GT],

$$|Dv| \leq C\eta^{-\frac{1}{\beta}} v$$

on Γ^+. Since

$$|D\eta| \leq C\eta^{1-\frac{1}{\beta}}\ ,\ |D^2\eta| \leq C\eta^{1-\frac{2}{\beta}},$$

we obtain

$$-\mathcal{L}^+(D^2w, Dw) \leq C\left(\eta^{-\frac{2}{\beta}} w + f^+\right) \tag{49}$$

a.e. on Γ^+.

By using Proposition 2.3 (a), together with (49), we obtain

$$\sup_{B_2} w \leq C\|\eta^{-\frac{2}{\beta}} w + f^+\|_{L^N(B_2)}.$$

Then we can complete the proof of Proposition 1.1, following the lines of the proof of Theorem 9.20 in [GT] (see the last three inequalities in this proof).

Finally, let us prove Lemma 4.1. Suppose that Lemma 4.1 is false, that is, there exists $\delta > 0$ such that for any $K > 0$ we can find an open set $\mathcal{O} \subset\subset B_{2-\delta}$, $x_0 \in \mathcal{O}$, a function $\varphi \in W^{2,N}(\mathcal{O})$ and a positive number ε for which $\varphi(x_0) = w(x_0), \varphi \geq w$ in $\mathcal{O}$, and

$$-\mathcal{M}^+(D^2\varphi) - K\,|D\varphi| \geq K(v+f) + \varepsilon \tag{50}$$

a.e. in $\mathcal{O}$.

Since $w = \eta v$, the function $\dfrac{\varphi}{\eta}$ is a legitimate test function for (47) in $\mathcal{O}$. We have, by (50),

$$\begin{aligned}
-\mathcal{L}^+(D^2\frac{\varphi}{\eta}, D\frac{\varphi}{\eta}) &= -\mathcal{M}^+(D^2\frac{\varphi}{\eta}, D\frac{\varphi}{\eta}) - b\left|D\frac{\varphi}{\eta}\right| \\
&\geq \frac{1}{\eta}\left(-\mathcal{M}^+(D^2\varphi) - b\,|D\varphi|\right) - 2\mathcal{M}^+\left(D\varphi \otimes D\frac{1}{\eta}\right) \\
&\quad -\mathcal{M}^+\left(\varphi D^2\frac{1}{\eta}\right) - C(\delta)b\varphi\,|D\eta| \\
&\geq \frac{1}{\eta}\left((K-b)\,|D\varphi| + K(v+f^+) + \varepsilon\right) \\
&\quad -C(\delta)\frac{N}{c_0}\left(|D\varphi| + \varphi\right) \\
&\geq \frac{K}{2}v - C(\delta)\frac{N}{c_0}\frac{\varphi}{\eta} + \nu v + f^+ + \varepsilon
\end{aligned} \tag{51}$$

a.e. in $\mathcal{O}$, provided $K = K(\delta, N, c_0, \nu, \beta)$ is large enough. More precisely, we take

$$K = 1 + 2\nu + b + 2C(\delta)Nc_0^{-1}.$$

Since v and φ are continuous functions, there exists an open neighbourhood $\mathcal{O}_1$ of x_0, with $\mathcal{O}_1 \subset \mathcal{O}$, such that

$$v \geq \frac{\varphi}{\eta} - \frac{\varepsilon}{2K} \qquad \text{in } \mathcal{O}_1.$$

Hence (51) yields

$$-\mathcal{L}^+(D^2\frac{\varphi}{\eta}, D\frac{\varphi}{\eta}) \geq \nu v + f^+ + \frac{\varepsilon}{2}$$

a.e. in $\mathcal{O}_1$. This contradicts (47). $\square$

5 Proof of the Harnack Inequality

5.1 The case of a fully coupled system

In this section we prove the Harnack and the weak Harnack inequalities (that is, Theorem 1.2 and Proposition 1.2) in the particular case when the system (1) is fully coupled. The proof is based on the following lemma.

Lemma 5.1 *Let the conditions of Proposition 1.2 hold. Let $k, l \in \{1, \ldots, n\}$, $k \neq l$, be such that $c_{kl} \not\equiv 0$ in B_1. Then*

$$\inf_{B_2} u_l \leq C \left(\inf_{B_1} u_k + \|f_k^+\|_{L^N(B_2)} \right), \tag{52}$$

where the constant C is as in Theorem 1.2. If, in addition, (1) is fully coupled, then

$$\inf_{B_2} u_l \leq C \left(\inf_{B_1} u_k + \|f^+\|_{L^N(B_2)} \right), \tag{53}$$

for any $k, l \in \{1, \ldots, n\}$, with $k \neq l$.

In the proof of Lemma 5.1 we use the following equivalent definition of full coupling, apparently due to Chen and Zhao.

Lemma 5.2 *The system (1) is fully coupled in Ω if and only if for any $k, l \in \{1, \ldots, n\}$, with $k \neq l$, we can find a sequence of indices $\{i_j\}_{j=0}^r$, such that $i_j \neq i_{j+1}$ for all $j = 0, \ldots, r-1, i_0 = k, i_r = l$, and*

$$c_{i_j i_{j+1}} \not\equiv 0 \quad \text{in } \Omega,$$

for all $j = 0, \ldots, r-1$.

We shall also use the following simple lemma.

Lemma 5.3 *Let $a_i, i = 1, \ldots, n$, be non-negative real numbers and let $\alpha > 0$. Then*

$$\min\{1, n^{\alpha-1}\} \sum_{i=1}^n a_i^\alpha \leq \left(\sum_{i=1}^n a_i \right)^\alpha \leq \max\{1, n^{\alpha-1}\} \sum_{i=1}^n a_i^\alpha. \tag{54}$$

Proof of Lemma 5.3. Without restricting the generality we can suppose that all a_i are positive.

First suppose that $\alpha > 1$. Since the function $g(x) = x^\alpha$ is convex, we have

$$g\left(\sum_{i=1}^{n} \frac{a_i}{n}\right) \leq \frac{1}{n} \sum_{i=1}^{n} g(a_i),$$

from which we get the right inequality in (54). Since $b^\alpha < b$ for any $b \in (0,1)$, we have

$$\frac{a_i^\alpha}{\left(\sum\limits_{j=1}^{n} a_j\right)^\alpha} \leq \frac{a_i}{\sum\limits_{j=1}^{n} a_j}, \tag{55}$$

and the left inequality follows, by summing (55) from 1 to n.

If $\alpha < 1$ the left inequality follows from the concavity of g, whence the right inequality follows from the fact that $b^\alpha > b$ for any $b \in (0,1)$. $\square$

Proof of Lemma 5.1. We can suppose that

$$\inf_{B_2} u_l > 0$$

(otherwise (52) is trivial).

We begin with the case when $c_{kl} \not\equiv 0$ in B_1. By using the cooperativeness of (1), we obtain from the k-th equation in (1)

$$\left(L_k - c_{kk}^-\right) u_k + c_{kl} u_l \leq f_k.$$

Next, we take a solution of the following problem

$$\begin{cases} \left(-L_k + c_{kk}^-\right) w & \leq & c_{kl} \inf\limits_{B_2} u_l & \text{a.e. in} & B_2 \\ w & \in & W^{2,q}_{\mathrm{loc}}(B_2) \cap C(\overline{B_2}) & \text{for all} & q > 1 \\ w & = & 0 & \text{on} & \partial B_2. \end{cases}$$

For instance, the function w can be taken to be the solution of the following equation

$$\begin{cases} -\mathcal{L}^-(D^2 w, Dw) + \left|c_{kk}^-\right|_{L^\infty(B_2)} w & = & c_{kl} \inf\limits_{B_2} u_l & \text{in} & B_2 \\ w & = & 0 & \text{on} & \partial B_2. \end{cases} \tag{56}$$

(this problem is solvable, by Proposition 2.5). We obtain, with the help of Lemma 2.2,

$$\begin{aligned} \left(L_k - c_{kk}^-\right)(w - u_k) & \geq c_{kl}\left(-\inf_{B_2} u_l + u_l\right) - f_k \\ & \geq -f_k \qquad \text{in } B_2. \end{aligned}$$

Hence, by Lemma 2.1,

$$\mathcal{L}^+(D^2(w-u_k), D(w-u_k)) - c_{kk}^-(w-u_k) \geq -f_k \qquad \text{in } \ B_2.$$

Since $w - u_k \leq 0$ on ∂B_2, we can use the ABP estimate for elliptic equations (Proposition 2.3 (b)) to obtain

$$-\inf_{B_2}(u_k - w) = \sup_{B_2}(w - u_k) \leq C\|f_k^+\|_{L^N(B_2)}.$$

Hence

$$u_k(x) \geq \inf_{B_1} w - C\|f_k^+\|_{L^N(B_2)}, \tag{57}$$

for all $x \in B_1$. We now use the following consequence of a theorem by Krylov (Theorem 12 on p. 129 in [K]), in the form which has been stated in [BNV].

Theorem 5.1 *Let L_0 be an uniformly elliptic operator with bounded measurable coefficients in the form (28), and suppose $c \in L^\infty(B_2)$. Suppose c_0 is an ellipticity constant for L_0, and ν is an upper bound for the L^∞-norms of c and the first order coefficients of L_0. Let $w \in W^{2,N}_{\mathrm{loc}}(B_2)$ be a positive function satisfying $L_0 w + cw \leq 0$ a.e. in B_2 and $L_0 w + cw \leq -\rho$ a.e. in a closed subset $\omega \subset B_2$, for some $\rho > 0$. Then there exists a constant $m > 0$, depending only on N, c_0, ν, and a positive lower bound on* meas $(\omega) > 0$, *such that*

$$\inf_{B_1} w \geq m\rho. \tag{58}$$

Since $c_{kl} \geq 0$ and $c_{kl} \not\equiv 0$ in B_1, we can find $\rho > 0$ such that the set

$$\omega = \{x \in B_1 \ : \ c_{kl}(x) \geq \rho\}$$

contains a closed set of positive measure, which can be taken arbitrarily close to meas(ω). We are now in a position to apply Theorem 5.1 to the operator $L_0^- - |c_{kk}^-|$, where L_0^- is the linear operator defined in Lemma 2.1, satisfying

$$L_0^- w = \mathcal{L}^-(D^2 w, Dw).$$

We infer from (57) and (58)

$$\inf_{B_1} u_k \geq m \inf_{B_2} u_l - C\|f_k^+\|_{L^N(B_2)},$$

which is the statement of Lemma 5.1.

Finally, if (1) is fully coupled and $c_{kl} \equiv 0$ in B_1, we take a sequence $\{i_j\}_{j=0}^r$ as in Lemma 5.2 and a sequence of nested balls $\{B_{\alpha_j}\}_{j=0}^r$, $\alpha_j = 1 + \dfrac{j}{r}$. Then, as above,

$$\inf_{B_{\alpha_{j+1}}} u_{i_{j+1}} \leq C\left(\inf_{B_{\alpha_j}} u_{i_j} + \|f_{i_j}^+\|_{L^N(B_2)}\right). \tag{59}$$

Lemma 5.1 follows by iterating (59) r times (note that $r \leq n(n-1)$). $\square$

Next, using the cooperativeness of (1), we have

$$\left(L_k - c_{kk}^-\right) u_k \leq f_k \qquad \text{in} \;\; B_2, \tag{60}$$

for all $k = 1, \ldots, n$. The weak Harnack inequality for scalar equations (Proposition 2.4) yields

$$|u_k|_{p_k,2} \leq C \left(\inf_{B_2} u_k + \|f\|_{L^N(B_3)} \right), \tag{61}$$

where p_k and C are positive constants which depend only on N, c_0 and ν.

We set $p = \min\{p_1, \ldots, p_n\}$ and note that $|\cdot|_{p,R}$ is non-decreasing in p. We obtain, with the help of Lemmas 5.1 and 5.3,

$$\begin{aligned}
|u_1 \vee \ldots \vee u_n|_{p,2} &\leq |u_1 + \ldots + u_n|_{p,2} \\
&\leq \max\{n^{1-\frac{1}{p}}, 1\} \sum_{i=1}^n |u_i|_{p,2} \\
&\leq C \left(\sum_{i=1}^n \inf_{B_2} u_i + \|f\|_{L^N(B_3)} \right) \\
&\leq C \left(\min_{1 \leq i \leq n} \inf_{B_1} u_i + \|f\|_{L^N(B_3)} \right) \\
&= C \left(\inf_{B_1} u_1 \wedge \ldots \wedge u_n + \|f\|_{L^N(B_3)} \right).
\end{aligned}$$

This concludes the proof of Proposition 1.2 in the case when (1) is fully coupled. $\square$

The full Harnack inequality (Theorem 1.2) is an immediate consequence of Propositions 1.1 and 1.2.

5.2 The General Case

The proofs of the Harnack and the weak Harnack inequalities in the general case will be carried out via an induction argument. We use induction with respect to m, where, we recall, m is the number of irreducible blocks which appear when we write the matrix $\mathcal{C}$ in the form (7).

The case $m = 1$ was studied in the previous section. Let us suppose Theorem 1.2 and Proposition 1.2 hold for systems with at most $m-1$ fully coupled subsystems, and let us have a system with m fully coupled subsystems.

Note that, by the induction hypothesis, Theorem 1.2 and Proposition 1.2 hold for the system formed by the first s_{m-1} equations in (1). It remains to prove (8), (9), (14) and (15) for $k = m$.

The inequality (14) is a consequence of the argument in the previous section and the fact that (52) and (60) hold for $k, l \in S_m$.

Let us prove (15). Fix $l \in \{1, \ldots, m-1\}$ such that $\mathcal{C}_{ml} \not\equiv 0$ in B_1. Let for example $c_{i_0 j_0} \not\equiv 0$ in B_1, for some $i_0 \in S_m, j_0 \in S_l$. Then, by Lemma 5.1,

$$\inf_{B_{\frac{3}{2}}} u_{j_0} \leq C \left(\inf_{B_2} u_{i_0} + \|f_{i_0}^+\|_{L^N(B_2)} \right)$$

Using Lemma 5.1, (61) and the induction hypothesis, we obtain

$$\begin{aligned}
|\overline{u_m} \vee \overline{u_l}|_{p,2} &\leq C\left(|\overline{u_m}|_{p,2} + |\overline{u_l}|_{p,2}\right) \\
&\leq C \left(\sum_{i \in S_m} \inf_{B_2} u_i + \min_{j \in S_l} \inf_{B_{\frac{3}{2}}} u_j + \|f^+\|_{L^N(B_3)} \right) \\
&\leq C \left(\sum_{i \in S_m} \inf_{B_2} u_i + \inf_{B_{\frac{3}{2}}} u_{j_0} + \|f^+\|_{L^N(B_3)} \right) \\
&\leq C \left(\min_{i \in S_m} \inf_{B_1} u_i + \|f^+\|_{L^N(B_3)} \right) \\
&= C \left(\inf_{B_1} \underline{u_m} + \|f^+\|_{L^N(B_3)} \right),
\end{aligned}$$

which proves (15).

Finally, let us prove (8) and (9). We distinguish two cases.

Case 1. There exists a number $l \in \{1, \ldots, m-1\}$ such that $\mathcal{C}_{kl} \equiv 0$ in B_1, for all $k > l$.

In this case we remove from (1) the equations with numbers in S_l and obtain a system to which the induction hypothesis applies.

Case 2. For all $l \in \{1, \ldots, m-1\}$ there exists $k > l$ such that $\mathcal{C}_{kl} \not\equiv 0$ in B_1.

In this case we can even prove that

$$\sup_{B_1} u_1 \vee \ldots \vee u_n \leq C \left(\inf_{B_1} \underline{u_m} + \|f\|_{L^N(B_3)} \right). \tag{62}$$

In view of Proposition 1.1 it suffices to prove that

$$|\overline{u_l}|_{p,2} \leq C \left(\inf_{B_1} \underline{u_m} + \|f\|_{L^N(B_3)} \right), \tag{63}$$

for all $l \in \{1, \dots, m\}$. For $l = m$ this follows from (14). On the other hand, the assumption of Case 2 implies, by (15), that for all $l \in \{1, \dots, m-1\}$ there exists $k > l$ such that

$$|\overline{u_l}|_{p,2} \le C \left(\inf_{B_1} \underline{u_k} + \|f\|_{L^N(B_3)} \right).$$

Finally, taking at most $n-1$ nested balls between B_1 and B_2 and applying the latter inequality between each two of them, we obtain (63). $\square$

Harnack Type Estimates for Linear Elliptic Systems and Applications

Part II : Applications

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In the second part of the paper we give several applications of the results obtained in Part I. We prove a maximum principle in unbounded domains and a sharp strong maximum principle for cooperative systems. The most important application is the existence of a principal eigenvalue and a principal eigenfunction of a fully coupled system. This result permits us to obtain a necessary and sufficient condition for a cooperative (not necessarily fully coupled) system to satisfy the maximum principle. Finally, we show how our results can be applied to give Harnack type estimates for a class of higher order elliptic equations, including the biharmonic and the polyharmonic equation. We show the existence of a principal eigenvalue and a principal eigenfunction for these equations, in a sense which seems to be new.

6 A Maximum Principle in Unbounded Domains

We say that the operator $L+\mathcal{C}$ satisfies the maximum principle in Ω provided for any $U \in C(\overline{\Omega}, \mathbb{R}^n)$,

$$\left\{ \begin{array}{rcll} LU + \mathcal{C}U & \leq & 0 & \text{in } \Omega \\ U & \geq & 0 & \text{on } \partial\Omega \end{array} \right. \tag{64}$$

implies $U \geq 0$ in Ω. When Ω is bounded, it is well-known that the assumptions of Theorem 1.1 (the ABP estimate) are sufficient to ensure that $L + \mathcal{C}$ satisfies the maximum principle in Ω (see [FM1]). This fact is a very particular case of Theorem 1.1.

The ABP estimate can also be used to derive a maximum principle in unbounded domains.

Proposition 6.1 *Let $\Omega \subseteq \mathbb{R}^N$ be a domain (bounded or unbounded). Suppose (H1)-(H4) hold and*

$$\sum_{j=1}^{n} c_{ij}(x) \leq -\delta < 0 \qquad a.e. \text{ in } \Omega \tag{65}$$

for all $i = 1, \ldots, n$, and some $\delta > 0$. Then there exists $\varepsilon_0 = \varepsilon_0(\delta, N, c_0, \nu) > 0$ such that

$$\begin{cases} LU + \mathcal{C}U & \leq & 0 & \text{in} & \Omega, & \\ U & \geq & 0 & \text{on} & \partial\Omega, & \text{if } \partial\Omega \neq \emptyset \\ u_1^-(x) \vee \ldots \vee u_n^-(x) & \leq & Ce^{\varepsilon_0|x|} & \text{in} & \Omega, & \text{for some } C > 0, \end{cases}$$

implies $U \geq 0$ in Ω.

Proof. We take $\beta > 0$ such that $2Nc_0^{-1}\beta^2 + (1+\nu)N\beta < \delta$ and set $\varepsilon_0 = \beta/2$. We make the change of functions $\overline{u}_i = \dfrac{u_i}{g}$, with $g(x) = \cosh\beta x_1 \ldots \cosh\beta x_N$. Then $\overline{U} = (\overline{u}_1, \ldots, \overline{u}_n)$ satisfies

$$\overline{L}\,\overline{U} + \overline{\mathcal{C}}\,\overline{U} \leq 0 \quad \text{in} \quad \Omega,$$

where $\overline{L} = (\overline{L}_1, \ldots, \overline{L}_n)$,

$$\overline{L}_k = L_k + 2\beta \sum_{1 \leq i,j \leq N} a_{ij}^k(x) \tanh\beta x_i \frac{\partial}{\partial x_j}$$

and

$$\overline{\mathcal{C}} = \mathcal{C} + \operatorname{diag}\left(\frac{L_1 g}{g}, \ldots, \frac{L_n g}{g}\right).$$

More precisely, $\overline{\mathcal{C}} = (\overline{c}_{kj})_{k,j=1}^n$, with

$$\begin{aligned} \overline{c}_{kj} &= c_{kj} + \delta_{kj}\left(\beta^2 \sum_{\substack{1 \leq l,m \leq N \\ l \neq m}} a_{lm}^k \tanh\beta x_l \tanh\beta x_m\right. \\ &+ \left.\beta^2 \sum_{1 \leq l \leq N} a_{ll}^k + \beta \sum_{1 \leq l \leq N} b_l^k \tanh\beta x_l\right). \end{aligned}$$

Note that, because of the choice of β and ε_0, $\overline{\mathcal{C}}$ satisfies condition (4) and $\overline{U}^- \to 0$ as $|x| \to +\infty$, $x \in \Omega$. By applying ABP inequality (Theorem 1.1, part (b)) to $-\overline{U}$ in balls of increasing radii, we obtain the conclusion of Proposition 6.1. □

7 Strong Maximum Principle

An immediate consequence of the weak Harnack inequality (Proposition 1.2) is the following

Proposition 7.1 *Assume (H1)–(H4) hold. Suppose* $U \in C(\overline{\Omega}, \mathbb{R}^n)$ *satisfies*

$$\begin{cases} LU + \mathcal{C}U & \leq & 0 & \text{in } \Omega \\ U & \geq & 0 & \text{in } \Omega. \end{cases}$$

Let $1 \leq k \leq m$ *and suppose there exists a point* $x_0 \in \Omega$ *and an index* $i \in S_k$ *such that* $u_i(x_0) = 0$*. Then* $u_j \equiv 0$ *for all* $j \in S_k$*. If, in addition,* $1 \leq k < l \leq m$ *are such that there exists a sequence* $\{i_j\}_{j=0}^r$ *for which*

$$k = i_0 > i_1 > \ldots > i_r = l \quad \text{and} \quad \mathcal{C}_{i_j i_{j+1}} \not\equiv 0 \quad \text{in } \Omega \tag{66}$$

(recall $\mathcal{C}$ *is written in the form (7), see Part I), then* $u_j \equiv 0$ *in* Ω*, for all* $j \in \bigcup_{j=0}^{r} S_{i_j}$*.*

Remark 1. In the particular case when the system is fully coupled ($m = 1$), Proposition 7.1 reduces to the usual strong maximum principle (see [S]), which states that $u_i(x_0) = 0$ for some $x_0 \in \Omega$ and some $i \in \{1, \ldots, n\}$ implies $U \equiv 0$ in Ω.

Remark 2. The strong maximum principle given by Proposition 7.1 is sharp in the sense that if a sequence as in (66) does not exist then, clearly, the system does not relate the functions with indices in S_k to the functions with indices in S_l.

8 Existence of a Principal Eigenfunction for a Fully Coupled System

Throughout this and the next section we suppose that (H1)-(H3) hold and, in addition,

$$a_{ij}^k \in C(\Omega),$$

for all $i, j \in \{1, \ldots, N\}, k \in \{1, \ldots, n\}$. All functions considered belong to $W^{2,q}_{\text{loc}}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$, for all $q < \infty$ (except otherwise stated), so that, in contrast to the rest of the paper, *all equalities and inequalities hold almost everywhere.*

For simplicity we suppose that Ω is regular (for example, Ω satisfies a uniform exterior cone condition). All our results can be extended to arbitrary domains, by using the arguments in [BNV].

We set

$$\lambda_1 = \lambda_1(L+\mathcal{C}) = \sup\{\lambda \in \mathbb{R} \quad | \quad \text{there exists } \ \Psi \in W^{2,N}_{\text{loc}}(\Omega, \mathbb{R}^n) \ \text{ such that} \\ \Psi > 0 \ \text{ and } \ (L+\mathcal{C}+\lambda I)\Psi \leq 0 \ \text{ in } \ \Omega\}.$$

Our main result in this section is the following theorem.

Theorem 8.1 *We suppose that $\mathcal{C}$ is an irreducible matrix (see Part I, Definition 1.1).*
(a) There exists a function $\Phi_1 \in W^{2,q}_{\text{loc}}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n), \forall q < \infty$, such that

$$\begin{cases} (L+\mathcal{C}+\lambda_1 I)\Phi_1 &= 0 \qquad \text{in } \ \Omega \\ \Phi_1 &> 0 \qquad \text{in } \ \Omega \\ \Phi_1 &= 0 \qquad \text{on } \ \partial\Omega. \end{cases}$$

(b) There are no eigenvalues of $-(L+\mathcal{C})$ in the interval $(-\infty, \lambda_1)$; the vector Φ_1 spans Ker $(L+\mathcal{C}+\lambda_1 I)$ in $W^{2,N}_{\text{loc}}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ under the Dirichlet boundary condition.
(c) Assume there is a function $\Psi \in W^{2,N}_{\text{loc}}(\Omega, \mathbb{R}^n)$ such that

$$\Psi > 0 \qquad \text{and} \qquad (L+\mathcal{C})\Psi \leq 0 \ \text{ in } \ \Omega.$$

Then either $\lambda_1 > 0$ or $\lambda_1 = 0$ and $\Psi = \text{const}.\Phi_1$.
(d) If $\overline{\Psi} \in W^{2,N}_{\text{loc}}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ satisfies

$$\begin{cases} (L+\mathcal{C}+\lambda_1 I)\overline{\Psi} &\leq 0 \qquad \text{in } \ \Omega \\ \overline{\Psi} &\geq 0 \qquad \text{on } \ \partial\Omega \end{cases}$$

then $\overline{\Psi} = \text{const}.\Phi_1$.
(e) If we normalize $\Phi_1 = (\phi_{1,1}, \ldots, \phi_{1,n})$ in such a way that

$$\min_{1 \leq i \leq n} \phi_{1,i}(x_0) = 1$$

for some $x_0 \in \Omega$, then

$$\sup_{\Omega} \Phi_1 := \sup_{\Omega} \phi_{1,1} \vee \ldots \vee \phi_{1,n} \leq C,$$

where C depends only on x_0, Ω and the same quantities as the constant which appears in the Harnack inequality (Theorem 1.2 in Part I).

Remark 1. Supposing *a priori* that $\lambda_1 > 0$ (he actually uses a hypothesis of type (Ψ) and proves his hypothesis is equivalent to $\lambda_1 > 0$), Sweers proved parts (a) and (b) of Theorem 8.1 in [S] (his proof relies on the Krein-Rutman technique ; see also [BMS] for the case of a non-regular domain). We are going to use this result in the proof of Theorem 8.1.

Remark 2. If the boundary of Ω is not regular the principal eigenfunction may not belong to $C(\overline{\Omega})$ and $\Phi_1 = 0$ on $\partial\Omega$ only in a certain sense (as in [BNV] and [BMS]).

Remark 3. In [BNV] Berestycki, Nirenberg and Varadhan made a deep study of the properties of the principal eigenvalue and the principal eigenfunction of a scalar elliptic operator in a general domain. The basic tools they used are the ABP and the Harnack-Krylov-Safonov inequalities for scalar equations. Since we now have such inequalities for cooperative elliptic systems, it is only a matter of technique to show that all results in [BNV] have their analogues for systems. In this and the next section we present some of these analogues (and often merely adapt the proofs in [BNV] to the case of a system). To extend to systems the rest of the results in [BNV] is left to the interested reader.

Proof of Theorem 8.1. It follows from the definition of λ_1 that for any $\lambda < \lambda_1$ the matrix operator $L + \mathcal{C} + \lambda I$ satisfies the hypotheses of Theorem 1.1 in [S]. This theorem implies the existence of couple $(\mu_\lambda, \Phi_\lambda)$ such that $\Phi_\lambda \in W^{2,N}_{\text{loc}}(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$, $\mu_\lambda > 0$ and

$$\left\{\begin{array}{rcll} (L + \mathcal{C} + (\lambda + \mu_\lambda)I)\Phi_\lambda & = & 0 & \text{in } \Omega \\ \Phi_\lambda & > & 0 & \text{in } \Omega \\ \Phi_\lambda & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{67}$$

We claim that

$$\lambda < \lambda + \mu_\lambda \leq \lambda_1 \leq \frac{C_1}{R^2}, \tag{68}$$

where $C_1 = C_1(N, c_0, \nu)$ and $R > 0$ is such that Ω contains a ball of radius R.

The first inequality in (68) is obvious. The second inequality follows from the definition of λ_1 and (67). The last inequality in (68) is known to hold if λ_1 is replaced by $\lambda_1(L_i + c_{ii})$ – the principal eigenvalue of the *scalar* operator $L_i + c_{ii}$ in Ω, for any $i = 1, \ldots, n$ (this is Lemma 1.1, page 51 in [BNV]). On the other hand, since $\mathcal{C}$ is cooperative,

$$(L + \mathcal{C} + \lambda I)\Phi \leq 0 \quad \text{implies} \quad (L_i + c_{ii} + \lambda)\phi_i \leq 0$$

$(\Phi = (\phi_1, \ldots, \phi_n))$, hence $\lambda_1 \leq \lambda_1(L_i + c_{ii})$, for all $i = 1, \ldots, n$.

We now fix $\delta = \delta(N, c_0, \nu, \Omega)$ such that

$$\delta < \left\{ 2C_{\mathrm{ABP}} \left(\nu + \frac{C_1}{R^2} \right) \right\}^{-N}$$

(C_{ABP} is the constant which appears in Part I, Theorem 1.1 (b)).

Let $x_0 \in \Omega$. We normalize the vector $\Phi_\lambda = (\phi_{\lambda,1}, \ldots, \phi_{\lambda,n})$ by dividing the equation in (67) by $\min\limits_{1 \le i \le n} \phi_{\lambda,i}(x_0) > 0$, so that we can suppose

$$\phi_{\lambda,i}(x_0) \ge 1, \qquad \text{for all } \ i = 1, \ldots, n,$$

and

$$\phi_{\lambda,1} \wedge \ldots \wedge \phi_{\lambda,n}(x_0) = 1. \tag{69}$$

We take a compact set $K \subset \Omega$ such that we have $x_0 \in K, B_R \subset K$, $\Lambda(K) = \Lambda(\Omega)$ (Λ is defined in Part I, page 5), and

$$\operatorname{meas}(\Omega \setminus K) < \delta.$$

It follows from (69) and our Harnack inequality (Corollary 1.1 in Part I) that

$$\sup_K \Phi_\lambda \le C_2 \tag{70}$$

(the constant C, indexed or not, depends on the appropriate quantities, in particular, C is independent of λ).

Set $e = (1, 1, \ldots, 1)$ and $\overline{\Phi}_\lambda = \Phi_\lambda - C_2 e$, so that

$$\overline{\Phi}_\lambda \le 0 \qquad \text{on } \ \partial(\Omega \setminus K). \tag{71}$$

We have, by (67) and (68),

$$\begin{aligned} (L + \mathcal{C} - \nu I)\overline{\Phi}_\lambda &= -(\nu + \lambda + \mu_\lambda)\Phi_\lambda - C_2(\mathcal{C} - \nu I)e \\ &\ge -\left(\nu + \frac{C_1}{R^2} \right) \Phi_\lambda \qquad \text{in } \ \Omega \setminus K, \end{aligned} \tag{72}$$

where we have used the fact that (H2) implies $(\mathcal{C} - \nu I)e \le 0$. The latter inequality shows that the operator in the left-hand side of (72) satisfies the hypotheses of the ABP estimate (Theorem 1.1 (b)). Applying this estimate to (72), we obtain

$$\begin{aligned} \sup_{\Omega \setminus K} \Phi_\lambda - C_2 &\le C_{\mathrm{ABP}} \left(\nu + \frac{C_1}{R^2} \right) \delta^{\frac{1}{N}} \sup_{\Omega \setminus K} \Phi_\lambda \\ &\le \frac{1}{2} \sup_{\Omega \setminus K} \Phi_\lambda. \end{aligned} \tag{73}$$

By combining (70) and (73) we obtain

$$\|\Phi_\lambda\|_{L^\infty(\Omega)} = \sup_\Omega \Phi_\lambda \le C_3.$$

Since

$$L_k\phi_{\lambda,k} = -\sum_{j=1}^n (c_{kj} - \delta_{kj}(\lambda + \mu_\lambda))\phi_{\lambda,j},$$

we see that $L_k\phi_{\lambda,k}$ remain bounded in $L^\infty(\Omega)$ as $\lambda \to \lambda_1$, for all $k = 1, \dots, n$. Classical interior elliptic estimates for scalar equations imply

$$\|\Phi_\lambda\|_{W^{2,q}(\Omega')} \le C, \tag{74}$$

for any $1 < q < \infty$ and any $\Omega' \subset\subset \Omega$.

We infer from (74) that there exists a sequence $\{\lambda^{(j)}\}_{j=1}^\infty$ and a function Φ_1 such that

$$\lambda^{(j)} \to \lambda_1 \quad \text{and} \quad \Phi^{(j)} = \Phi_{\lambda^{(j)}} \to \Phi_1$$

weakly in $W^{2,q}_{\mathrm{loc}}(\Omega), 1 < q < \infty$, and uniformly in any compact subset of Ω. It follows that Φ_1 solves the equation

$$(L + \mathcal{C} + \lambda_1 I)\Phi_1 = 0 \qquad \text{in } \Omega$$

(note that (68) implies $\lambda^{(j)} + \mu_{\lambda^{(j)}} \to \lambda_1$), and

$$0 \le \Phi_1 \le C_3 e \qquad \text{in } \Omega.$$

Since, by (69), $\Phi_1(x_0) \ge e$, the strong maximum principle implies $\Phi_1 > 0$ in Ω.

Finally,

$$\begin{array}{rcll} L\Phi^{(j)} & = & -\mathcal{C}\Phi^{(j)} - (\lambda^{(j)} + \mu_{\lambda^{(j)}})\Phi^{(j)} & \\ & \ge & -\nu C_3 e - \dfrac{C_1 C_3}{R^2} e = -C_4 e & \text{in } \Omega, \end{array}$$

so the usual maximum principle for scalar equations implies

$$\Phi^{(j)} \le C_4 \Phi_0 \qquad \text{in } \Omega,$$

where Φ_0 is the solution of the problem

$$\left\{ \begin{array}{rcll} L\Phi_0 & = & -e & \text{in } \Omega \\ \Phi_0 & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

Hence

$$0 < \Phi_1 \le C_4 \Phi_0 \qquad \text{in } \Omega,$$

so $\Phi_1 \in C(\overline{\Omega})$ and $\Phi_1 = 0$ on $\partial\Omega$.

Statements (a) and (e) of Theorem 8.1 are proved.

The first part of statement (b) follows from the result of Sweers (Theorem 1.1 in [S]). Indeed, if $\overline{\lambda} < \lambda_1$ is an eigenvalue for $-(L+\mathcal{C})$ then the operator $L+\mathcal{C}+\frac{\lambda_1+\overline{\lambda}}{2}I$ satisfies the hypotheses of Theorem 1.1 in [S], and hence does not have negative eigenvalues - a contradiction. The second part of (b) is a particular case of (d).

Let us prove statements (c) and (d). It is clear that the assumptions in (c) imply $\lambda_1 \geq 0$. If $\lambda_1 = 0$, set

$$\tau = \sup\{t \in \mathbb{R} \mid \Psi \geq t\Phi_1 \text{ in } \Omega\}. \tag{75}$$

By continuity $\Psi \geq \tau\Phi_1$. If $\Psi = \tau\Phi_1$, we are done. If not, the strong maximum principle, applied to

$$(L+\mathcal{C})(\Psi - \tau\Phi_1) \leq 0,$$

implies $\Psi > \tau\Phi_1$ in Ω. We now make use of the Maximum Principle in "small domains" obtained by de Figueiredo in [DF] (this result is stated in Corollary 9.1 below). It follows from this result that if a compact $K \subset \Omega$ is such that $\text{meas}(\Omega \setminus K)$ is small enough, then $L+\mathcal{C}$ satisfies the maximum principle in $\Omega \setminus K$. Since K is compact, there exists $\varepsilon > 0$ such that $\Psi \geq (\tau+\varepsilon)\Phi_1$ in K. Since

$$\begin{cases} (L+\mathcal{C})(\Psi - (\tau+\varepsilon)\Phi_1) \leq 0 & \text{in } \Omega \setminus K \\ \Psi - (\tau+\varepsilon)\Phi_1 \geq 0 & \text{on } \partial(\Omega \setminus K), \end{cases} \tag{76}$$

we get, by the maximum principle, $\Psi \geq (\tau+\varepsilon)\Phi_1$ in Ω, which contradicts (75).

Finally, let us prove (d). We can suppose that $\lambda_1 = 0$ (replace $\mathcal{C}$ by $\mathcal{C}+\lambda_1 I$). It suffices to find $\alpha > 0$ such that

$$z = \overline{\psi} + \alpha\phi_1 > 0 \quad \text{in} \quad \Omega$$

(z then satisfies the assumption in (c)). We fix a compact $K \subset \Omega$ such that $L+\mathcal{C}$ satisfies the Maximum Principle in $\Omega \setminus K$ and we take $\alpha > 0$ such that $z > 0$ in K. Since

$$\begin{cases} (L+\mathcal{C})\, z \leq 0 & \text{in } \Omega \setminus K \\ z \geq 0 & \text{on } \partial\,(\Omega \setminus K), \end{cases}$$

we get $z > 0$ in $\Omega \setminus K$.

Theorem 8.1 is proved. $\square$

At the end of this section we recall the following result of Sweers.

Theorem 8.2 *Suppose $\mathcal{C}$ is irreducible. Let $F \in L^q(\Omega, \mathbb{R}^n), q \geq N$. If $\lambda_1 > 0$ then there exists a unique solution in $W^{2,q}_{\text{loc}}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ of the problem*

$$\begin{cases} LU + \mathcal{C}U = -F & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $F \geq 0$ in Ω implies $U \geq 0$ in Ω ; $F = (f_1, \ldots, f_n) \geq 0$ and $f_i \not\equiv 0$, for some i, imply $F > 0$.

9 Necessary and Sufficient Conditions for the Maximum Principle

.

The existence of a principal eigenfunction permits us to show that condition (Ψ) in Theorem 1.1 (a), which has long been known to be sufficient for the maximum principle, is also necessary for its validity. Our result, applied to a fully coupled system, says the maximum principle holds if and only if the principal eigenvalue of the matrix operator $L + \mathcal{C}$ is positive. This result contains the well-known necessary and sufficient condition for a scalar operator to verify the maximum principle.

Let the matrix $\mathcal{C}$ be written in the form

$$\mathcal{C} = (\mathcal{C}_{kl})^m_{k,l=1} \tag{77}$$

in Ω, where, we recall, $1 \leq m \leq n$, $\mathcal{C}_{kl}$ are $t_k \times t_l$ matrices, $\sum\limits_{k=1}^{m} t_k = n$, $\mathcal{C}_{kk}$ is an irreducible matrix for all $k = 1, \ldots, m$, and $\mathcal{C}_{kl} \equiv 0$ for all $k, l \in \{1, \ldots, m\}$ with $k < l$. We have set $s_0 = 0, s_k = \sum\limits_{i=1}^{k} t_i$, and $S_k = \{s_{k-1} + 1, \ldots, s_k\}$.

Let $\lambda_1^{(k)}$ and $\Phi_1^{(k)}$ be the principal eigenvalue and eigenfunction of the operator $L^k + \mathcal{C}_{kk}$ in Ω, where

$$L^k = \text{diag}\,(L_{s_{k-1}}, \ldots, L_{s_k}) \quad \text{and} \quad \Phi_1^{(k)} \in W^{2,q}_{\text{loc}}(\Omega, \mathbb{R}^{t_k}) \cap C(\overline{\Omega}, \mathbb{R}^{t_k}),$$

for all $q < \infty$. The existence of $\lambda_1^{(k)}$ and $\Phi_1^{(k)}$ is ensured by Theorem 8.1.

Theorem 9.1 *Let (H1)-(H3) hold and let a^k_{ij} be continuous functions, for all i, j, k. The following are equivalent :*

(i) *$L + \mathcal{C}$ satisfies the maximum principle in Ω, in the sense described in the beginning of Section 6 ;*

(ii) $\lambda_1^{(k)} > 0$*, for all* $k \in \{1, \dots, m\}$*.*

(iii) there exists a function $\Psi > 0$ *in* Ω *such that* $(L + \mathcal{C})\Psi \le 0$ *in* Ω*, and either* $\Psi \not\equiv 0$ *on* $\partial\Omega$ *or* $(L + \mathcal{C})\Psi \not\equiv 0$ *in* Ω *(component-wise).*

Proof. *(i)⇒(ii).* Suppose there exists $k \in \{1, \dots, m\}$ such that $\lambda_1^{(k)} \le 0$. Set

$$\Psi = (0, \dots, 0, \Phi_1^{(k)}, 0, \dots, 0)$$

(the only non-zero coordinates of Ψ are those with indices in S_k). Then

$$(L + \mathcal{C})\Psi = (0, \dots, 0, -\lambda_1^{(k)}\Phi_1^{(k)}, \mathcal{C}_{k+1k}\Phi_1^{(k)}, \dots, \mathcal{C}_{mk}\Phi_1^{(k)}),$$

so, by (H3),

$$\begin{cases} (L + \mathcal{C})\Psi & \ge & 0 & \text{in } \Omega \\ \Psi & = & 0 & \text{on } \partial\Omega, \end{cases}$$

but $\Psi \not\le 0$ in Ω, which contradicts the maximum principle.

(ii)⇒(iii). We use a recurrent procedure to construct $\Psi = (\Psi^{(1)}, \dots, \Psi^{(m)})$ ($\Psi^{(k)}$ consists of t_k components). Set $\Psi^{(1)} = \Phi_1^{(1)}$. If $\mathcal{C}_{21} \equiv 0$ we take $\Psi^{(2)} = \Phi_1^{(2)}$. If $\mathcal{C}_{21} \not\equiv 0$ we take $\Psi^{(2)}$ to be the solution of the problem

$$\begin{cases} \left(L^2 + \mathcal{C}_{22} + \dfrac{\lambda_1^{(2)}}{2}\right)\Psi^{(2)} & = & -\mathcal{C}_{21}\Psi^{(1)} & \text{in} & \Omega \\ \Psi^{(2)} & > & 0 & \text{in} & \Omega \\ \Psi^{(2)} & = & 0 & \text{on} & \partial\Omega. \end{cases}$$

This boundary value problem is solvable, by Theorem 8.2.

Finally, when we have constructed $\Psi^{(1)}, \dots, \Psi^{(m-1)}$, we take $\Psi^{(m)}$ to be either $\Phi_1^{(m)}$, in case $\mathcal{C}_{mk} \equiv 0$ for all $k = 1, \dots, m-1$, or the positive solution of the problem

$$\begin{cases} \left(L^m + \mathcal{C}_{mm} + \dfrac{\lambda_1^{(m)}}{2}\right)\Psi^{(m)} & = & -\sum\limits_{k=1}^{m-1}\mathcal{C}_{mk}\Psi^{(k)} & \text{in} & \Omega \\ \Psi^{(m)} & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{78}$$

in case the right-hand side in (78) is not identically zero.

Then we have $\Psi > 0$ and

$$(L + \mathcal{C})\Psi \le \left(-\lambda_1^{(1)}\Phi_1^{(1)}, -\frac{\lambda_1^{(2)}}{2}\Phi_1^{(2)}, \dots, -\frac{\lambda_1^{(m)}}{2}\Phi_1^{(m)}\right) < 0 \qquad \text{in } \Omega.$$

(iii)⇒(ii). Let us note $\Psi = (\Psi^{(1)}, \ldots, \Psi^{(m)}) > 0$. Since (H3) holds, we have

$$(L^k + \mathcal{C}_{kk})\Psi^{(k)} \leq 0,$$

and either $\Psi^{(k)} \not\equiv 0$ on $\partial\Omega$ or $(L^k + \mathcal{C}_{kk})\Psi^{(k)} \not\equiv 0$ in Ω, for all $k = 1, \ldots, m$. It follows from Theorem 8.1 (c) that either $\lambda_1^{(k)} > 0$ or $\lambda_1^{(k)} = 0$ and, in the latter case, $\Psi^{(k)} = \text{const.}\Phi_1^{(k)}$. The last equality is impossible, by (iii).

(ii)⇒(i). We show that (iii) is equivalent to condition (Ψ) in the ABP estimate (Theorem 1.1 (a)). Then the conclusion is immediate, since the maximum principle is a particular case of the ABP estimate.

Lemma 9.1 *Suppose $\lambda_1^{(k)} > 0$, for all $k = 1, \ldots, m$. Then there exists a function $\overline{\Psi} \in W^{2,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ such that*

$$\begin{cases} (L + \mathcal{C})\overline{\Psi} & \leq & 0 & \textit{in} & \Omega \\ \overline{\Psi} & \geq & e & \textit{in} & \Omega. \end{cases}$$

Proof. Since (ii) implies (iii) we can consider the vector Ψ given by (iii). Then we can adapt to our situation the proof of Lemma 6.1 in [BNV]. Let us sketch the argument.

We take a compact set $K \subset \Omega$ and solve the following n scalar equations

$$\begin{cases} LW & = & -2\nu e & \text{in} & \Omega \setminus K \\ LW & = & 0 & \text{in} & K \\ W & = & 0 & \text{on} & \partial\Omega. \end{cases}$$

As in [BNV], by taking K sufficiently close to Ω, we can ensure that $W < e$ in Ω. Taking ε_0 such that $W \geq \varepsilon_0 e$ on K and setting

$$A = \max_{1 \leq k \leq m} \frac{4\nu}{\lambda_1^{(k)}\varepsilon_0} \quad \text{and} \quad \overline{\Psi} = e + W + A\Psi,$$

we get $(L + \mathcal{C})\overline{\Psi} \leq 0$ in Ω, as in [BNV].

Theorem 9.1 is proved. □

Finally, we give a lower bound for the principal eigenvalue, analogous to the estimate in Lemma 4.1 in [BNV].

Proposition 9.1 *Under the conditions of Theorem 8.1, set*

$$b_1 = \max_{1 \leq j \leq n} \left\{ \sum_{i \neq j} \|c_{ij}\|_{L^\infty(\Omega)} + \|c_{jj}^+\|_{L^\infty(\Omega)} \right\}.$$

Then

$$\lambda_1 \geq \frac{1}{C_{\mathrm{ABP}}|\Omega|^{\frac{1}{N}}} - b_1.$$

Proof. Apply the ABP estimate to

$$(L + \mathcal{C} - b_1 I)\Phi_1 = -(b_1 + \lambda_1)\Phi_1.$$

The maximum principle in small domains is a consequence of Proposition 9.1 and Theorem 9.1.

Corollary 9.1 *Suppose (H1)–(H3) hold. For any $d > 0$ there exists a positive number $\delta = \delta(N, c_0, \nu, d)$ such that*

$$\operatorname{diam}\Omega < d \quad \textit{and} \quad |\Omega| < \delta$$

imply that the operator $L + \mathcal{C}$ satisfies the maximum principle in Ω.

10 Higher Order Equations. The Biharmonic and the Polyharmonic Equations.

Our results give ABP and Harnack estimates for equations of arbitrary order in the form

$$(-L_n - c_n)\dots(-L_1 - c_1)u - cu = f \qquad \text{in } \Omega. \tag{79}$$

It is easy to see that (79) is equivalent to the system

$$LU + \mathcal{C}U = F, \tag{80}$$

where

$$\mathcal{C} = \begin{pmatrix} c_1 & 1 & 0 & \dots & 0 \\ 0 & c_2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ c & 0 & 0 & \dots & c_n \end{pmatrix}, \; U = \begin{pmatrix} u \\ (-L_1 - c_1)u \\ \vdots \\ (-L_{n-1} - c_{n-1})\dots(-L_1 - c_1)u \end{pmatrix}$$

and $F = (0, \dots, 0, f)$. Note that the matrix $\mathcal{C}$ is fully coupled when $c \not\equiv 0$ and is in triangular form when $c \equiv 0$.

The simplest and most studied example for (79) is the biharmonic equation

$$-\Delta\left(a(x)\Delta u\right) + c(x)u = f(x) \quad \text{in } \Omega. \tag{81}$$

which corresponds to the system

$$\begin{cases} a(x)\Delta u + v &= 0 \\ \Delta v + c(x)u &= f(x). \end{cases} \tag{82}$$

We have the following Harnack estimate for equation (79). We give separately the Harnack estimate for the polyharmonic equation.

Theorem 10.1 *(a) Suppose that $u \in C^{2n-2}(\Omega)$ satisfies the equation*

$$(-\Delta)^n u = f$$

in the viscosity sense in Ω and

$$(-\Delta)^k u \geq 0$$

in a ball $B_{3R} \subset \Omega$, for $k = 0, \ldots, n-1$.

Then

$$\sup_{B_R} u \leq C \left(\inf_{B_R} u + R \|f\|_{L^N(B_{3R})} \right),$$

where $C = C(n, N)$.

(b) Let $c \in L^\infty$ and $c_i \in L^\infty(\Omega), i = 1, \ldots, n$, be functions such that

$$c \not\equiv 0, \;\; 0 \leq c \leq \nu, \;\; \max_{1 \leq k \leq n} \|c_i\|_{L^\infty(\Omega)} \leq \nu.$$

Let $u \in C^{2n-2}(\Omega)$ be a non-negative viscosity solution of

$$(-L_n - c_n) \ldots (-L_1 - c_1)u - cu = f \qquad \text{in } \Omega \tag{83}$$

such that

$$(-L_k - c_k) \ldots (-L_1 - c_1)u \geq 0$$

for all $k = 1, \ldots, n-1$.

Then

$$\sup_{B_R} u \leq C \left(\inf_{B_R} u \wedge \min_{1 \leq k \leq n-1} \{(-L_k - c_k) \ldots (-L_1 - c_1)u\} + R \|f\|_{L^N(B_{3R})} \right).$$

Remark 1. It is easy to see that the hypothesis on the sign of $(-\Delta)^k u$ cannot be omitted (take for example $u = |x|^2$ in the unit ball ; then the weak Harnack inequality fails, since $\Delta^2 u = 0, u \geq 0, u(0) = 0$, but $u \not\equiv 0$).

Recently, using Green functions, Grunau and Sweers (see [GS1], Theorem 5.1) obtained related results for classical solutions of (79), in the case when the domain is a ball and all derivatives of u of order smaller than n vanish on $\partial\Omega$. In [GS2] they used this result to obtain a local maximum principle for equations of order $2n$, provided the lower order coefficients are sufficiently small.

Theorem 8.1 permits us to define a principal eigenvalue and a principal eigenfunction for the polyharmonic operator, in an appropriate sense.

Theorem 10.2 *(a) Let $c \in L^\infty(\Omega)$, $0 \le c \le \nu$. There exists a real number $\lambda_1 = \lambda_1^{(c)}$ and a function $\phi_1 = \phi_1^{(c)}$ in $W^{2n,q}_{\rm loc}(\Omega) \cap C^{2n-2}(\overline{\Omega}), \forall\, q < \infty$, such that*

$$\begin{aligned} (-\Delta - \lambda_1)^n \phi_1 - c\phi_1 &= 0 \qquad \text{in } \Omega && (84)\\ \phi_1 > 0,\ (-\Delta - \lambda_1)^k \phi_1 &> 0 \qquad \text{in } \Omega && (85)\\ \phi_1 = 0,\ (-\Delta - \lambda_1)^k \phi_1 &= 0 \qquad \text{on } \partial\Omega, && (86) \end{aligned}$$

for $k = 1, \dots, n-1$. There are no eigenvalues (with the Dirichlet condition (86)) smaller than λ_1 and (λ_1, ϕ_1) is the unique couple (up to a normalization of ϕ_1), which satisfies (84), (85) and (86).
(b) $\lambda_1^{(c)} > 0$ is a necessary and sufficient condition for the polyharmonic operator to satisfy the maximum principle, in the following sense

$$\begin{cases} (-\Delta)^n u - cu \le 0 & \text{in } \Omega \\ (-\Delta)^k u \le 0 & \text{on } \partial\Omega, \qquad k = 0, \dots, n-1, \end{cases}$$

implies $(-\Delta)^k u \le 0$ in Ω, for $k = 0, \dots, n-1$.
(c) Let $\phi_1^{(c)}$ be normalized so that $\phi_1^{(c)}(x_0) = 1$, for some $x_0 \in \Omega$. Then

$$\sup_\Omega \phi_1^{(c)} \le C(n, N, \nu, \Omega).$$

(d) If $\{c_j\}_{j=1}^\infty$ is a sequence such that $\{c_j\}$ is bounded in $L^\infty(\Omega)$ and $\{c_j\}$ tends to zero a.e. in Ω, then $\lambda_1^{(c_j)}$ tends to the principal eigenvalue of the Laplacian in Ω and

$$\begin{aligned} \phi_1^{(c_j)} &\to \phi_1(\Delta) \\ (-\Delta - \lambda_1^{(c_j)})^k \phi_1^{(c_j)} &\to 0, \qquad k = 1, \dots, n-1, \end{aligned}$$

weakly in $W^{2,q}_{\rm loc}(\Omega)$ and strongly in $C(\overline{\Omega})$ (the functions $\phi_1^{(c_j)}$ are assumed to be normalized as in (c)).

Proof. Only part (d) needs a proof. Suppose for simplicity $n = 2$ (the case $n > 2$ is very much the same).

Set $\lambda^j = \lambda_1^{(c_j)}$ and $\phi_j = \phi_1^{(c_j)}$. It follows from (68) and Proposition 9.1 that the sequence $\{\lambda^j\}$ is bounded. Hence (up to a subsequence) it converges to a number $\overline{\lambda}$.

We have

$$\left\{ \begin{array}{rcl} (-\Delta-\lambda^j)u_j &=& v_j \\ (-\Delta-\lambda^j)v_j &=& c_j u_j, \end{array} \right. \tag{87}$$

where $u_j = \phi_j$ and $v_j = (-\Delta - \lambda^j)\phi_j$.

Applying the Theorem of Krylov we already used in Part I (Theorem 5.1 on page 21), we obtain from the first equation in (87)

$$\inf_K u_j \geq C(N, \nu, K, \Omega) \inf_K v_j, \tag{88}$$

for any compact subset $K \subset \Omega$.

Applied to (87), our local maximum principle (Proposition 1.1) yields

$$\sup_{B_R} u_j \vee v_j \leq C\left(|u_j|_{p,2R} + |v_j|_{p,2R}\right), \tag{89}$$

for any $p > 0$, any $B_{2R} \subset \Omega$, with $C = C(p, N, \nu R^2)$. The weak Harnack inequality for scalar equations (Theorem 9.22 in [GT]) applied to

$$(-\Delta - \lambda^j)u_j \geq 0 \quad \text{and} \quad (-\Delta - \lambda^j)v_j \geq 0$$

then gives

$$\begin{array}{rcl} \sup\limits_{B_R} u_j \vee v_j &\leq& C\left(\inf\limits_{B_R} u_j + \inf\limits_{B_R} v_j\right) \\ &\leq& C \inf\limits_{B_R} u_j = C(N, \nu, \Omega) \end{array}$$

if $x_0 \in B_R$, where we have used (88), (89), and $u_j(x_0) = 1$. Hence, for any compact set $K \subset \Omega$,

$$\sup_K u_j \vee v_j \leq C(N, \nu, K, \Omega).$$

Taking K close enough to Ω and applying the ABP inequality in $\Omega \setminus K$, we get, as in the proof of Theorem 8.1

$$\|u_j \vee v_j\|_{L^\infty(\Omega)} \leq C(N, \nu, \Omega).$$

Then elliptic theory implies

$$u_j \to u \qquad \text{and} \qquad v_j \to v$$

weakly in $W^{2,q}_{\mathrm{loc}}(\Omega), \forall q < \infty$, and strongly in $C(\overline{\Omega})$.

Passing to the limit in (87), we obtain

$$\begin{cases} (-\Delta - \overline{\lambda})u & = & v & \text{in} & \Omega \\ (-\Delta - \overline{\lambda})v & = & 0 & \text{in} & \Omega \\ u = v & = & 0 & \text{on} & \partial\Omega. \end{cases} \tag{90}$$

Since $v \geq 0$, we infer from the second equation in (90) that either $v \equiv 0$ or $\overline{\lambda} = \lambda_1(\Delta)$ and $v = \phi_1(\Delta) > 0$. In the second case we obtain a contradiction with the first equation, since

$$\begin{cases} (-\Delta - \lambda_1(\Delta))u & > & 0 & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega. \end{cases}$$

has no solution. Hence

$$\begin{cases} (-\Delta - \overline{\lambda})u & = & 0 & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega. \end{cases}$$

Since $u \geq 0$ and $u(x_0) = 1$, we get $\overline{\lambda} = \lambda_1(\Delta)$ and $u = \phi_1(\Delta)$.

Theorem 10.2 is proved. □

More generally, for any n elliptic operators $L_1, \ldots, L_n$, satisfying (H1)-(H2), and any $c \geq 0$, there exists a couple (λ_1, ϕ_1) such that

$$(-L_n - \lambda_1) \ldots (-L_1 - \lambda_1) - c\phi_1 = 0,$$

with the appropriate positivity and boundary conditions.

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