Minimizers of energy functionals under not very integrable constraints

C. Léonard

Equipe Modal-X
Université de Paris X, Bâtiment G. 200, Av. de la République. 92001 Nanterre Cedex, France

Centre de Mathématiques Appliquées. Ecole Polytechnique.
91128 Palaiseau Cedex, France

E-mail: leonard@modalx.u-paris10.fr

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Abstract. We consider a general class of problems of minimization of convex integral functionals (maximization of entropy) subject to linear constraints. Under general assumptions, the minimizing solutions are characterized. Our results improve previous literature on the subject in the following directions:

- necessary and sufficient conditions for the shape of the minimizers are proved
- without constraint qualification
- under infinitely many linear constraints subject to natural integrability conditions (no topological restrictions).

This paper extends the author's previous results in [Lé2] by relaxing some integrability conditions on the constraint. As a consequence, the minimizers may admit a singular component. As an illustration, we solve the problem of minimizing a natural extension of the relative entropy under a general linear constraint. An example of I. Csiszár is revisited. Our proofs mainly rely on convex duality.

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1. Introduction

Let us first consider the energy functionals defined on the space: $\mathcal{M}$, of the signed measures on the measure space $(\Omega, \mathcal{A})$ which are of the following form

$$I(Q) = \int_\Omega \gamma^* \left(\frac{dQ}{dR} \right) dR \in [0, +\infty], \quad Q \in \mathcal{M}$$

if Q is absolutely continuous with respect to the given nonnegative reference measure R, and $I(Q) = +\infty$ otherwise. The function $\gamma^* : \mathbb{R} \to [0, \infty]$ is the convex conjugate of a function γ, hence it is convex and lower semicontinuous. The energy functional I is sometimes called γ^*-divergence or γ^*-entropy. With the special choice: $\gamma(s)=e^s-s-1$, we get $\gamma^*(t)=(t+1)\log(t+1)-t$ and, P,R being probability measures, $I(P-R)$ is the relative entropy of P with respect to R which is also called Kullback information or Boltzmann-Shannon entropy of P with respect to R.

Let $A: M(\Omega)\to\mathcal{X}$ be a linear operator on $M(\Omega)$ with its values in a vector space $\mathcal{X}$; $A(Q)=x$ is the expression of a linear constraint. We are concerned with the minimum energy problem

$$\text{minimize } Q\in\mathcal{M}\mapsto I(Q) \text{ subject to } A(Q)=x. \tag{1.1}$$

Note that a solution to (1.1) is absolutely continuous with respect to R. Such a minimization problem is sometimes called a maximum entropy problem: $-I$ may be seen as an entropy.

The classical moment constraint: $\int_\Omega f_k\,dQ=x_k$, $f_k:\Omega\to\mathrm{I\!R}$, $1\le k\le n$ corresponds to $\mathcal{X}=\mathrm{I\!R}^n$ and $A(Q)=\left(\int_\Omega f_k\,dQ\right)_{1\le k\le n}$. It naturally extends to infinitely many moments with

$$A(Q)=\left(\int_\Omega f_k\,dQ\right)_{k\in K}\in\mathcal{X}=\mathrm{I\!R}^K$$

where K is an arbitrary index set. Therefore, the problem (1.1) leads to solutions of some moment problems under the additional constraint: $I(Q)<\infty$, which implies in particular that Q is absolutely continuous with respect to R. It is extensively studied in the literature, among others, let us refer to: [BL1-5], [Cs1-3], [DCG], [GG], [LV], [TV] and more recently [CGG].

If the constraint operator A is regular enough, for instance in the above moment problem: if the constraint functions f_k are integrable enough, the minimizer of (1.1) is attained in $\mathcal{M}$, provided that $\inf\{I(Q)\,;\,Q\in\mathcal{M},A(Q)=x\}<\infty$. On the other hand, if A doesn't carry enough integrability, no minimizer may be attained in $\mathcal{M}$. In the present paper, such "bad" constraints are considered.

It is worth considering some extension of (1.1) of the following form

$$\text{minimize } \ell\mapsto\overline{I}(\ell) \text{ subject to } A(\ell)=x,\ell\in\mathcal{N} \tag{1.2}$$

where $\overline{I}$ is an extension of I to some vector space $\mathcal{N}$ which contains the effective domain of I. For a more precise description of this extended minimization problem see $(\overline{P}_x)$ in Section 2, and for its exact statement see (5.3) in Section 5. It appears that under not very integrable constraints (see (2.3) for the exact statement), the minimizers of (1.2) are attained in $\mathcal{N}$, for suitable $\overline{I}$ and $\mathcal{N}$.

The aim of this article. The aim of this article is to give an exact characterization (without approximating sequences) of the solutions to (1.2). In [Lé2], the author had already obtained the characterization of the solutions to (1.1) under more integrable constraints.

A connection with large deviations. The case of the extended relative entropy is of special interest. It arises naturally in the theory of large deviations (see [DZ]). More precisely, the minimization problem (1.2) in this context (see (2.6)) is a consequence of the contraction principle applied to an extended Sanov's theorem to obtain the usual statement of Cramér's theorem (see [LN], [Lé5]). This is closely connected with the work of I. Csiszár about the generalized I-projections. Indeed, it appears that the absolutely continuous part of the minimizers of (2.6) are precisely the generalized I-projections which were discovered by I. Csiszár ([Cs2], [Cs3]). Another example is given by the large deviations of renormalized Levy processes (with very frequent and very small jumps) in the situation where the Levy kernel only integrates some (but not every) exponential moments of the jumps (see [Lé4]). Applying the contraction principle to the renormalized underlying Poisson measures to obtain the large deviation principle for the Levy process, leads us to a minimization problem (2.6).

The outline of the paper is postponed at the end of Section 2.

2. Motivations and description of the main results

Let R be a σ-finite nonnegative measure on a measurable space $(\Omega, \mathcal{A})$. Let $\gamma : \mathbb{R} \to [0, \infty]$ be a function which satisfies

(2.1) γ is convex, lower semicontinuous, $\gamma(\mathbb{R}) \subset [0, +\infty]$, $0 < \gamma(-a) + \gamma(a) < \infty$ for some $a > 0$ and $\gamma(0) = 0$.

The function $\gamma_o(s) = \max(\gamma(-s), \gamma(s))$ and its convex conjugate γ_o^* are Young functions and one can consider the Orlicz spaces

$$L_{\gamma_o}(R) = \left\{ u : \Omega \to \mathbb{R} \,;\, \int_\Omega \gamma_o(\lambda u)\, dR < \infty, \text{ for some } \lambda > 0 \right\}$$
$$L_{\gamma_o^*}(R) = \left\{ f : \Omega \to \mathbb{R} \,;\, \int_\Omega \gamma_o^*(\lambda f)\, dR < \infty, \text{ for some } \lambda > 0 \right\}$$

Let us consider the convex conjugate γ^* of γ and the *energy functional*

$$I : Q \in L_{\gamma_o^*} \cdot R \mapsto \int_\Omega \gamma^* \left(\frac{dQ}{dR} \right) dR \in [0, \infty]$$

Let us consider the *constraint function* $\varphi : \Omega \to \mathcal{X}$ with values in a vector space $\mathcal{X}$ in separating duality with another vector space $\mathcal{Y}$. Let us assume that for all $y \in \mathcal{Y}$, $\langle y, \varphi(\cdot) \rangle$ stands in $L_{\gamma_o}(R)$. This allows us to define for all $Q \in L_{\gamma_o^*} \cdot R$ and $x \in \mathcal{X}$ the linear constraint

$$\int_\Omega \varphi \, dQ = x \overset{\text{def}}{\iff} \forall y \in \mathcal{Y}, \int_\Omega \langle y, \varphi(\omega) \rangle \, Q(d\omega) = \langle x, y \rangle$$

For all $x \in \mathcal{X}$, let us state the following problem

(P_x) minimize $Q \mapsto I(Q)$ subject to $\int_\Omega \varphi \, dQ = x, Q \in L_{\gamma_o^*} \cdot R$

The dual problem which is formally associated with this primal problem is

(D_x) maximize $y \in \mathcal{Y} \mapsto \langle x, y \rangle - \int_\Omega \gamma(\langle y, \varphi(\omega) \rangle \, R(d\omega) \in [-\infty, \infty[$

As γ is differentiable, γ^* is strictly convex and (P_x) admits at most one solution Q_x. If y_x maximizes (D_x), we have $Q_x = \gamma'(\langle y_x, \varphi \rangle) \cdot R$. But in general, the supremum in (D_x) is not attained in $\mathcal{Y}$ and the previous equality holds only on a formal level.

We introduce the function $\Lambda^*(x) = \sup(D_x) \in [0,\infty]$, $x \in \mathcal{X}$. We denote $\mathcal{X}^*$ the algebraic dual space of $\mathcal{X}$ and $\partial\Lambda^*(x)$ the subdifferential of Λ^* at x for the duality $\langle \mathcal{X}, \mathcal{X}^* \rangle$.

Definition 2.1. *A vector $x \in \mathcal{X}$ is said to be an admissible constraint, if $\Lambda^*(x) < \infty$.*
A vector $x \in \mathcal{X}$ is said to be a subgradient constraint, if $\Lambda^*(x) < \infty$ and $\partial\Lambda^*(x)$ is not empty.

As the duality $\langle \mathcal{X}, \mathcal{X}^* \rangle$ is very rich, $\partial\Lambda^*(x)$ is not frequently empty. In particular, any x in the relative interior of the effective domain: *dom* Λ^*, of Λ^* is a subgradient constraint. Assuming that φ is a *quite integrable constraint* function in the following sense

$$\text{for all } y \in \mathcal{Y}, \int_\Omega \gamma_o(\langle y, \varphi \rangle)\, dR < \infty \tag{2.2}$$

it is proved in [Lé2] and [Lé3] that (P_x) admits a solution if and only if x is admissible. It is also shown in [Lé2] that if x is a subgradient constraint, the minimizer of (P_x) has the form $Q_x = \gamma'(\langle z_x, \varphi \rangle) \cdot R$ where $z_x \in \mathcal{X}^*$ is such that $\langle z_x, \varphi \rangle$ satisfies some integrability conditions and can be approximated by some $\langle y, \varphi \rangle$'s with y in $\mathcal{Y}$. More, the z's in $\mathcal{X}^*$ such that $\gamma'(\langle z, \varphi \rangle) \cdot R$ is the solution to some problem $(P_{x(z)})$ are characterized. We shall recover this result as a particular case of Theorem 7.6 below.

Such z's are called admissible dual parameters. In case γ is an even function (to make statements easier), $\kappa := \lim_{s\to\infty} \gamma'(s) < \infty$ and $\gamma^*(\kappa) < \infty$, the characterization of the minimizer Q_x of (P_x) is obtained in [Lé2] for any admissible constraint x (not necessarily subgradient). For the description of Q_x, one needs to consider a wider class of admissible parameters: Partly composed with an infinite force field which "pushes" x onto the relative boundary of *dom* Λ^*. This characterization is recalled below in Theorem 7.9.

The aim of this paper is to obtain analogous characterizations of minimizers as in [Lé2], when φ is only a *not very integrable constraint* function in the following sense

$$\text{for all } y \in \mathcal{Y}, \text{ there exists } \lambda > 0 \text{ such that } \int_\Omega \gamma_o(\lambda\langle y, \varphi \rangle)\, dR < \infty \tag{2.3}$$

This integrability assumption is equivalent to: $\langle y, \varphi \rangle$ is in L_{γ_o} for all $y \in \mathcal{Y}$. Thus, the notation $\int_\Omega \varphi\, dQ$ is still meaningful under (2.3). Under this assumption, it appears that (P_x) doesn't always admit a solution in $L_{\gamma_o^*} \cdot R$ (see for instance Csiszár's example in Section 10 below). It is worth introducing the following extension of (P_x) :

$$(\overline{P}_x) \qquad \text{minimize } \ell \mapsto \overline{I}(\ell) \text{ subject to } \int_\Omega \varphi\, d\ell = x, \ell \in L'_{\gamma_o}$$

where L'_{γ_o} is the topological dual space of the Orlicz space $(L_{\gamma_o}, \|\cdot\|_{\gamma_o})$,

$$\int_\Omega \varphi\, d\ell \overset{\text{def}}{\iff} \Big\langle \ell, \langle y, \varphi \rangle \Big\rangle_{L'_{\gamma_o}, L_{\gamma_o}} = \langle x, y \rangle_{\mathcal{X},\mathcal{Y}}, \ \forall y \in \mathcal{Y} \tag{2.4}$$

and the energy functional $\overline{I}$ is given for any $\ell \in L'_{\gamma_o}$ by

$$\begin{aligned} \overline{I}(\ell) &= I(\ell^a) + I_s(\ell^s) \in [0,\infty] \text{ with} \\ I_s(\ell^s) &= \sup\{\langle \ell^s, u \rangle\,;\, u \in L_{\gamma_o}, \int_\Omega \gamma(u)\, dR < \infty\} \in [0,\infty] \end{aligned}$$

where $\ell = \ell^a + \ell^s$ is the decomposition in absolutely continuous and singular parts of $\ell \in L'_{\gamma_o}$. This unique decomposition is proved at Theorem 4.7. Note that $\overline{I}$ is convex but not strictly convex since I_s is positively homogeneous. As a consequence, $(\overline{P}_x)$ may admit infinitely many solutions. The main reason for considering this extension of (P_x) is that the recession function of I in the algebraic dual space $L^*_{\gamma_o}$ of L_{γ_o} has an effective domain included in the topological dual space L'_{γ_o} of L_{γ_o} and is equal to I_s on its domain.

For the moment, let us point out the following facts about duality in Orlicz spaces. Due to Hölder's inequality in Orlicz spaces, $L_{\gamma_o^*} \cdot R$ is included in L'_{γ_o}. If γ_o grows too fast, this inclusion may be strict. In such a situation, $L_{\gamma_o^*} \cdot R$ admits a supplementary subspace in L'_{γ_o}. Roughly speaking, ℓ^a belongs to $L_{\gamma_o^*} \cdot R$ and ℓ^s belongs to some supplementary subspace. More precisely, in the case where γ_o is a finite function ($dom\ \gamma_o = \mathbb{R}$), considering the following subspace of L_{γ_o} :

$$M_{\gamma_o} = \left\{ u \in L_{\gamma_o} ; \int_\Omega \gamma_o(\lambda u)\, dR, \forall \lambda > 0 \right\},$$

we have: $M'_{\gamma_o} \simeq L_{\gamma_o^*} \cdot R$ (it is the extension of the theorem of representation of the dual space of L_p for $1 \leq p < \infty$). Let N_{γ_o} be the factor space $N_{\gamma_o} = L_{\gamma_o}/M_{\gamma_o}$. One can represent $L'_{\gamma_o} \simeq M'_{\gamma_o} \oplus N'_{\gamma_o} \simeq (L_{\gamma_o^*} \cdot R) \oplus N'_{\gamma_o}$. When M_{γ_o} is strictly included in L_{γ_o}, N_{γ_o} is not reduced to zero and the above decomposition is nontrivial. The space N'_{γ_o} is the annihilator of M_{γ_o}. Its elements are singular in several equivalent senses (see Proposition 4.2 below).
If γ_o isn't finite everywhere, $M_{\gamma_o} = \{0\}$ and the previous sketch of reasoning doesn't hold anymore. In case R is a bounded measure, we have $L_{\gamma_o} = L_\infty$ and the decomposition $L'_\infty \simeq L_1 \oplus L^s_\infty$ is a known result (see [Ro2]).

We have already noticed that (2.3) is equivalent to $\langle y, \varphi \rangle \in L_{\gamma_o}$, for all $y \in \mathcal{Y}$. Let us also note that (2.2) is equivalent to $\langle y, \varphi \rangle \in M_{\gamma_o}$, for all $y \in \mathcal{Y}$. The relevance of the extension of I by $\overline{I}$ is reflected in the comparison of the following identities (when γ_o is finite everywhere)

$$\begin{aligned} I(\ell) &= \sup_{u \in M_{\gamma_o}} \left\{ \langle \ell, u \rangle - \int_\Omega \gamma(u)\, dR \right\} \\ \overline{I}(\ell) &= \sup_{u \in L_{\gamma_o}} \left\{ \langle \ell, u \rangle - \int_\Omega \gamma(u)\, dR \right\} \end{aligned} \tag{2.5}$$

for all $\ell \in L'_{\gamma_o}$ (see Proposition 5.1 below). As a consequence, both primal problems (P_x) and $(\overline{P}_x)$ admit the same dual problem (D_x) on $\mathcal{Y}$. In particular, it is proved in [Lé3] that $(\overline{P}_x)$ admits a solution if and only if $\Lambda^*(x) < \infty$. On the other hand, the extension of (D_x) in order to reach the dual attainment is wider for $(\overline{P}_x)$ than for (P_x) : The dual parameters of the subgradient constraints for $(\overline{P}_x)$ are described with new ingredients which allow the characterization of the singular parts of the minimizers. Roughly speaking, these singular parts are "essentially" the directions of support hyperplanes of the convex set $\{u \in L_{\gamma_o} ; \int_\Omega \gamma(u)\, dR < \infty\}$ and the new ingredients describe the contact points. Hence, they are typically points of the relative boundary of the closure (in some topological sense) of $\{u \in L_{\gamma_o} ; \int_\Omega \gamma(u)\, dR < \infty\}$. This result is stated in Theorem 7.6.

Extended relative entropy. The original motivation of the present paper was the characterization of the minimizers of the relative entropy subject to not very integrable constraints. It enters our framework provided that simple transformations are done. Let $\mathcal{P}$ be the set of all probability measures on $(\Omega,\mathcal{A})$. For any R and P in $\mathcal{P}$, the relative entropy of P with respect to R is $H(P\mid R)=\int_\Omega \log(\frac{dP}{dR})\,dP$ if $P\ll R$ and $+\infty$ otherwise. Taking $\gamma(s)=\rho(s):=e^s-s-1$, one sees that $H(P\mid R)=\int_\Omega \rho^*(\frac{d(P-R)}{dR})\,dR$. The corresponding function γ_o is denoted by $\tau(s)=e^{|s|}-|s|-1$, $s\in\mathbb{R}$. Its convex conjugate is $\tau^*(t)=(|t|+1)\log(|t|+1)-|t|$, $t\in\mathbb{R}$. The effective domain of $H(\cdot\mid R)$ is included in $\mathcal{P}\cap(L_{\tau^*}\cdot R)$. Considering for any $\ell\in L_{\tau^*}$, $I(\ell)=\int_\Omega\rho^*(\frac{d\ell}{dR})\,dR$, we obtain that for all P in $\mathcal{P}\cap(L_{\tau^*}\cdot R)$, $H(P\mid R)=I(P-R)$. In this situation, the minimization problem $(\overline{P}_x)$ corresponds to

$$\text{minimize } Q\in\mathcal{Q}_o\mapsto \overline{H}(Q\mid R) \text{ subject to } \int_\Omega \varphi\,dQ=x \tag{2.6}$$

where $\mathcal{Q}_o=\{Q\,;\,Q\in L'_\tau, Q\geq 0, \langle Q,\mathbb{I}\rangle=1\}$ and for all $Q\in\mathcal{Q}_o$, $\overline{H}(Q\mid R)=H(Q^a\mid R)+\sup\{\langle Q^s,u\rangle\,;\,u\geq 0,\int e^u\,dR<\infty\}$ is the extended relative entropy of Q with respect to R. Under the assumption that for all $y\in\mathcal{Y}$, $\int_\Omega e^{\lambda|\langle y,\varphi\rangle|}\,dR<\infty$ for some $\lambda>0$, the characterization of the minimizers is stated in Theorem 10.4. The dual parameters are built with all the above ingredients: finite and infinite force fields and the contact points of the singular support hyperplanes. The appearance of a singular component is a consequence of the fast growth of e^s-s-1 around $+\infty$. On the other hand, the appearance of an infinite force field is a consequence of the linear growth of e^s-s-1 around $-\infty$.

Outline of the paper. The identities (2.5) indicate that a natural approach to these minimization problems is in terms of convex duality. Denoting $\Phi(u)=\int_\Omega\gamma(u)\,dR$, one defines a convex function such that the energy functionals I and $\overline{I}$ are its convex conjugates: Φ^*, for different dualities.

At Section 3, the minimization of a general convex conjugate Φ^* under a general linear constraint is handled. The main result of this section is Theorem 3.2. It states an abstract characterization of the minimizers corresponding to a subgradient constraint, provided a dual equality holds. In our context, the dual equality $\inf(\overline{P}_x)=\sup(D_x)$ has been proved in [Lé3].

At Section 4, we study the duality of Orlicz spaces. Most of the results which are stated there, are already known. Nevertheless, they are not easily accessible in the literature and sometimes their proofs are deficient. The aim of this section is an attempt to remedy this fact. Several proofs seem to be new. The main results of this section are Proposition 4.2, which seems to be new, and Theorem 4.7.

At Section 5, it is shown that the abstract result of Section 3 applies to the framework of Orlicz spaces which is pertinent for our purpose. The transcription of Theorem 3.2 is stated in Theorem 5.4: For any subgradient constraint x, the set of solutions of $(\overline{P}_x)$ is included in the subdifferential (evaluated at a dual parameter) of some function $\overline{\Phi}$ defined on the algebraic bidual space $L^{**}_{\gamma_o}$ of L_{γ_o}. This function $\overline{\Phi}$ is a convex lower semicontinuous extension of $\Phi(u)=\int_\Omega\gamma(u)\,dR$.

The aim of Section 6 is to compute $\overline{\Phi}$, iterating the convex conjugacy on Φ. The pertinence of the use of Orlicz spaces mainly remains in the fact that the effective domain of $\overline{\Phi}$ is included in the topological bidual space L''_{γ_o} of L_{γ_o}, when γ is an even function (to simplify the statement). The main results of this section are Propositions 6.6 and 6.9.

At Sections 7 and 9, we deduce characterizations of the solutions of $(\overline{P}_x)$, from the previous results. They are stated in Theorems 7.6 and 9.2.

Section 8 is devoted to the description of the singular parts of the minimizers of $(\overline{P}_x)$.

Finally, at Section 10 we consider the minimization of the extended relative entropy. The main result of this section is Theorem 10.4. It is illustrated with an example proposed by I. Csiszár in [Cs2] in connection with the generalized I-projections.

3. Minimization of a convex conjugate

In this section, we give a new presentation of some results of [Lé1]. The abstract Theorem 3.2 is essential in our approach to derive a representation of the minimizers.

In this paper, it is understood that all the convex functions are proper convex, that is: convex and $]-\infty,+\infty]$-valued with at least one finite value. The geometric interior (core) of a subset A of X is the set of those $a \in A$ such that for any $x \in X$, there exists $\lambda > 0$ satisfying $[a, \lambda x[\subset A$. The affine hull of A : *aff* A, is the smallest affine space containing A. The relative interior of A : *ri* A, is the geometric interior of A considered as a subset of its affine hull: $\mathit{ri}\, A = \{a \in A ; \forall x \in \mathit{aff}\, A, \exists \lambda > 0, [a, \lambda x[\subset A\}$. The relative boundary of A : *rb* A, is defined by: $\mathit{rb}\, A = A \setminus \mathit{ri}\, A$. Note that *rb* A is included in A rather than in its closure. The effective domain of f is $\mathit{dom}\, f := \{x \in X ; f(x) < +\infty\}$. The relative interior of *dom* f is denoted by *ridom* f and the relative boundary of *dom* f is denoted by *rbdom* f

Let U be a vector space and V a vector subspace of U. The algebraic dual and bidual spaces of U and V are denoted by: U^*, U^{**}, V^* and V^{**}. We consider a nonnegative convex function

$$\Phi : u \in U \mapsto \Phi(u) \in [0, \infty]$$

such that $\Phi(0) = 0$. Its conjugates are

$$\begin{array}{rcllll} \Phi^* & : & \ell \in U^* & \mapsto & \sup_{u \in U} \{\langle \ell, u\rangle - \Phi(u)\} & \in [0,\infty], \\ \overline{\Phi} & : & \zeta \in U^{**} & \mapsto & \sup_{\ell \in U^*} \{\langle \zeta, \ell\rangle - \Phi^*(\ell)\} & \in [0,\infty]. \end{array}$$

Let us consider the relations between the vector spaces. We define the equivalence relation on U^* : $\ell \sim \ell'$ for any $\ell, \ell' \in U^*$ if and only if $\ell(u) = \ell'(u), \forall u \in V$. In other words: $\ell \sim \ell' \iff \ell_{|V} = \ell'_{|V}$. We identify V^* with the factor space:

$$V^* = U^*/\sim \tag{3.1}$$

and $\dot{\ell} \in V^*$ stands for the equivalence class of $\ell \in U^*$. Therefore, one can identify V^{**} with a vector subspace of U^{**} : $V^{**} \subset U^{**}$ as follows. For any $\zeta \in U^{**}$,

$$\zeta \in V^{**} \iff \Big(\forall \ell, \ell' \in U^*, \ell \sim \ell' \implies \langle \zeta, \ell - \ell'\rangle = 0\Big).$$

In this article, we are going to solve a minimization problem of the type

$$\text{minimize } \Phi^*(\ell) \text{ subject to } \ell \in \alpha \tag{3.2}$$

with $\alpha \in V^*$. Since α is an affine subspace of U^*, $\ell \in \alpha$ is a linear constraint. In order to solve (3.2), let us introduce the restriction Ψ of Φ to $V \subset U$:

$$\Psi : u \in V \mapsto \Phi(u) \in [0, \infty].$$

Its conjugates are

$$\begin{array}{llll} \Psi^* & : \alpha \in V^* & \mapsto \sup_{u \in V}\{\langle \alpha, u\rangle - \Phi(u)\} & \in [0,\infty], \\ \overline{\Psi} & : \zeta \in V^{**} & \mapsto \sup_{\alpha \in V^*}\{\langle \zeta, \alpha\rangle - \Psi^*(\alpha)\} & \in [0,\infty]. \end{array}$$

Let us introduce the following algebraic subdifferentials:

$$\begin{aligned} \partial\Psi^*(\alpha) &= \{\zeta \in V^{**}\,;\, \Psi^*(\alpha+\eta) \geq \Psi^*(\alpha) + \langle \zeta, \eta\rangle, \forall \eta \in V^*\}, \alpha \in dom\ \Psi^* \\ \partial\overline{\Psi}(\zeta) &= \{\alpha \in V^*\,;\, \overline{\Psi}(\zeta+v) \geq \overline{\Psi}(\zeta) + \langle \alpha, v\rangle, \forall v \in V\}, \zeta \in dom\ \overline{\Psi} \\ \partial\overline{\Phi}(\zeta) &= \{\ell \in U^*\,;\, \overline{\Phi}(\zeta+u) \geq \overline{\Phi}(\zeta) + \langle \ell, u\rangle, \forall u \in U\}, \zeta \in dom\ \overline{\Phi} \end{aligned}$$

In order to write an expression for $\overline{\Psi}$, let us consider the closure $\overline{V}$ of V in U^{**} with respect to the $*$-weak topology $\sigma(U^{**}, U^*)$. We have $V \subset \overline{V} \subset V^{**} \subset U^{**}$ and $\overline{V}$ is also the $\sigma(V^{**}, V^*)$-closure of V in V^{**} (see [Lé1], Lemma 3.1). In ([Lé1], Proposition 3.3), it is proved that for any $\zeta \in V^{**}$:

$$\overline{\Psi}(\zeta) = \begin{cases} \overline{\Phi}(\zeta) & \text{if } \zeta \in \overline{V} \\ +\infty & \text{otherwise} \end{cases} \tag{3.3}$$

For any $\alpha \in V^*$, let $\mathcal{S}(\alpha)$ stand for the set of the solutions to (3.2).

Lemma 3.1.

(a) *For any* $\alpha \in V^*, \Psi^*(\alpha) \leq \inf\{\Phi^*(\ell)\,;\, \ell \in \alpha\}$

(b) *For any* $\alpha \in V^*, \Psi^*(\alpha) = \infty \Rightarrow \mathcal{S}(\alpha) = \emptyset$

(c) *For any* $\ell \in U^*, \Psi^*(\dot{\ell}) = \Phi^*(\ell) < \infty \Rightarrow \ell \in \mathcal{S}(\dot{\ell})$

(d) *For any* $\alpha \in V^*$ *such that* $\Psi^*(\alpha) = \inf\{\Phi^*(\ell)\,;\, \ell \in \alpha\} < \infty$ *and* $\partial\Psi^*(\alpha) \neq \emptyset$, *we have*

$$\mathcal{S}(\alpha) \subset \partial\overline{\Phi}(\zeta),\ \forall \zeta \in \partial\Psi^*(\alpha)$$

(e) *For all* $\zeta \in \overline{V}, \ell \in U^*, \ell \in \partial\overline{\Phi}(\zeta) \Rightarrow \Psi^*(\dot{\ell}) = \Phi^*(\ell) < \infty$

Theorem 3.2. *For any* $\alpha \in V^*$ *such that* $\Psi^*(\alpha) = \inf\{\Phi^*(\ell)\,;\, \ell \in \alpha\} < \infty$ *and* $\partial\Psi^*(\alpha) \neq \emptyset$, *we have:* $\mathcal{S}(\alpha) = \alpha \cap \partial\overline{\Phi}(\zeta)$, *for all* $\zeta \in \partial\Psi^*(\alpha)$.

As $V^{**} \subset U^{**}$, $\overline{\Phi}(\zeta)$ is meaningful for every $\zeta \in V^{**}$ and in particular for $\zeta \in \partial\Psi^*(\alpha) \subset V^{**}$.

The main interest of this result is that any α in *ridom* Ψ^* satisfies $\partial\Psi^*(\alpha) \neq \emptyset$. This is a consequence of the geometric version of Hahn-Banach's theorem. It is the reason why algebraic duality is considered here. Of course, the price to pay for such a general result is the computation of $\overline{\Phi}$.

Proof of Theorem 3.2. Let us admit the lemma for a while. As $\mathcal{S}(\alpha) \subset \alpha$, (d) leads us to: $\mathcal{S}(\alpha) \subset \alpha \cap \partial\overline{\Phi}(\zeta)$ for all $\zeta \in \partial\Psi^*(\alpha)$.

Let us prove the converse inclusion. As $\zeta \in \partial\Psi^*(\alpha) \Rightarrow \overline{\Psi}(\zeta) < \infty \Rightarrow \zeta \in \overline{V}$ (see (3.3)) and $\ell \in \alpha \Leftrightarrow \dot{\ell} = \alpha$, with (e) and (c) one obtains that for any $\ell \in U^*, \zeta \in \partial\Psi^*(\alpha)$: $\ell \in \alpha \cap \partial\overline{\Phi}(\zeta) \Rightarrow \Psi^*(\alpha) = \Phi^*(\ell) < \infty \Rightarrow \ell \in \mathcal{S}(\alpha)$. ■

Proof of Lemma 3.1.

Proof of (a). For any $\ell \in \alpha$, we have: $\Psi^*(\alpha) = \sup_{v\in V}\{\langle\alpha, v\rangle - \Psi(v)\} = \sup_{v\in V}\{\langle\ell, v\rangle - \Phi(v)\} \leq \sup_{u\in U}\{\langle\ell, u\rangle - \Phi(u)\} = \Phi^*(\ell)$, from which it follows that $\Psi^*(\alpha) \leq \inf_{\ell\in\alpha} \Phi^*(\ell)$.

(b) and (c) are direct consequences of (a).

Proof of (d). Let us begin noting that any $\zeta \in \partial\Psi^*(\alpha)$ belongs to *dom* $\overline{\Psi}$ and therefore to $\overline{V}$. As a consequence, $\overline{\Psi}(\zeta) = \overline{\Phi}(\zeta)$ (see (3.3)) and $\langle\zeta, \alpha\rangle = \langle\zeta, \ell\rangle$ for all $\ell \in \alpha$. Indeed, let (v_α) be a generalized sequence in V converging to $\zeta \in \overline{V}$ for $\sigma(V^{**}, V^*)$. Then, $\langle\zeta, \alpha\rangle = \lim_\alpha\langle v_\alpha, \alpha\rangle = \lim_\alpha\langle v_\alpha, \ell\rangle$ for any $\ell \in \alpha$.
Let ℓ stand in $\mathcal{S}(\alpha)$. As $\Psi^*(\alpha) = \inf\{\Phi^*(\ell)\,;\, \ell \in \alpha\} < \infty$, we have $\Phi^*(\ell) = \Psi^*(\alpha)$. Then, for any $\zeta \in \partial\Psi^*(\alpha)$, we get $\Phi^*(\ell) = \Psi^*(\alpha) = \langle\zeta, \alpha\rangle - \overline{\Psi}(\zeta) = \langle\zeta, \ell\rangle - \overline{\Phi}(\zeta)$. This implies that $\ell \in \partial\overline{\Phi}(\zeta)$, which is the desired result.

Proof of (e). For any $\zeta \in \overline{V}, \ell \in U^*$, $\ell \in \partial\overline{\Phi}(\zeta) \Rightarrow \forall u \in U, \overline{\Phi}(\zeta + u) \geq \overline{\Phi}(\zeta) + \langle\ell, u\rangle \Rightarrow \forall v \in V, \overline{\Phi}(\zeta + v) \geq \overline{\Phi}(\zeta) + \langle\ell, v\rangle \Rightarrow \forall v \in V, \overline{\Psi}(\zeta + v) \geq \overline{\Psi}(\zeta) + \langle\dot{\ell}, v\rangle$ since $\zeta \in \overline{V}$ and $\zeta + v \in \overline{V}$. This means that $\dot{\ell} \in \partial\overline{\Psi}(\zeta)$. Therefore, $\infty > \Psi^*(\dot{\ell}) = \langle\zeta, \dot{\ell}\rangle - \overline{\Psi}(\zeta) = \langle\zeta, \ell\rangle - \overline{\Phi}(\zeta)$ (since $\zeta \in \overline{V}$) and because $\ell \in \partial\overline{\Phi}(\zeta)$, one gets $\Psi^*(\dot{\ell}) = \langle\zeta, \ell\rangle - \overline{\Phi}(\zeta) = \Phi^*(\ell) < \infty$ which is the desired result. ■

4. Duality in Orlicz spaces

The function $\theta : \mathbb{R} \to [0, +\infty]$ is called a Young function if it is convex, even and satisfies $\theta(0) = 0$, $\lim_{x\to\infty} \theta(x) = +\infty$ and there exists $x_o > 0$ such that $0 \leq \theta(x_o) < \infty$.
Let Ω be an arbitrary set, $\mathcal{A}$ be a σ-field of subsets of Ω and let R be a nonnegative σ-finite measure on $\mathcal{A}$. In this section, all the numerical functions on Ω are $\mathcal{A}$-measurable and R-almost everywhere equal functions are identified.
The Orlicz space associated with θ is defined by: $L_\theta := \{f : \Omega \to \mathbb{R}\,; \|f\|_\theta < +\infty\}$ with $\|f\|_\theta = \inf\left\{\beta > 0\,;\, \int_\Omega \theta\left(\frac{|f(\omega)|}{\beta}\right) R(d\omega) \leq 1\right\}$. The function $\|\cdot\|_\theta$ is a norm (the Luxemburg norm) and $L_\theta = \{f : \Omega \to \mathbb{R}\,; \exists\lambda_o > 0, \int_\Omega \theta(\lambda_o f)\, dR < \infty\}$. A subspace of interest is $M_\theta := \{f : \Omega \to \mathbb{R}\,; \forall\lambda > 0, \int_\Omega \theta(\lambda f)\, dR < \infty\}$. Of course: $M_\theta \subset L_\theta$. The function θ is said to satisfy the Δ_2-condition if

$$\text{there exist } C > 0, t_o \geq 0 \text{ such that } \forall t \geq t_o, \theta(2t) \leq C\theta(t) \tag{4.1}$$

If $t_o = 0$, the Δ_2-condition is said to be global. When R is bounded, in order that $M_\theta = L_\theta$, it is enough that θ satisfies the Δ_2-condition. When R is unbounded, this equality still holds if the Δ_2-condition is global.

Note that if $\theta(t_o) = \infty$ for some $t_o > 0$, M_θ reduces to the null space. If in addition R is bounded, L_θ is L_∞.
On the other hand, when θ is a finite function, M_θ contains all the bounded functions.

Duality in Orlicz spaces is intimately linked with the convex conjugacy. The convex conjugate θ^* of θ is also a Young function so that one may consider the Orlicz space L_{θ^*}.

Result. (Hölder's inequality). *For any $f \in L_\theta$ and $g \in L_{\theta^*}$, fg belongs to L_1 and $\int_\Omega |fg|\, dR \leq 2\|f\|_\theta \|g\|_{\theta^*}$.*

Proof. Young's inequality: $xy \leq \theta(x)+\theta^*(y), \forall x,y$, gives: $|fg|/ab \leq \theta(|f|/a)+\theta^*(|g|/b), \forall a,b>0$. One concludes, choosing $a=\|f\|_\theta$ and $b=\|g\|_{\theta^*}$ (if $0<a,b<\infty$) and integrating. ■

Therefore, any g in L_{θ^*} defines a continuous linear form on L_θ for the duality bracket: $\langle f,g\rangle = \int_\Omega fg\, dR, \quad \forall f \in L_\theta$.

Let L'_θ and M'_θ denote the topological dual spaces of L_θ and M_θ endowed with the norm $\|\cdot\|_\theta$. We have: $L_{\theta^*} \subset L'_\theta$. But in the general case, L'_θ may be larger than L_{θ^*}. Nevertheless, we have the following representation.

Result. (Representation of M'_θ). *It is assumed that θ is a finite function. We have $M'_\theta \simeq L_{\theta^*}$. This means that for any $\chi \in M'_\theta$, there exists a unique $g_\chi \in L_{\theta^*}$ such that $\chi(f) = \int_\Omega f g_\chi\, dR$, for all $f \in M_\theta$. More, the dual norm on M'_θ and $\|\cdot\|_{\theta^*}$ are equivalent norms.*

Proof. For any pointwise nonincreasing sequence (f_k) in M_θ converging to zero, by the dominated convergence theorem we obtain $(\star)$ $\lim_{k\to\infty} \int_\Omega \theta(\lambda f_k)\, dR = 0$, for any $\lambda > 0$. This implies that for any $f \in M_\theta$, $(\dagger)$ $\lim_{k\to\infty} \mathbb{1}_{|f|\leq k} f = f$ and $(\ddagger)$ $\lim_{p\to\infty} \mathbb{1}_{\Omega_p} f = f$ where the limits are taken with respect to $\|\cdot\|_\theta$ and (Ω_p) is an R-localizing sequence (see below). Note that in the argument $(\star)$, it is essential that f belongs to M_θ rather than L_θ.
When $R(\Omega) < \infty$, thanks to $(\star)$, $E \in \mathcal{A} \mapsto \chi(\mathbb{1}_E)$ is real measure. As it is absolutely continuous with respect to R, by Radon-Nikodym's theorem, there exists a function $g_\chi \in L_1$ such that $\chi(\mathbb{1}_E) = \int_\Omega \mathbb{1}_E g_\chi\, dR$, $\forall E \in \mathcal{A}$. Since a consequence of $(\dagger)$ is that the simple functions are dense in M_θ, it follows that $\chi(f) = \int_\Omega g_\chi f\, dR, \forall f \in M_\theta$. The limit $(\ddagger)$ together with the monotone convergence theorem allow the extension to the case where R is σ-finite. Moreover, it follows from: $\int_\Omega |g_\chi f|\, dR < \infty, \forall f \in M_\theta$, that $g_\chi \in L_{\theta^*}$ (see [RaR], Proposition 4.1.1). ■

Recall that a linear form ℓ on L_θ is relatively bounded if: $\forall v \in L_\theta, v \geq 0$, $\sup\{|\ell(u)|\,;\, u \in L_\theta, |u| \leq v\} < \infty$. We denote L^b_θ the space of all these forms.

Lemma 4.1. *The topological dual space L'_θ of L_θ is included in L^b_θ.*

Proof. Let $\ell \in L'_\theta$ and $u,v \in L_\theta$ be such that $|u| \leq v$. Then, $\|u\|_\theta \leq \|v\|_\theta$ and $|\ell(u)| \leq \|\ell\| \|v\|_\theta$. ■

Since L^b_θ is a Riesz space (see [Bou]), any $\ell \in L^b_\theta$ admits a unique decomposition in nonnegative and nonpositive parts $\ell = \ell_+ - \ell_-$, $|\ell| = \ell_+ + \ell_-$ is meaningful and $\ell_1 \wedge \ell_2$ stands for the infimum operation in L^b_θ. In order to state a representation result for L'_θ, we introduce the following definitions.

Definitions. *A sequence $(\Omega_p)_{p\geq 1}$ in $\mathcal{A}$ is said to be R-localizing if it is nondecreasing, $\bigcup_p \Omega_p = \Omega$ and $R(\Omega_p) < \infty$, $\forall p \geq 1$.*

A form $\ell \in L^b_\theta$ is said to be R-absolutely continuous if there exists $g \in L_{\theta^*}$ such that $\langle \ell, f\rangle = \int_\Omega fg\, dR$, for all $f \in L_\theta$.

A form $\ell \in L^b_\theta$ is said to be L_{θ^*}-singular if for any $g \in L_{\theta^*}, |\ell| \wedge |g\cdot R| = 0$.

A form $\ell \in L_\theta^b$ is said to be R-singular *if for any R-localizing sequence $(\Omega_p)_{p\geq 1}$, there exists a nonincreasing sequence $(A_k)_{k\geq 1}$ in $\mathcal{A}$ such that $\lim_{k\to\infty} R(A_k) = 0$ and $\langle |\ell|, \mathbb{I}_{(\Omega_p\setminus A_k)}\rangle = 0,\ \forall p, k$.*

A form $\ell \in L_\theta^b$ is said to be strongly R-singular *if for any R-localizing sequence $(\Omega_p)_{p\geq 1}$ and any $f \geq 0$ with $\mathbb{I}_{\Omega_p} f \in L_\theta$, $\forall p \geq 1$, there exists a nonincreasing sequence $(A_k)_{k\geq 1}$ in $\mathcal{A}$ such that: $\lim_{k\to\infty} R(A_k) = 0$ and $\langle |\ell|, \mathbb{I}_{(\Omega_p\setminus A_k)} f\rangle = 0,\ \forall p, k \geq 1$.*

Proposition 4.2. *Let us assume that θ is a* finite *function. For any $\ell \in L'_\theta$, the following properties are equivalent.*

(a) $\langle \ell, f\rangle = 0, \forall f \in M_\theta$.

(b) ℓ *is L_{θ^*}-singular.*

(c) ℓ *is strongly R-singular.*

(d) ℓ *is R-singular.*

Proof. $(a) \Rightarrow (b)$. For any nonnegative $f \in L_\theta$, denoting $B_n = \{f \leq n\} \cap \Omega_n$, as $\mathbb{I}_{B_n} f \in M_\theta$, (a) implies that $\langle \ell_+, \mathbb{I}_{B_n} f\rangle = \sup\{\langle \ell, h\rangle\,;\, 0 \leq h \leq \mathbb{I}_{B_n} f, h \in L_\theta\} = 0$ since $0 \leq h \leq \mathbb{I}_{B_n} f \in M_\theta \Rightarrow h \in M_\theta$. By dominated convergence, for any $g \in L_{\theta^*}$, $\lim_{n\to\infty} \int_\Omega g \mathbb{I}_{B_n^c} f\, dR = 0$. As $f = \mathbb{I}_{B_n} f + \mathbb{I}_{B_n^c} f$, for any n, it follows that $0 \leq \langle \ell_+ \wedge |g\cdot R|, f\rangle = \inf\{\langle \ell_+, u\rangle + \int_\Omega |g| v\, dR\,;\, u, v \geq 0, u + v = f\} \leq \lim_{n\to\infty}(\langle \ell_+, \mathbb{I}_{B_n} f\rangle + \int_\Omega g \mathbb{I}_{B_n^c} f\, dR) = 0$. We have obtained $\ell_+ \wedge |g\cdot R|$. A similar work for ℓ_- leads us to the desired result.

$(b) \Rightarrow (c)$. It is enough to prove it with $\ell \geq 0$. Let us first suppose that R is bounded. Choosing $g = 1$ in (b), we have $\ell \wedge 1 = 0$ which is equivalent to: $\forall \varepsilon > 0, \exists 0 \leq f_\varepsilon \leq 1$, such that $\int_\Omega (1 - f_\varepsilon)\, dR \leq \varepsilon$ and $\langle \ell, f_\varepsilon\rangle \leq \varepsilon$. But, for any $\lambda > 0$, $\theta(\lambda t) \leq \theta(\lambda) t$, $\forall 0 \leq t \leq 1$. Hence, for all $\lambda > 0$, $\int_\Omega \theta(\lambda(1 - f_\varepsilon))\, dR \leq \theta(\lambda) \int_\Omega (1 - f_\varepsilon)\, dR \leq \theta(\lambda)\varepsilon$. Since it is assumed that $\theta(\lambda) < \infty$, it follows that $\lim_{\varepsilon\to 0} \|1 - f_\varepsilon\|_\theta = 0$. One concludes with $\langle \ell, 1\rangle = \langle \ell, f_\varepsilon\rangle + \langle \ell, 1 - f_\varepsilon\rangle \leq \varepsilon + \langle \ell, 1 - f_\varepsilon\rangle$, $\forall \varepsilon > 0$, that $\langle \ell, 1\rangle = 0$. It follows that $\langle \ell, f\rangle = 0$ for any simple function f, and therefore for any $f \in M_\theta$ (we have already noted that the simple functions are dense in M_θ when R is bounded). This is (a) when R is bounded, which in turn gives (c) in the general case, taking $A_k = \{f \geq k\}$.

$(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (a)$. It is enough to consider $\ell \geq 0$. Let Ω_p and A_k as in (d). We have $\langle \ell, \mathbb{I}_{\Omega_p}\rangle = \langle \ell, \mathbb{I}_{\Omega_p \cap A_k}\rangle + \langle \ell, \mathbb{I}_{\Omega_p\setminus A_k}\rangle = \langle \ell, \mathbb{I}_{\Omega_p\cap A_k}\rangle$, by (d). By dominated convergence, for any $\lambda > 0$: $\lim_{k\to\infty} \int_\Omega \theta(\lambda \mathbb{I}_{\Omega_p\cap A_k})\, dR = 0$. This implies that $\mathbb{I}_{\Omega_p\cap A_k}$ tends to zero in L_θ. As ℓ is continuous, we obtain: $\langle \ell, \mathbb{I}_{\Omega_p}\rangle = 0$. It is already seen that this implies that $\langle \ell, f\mathbb{I}_{\Omega_p}\rangle = 0$, for any $f \in M_\theta$. Similarly, $\mathbb{I}_{\Omega_p} f$ tends to f in L_θ whenever f belongs to M_θ and it follows that $\langle \ell, f\rangle = \lim_{p\to\infty}\langle \ell, f\mathbb{I}_{\Omega_p}\rangle = 0$. ■

Let L_θ^s stand for the space of all L_{θ^*}-singular forms in L'_θ.

Theorem 4.3. (Representation of L'_θ). *Let θ be any* finite *Young function. Any $\ell \in L'_\theta$ is uniquely decomposed as $\ell = \ell^a + \ell^s$ with ℓ^a in $L_{\theta^*}\cdot R$ and ℓ^s in L_θ^s. This means that L'_θ is isomorphic to the direct sum $L'_\theta \simeq L_{\theta^*} \oplus L_\theta^s$.*

Proof. This result already appeared in ([Koz], Theorem 2.2), let us recall its proof. There exists a unique $g_\ell \in L_{\theta^*}$ such that for any $f \in M_\theta$, $\langle \ell, f\rangle = \int_\Omega g_\ell f\, dR$. Let us define $\langle \ell^s, f\rangle :=$

$\langle \ell, f\rangle - \int_\Omega g_\ell f\, dR, f\in L_\theta$. Note that, by Hölder's inequality, $g_\ell f$ is in L_1. Clearly, $\langle \ell^s, f\rangle = 0$ for all $f\in M_\theta$. Hence, by Proposition 4.2, ℓ^s is singular. One obtains the decomposition taking $\ell^a = g_\ell \cdot R$. Let us show that the decomposition is unique. Suppose that $\ell = \ell_1^a + \ell_1^s = \ell_2^a + \ell_2^s$. By Proposition 4.2, for any $f\in M_\theta$, $\langle \ell_1^a, f\rangle = \int_\Omega g_1 f\, dR = \int_\Omega g_2 f\, dR = \langle \ell_2^a, f\rangle$. But this uniquely extends to L_θ, i.e. $\ell_1^a = \ell_2^a$ since for any $f\in L_\theta$, $\int_\Omega g_1 f\, dR = \lim_{k\to\infty} \int_\Omega g_1 \mathbb{I}_{|f|\le k} f\, dR$ by dominated convergence. The uniqueness follows. ■

More about duality in Orlicz spaces. In the case where θ is non-finite, i.e. $\theta(t_o) = \infty$ for some $t_o > 0$, $M_\theta = \{0\}$ and the above proof breaks down. A representation result for L'_∞ is well-known (see for instance [CaV], Theorem VIII.5). This corresponds to a non-finite function θ with $\theta(t) = 0, \forall 0\le t\le t_1$, for some $t_1 > 0$. It doesn't seem that a representation result already appeared in the literature in the alternate case where θ is non-finite, $\theta(t) = 0 \Leftrightarrow t = 0$ and $R(\Omega) = \infty$.
Our aim in this subsection is to state in Theorem 4.7 a representation result for L'_θ with a general Young function θ. More precisely, we are going to describe the space L^b_θ. This result is an extension of the result of representation of L'_∞. Our proof is partly an adaptation of classical proofs on the subject (see [CaV], Chapter VIII).

Lemma 4.4. *The space L_{θ^*} is a band in L^b_θ.*

This means that
(a) for any $\ell\in L^b_\theta, f\in L_{\theta^*}, f\ge 0$, $|\ell|\le f \Rightarrow \ell\in L_{\theta^*}$ and
(b) for any nonempty $H\subset L_{\theta^*}$ such that there exists $\ell\in L^b_\theta$ which is greater than any element of H, then the supremum of H in L^b_θ belongs to L_{θ^*}.

Proof. Let us show (a). It is enough to prove it with $\ell\ge 0$. As, $0\le \langle \ell, \mathbb{I}_A\rangle \le \int_A f\, dR$ for any measurable A, we obtain that ℓ is a measure which is absolutely continuous with respect to R. Therefore, $0\le \frac{d\ell}{dR}\le f\in L_{\theta^*}$ and $\frac{d\ell}{dR}\in L_{\theta^*}$.
Let us show (b). Let $H\subset L_{\theta^*}$ and $\ell\in L^b_\theta$ which is greater than any element of H. We have to prove that $\sup H\in L_{\theta^*}$.
Without loss of generality, one can assume that H is a directed set (for this argument see ([Bou], Ch. 2, §1, Proposition 1) and that the elements of H are nonnegative (indeed, let $f_o\in H$, then $H_o = \{f\in H\,;\, f\ge f_o\}$ has the same supremum than H and $\sup H_o = f_o + \sup(H_o - f_o)$).
For any nonnegative measurable function u, let us define $\lambda(u) = \sup\left\{\int_\Omega uf\, dR\,;\, f\in H\right\} \in [0,\infty], \quad u\ge 0$ and show that $\lambda(u+v) = \lambda(u) + \lambda(v), \quad \forall u, v\ge 0$.
Clearly, $\lambda(u+v)\le \lambda(u)+\lambda(v)$. Conversely, for any $\varepsilon > 0$, there exist $f, g\in H$ such that: $\int_\Omega uf\, dR \ge \lambda(u) - \varepsilon$ and $\int_\Omega vg\, dR\ge \lambda(v) - \varepsilon$. Since H is a directed set, $f\vee g$ belongs to H, so that $\lambda(u+v)\ge \int_\Omega (u+v)(f\vee g)\, dR \ge \lambda(u)+\lambda(v) - 2\varepsilon$, which yields the desired equality.

If $(u_n)_{n\ge 1}$ is a nondecreasing sequence of nonnegative measurable functions which converges pointwise towards u, then
$\lambda(u) = \sup\left\{\int_\Omega \left(\sup_{n\ge 1} u_n\right) f\, dR\,;\, f\in H\right\} = \sup_{f\in H}\sup_{n\ge 1}\int_\Omega u_n f\, dR$ (monotone convergence) $= \sup_{n\ge 1}\sup_{f\in H}\int_\Omega u_n f\, dR = \sup_{n\ge 1}\lambda(u_n)$. Therefore, for any $A\in\mathcal{A}$, $\tilde\lambda(A) := \lambda(\mathbb{I}_A)$ defines a measure on $\mathcal{A}$ and for any nonnegative measurable function $u : \lambda(u) = \int_\Omega u\, d\tilde\lambda \in [0,\infty]$, since the Lebesgue integral of a nonnegative function u is the supremum of the integrals of the simple functions which are bounded above by u.

On the other hand, for any $A \in \mathcal{A}$, $R(A) = 0 \Longrightarrow \tilde{\lambda}(A) = 0$. Hence, there exists $h \in L_{1,loc}$ $(h \geq 0)$ such that $\tilde{\lambda} = h \cdot R$. Of course, h majorizes H since: $\forall A \in \mathcal{A}, \forall f \in H, \int_A h\,dR \geq \int_A f\,dR$. As H is bounded above by $\ell \in L_\theta^b$, for any $u \in L_\theta, u \geq 0$, we have $0 \leq \lambda(u) \leq \ell(u)$. Consequently, for any $u \in L_\theta$, we have: $\left|\int_\Omega hu\,dR\right| \leq \int_\Omega h|u|\,dR = \lambda(|u|) \leq \ell(|u|) < +\infty$. This proves that h belongs to L_{θ^*} (see [RaR], Proposition 4.1.1).

Finally, for any $m \in L_\theta^b$ such that $\int_\Omega uf\,dR \leq m(u), \forall f \in H, \forall u \in L_\theta, u \geq 0$, we have: $\int_\Omega uh\,dR = \lambda(u) = \sup_{f\in H} \int_\Omega uf\,dR \leq m(u), \quad \forall u \in L_\theta, u \geq 0$. Therefore h is the least upper bound of H, i.e.: $h = \sup H \in L_{\theta^*}$. Which is the desired result. ■

The Riesz theorem (see [Bou], Ch. 2, §1, Thm 1) together with Lemma 4.4 tells us that L_θ^b is the direct sum of the band L_{θ^*} and its complementary band. In other words:

Proposition 4.5. *Any $\ell \in L_\theta^b$ is uniquely decomposed into the sum $\ell = \ell^a + \ell^s$ where ℓ^a belongs to $L_{\theta^*} \cdot R$ and ℓ^s is L_{θ^*}-singular.*

Now, let us prove that the L_{θ^*}-singular part ℓ^s is also strongly R-singular.

Proposition 4.6. *Any $\ell \in L_\theta^b$ which is L_{θ^*}-singular is strongly R-singular.*

Proof. It is enough to prove the proposition with $\ell \geq 0$.
Let us first assume that R is a bounded measure. ℓ is L_{θ^*}-singular is equivalent to: $\forall \varepsilon > 0, \forall g \in L_{\theta^*}, g \geq 0, \ \forall w \in L_\theta, w \geq 0, \exists w_1, w_2 \in L_\theta, w_1, w_2 \geq 0$ such that $w = w_1 + w_2$ and $\langle \ell, w_1 \rangle + \langle g, w_2 \rangle \leq \varepsilon$.
In the special case where $g = 1 \in L_{\theta^*}$, taking $w = 1 + f \in L_\theta$ (note that since R is bounded, $1 \in L_{\theta^*}$ and $1 \in L_\theta$), one gets $\forall \varepsilon > 0, \exists w_1, w_2 \geq 0$ such that $\begin{cases} w_1 + w_2 = 1 + f \\ \langle \ell, w_1 \rangle + \langle 1, w_2 \rangle \leq \varepsilon. \end{cases}$ Therefore, $\varepsilon \geq \langle \ell, w_1 \rangle \geq \langle \ell, \frac{f}{2} \mathbb{I}_{\{w_1 \geq \frac{f}{2}\}} \rangle$, so that

$$\langle \ell, f \mathbb{I}_{\{w_1 \geq \frac{f}{2}\}} \rangle \leq 2\varepsilon. \tag{4.2}$$

On the other hand, we have $K := \int_\Omega (1+f)\,dR < \infty$. But, $w_1 = 1 + f - w_2$ and $\int_\Omega w_2\,dR = \langle 1, w_2 \rangle \leq \varepsilon$, so that $\int_\Omega w_1\,dR = K - \int_\Omega w_2\,dR \geq K - \varepsilon$. As, $1 + f - \mathbb{I}_{\{w_1 < \frac{f}{2}\}} \frac{1+f}{2} \geq w_1$, we get $K - \frac{1}{2} \int_\Omega \mathbb{I}_{\{w_1 < \frac{f}{2}\}} (1+f)\,dR \geq K - \varepsilon$ so that

$$R\left(w_1 < \frac{f}{2}\right) \leq \int_\Omega \mathbb{I}_{\{w_1 < \frac{f}{2}\}} (1+f)\,dR \leq 2\varepsilon. \tag{4.3}$$

With $\varepsilon = \frac{1}{2n}$, let us denote $B_n = \{w_1 < \frac{f}{2}\}$ (w_1 depends on ε), so that for any $n \geq 1$, with (4.2) and (4.3), we obtain: $\langle \ell, \mathbb{I}_{(\Omega \setminus B_n)} f \rangle \leq \frac{1}{n}$ and $R(B_n) \leq \frac{1}{n}$.
Let us define $A_k := \bigcup_{n \geq k} B_{n^2}$, so that $(A_k)_{k \geq 1}$ is a nonincreasing sequence in $\mathcal{A}$ which satisfies $R(A_k) \leq \sum_{n \geq k} R(B_{n^2}) \leq \sum_{n \geq k} \frac{1}{n^2} \underset{k \to \infty}{\longrightarrow} 0$.
On the other hand, $\langle \ell, \mathbb{I}_{(\Omega \setminus A_k)} f \rangle \leq \langle \ell, \mathbb{I}_{(\Omega \setminus B_{n^2})} f \rangle \leq \frac{1}{n^2}, \ \forall n \geq k$. Consequently: $\langle \ell, \mathbb{I}_{(\Omega \setminus A_k)} f \rangle = 0$. This completes the proof of the proposition when R is bounded.

Now, let us consider the general case where R is σ-finite. Let $(\Omega_p)_{p \geq 1}$ be an R-localizing sequence. We define $\Omega_1' = \Omega_1$ and for any $p \geq 2$, $\Omega_p' = \Omega_p \setminus \Omega_{p-1}$. We have just proved that for any $p \geq 1$, there exists a nonincreasing sequence $(A_k^p)_{k\geq 1}$ in $\mathcal{A}$ such that for all $k \geq 1$, $A_k^p \subset \Omega'_p, R(A_k^p) \leq \frac{2^{-p}}{k}$ and $\langle |\ell|, \mathbb{I}_{(\Omega'_p \setminus A_k^p)} f\rangle = 0$.

Putting $A_k := \bigcup_{p\geq 1} A_k^p$, one obtains that for all $k \geq 1$, $R(A_k) = \sum_{p\geq 1} R(A_k^p) \leq \frac{1}{k}\sum_{p\geq 1} 2^{-p} = \frac{1}{k} \underset{k\to\infty}{\longrightarrow} 0$ and for any $p_o \geq 1$, as $\Omega_{p_o} \setminus A_k = \bigcup_{p=1}^{p_o} (\Omega'_p \setminus A_k^p)$, we get $\langle |\ell|, \mathbb{I}_{(\Omega_{p_o}\setminus A_k)} f\rangle = \sum_{p=1}^{p_o} \langle |\ell|, \mathbb{I}_{(\Omega'_p\setminus A_k^p)} f\rangle = 0$. This completes the proof of the proposition. ∎

Let us sum the previous results up.

Theorem 4.7. (Representation of L'_θ). *Let θ be any Young function. Any $\ell \in L'_\theta$ is uniquely decomposed as*

$$\ell = \ell^a + \ell^s \tag{4.4}$$

with ℓ^a in $L_{\theta^*} \cdot R$ and ℓ^s in L_θ^s (the space of all continuous L_{θ^*}-singular forms on L_θ). This means that L'_θ is the direct sum $L'_\theta = (L_{\theta^*} \cdot R) \oplus L_\theta^s$.
Moreover, ℓ^s is strongly R-singular.

If θ satisfies the Δ_2-condition $L'_\theta = L_{\theta^*} \cdot R$, so that L_θ^s reduces to the null vector space.

In the decomposition (4.4), ℓ^a is called the absolutely continuous part of ℓ while ℓ^s is its singular part.

5. The minimization problem

In this section, we state precisely the minimization problem to be studied and establish the connections with Sections 3 and 4.

The energy functional. Let $(\Omega, \mathcal{A})$ be a measure space, R be a σ-finite measure on $\mathcal{A}$ and $\gamma : \mathbb{R} \to [0, \infty]$ a function which satisfies (2.1). In the sequel, any function on Ω is assumed to be measurable and R-a.e. equal functions are identified. We define the function space

$$U := \left\{u : \Omega \to \mathbb{R}\,;\ \int_\Omega \gamma(\lambda u)\, dR + \int_\Omega \gamma(-\lambda u)\, dR < \infty,\ \text{for some } \lambda > 0\right\} = L_{\gamma_o} \tag{5.1}$$

where $\gamma_o(s) = \max(\gamma(s), \gamma(-s)),\ s \in \mathbb{R}$. It is assumed that 0 stands in the interior of *dom* γ_o to avoid the trivial situation: $U = \{0\}$. Let us define the convex integral functional

$$\Phi : u \in U \mapsto \int_\Omega \gamma(u)\, dR \in [0, \infty]. \tag{5.2}$$

Note that γ is a normal integrand in the sense of Rockafellar.

We denote $\ell^a = \frac{d\ell^a}{dR} \cdot R$ with $\frac{d\ell^a}{dR} \in L_{\gamma_o^*}$ (see (4.4)). The energy functional to be studied is Φ^*, its analytic expression is stated in the following proposition. This result already appeared in ([Lé1], Theorem 6.2) in the case where γ is a finite function.

Proposition 5.1. *Under (2.1), (5.1) and (5.2), the convex conjugate Φ^* of Φ for the duality (U, U^*) is given for any $\ell \in U^*$, by*

$$\Phi^*(\ell) = \begin{cases} \int_\Omega \gamma^* \left(\frac{d\ell^a}{dR}\right) dR + \sup\{\langle \ell^s, u\rangle \,;\, u \in U, \Phi(u) < \infty\} & \text{if } \ell \in L'_{\gamma_o} \\ +\infty & \text{otherwise} \end{cases}$$

where γ^* is the convex conjugate of γ and $\ell \in L'_{\gamma_o}$ is decomposed as in (4.4).

Proof. It relies on Lemmas 6.1 and 6.4 which are stated and proved in the next section. As $\gamma \leq \gamma_o$, we have $\Phi = I_\gamma \leq I_{\gamma_o}$ and $I^*_{\gamma_o} \leq \Phi^*$, hence: *dom* $\Phi^* \subset$ *dom* $I^*_{\gamma_o}$. But, *dom* $I^*_{\gamma_o} \subset L'_{\gamma_o}$ by Lemma 6.4.(b). Now, the result follows from Lemma 6.1 with $\theta = \gamma_o$. ■

The linear constrainsts. We consider a vector space $\mathcal{Y}$ and its algebraic dual and bidual spaces $\mathcal{Y}^* = \mathcal{X}$ and $\mathcal{X}^*$. The space $\mathcal{X}$ is endowed with the smallest σ-field which makes all the functions $x \in \mathcal{X} \mapsto \langle y, x\rangle \in \mathbb{R}$, $y \in \mathcal{Y}$, measurable. Let $\varphi : \Omega \to \mathcal{X}$ be a measurable function which satisfies (2.3). In order to define the constraints via (3.1), we state $V = \big\{(\omega \in \Omega \mapsto \langle y, \varphi(\omega)\rangle \in \mathbb{R})\,;\, y \in \mathcal{Y}\big\}$. The assumption (2.3) is equivalent to $V \subset L_{\gamma_o} = U$. We assume that the spaces V and $\mathcal{Y}$ are in one-one correspondence. It is not a restriction: considering the factor space of $Y = \mathcal{Y}/\sim$ with respect to the equivalence relation $y_1 \sim y_2 \overset{\text{def}}{\iff} \langle y_1, \varphi\rangle = \langle y_2, \varphi\rangle$, one still has $\varphi(\omega) \in Y^*, \forall \omega \in \Omega$, up to some isomorphism. These one-one mappings are

$$\pi : \begin{cases} \mathcal{Y} & \to & V \\ y & \mapsto & v_y \end{cases} \quad v_y = \langle y, \varphi\rangle \qquad \pi^{-1} : \begin{cases} V & \to & \mathcal{Y} \\ v & \mapsto & y_v \end{cases} \quad y_v \text{ such that } v = \langle y_v, \varphi\rangle$$

Considering the transposition and the iterated transposition of π and π^{-1}, one obtains

$$\pi^t : \begin{cases} V^* & \to & \mathcal{X} \\ \alpha & \mapsto & x_\alpha \end{cases} \quad \langle x_\alpha, y\rangle := \langle \alpha, v_y\rangle, \forall y; \qquad \pi^{-t} : \begin{cases} \mathcal{X} & \to & V^* \\ x & \mapsto & \alpha_x \end{cases} \quad \langle \alpha_x, v\rangle := \langle x, y_v\rangle, \forall v$$

$$\pi^{tt} : \begin{cases} \mathcal{X}^* & \to & V^{**} \\ z & \mapsto & \zeta_z \end{cases} \quad \langle \zeta_z, \alpha\rangle := \langle z, x_\alpha\rangle, \forall \alpha; \quad \pi^{-tt} : \begin{cases} V^{**} & \to & \mathcal{X}^* \\ \zeta & \mapsto & z_\zeta \end{cases} \quad \langle z_\zeta, x\rangle := \langle \zeta, \alpha_x\rangle, \forall x$$

For any $\ell \in U^*, x \in \mathcal{X}$, we state the notation $\int_\Omega \varphi\, d\ell = x$ if and only if $\langle \ell, \langle y, \varphi\rangle\rangle_{U^*,U} = \langle x, y\rangle_{\mathcal{X},\mathcal{Y}}, \forall y \in \mathcal{Y}$. As $\int_\Omega \varphi\, d\ell = x \Leftrightarrow \ell \in \alpha_x$, it is the expression of a linear constraint (see (3.1)). By Proposition 5.2 below, the value of (P_x) and $(\overline{P}_x)$ is given by

$$\Lambda^*(x) = \sup_{y \in \mathcal{Y}} \left\{ \langle x, y\rangle - \int_\Omega \gamma(\langle y, \varphi(\omega)\rangle)\, R(d\omega) \right\} \in [0, \infty],\ x \in \mathcal{X}$$

It is the convex conjugate of

$$\Lambda(y) = \int_\Omega \gamma(\langle y, \varphi(\omega)\rangle)\, R(d\omega) \in [0, \infty],\ y \in \mathcal{Y}$$

Taking Ψ to be the restriction of Φ to V, we have

$$\Lambda = \Psi \circ \pi, \Psi = \Lambda \circ \pi^{-1} \qquad \Lambda^* = \Psi^* \circ \pi^{-t}, \Psi^* = \Lambda^* \circ \pi^t$$
$$\partial\Lambda^* = \pi^{-tt} \circ \partial\Psi^* \circ \pi^{-t} \qquad \partial\Psi^* = \pi^{tt} \circ \partial\Lambda^* \circ \pi^t.$$

Notations and assumptions. From now on, unless specified, U and Φ are always defined by (5.1) and (5.2), γ and φ always satisfy (2.1) and (2.3), while V and $\mathcal{Y}$ are in one-one correspondence.

Proposition 5.2. *For any $x \in \mathcal{X}$, we have $\inf\{\Phi^*(\ell)\,;\, \ell \in L'_{\gamma_o}, \int_\Omega \varphi\, d\ell = x\} = \Lambda^*(x)$. If the value is finite, the infimum is attained.*

Proof. See ([Lé3], Theorem 3.3). ■

Remark. In the case where *dom* γ is a proper subset of $\mathbb{R}$ and R is a bounded measure, we have: $L_{\gamma_o} = L_\infty$ and $L_{\gamma_o^*} = L_1$.

Noting that *dom* $\Phi^* \subset L'_{\gamma_o}$, one can rewrite Proposition 5.2 as follows.

Corollary 5.3. *For any $\alpha \in V^*$, $\inf\{\Phi^*(\ell)\,;\, \ell \in U^*, \ell \in \alpha\} = \Psi^*(\alpha)$. If the value is finite, the infimum is attained.*

We are now in position to state our minimization problem.

(5.3) **The minimization problem.** *Let $\gamma : \mathbb{R} \to [0,\infty]$ be a function which satisfies (2.1): it is convex, differentiable on the interior of its domain, such that $\gamma(0) = 0$, it is not identically zero and its effective domain is not reduced to $\{0\}$.*
Let us assume that the constraint function $\varphi : \Omega \to \mathcal{X}$ satisfies (2.3). This means that for all $y \in \mathcal{Y}$, $\langle y, \varphi\rangle$ is measurable and $\int_\Omega \gamma\big(\lambda\langle y,\varphi\rangle\big)\, dR + \int_\Omega \gamma\big(-\lambda\langle y,\varphi\rangle\big)\, dR < \infty$, for some $\lambda > 0$.
It is assumed that V and $\mathcal{Y}$ are in one-one correspondence.
The expression of the linear constraint $\int_\Omega \varphi\, d\ell = x$ is defined by (2.4).
For $x \in \mathcal{X}$, the minimization problem to be studied is

$$\left|\begin{array}{l} \text{minimize } \ell \in L'_{\gamma_o} \mapsto \int_\Omega \gamma^*\left(\frac{d\ell^a}{dR}\right) dR + \sup\{\langle \ell^s, u\rangle\,;\, u \in L_{\gamma_o}, \int_\Omega \gamma(u)\, dR < \infty\} \in [0,\infty] \\ \text{subject to } \int_\Omega \varphi\, d\ell = x \end{array}\right.$$

We denote $\mathcal{S}(x) \subset L'_{\gamma_o}$ the set of its solutions.

Let us recall Definition 2.1.

Definition. *A vector $x \in \mathcal{X}$ is said to be an* <u>*admissible constraint*</u>*, if $\Lambda^*(x) < \infty$.*

A vector $x \in \mathcal{X}$ is said to be a <u>*subgradient constraint*</u>*, if $\Lambda^*(x) < \infty$ and $\partial\Lambda^*(x)$ is not empty, or equivalently if $\Psi^*(\alpha_x) < \infty$ and $\partial\Psi^*(\alpha_x)$ is nonempty.*

The main result of this section is the following

Theorem 5.4. *For any $x \in \mathcal{X}$, $\mathcal{S}(x)$ is nonempty if and only if x is an admissible constraint.*
The set of solutions $\mathcal{S}(x)$ is a convex $\sigma(L'_{\gamma_o}, L_{\gamma_o})$-compact subset of L'_{γ_o}.
For any subgradient constraint $x \in \mathcal{X}$ (in particular for any $x \in$ ridom Λ^*), we have:
$\mathcal{S}(x) = \alpha_x \cap \partial\overline{\Phi}(\zeta_z)$, for all $z \in \partial\Lambda^*(x)$, where $\alpha_x = \pi^{-t}(x)$ and $\zeta_z = \pi^{tt}(z)$,
or equivalently: $\mathcal{S}(x) = \alpha_x \cap \partial\overline{\Phi}(\zeta)$ for all $\zeta \in \partial\Psi^*(\alpha_x)$.

Proof. Putting together Theorem 3.2, Proposition 5.1, Corollary 5.3 and the considerations about the connection between V and $\mathcal{Y}$, we obtain the first and last statements.

Being the subset of minimizers of a convex lower semicontinuous function under closed linear constraints, $\mathcal{S}(x)$ is a $\sigma(L'_{\gamma_o}, L_{\gamma_o})$-closed convex set. It is also compact since Φ^* has $\|\cdot\|_\gamma^*$-bounded level sets (see [Lé1], Corollary 2.2). ■

6. The biconjugate

Our aim in this section is to compute the convex biconjugate $\overline{\Phi}$ of Φ. With Theorem 5.4, this will lead us at Section 7, to the characterization of the solutions of (5.3).

One says that $\ell \in U^*$ satisfies $\ell \in L'_\theta$ if $L_\theta \subset U$ and $\ell_{|L_\theta} \in L'_\theta$. We denote $I_\gamma(f) = \int_\Omega \gamma(f)\, dR \in [0,\infty]$ and $I_{\gamma^*}(f) = \int_\Omega \gamma^*(f)\, dR \in [0,\infty]$. Let A be a subset of a vector space X in duality with Y. The indicator function of A is $\delta(x \mid A) = \begin{cases} 0 & \text{if } x \in A \\ \infty & \text{if } x \notin A \end{cases}, x \in X$ and its support function is $\delta^*(y \mid A) = \sup_{x\in X}\{\langle x,y\rangle - \delta(x \mid A)\} = \sup_{x\in A}\langle x,y\rangle, y \in Y$.

Lemma 6.1. *Let θ be a Young function such that dom $\Phi \subset L_\theta$ (for instance $\theta = \gamma_o$), then for any $\ell \in L'_\theta$, $\Phi^*(\ell) = I_{\gamma^*}(\frac{d\ell^a}{dR}) + \delta^*(\ell^s \mid \text{dom } \Phi)$ where $\ell = \ell^a + \ell^s$ is the decomposition (4.4).*

Proof. This result is ([Koz], Thm 2.6) when θ is a finite Young function, it is ([Ro2], Thm 1) when $L_\theta = L_\infty$. It doesn't seem that the case where θ is non-finite, $(\theta(t) = 0 \Leftrightarrow t = 0)$ and $R(\Omega) = \infty$ is treated in the literature. A proof for the general case is given below.
As for any $u \in L_\theta$, $\langle \ell, u\rangle - \Phi(u) = [\langle \ell^a, u\rangle - \Phi(u)] + [\langle \ell^s, u\rangle - \delta(u \mid \text{dom } \Phi)]$ and since dom $\Phi \subset L_\theta$, we have: $\Phi^*(\ell) = \sup_{u\in U}\{\langle \ell, u\rangle - \Phi(u)\} = \sup_{u\in L_\theta}\{\langle \ell, u\rangle - \Phi(u)\} \le \sup_{u\in L_\theta}\{\langle \ell^a, u\rangle - \Phi(u)\} + \sup_{u\in L_\theta}\{\langle \ell^s, u\rangle - \delta(u \mid \text{dom } \Phi)\} = \Phi^*(\ell^a) + \delta^*(\ell^s \mid \text{dom } \Phi)$.
Let us prove the converse inequality. Let $u_1, u_2 \in L_\theta$. As ℓ^s is strongly R-singular (Proposition 4.6), there exists a nonincreasing sequence $(A_k)_{k\ge 1}$ in $\mathcal{A}$ such that $\lim_{k\to\infty} R(A_k) = 0$ and $\langle |\ell^s|, (|u_1|+|u_2|)\mathbb{1}_{(\Omega_p\setminus A_k)}\rangle = 0, \forall p, k$. Considering the sequence $f_n = \mathbb{1}_{(\Omega_n\setminus A_n)}u_1 + \mathbb{1}_{(\Omega_n\setminus A_n)^c}u_2, n \ge 1$, we get $\langle \ell, f_n\rangle = \langle \ell^a, \mathbb{1}_{(\Omega_n\setminus A_n)}u_1\rangle + \langle \ell^s, u_2\rangle + \langle \ell^a, \mathbb{1}_{(\Omega_n\setminus A_n)^c}u_2\rangle$. As $\frac{d\ell^a}{dR} \in L_{\theta^*}, u_1, u_2 \in L_\theta$, by dominated convergence we obtain: $\lim_{n\to\infty}\langle \ell, f_n\rangle = \langle \ell^a, u_1\rangle + \langle \ell^s, u_2\rangle$.
More, as $\gamma(0) = 0$, $\Phi(f_n) = \Phi(\mathbb{1}_{(\Omega_n\setminus A_n)}u_1) + \Phi(\mathbb{1}_{(\Omega_n\setminus A_n)^c}u_2)$. By monotone convergence $\lim_{n\to\infty}\Phi(\mathbb{1}_{(\Omega_n\setminus A_n)}u_1) = \Phi(u_1)$ and by dominated convergence $\lim_{n\to\infty}\Phi(\mathbb{1}_{(\Omega_n\setminus A_n)^c}u_2) \le \delta(u_2 \mid \text{dom } \Phi)$. Consequently, $\limsup_{n\to\infty}\Phi(f_n) \le \Phi(u_1) + \delta(u_2 \mid \text{dom } \Phi)$. Finally, $\Phi^*(\ell) \ge \sup_{u_1,u_2\in L_\theta}\limsup_{n\to\infty}\{\langle \ell, f_n\rangle - \Phi(f_n)\} \ge \sup_{u_1,u_2\in L_\theta}\{\langle \ell^a, u_1\rangle + \langle \ell^s, u_2\rangle - \Phi(u_1) - \delta(u_2 \mid \text{dom } \Phi)\} = \Phi^*(\ell^a) + \delta^*(\ell^s \mid \text{dom } \Phi)$, which is the desired inequality.
One completes the proof with ([Ro1], Theorem 2) which states that $I_\gamma^*(g) = I_{\gamma^*}(g)$, for all $g \in L_{\theta^*}$. ■

During this proof, it has been shown that

$$\forall u_1, u_2 \in L_\theta, \ell \in L'_\theta, \exists (f_n)_{n\ge1} \text{ in } L_\theta \text{ such that } \lim_{n\to\infty}\langle \ell, f_n\rangle = \langle \ell^a, u_1\rangle + \langle \ell^s, u_2\rangle. \tag{6.1}$$

A special case: γ is even. The main reason for considering as a first step the case where γ is even, is provided by the proof of Lemma 6.3 below.
As γ is even, it is a Young function and we have $\gamma = \gamma_o$ so that $U = L_\gamma$.
The key result of this subsection is the following lemma.

Lemma 6.2.
(a) Suppose that there exists a norm $\|\cdot\|$ on U such that $\sup\{\Phi(u)\,;\,\|u\|\leq r\}\leq 1$, for some $r>0$. Then, Φ is $\|\cdot\|$-continuous on ridom Φ and dom $\Phi^*\subset U'$, where U' is the topological dual space of $(U,\|\cdot\|)$.

(b) If in addition, $0<\inf\{\Phi(u)\,;\,\|u\|=t_o\}<\infty$ for some $t_o>0$, Φ^* is $\|\cdot\|^*$-continuous on ridom Φ^* and dom $\overline{\Phi}\subset U''$, where $\|\cdot\|^*$ is the dual norm on U' and U'' is the topological bidual space of $(U,\|\cdot\|)$.

Proof. See ([Lé1], Lemma 2.1). ∎

We are going to consider the two cases where the property

$$\|u_o\|_\gamma>1 \text{ and } \Phi(u_o)<\infty \text{ for some } u_o\in L_\gamma \tag{6.2}$$

holds or fails.

Lemma 6.3. *Let us assume that γ is even. If (6.2) holds, then there exists $t_o>0$ such that $0<\inf\{\Phi(u)\,;\,u\in L_\gamma,\|u\|_\gamma=t\}<\infty$.*

Proof. Take $t_o=\|u_o\|_\gamma>1$. For any $u\in L_\gamma$, $\|u\|_\gamma=t_o\Rightarrow\forall 0<a<t_o,\int_\Omega\gamma(u/a)\,dR\geq 1\Rightarrow$ $\Phi(u)\geq 1$ (with $a=1$). Therefore, $1\leq\inf\{\Phi(u)\,;\,\|u\|_\gamma=t_o\}\leq\Phi(u_o)<\infty$. ∎

Lemma 6.4. *Let assume that γ is even.*
(a) Φ is $\|\cdot\|_\gamma$-continuous on ridom Φ.
(b) dom $\Phi^*\subset L'_\gamma$
(c) If (6.2) holds, Φ^* is $\|\cdot\|^*_\gamma$-continuous on ridom Φ^*
(d) If (6.2) holds, dom $\overline{\Phi}\subset L''_\gamma$, where for any $\zeta\in U^{**}$, $\zeta\in L''_\gamma$ means that $\zeta_{|L'_\gamma}\in L''_\gamma$.

Proof. To prove (a) and (b) it is enough to check that $\|\cdot\|_\gamma$ satisfies the assumption of Lemma 6.2.(a). With $r=\frac{1}{2}$, for any $u\in L_\gamma$, $\|u\|_\gamma\leq\frac{1}{2}\Rightarrow\forall\beta>\frac{1}{2}$, $\int_\Omega\gamma(u/\beta)\,dR\leq 1\Rightarrow\Phi(u)\leq 1$ (with $\beta=1$).
Proof of (c) and (d). By Lemma 6.3, the assumption of Lemma 6.2.(b) is satisfied for $\|\cdot\|_\gamma$. This completes the proof of the lemma. ∎

Lemma 6.5. *Let us assume that γ is even. If (6.2) fails, then $L_\gamma=L_\infty$ and dom $\Phi\subset\{u\in L_\infty\,;\,\|u\|_\gamma\leq 1\}$.*

Proof. Since (6.2) fails, for all $u\in L_\gamma$, we have: $\Phi(u)<\infty\Rightarrow\|u\|_\gamma\leq 1\Rightarrow\forall a<1,\int_\Omega\gamma(au)\,dR\leq 1$ $\Rightarrow\Phi(u)=\lim_{a\uparrow 1}\Phi(au)\leq 1$ (monotone convergence). Hence,

$$\Phi(L_\gamma)\subset[0,1]\cup\{\infty\}. \tag{6.3}$$

Suppose that: $\sup dom\,\gamma:=a_+=+\infty$. Let $A\in\mathcal{A}$ such that $0<R(A)<\infty$ (R is σ-finite and non-zero) and $n\geq 1$, let us take $u_n=n\mathbb{1}_A$. Then, $\Phi(u_n)=\gamma(n)R(A)<\infty$, $\forall n\geq 1$ and $\lim_{n\to\infty}\Phi(u_n)=\infty$. This contradicts (6.3) and proves that $a_+<\infty$.
Let us suppose that $R(\Omega)=\infty$ and that there exists $a_o>0$ such that $0<\gamma(a_o)<\infty$. Taking $u_p=a_o\mathbb{1}_{\Omega_p}$ where (Ω_p) is R-localizing, we obtain: $\Phi(u_p)=\gamma(a_o)R(\Omega_p)<\infty$, $\forall p\geq 1$ and

$\lim_{p\to\infty}\Phi(u_p)=+\infty$, which contradicts (6.3). Therefore, if (6.2) fails, we have $0<a_+<\infty$ and $[R(\Omega)<\infty$ or $\gamma=\delta(\cdot\mid[-a_+,a_+])]$. But, $[0<a_+<\infty$ and $R(\Omega)<\infty]\Rightarrow L_\gamma=L_\infty$ and $[0<a_+<\infty$ and $\gamma=\delta(\cdot\mid[-a_+,a_+])]\Rightarrow L_\gamma=L_\infty$, which is the desired result. ■

Notations. *Let us denote $N_\gamma:=L_\gamma/M_\gamma$, the factor space of L_γ with respect to the equivalence relation: $u_1\sim u_2 \overset{\text{def}}{\iff} u_1-u_2\in M_\gamma$.*
Let D_γ be the $\|\cdot\|_\gamma$-closure in N_γ of the set $\{\dot u\in N_\gamma\,;\, u\in dom\ I_\gamma\}$.
Let $\overline{D}_\gamma$ stand for the $\sigma(L_\gamma^{s\prime},L_\gamma^s)$-closure of $D_\gamma\subset N_\gamma\subset L_\gamma^{s\prime}$.
For any ζ in the algebraic dual space $L_\gamma^{\prime *}$ of L'_γ, let us denote the restrictions

$$\zeta_1=\zeta_{|L_{\gamma^*}}\in L^*_{\gamma^*}\ \text{and}\ \zeta_2=\zeta_{|L^s_\gamma}\in L^{s\,*}_\gamma.$$

If ζ_1 is continuous on L_{γ^*}, its decomposition (4.4) is $\zeta_1=\zeta_1^a+\zeta_1^s$, with $\zeta_1^a\in L_\gamma$ and $\zeta_1^s\in L^s_{\gamma^*}$.

If γ is a finite function, the dual spaces N'_γ and L^s_γ are isomorphic. If γ has infinite values, $M_\gamma=\{0\}$ and $N_\gamma=L_\gamma$.
We have $\Phi(u)=\Phi_1(u)+\Phi_2(\dot u), u\in L_\gamma$ where $\Phi_1(u)=I_\gamma(u), u\in L_\gamma$ and $\Phi_2(\dot u)=\delta(\dot u\mid D_\gamma), \dot u\in N_\gamma$ being the equivalence class of $u\in L_\gamma$. Their convex conjugates with respect to the dualities (L_γ,L_{γ^*}) and (N_γ,L^s_γ) are $\Phi_1^*(\ell)=I_{\gamma^*}(\ell),\ell\in L_{\gamma^*}$ and $\Phi_2^*(\ell)=\delta^*(\ell\mid D_\gamma),\ell\in L^s_\gamma$. Proposition 5.1 states that: $\Phi^*(\ell)=\Phi_1^*(\ell^a)+\Phi_2^*(\ell^s),\ell\in L'_\gamma$.

Proposition 6.6. *Let us assume that γ is even. Then for any $\zeta\in U^{**}$,*

$$\overline{\Phi}(\zeta)=\begin{cases} I_\gamma(\frac{d\zeta_1^a}{dR})+\delta^*(\zeta_1^s\mid dom\ I_{\gamma^*})+\delta(\zeta_2\mid\overline{D}_\gamma) & \text{if } \zeta\in L''_\gamma\\ \infty & \text{otherwise}\end{cases}$$

Moreover, $\overline{D}_\gamma$ is norm bounded in $L^{s\prime}_\gamma$ and $\delta^*(\ell\mid D_\gamma)<\infty$ for all ℓ in L^s_γ.

Proof. As $dom\ \Phi^*\subset L'_\gamma$ (by Lemma 6.4.(b)), for any $\zeta\in U^{**}$, $\overline{\Phi}(\zeta)=\sup_{\ell\in U^*}\{\langle\zeta,\ell\rangle-\Phi^*(\ell)\}=\sup_{\ell\in L'_\gamma}\{\langle\zeta,\ell\rangle-\Phi^*(\ell)\}$. Thanks to the decomposition $L'_\gamma=L_{\gamma^*}\oplus L^s_\gamma$ (see Theorem 4.7) and Proposition 5.1, for any $\zeta=(\zeta_1,\zeta_2)\in L'^*_\gamma=L^*_{\gamma^*}\oplus L^{s\,*}_\gamma$ we have: $\overline{\Phi}(\zeta)=\overline{\Phi}_1(\zeta_1)+\overline{\Phi}_2(\zeta_2)$ with $\overline{\Phi}_1(\zeta_1)=\sup_{f_1\in L_{\gamma^*}}\{\langle\zeta_1,f_1\rangle-I_{\gamma^*}(f_1)\}$ and $\overline{\Phi}_2(\zeta_2)=\sup_{\ell_2\in L^s_\gamma}\{\langle\zeta_2,\ell_2\rangle-\delta^*(\ell_2\mid dom\ I_\gamma)\}$.
(a) Let us first suppose that (6.2) holds. By Lemma 6.4.(d), $dom\ \overline{\Phi}\subset L''_\gamma$. Hence, one can apply Lemma 6.1 to I_{γ^*} : $\overline{\Phi}_1(\zeta_1)=I_\gamma(\frac{d\zeta_1^a}{dR})+\delta^*(\zeta_1^s\mid dom\ I_{\gamma^*})$, and by Lemma 6.7 below (applied with $X=N_\gamma, Y=L^s_\gamma$ and $Z=L^{s\prime}_\gamma$) : $\overline{\Phi}_2(\zeta_2)=\delta(\zeta_2\mid\overline{D}_\gamma)$.
As Lemma 6.4.(c) holds, Φ_2^* is bounded above on a non empty $\|\cdot\|^*_\gamma$-open ball. As Φ_2^* is positively homogeneous, we obtain that its domain is the whole space L^s_γ, which in turn is equivalent to: $\overline{D}_\gamma$ is $\sigma(L^{s\prime}_\gamma,L^s_\gamma)$-bounded, hence norm bounded in $L^{s\prime}_\gamma$ (by the uniform bound theorem).
(b) If (6.2) fails, by Lemma 6.5: $L_\gamma=L_\infty$ and $\{u\in L_\infty\,;\,\|u\|_\infty<a_+\}\subset dom\ \Phi\subset\{u\in L_\infty\,;\,\|u\|_\infty\le a_+\}$ with $0<a_+=\sup dom\ \gamma<\infty$. More, $\Phi_1^*(\ell^a)=I_{\gamma^*}(\frac{d\ell^a}{dR}),\forall\ell^a\in L_1$ and $\Phi_2^*(\ell^s)=\delta^*(\ell^s\mid D_\gamma),\forall\ell^s\in L^s_\infty$. Applying Lemma 6.2.(a) with $\Phi_1^*=I_{\gamma^*}$ and $\|\cdot\|=\|\cdot\|_1$, we obtain: $dom\ \overline{\Phi}_1\subset L'_1=L_\infty$ and by Lemma 6.1: $\overline{\Phi}_1(\zeta_1)=I_\gamma(\zeta_1),\forall\zeta_1\in L_\infty$.
We have $L_\gamma=N_\gamma=L_\infty$ with D_γ a $\|\cdot\|_\infty$-bounded convex set with a non empty interior. Therefore, Φ_2^* is a norm which is equivalent to the dual norm $\|\cdot\|^*_\infty$ of L'_∞, and $dom\ \Phi_2^*=L^s_\gamma$.
It follows that the condition of Lemma 6.2.(a) is fulfilled with Φ_2^* and $\|\cdot\|^*_\infty$. Therefore, $\zeta_2\in dom\ \overline{\Phi}_2$ implies that $\ell\in L'_\infty\mapsto\langle\zeta_2,\ell^s\rangle\in\mathbb{R}$ belongs to L''_∞. One concludes using Lemma 6.7 with $X=L_\infty$, $Y=L^s_\infty$, $Z=L^{s\prime}_\infty$, which states that $\overline{\Phi}_2(\zeta_2)=\delta(\zeta_2\mid\overline{D}_\gamma)$ for all $\zeta_2\in L^{s\prime}_\infty$. ■

Lemma 6.7. *Let (X, Y) and (Y, Z) two dual pairs such that $X \subset Z$ and Y separates Z. Let A be a subset of X and $\overline{co}A$ be its $\sigma(Z, Y)$-closed convex hull in Z.*
Then, for any $z \in Z$, $\overline{\delta}(z \mid A) := \sup_{y \in Y}\{\langle y, z\rangle - \delta^*(y \mid A)\} = \delta(z \mid \overline{co}A)$.

Proof. Let $z \in Z$, $\overline{\delta}(z \mid A) = \sup_{y \in Y}\{\langle y, z\rangle - \sup_{x \in A}\langle x, y\rangle\} = \sup_{y \in Y} \inf_{x \in A}\langle z - x, y\rangle$. With $y = 0$, it comes out that $\overline{\delta}(z \mid A) \geq 0$.
If there exists $y_o \in Y$ such that $\inf_{x \in A}\langle z - x, y_o\rangle > 0$, with $y = \lambda y_o$ and $\lambda \to +\infty$, it comes out that $\overline{\delta}(z \mid A) = +\infty$. Otherwise, we have

$$\forall y \in Y, \sup_{x \in A}\langle x - z, y\rangle \geq 0 \tag{6.4}$$

which implies: $\overline{\delta}(z \mid A) \leq 0$, hence: $\overline{\delta}(z \mid A) = 0$.
Now, let us show that (6.4) is equivalent to $z \in \overline{co}A$. By linearity of $x \mapsto \langle x, y\rangle$, and then by $\sigma(Z, Y)$-continuity of $z' \mapsto \langle z', y\rangle \in \mathbb{R}$, we obtain: $\sup_{x \in A}\langle x - z, y\rangle = \sup_{x \in \overline{co}A}\langle x - z, y\rangle = \sup_{z' \in \overline{co}A}\langle z' - z, y\rangle$. Therefore, non(6.4)$\Leftrightarrow \exists y_o \in Y, \sup_{z' \in \overline{co}A}\langle z' - z, y_o\rangle < 0 \Leftrightarrow \exists y_o \in Y, \sup_{z' \in \overline{co}A}\langle z', y_o\rangle < \langle z, y_o\rangle \Leftrightarrow z \notin \overline{co}A$ by Hahn-Banach theorem. ■

The general case: γ is not necessarily even. In this situation, let us consider the Young functions $\gamma_+(x) = \gamma(|x|)$ and $\gamma_-(x) = \gamma(-|x|)$, $x \in \mathbb{R}$, and the associated integral functionals $\Phi_+ = I_{\gamma_+}$ and $\Phi_- = I_{\gamma_-}$ defined on $U = L_{\gamma_o} = L_{\gamma_+} \cap L_{\gamma_-}$ where $\gamma_o = \max(\gamma_+, \gamma_-)$ as in Section 5.
Let us recall that there is a natural order on the algebraic dual space E^* of a Riesz vector space E which is defined by: $e^* \leq f^*$ if and only if $\langle e^*, e\rangle \leq \langle f^*, e\rangle$ for any $e \in E$ with $e \geq 0$. A linear form $e^* \in E^*$ is said to be relatively bounded if for any $f \in E$, $f \geq 0$, we have $\sup_{e\,;\,|e| \leq f} |\langle e^*, e\rangle| < +\infty$.
Although E^* may not be a Riesz space in general, the vector space: E^b, of all the relatively bounded linear forms on E is always a Riesz space. In particular, the elements of E^b admit a decomposition in nonnegative and nonpositive parts $e^* = e^*_+ - e^*_-$.

Proposition 6.8. *For any $\zeta \in U^{**}$, $\overline{\Phi}(\zeta) = \overline{\Phi_+}(\zeta_+) + \overline{\Phi_-}(\zeta_-)$ if $\zeta \in L'^b_{\gamma_o}$ and $\overline{\Phi}(\zeta) = \infty$ otherwise, where $L'^b_{\gamma_o} \subset U^{**}$ stands for the subspace of relatively bounded linear forms on L'_{γ_o} : $\zeta \in U^{**}$ belongs to $L'^b_{\gamma_o}$ if its restriction to $L'_{\gamma_o} \subset U^*$ is relatively bounded.*

Proof. See ([Lé1], Proposition 4.4), noting that the vector space spanned by *dom* Φ^* is L'_{γ_o} which is a Riesz space. ■

Remarks. Since the domains of Φ^* (*dom* $\Phi^* \subset L'_{\gamma_o}$) and $\overline{\Phi}$ consist of relatively bounded forms, one can consider their decompositions into nonnegative and nonpositive parts.
As $\overline{\Phi}(\zeta)$ only depends on the restriction of $\zeta \in U^{**}$ to $L'_{\gamma_o} \subset U^*$ (see [Lé1], Lemma 4.6), the above expression of $\overline{\Phi}(\zeta)$ is unambiguous.

Proposition 6.9. *For any $\ell \in \partial\overline{\Phi}(\zeta)$ with $\zeta \in$ dom $\overline{\Phi}$, we have $\ell \in L^b_{\gamma_o}$ and $\zeta \in L'^b_{\gamma_o}$.*
Moreover, $\ell_+ \in \partial\overline{\Phi_+}(\zeta_+)$and $\ell_- \in \partial\overline{\Phi_-}(\zeta_-)$.

Proof. The first statement comes from *dom* $\Phi^* \subset L'_{\gamma_o} \subset L^b_{\gamma_o}$ and *dom* $\overline{\Phi} \subset L'^b_{\gamma_o}$.
Let us prove the second statement. Since $\langle \ell_+, \zeta_+\rangle \leq \Phi^*_+(\ell_+) + \overline{\Phi_+}(\zeta_+)$ and $\langle \ell_-, \zeta_-\rangle \leq \Phi^*_-(\ell_-) + \overline{\Phi_-}(\zeta_-)$, we have $\langle \ell, \zeta\rangle = \langle \ell_+ - \ell_-, \zeta_+ - \zeta_-\rangle = \langle \ell_+, \zeta_+\rangle + \langle \ell_-, \zeta_-\rangle - [\langle \ell_+, \zeta_-\rangle + \langle \ell_-, \zeta_+\rangle] \leq$

$\langle \ell_+, \zeta_+\rangle + \langle \ell_-, \zeta_-\rangle \le \Phi_+^*(\ell_+) + \Phi_-^*(\ell_-) + \overline{\Phi_+}(\zeta_+) + \overline{\Phi_-}(\zeta_-) = \Phi^*(\ell) + \overline{\Phi}(\zeta) = \langle \ell, \zeta\rangle$. Hence, $\langle \ell_+, \zeta_-\rangle = \langle \ell_-, \zeta_+\rangle = 0$.

Suppose that $\langle \ell_+, \zeta_+\rangle < \Phi_+^*(\ell_+) + \overline{\Phi_+}(\zeta_+)$. Then, $\langle \ell, \zeta\rangle = \langle \ell_+, \zeta_+\rangle + \langle \ell_-, \zeta_-\rangle < \Phi_+^*(\ell_+) + \overline{\Phi_+}(\zeta_+) + \Phi_-^*(\ell_-) + \overline{\Phi_-}(\zeta_-) = \Phi^*(\ell) + \overline{\Phi}(\zeta) = \langle \ell, \zeta\rangle$, which is impossible. Therefore, $\langle \ell_+, \zeta_+\rangle = \Phi_+^*(\ell_+) + \overline{\Phi_+}(\zeta_+)$, that is $\ell_+ \in \partial\overline{\Phi_+}(\zeta_+)$ (and similarly for the negative index). ■

As γ_+ and γ_- are even functions, Propositions 6.6, 6.8 and 6.9 lead us to the complete expression of $\overline{\Phi}$ and of its subgradients.

7. Characterization of the minimizers I

The aim of this section is to compute and describe the subdifferential $\partial\overline{\Phi}(\zeta)$. Because of Theorem 5.4, this will lead us to the statement in Theorem 7.6 of a characterization of the solutions to (5.3). In this section, it is assumed that <u>γ is an even function</u>. The general case will follow from Proposition 6.9.

The subdifferential $\partial\overline{\Phi}(\zeta)$. For any $\ell \in L_\gamma^*$, ℓ belongs to $\partial\overline{\Phi}(\zeta)$ if and only if for all $u \in L_\gamma$, $\overline{\Phi}(\zeta+u) - \overline{\Phi}(\zeta) \ge \langle \ell, u\rangle$. By Proposition 6.6, for all $u \in L_\gamma$, $\zeta \in dom\, \overline{\Phi}$, $\overline{\Phi}(\zeta+u) - \overline{\Phi}(\zeta) = I_\gamma(\zeta_1^a + u) - I_\gamma(\zeta_1^a) + \delta^*((\zeta_1+u)^s \mid dom\, I_{\gamma^*}) - \delta^*(\zeta_1^s \mid dom\, I_{\gamma^*}) + \delta(\zeta_2 + u \mid \overline{D}_\gamma) - \delta(\zeta_2 \mid \overline{D}_\gamma)$. But $(\zeta_1+u)^s = \zeta_1^s$ for all $u \in L_\gamma$. Hence, the difference of δ^* vanishes. As $\overline{\Phi}(\zeta) < \infty$, we have: $\delta(\zeta_2 \mid \overline{D}_\gamma) = 0$. It comes out that for all $u \in L_\gamma$, $\zeta \in dom\, \overline{\Phi}$,

$$\overline{\Phi}(\zeta+u) - \overline{\Phi}(\zeta) = I_\gamma(\zeta_1^a + u) - I_\gamma(\zeta_1^a) + \delta(\zeta_2 + u \mid \overline{D}_\gamma). \tag{7.1}$$

Lemma 7.1. *Let $\zeta \in dom\, \overline{\Phi}$ be such that $\partial\overline{\Phi}(\zeta)$ is nonempty.*
(a) We have $0 \le \int_\Omega \zeta_1^a \gamma'(\zeta_1^a)\, dR < \infty$ and $0 \le \int_\Omega \gamma^* \circ \gamma'(\zeta_1^a)\, dR < \infty$. In particular, $\int_\Omega u\gamma'(\zeta_1^a)\, dR$ is meaningful for any $u \in L_\gamma$.
(b) For any $u \in L_\gamma$ such that $\zeta_1^a + t_o u \in dom\, I_\gamma$ for some $t_o > 0$, we have: $\lim_{t\downarrow 0} \frac{1}{t}[I_\gamma(\zeta_1^a + tu) - I_\gamma(\zeta_1^a)] = \int_\Omega u\gamma'(\zeta_1^a)\, dR$.

<u>Proof</u>. (a) is ([Lé2], Lemma 3.2), noting that its proof still holds when γ is not finite and the restrictive assumption (3.1) in [Lé2] is useless for this result.

(b) Let us introduce $g_t = [\gamma(\zeta_1^a + tu) - \gamma(\zeta_1^a) - tu\gamma'(\zeta_1^a)]/t$, $t > 0$. As γ is a convex function, $t \mapsto g_t$ is nondecreasing and $\lim_{t\downarrow 0} g_t = 0$. The result follows from (a) and the dominated convergence theorem. ■

Definitions of the singular cones. *Let us denote for any $\zeta \in L_\gamma''$:*

$$\begin{aligned} K(\zeta_1^a, \zeta_2) &= \{\ell \in L_\gamma' \,;\, \langle \ell, u\rangle \le 0, \forall u \in L_\gamma, \zeta_1^a + u \in dom\, I_\gamma, \zeta_2 + \dot{u} \in \overline{D}_\gamma\}, \\ K_1(\zeta_1^a) &= \{\ell \in L_\gamma' \,;\, \langle \ell, u\rangle \le 0, \forall u \in L_\gamma, \zeta_1^a + u \in dom\, I_\gamma\} \text{ and} \\ K_2(\zeta_2) &= \{\ell \in L_\gamma' \,;\, \langle \ell, u\rangle \le 0, \forall u \in L_\gamma, \zeta_2 + \dot{u} \in \overline{D}_\gamma\} \end{aligned}$$

These are $\sigma(L_\gamma', L_\gamma)$-closed convex cones with vertex 0. $K_1(\zeta_1^a)$ and $K_2(\zeta_2)$ are respectively the polar cones (for the duality (L_γ', L_γ)) of the cones: $\{u \in L_\gamma \,;\, \exists t > 0, \zeta_1^a + tu \in dom\, I_\gamma\}$ and $\{u \in L_\gamma \,;\, \exists t > 0, \zeta_2 + t\dot{u} \in \overline{D}_\gamma\}$. More,

$$K(\zeta_1^a, \zeta_2) = cl(K_1(\zeta_1^a) + K_2(\zeta_2))$$

where the closure is in $\sigma(L'_\gamma, L_\gamma)$. For a proof, see ([Ro3], Corollary 16.4.2).

These cones are called singular because of Lemma 8.1 below. With this definition in hand, we are in position to state the following

Proposition 7.2. *Let $\zeta \in dom\ \overline{\Phi}$ be such that $\partial\overline{\Phi}(\zeta)$ is nonempty. Then, $\partial\overline{\Phi}(\zeta) = \gamma'(\zeta_1^a)\cdot R + K(\zeta)$.*

Proof.
1- Let us show that $\partial\overline{\Phi}(\zeta) \subset \gamma'(\zeta_1^a)\cdot R + K(\zeta)$. Let $\ell \in \partial\overline{\Phi}(\zeta)$. We denote $\ell' = \ell - \gamma'(\zeta_1^a)\cdot R \in L'_\gamma$. Then, by (7.1), for all $t > 0$ and $u \in L_\gamma$: $\frac{1}{t}[\overline{\Phi}(\zeta + tu) - \overline{\Phi}(\zeta)] = \frac{1}{t}[I_\gamma(\zeta_1^a + tu) - I_\gamma(\zeta_1^a)] + \frac{1}{t}\delta(\zeta_2 + tu \mid \overline{D}_\gamma) \geq \frac{1}{t}\langle \ell, tu\rangle = \langle \ell, u\rangle = \int_\Omega u\gamma'(\zeta_1^a)\, dR + \langle \ell', u\rangle$. But for any $u \in L_\gamma$ such that $\zeta_1^a + t_o u \in dom\ I_\gamma$ for some $t_o > 0$, we have by Lemma 7.1: $\lim_{t\to 0}\frac{1}{t}[I_\gamma(\zeta_1^a + tu) - I_\gamma(\zeta_1^a)] = \int_\Omega u\gamma'(\zeta_1^a)\, dR$. Therefore, $\liminf_{t\to 0}\frac{1}{t}\delta(\zeta_2 + tu \mid \overline{D}_\gamma) \geq \langle \ell', u\rangle$. In particular, ℓ' satisfies

$$\langle \ell', u\rangle \leq 0 \text{ for all } u \in L_\gamma \text{ such that } \zeta_1^a + u \in dom\ I_\gamma \text{ and } \zeta_2 + \dot{u} \in \overline{D}_\gamma. \tag{7.2}$$

2- Let us prove the converse inclusion: $\partial\overline{\Phi}(\zeta) \supset \gamma'(\zeta_1^a)\cdot R + K(\zeta)$. Let $\ell = \gamma'(\zeta_1^a)\cdot R + \ell'$ where ℓ' satisfies (7.2). By (7.1), for any $u \in L_\gamma$ such that $\zeta_1^a + u \in dom\ I_\gamma$ and $\zeta_2 + \in \dot{u}\overline{D}_\gamma$, we have: $\overline{\Phi}(\zeta + u) - \overline{\Phi}(\zeta) \geq \int_\Omega u\gamma'(\zeta_1^a)\, dR \geq \int_\Omega u\gamma'(\zeta_1^a)\, dR + \langle \ell', u\rangle = \langle \ell, u\rangle$. The first inequality is a consequence of Lemma 7.1, the second one follows from $\langle \ell', u\rangle \leq 0$.
On the other hand, if $u \in L_\gamma$ is such that $\zeta_1^a + u \notin dom\ I_\gamma$ or $\zeta_2 + u \notin \overline{D}_\gamma$, then $\overline{\Phi}(\zeta + u) = \infty$ and the above inequality is trivial. Therefore, for all $u \in L_\gamma$, $\overline{\Phi}(\zeta + u) - \overline{\Phi}(\zeta) \geq \langle \ell, u\rangle$. ■

Corollary 7.3. *Let $\zeta \in dom\ \overline{\Phi}$ be such that $\partial\overline{\Phi}(\zeta)$ is nonempty. Then, $\partial\overline{\Phi}(\zeta)$ is reduced to its vertex: $\partial\overline{\Phi}(\zeta) = \{\gamma'(\zeta_1^a)\cdot R\}$, if and only if ζ_1^a stands in ridom I_γ and for all $u \in L_\gamma$, there exists $t > 0$ such that $\zeta_2 + tu, \zeta_2 - tu \in \overline{D}_\gamma$.*
In particular, if γ satisfies the Δ_2-condition (4.1) (globally if R is unbounded), we have: dom $\overline{\Phi} = L'_{\gamma^*} \simeq L_\gamma \oplus L^s_{\gamma^*}$ and $\partial\overline{\Phi}(\zeta) = \{\gamma'(\zeta^a)\cdot R\}$.

Proof. The first statement is a direct consequence of Proposition 7.2. The last one holds since $M_\gamma = L_\gamma = dom\ I_\gamma$ and $N_\gamma = \{0\}$ under our assumption. ■

Admissible dual parameters. We have $dom\ \overline{\Psi} = \overline{V} \cap dom\ \overline{\Phi}$ with $dom\ \overline{\Phi} \subset U^{**}$ and $\overline{V}$ the $\sigma(U^{**}, U^*)$-closure of V in U^{**}. For any $\zeta \in U^{**}$, one writes $\zeta \in L''_\gamma$ to signify that the restriction to L'_γ of ζ stands in L''_γ. More, we have: for all $\zeta, \xi \in U^{**}$, if ζ and ξ match on L'_γ, then $\overline{\Phi}(\zeta) = \overline{\Phi}(\xi)$. Therefore, we can identify any element of $dom\ \overline{\Phi}$ with its restriction to L'_γ : $dom\ \overline{\Phi} \subset L''_\gamma$.
As $\overline{\Psi}$ is the "restriction" of $\overline{\Phi}$ to $\overline{V}$, we also have: for all $\zeta, \xi \in U^{**}$, if ζ and ξ match on L'_γ, then $\overline{\Psi}(\zeta) = \overline{\Psi}(\xi)$. Hence, one can identify $dom\ \overline{\Psi}$ with $dom\ \overline{\Phi} \cap \widetilde{V} \subset L''_\gamma$, where $\widetilde{V}$ is the $\sigma(L''_\gamma, L'_\gamma)$-closure of V in L''_γ. It is clear that for any $\zeta \in dom\ \overline{\Phi}$, $\zeta_1^a \in dom\ I_\gamma \subset L_\gamma$ and $\zeta_2 \in \overline{D}_\gamma \subset L^s_\gamma$. Taking these identifications into account, we have shown the following

Proposition 7.4. *For all $\zeta \in U^{**}$, denoting ζ_1 and ζ_2 the restrictions of ζ to L_{γ^*} and L^s_γ we have: $\zeta \in dom\ \overline{\Psi}$ and $\zeta_1^s = 0$ if and only if $\zeta_1 \in dom\ I_\gamma$, $\zeta_2 \in \overline{D}_\gamma$ and for all $\varepsilon > 0$, $K \geq 1$, $f_1, \ldots, f_K \in L_{\gamma^*}$, $\ell_1, \ldots, \ell_K \in L^s_\gamma$, there exists $v \in V$ such that $\sum_{k\leq K} |\int_\Omega f_k(v - \zeta_1)\, dR| + |\langle \ell_k, (v - \zeta_2)\rangle| \leq \varepsilon$.*

In ([Lé2], Lemma 3.1), it is proved that ζ_1 can be written as $\zeta_1 = \langle \bar{z}_1, \varphi\rangle$ for some $\bar{z}_1 \in \mathcal{X}^*$.

Definition 7.5. *Let $\bar{z}_1$ be a linear form on $\mathcal{X}$. Under the assumption that γ is even, one says that $(\bar{z}_1, \zeta_2)$ is an* admissible dual parameter, *if the following properties are satisfied.*

1. $\langle \bar{z}_1, \varphi \rangle$ *is a measurable function*
2. $\int_\Omega \gamma(\langle \bar{z}_1, \varphi \rangle)\, dR < \infty$ *and* $\int_\Omega \gamma^* \circ \gamma'(\langle \bar{z}_1, \varphi \rangle)\, dR < \infty$
3. ζ_2 *stands in* $L_\gamma^{s\prime}$
4. *for all* $\varepsilon > 0$, $K \geq 1$, $f_k \in L_{\gamma^*}$, $\ell_k \in L_\gamma^s$ $k \leq K$, *there exists* $y \in \mathcal{Y}$ *such that* $\int_\Omega \gamma(\langle y, \varphi \rangle)\, dR < \infty$ *and for all* $k \leq K$, $|\int_\Omega (\langle \bar{z}_1, \varphi \rangle - \langle y, \varphi \rangle) f_k\, dR| \leq \varepsilon$, $|\langle \zeta_2 - \langle y, \varphi \rangle, \ell_k \rangle| \leq \varepsilon$.

Minimizers associated with a subgradient constraint. In the case where the constraint is subgradient, one can apply Theorem 5.4 to state a characterization of the solutions to the minimization problem (5.3).

Theorem 7.6. *(γ is an even function).*
Let us consider the minimization problem (5.3). The set of minimizers $\mathcal{S}(x)$ is nonempty if and only if x is admissible.
Let x be a subgradient constraint. One can associate with it an admissible dual parameter $(\bar{z}_1, \zeta_2)$ in the sense of Definition 7.5, such that

$$x = \int_\Omega \varphi \gamma'(\langle \bar{z}_1, \varphi \rangle)\, dR + \langle \varphi, \tilde{\ell} \rangle, \text{ for some } \tilde{\ell} \in K(\langle \bar{z}_1, \varphi \rangle, \zeta_2). \tag{7.3}$$

Moreover, the set of minimizers is

$$\mathcal{S}(x) = \{\gamma'(\langle \bar{z}_1, \varphi \rangle) \cdot R + \tilde{\ell};\ \tilde{\ell} \in K(\langle \bar{z}_1, \varphi \rangle, \zeta_2) \text{ such that (7.3) holds}\} \tag{7.4}$$

Conversely, for any admissible dual parameter $(\bar{z}_1, \zeta_2)$, x being defined by (7.3), we have $\Lambda^*(x) < \infty$. If in addition x is subgradient, then $\mathcal{S}(x)$ is given by (7.4).

Proof. It is a direct consequence of Theorem 5.4, Proposition 7.2 and Proposition 7.4. ∎

What happens when x is not a subgradient constraint. From now on, we have only considered subgradient constraints x. In this subsection, we examine the situation where x stands on *rbdom* Λ^* : the relative boundary of *dom* Λ^*.

Proposition 7.7. *Let us assume that*

$$\text{if } \gamma^* \text{ isn't } \Delta_2, \text{ then } \gamma \text{ is } \Delta_2 \text{ (globally, if } R \text{ is unbounded)} \tag{7.5}$$

With any x on the relative boundary of dom Λ^*, one can associate a measurable function $z_1 : \mathcal{X} \to [-\infty, +\infty]$, which is the pointwise limit of a sequence in $\mathcal{Y} \subset \mathcal{X}$ such that

(a) $x = \int_\Omega \varphi \gamma'(z_1 \circ \varphi)\, dR$ *and*

(b) $\Lambda^*(x) = \Phi^*\Big(\gamma'(z_1 \circ \varphi) \cdot R\Big) = \int_\Omega \gamma^*\left[\gamma'(z_1 \circ \varphi)\right] dR < \infty.$

Remark. Note that if γ^* is Δ_2, then the domain of Φ^* is the whole space L_{γ^*}, and *rbdom* Λ^* is empty. Therefore, under (7.5), if *rbdom* Λ^* is nonempty, γ is Δ_2 and it follows from Corollary 7.3 that $\partial\overline{\Phi}(\zeta)$ is reduced to $\gamma'(\zeta_1^a) \cdot R$.

Proof. See ([Lé2], Proposition 3.4). Thanks to the above remark, let us assume that γ is Δ_2. The method consists of approaching the class $\alpha_x \in V^*$ by the sequence $\alpha_n = (1-1/n)\alpha_x$. As α_n stands in the relative interior of *dom* Ψ^*, it admits a representation $\gamma'(\zeta_n)\cdot R \in \alpha_n$ with $\zeta_n \in L_\gamma$. Letting n tend to infinity leads us to the desired result. Under (7.5), the last argument of ([Lé2], Proposition 3.4) is preserved, since $L_\gamma = M_\gamma$. ■

It is important to note that z_1 may have infinite values. We define

$$\kappa = \lim_{t\to\infty} \gamma'(t) \in]0,\infty]$$

In the above proposition, it is understood that $\gamma'(+\infty) = \kappa$ and $\gamma'(-\infty) = -\kappa$. Therefore, $\Lambda^*(x) < \infty$ implies the following:

- as $\kappa > 0$, we have $R(z_1 \circ \varphi = \infty) < \infty$
- if $\kappa = \infty$, then $R(z_1 \circ \varphi = \infty) = 0$.

In the special situation where $\gamma^*(\kappa) < \infty$, it is proved in [Lé2] that a counterpart of Theorem 7.6 holds for the boundary admissible constraints. It is stated below in Theorem 7.9. Note that when $\gamma^*(\kappa) < \infty$, γ^* is nonfinite since its domain is $[-\kappa,\kappa]$. The description of the admissible dual parameters requires the introduction of the notion of admissible force field.

Let $\mathcal{J}$ be a totally ordered countable index set which admits a smaller element: $\flat$. We consider a family $(n) = (\langle n^j,\varphi\rangle)_{j\in\mathcal{J}}$ of measurable linear forms on $\mathcal{X}$. For any $j \in \mathcal{J}$, let us denote $\mathcal{T}_+^j = \{\langle n^j,\varphi\rangle > 0\}\bigcap \cap_{i<j}\{\langle n^i,\varphi\rangle = 0\}$ and $\mathcal{T}_-^j = \{\langle n^j,\varphi\rangle < 0\}\bigcap \cap_{i<j}\{\langle n^i,\varphi\rangle = 0\}$ with the convention: $\bigcap_{i<\flat}\{\langle n^i,\varphi\rangle = 0\} = \mathcal{X}$, so that $\mathcal{T}_+^\flat = \{n^\flat > 0\}$ and $\mathcal{T}_-^\flat = \{n^\flat < 0\}$. We define

$$\mathcal{S} = \bigcap_{j\in\mathcal{J}} \{\langle n^j,\varphi\rangle = 0\}, \quad \mathcal{T}_+ = \bigcup_{j\in\mathcal{J}} \mathcal{T}_+^j \quad \text{and} \quad \mathcal{T}_- = \bigcup_{j\in\mathcal{J}} \mathcal{T}_-^j \tag{7.6}$$

Up to a R-negligible set, $\mathcal{S}, \mathcal{T}_+$ and $\mathcal{T}_-$ form a measurable partition of Ω.
Let us introduce a notation for the force fields. Let $\bar{z}$ be a measurable linear form on $\mathcal{X}$ and $(n) = (\langle n^j,\varphi\rangle)_{j\in\mathcal{J}}$ as above. We define the application $\langle \bar{z} + \infty\cdot(n),\varphi\rangle : \Omega \to [-\infty,+\infty]$, for any $\omega \in \Omega$, by

$$\langle \bar{z} + \infty\cdot(n),\varphi(\omega)\rangle = \begin{cases} +\infty & \text{if } \omega \in \mathcal{T}_+ \\ -\infty & \text{if } \omega \in \mathcal{T}_- \\ \langle \bar{z},x\rangle & \text{if } \omega \in \mathcal{S}. \end{cases}$$

It is a measurable application. If $(n) = 0$, $\bar{z} + \infty\cdot(n) = \bar{z}$ has no infinite value.

Definition 7.8. *Let $\bar{z}_1 + \infty(n)$ be a force field. Its infinite component: $\infty(n)$, determines the measurable subsets $\mathcal{S}$, $\mathcal{T}_+$ and $\mathcal{T}_-$ (see (7.6)). Its finite component $\bar{z}_1$ is a linear form on $\mathcal{X}$.*
Under the assumption that γ is even and $\gamma^*(\kappa) < \infty$, $\bar{z}_1 + \infty(n)$ is said to be an admissible force field, if the following properties are satisfied.

1. *$\langle \bar{z}_1,\varphi\rangle$ is a measurable function whose support is included in $\mathcal{S}$.*
2. *$\int_{\mathcal{S}} \gamma(\langle \bar{z}_1,\varphi\rangle)\,dR < \infty$ and $\int_{\mathcal{S}} \gamma^*\circ\gamma'(\langle \bar{z}_1,\varphi\rangle)\,dR < \infty$*
3. *for any $j \in \mathcal{J}$, there exists $\alpha > 0$ such that $\int_{\cap_{i<j}\{n^i=0\}} \gamma(\alpha\langle n^j,\varphi\rangle)\,dR < \infty$*
4. *$R(\mathcal{T}_+) < \infty$ and $R(\mathcal{T}_-) < \infty$*

5. *for all* $\varepsilon > 0$, $K \geq 1$, $f_k \in L_{\gamma^*}$, $k \leq K$, *there exists* $y \in \mathcal{Y}$ *such that* $\int_\Omega \gamma(\langle y, \varphi \rangle)\, dR < \infty$ *and for all* $k \leq K$, $|\int_{\mathcal{S}}(\langle \bar{z}_1, \varphi\rangle - \langle y, \varphi\rangle) f_k\, dR| \leq \varepsilon$

With this definition in hand, we are ready to state the following result.

Theorem 7.9. *(γ is an even function). Let us assume that $\gamma^*(\kappa) < \infty$.*
Let x be any admissible constraint. One can associate with it an admissible force field $z_1 = \bar{z}_1 + \infty(n)$, in the sense of Definition 7.8, such that

$$x = \int_\Omega \varphi \gamma'(\langle z_1, \varphi \rangle)\, dR \tag{7.7}$$

Moreover, the set of minimizers is

$$\mathcal{S}(x) = \{\gamma'(\langle z_1, \varphi\rangle) \cdot R\} \tag{7.8}$$

Conversely, for any admissible force field z_1, x being defined by (7.7), we have $\Lambda^*(x) < \infty$ and $\mathcal{S}(x)$ is given by (7.8).
Furthermore, any admissible constraint is interior if and only if the force field is finite R-a.e..

<u>Proof</u>. See ([Lé2], Theorems 4.4 and 4.5). ■

If x stands in the relative geometric interior of the effective domain of Λ^*, $z_1 = \bar{z}_1$ has no infinite component. If it stands on the geometric boundary of the effective domain of Λ^*, the field of ordered collections of outward normal vector (n) characterizes the minimal face of the boundary on which x stands. Note that x is in the relative geometric interior of this face and $\bar{z}_1$ characterizes x in this face. For more details, see ([Lé2], Section 4).

8. Singular part of the minimizers

In this section, it is assumed that <u>γ is an even function</u>. The general case will follow from Proposition 6.9.
Let us recall that a differentiable convex function γ on $\mathbb{R}$ is said to be *steep* if it is finite or if $\sup\{\gamma'(t)\,;\, t \in dom\, \gamma\} = \infty$.

Lemma 8.1. *The function γ^* is stricly convex at infinity if and only if γ is steep.*
Under this assumption, for any $\zeta \in dom\, \overline{\Phi}$ such that $\partial\overline{\Phi}(\zeta)$ is nonempty, we have: $K(\zeta) \subset L^s_\gamma$.

<u>Proof</u>. The first statement is easy.
Suppose that there exists $k \in K(\zeta)$ such that k is absolutely continuous and different from zero. Because of Proposition 7.2, for all $t \geq 0$, $\ell_t = \gamma'(\zeta_1^a) \cdot R + tk$ is absolutely continuous and belongs to $\partial\overline{\Phi}(\zeta)$. With Proposition 5.1, we obtain: $\Phi^*(\ell_t) = \int_\Omega \gamma^*(\gamma'(\zeta_1^a) + t\frac{dk}{dR})\, dR < \infty$, $\forall t \geq 0$. On the other hand, $\ell_0, \ell_t \in \partial\overline{\Phi}(\zeta)$ implies that $\Phi^*(\ell_t) = \Phi^*(\ell_0) + \langle \zeta, \ell_t - \ell_0\rangle = \Phi^*(\ell_0) + t\langle\zeta, k\rangle$, $\forall t \geq 0$. These two expressions of $\Phi^*(\ell_t)$ imply that γ^* is affine at infinity. ■

The gauge function of $\mathcal{D}_\gamma$. Let $\mathcal{D}_\gamma$ stand for the $\|\cdot\|_\gamma$-closure of *dom* I_γ in L_γ. It is a convex set. As the $\|\cdot\|_\gamma$-interior of *dom* I_γ is nonempty, its topological closure: $\mathcal{D}_\gamma$, is equal to its geometric closure lin(*dom* I_γ) : the set of linearly accessible points from *dom* I_γ. As a consequence, the gauge functions of $dom\ I_\gamma$ and $\mathcal{D}_\gamma$ are equal. We have: $p_{\mathcal{D}_\gamma}(u) = \inf\{\lambda > 0\,;\, u/\lambda \in \mathcal{D}_\gamma\}$, for any $u \in L_\gamma$. As the $\|\cdot\|_\gamma$-interior of $dom\ I_\gamma$ is nonempty, $p_{\mathcal{D}_\gamma}(u)$ is finite for all $u \in L_\gamma$. Note that $p_{\mathcal{D}_\gamma}(u) \le 1 \Leftrightarrow \int_\Omega \gamma(u/\lambda)\, dR < \infty, \forall \lambda > 1 \Leftrightarrow u \in \mathcal{D}_\gamma$.

A description of $K_1(u)$.

Lemma 8.2. *Let $u \in \mathcal{D}_\gamma$, $u \ge 0$. For all $h \in L_\gamma$, $h \ge 0$, there exists $t_o > 0$ which only depends on h such that $u - t_o h$ belongs to $\mathcal{D}_\gamma$.*

Proof. For any $t > 0, \lambda > 1$, $\int_\Omega \gamma(\frac{u-th}{\lambda})\, dR \le \int_\Omega \gamma(\frac{(u-th)_+}{\lambda})\, dR + \int_\Omega \gamma((u-th)_-)\, dR \le \int_\Omega \gamma(u/\lambda)\, dR + \int_\Omega \gamma(th)\, dR$. But $\int_\Omega \gamma(u/\lambda)\, dR < \infty$ and $\int_\Omega \gamma(t_o h)\, dR < \infty$ for $t_o > 0$ small enough. Therefore, for all $\lambda > 1$, $\int_\Omega \gamma(\frac{u-t_o h}{\lambda})\, dR < \infty$. That is $p_{\mathcal{D}_\gamma}(u - t_o h) \le 1$, which is the desired result. ■

Corollary 8.3. *For any $u \in \mathcal{D}_\gamma$ and $\ell \in L'_\gamma$, we have: $(u \ge 0$ and $\ell \in K_1(u)) \Rightarrow \ell \ge 0$.*

Proof. For all $h \in L_\gamma, h \ge 0$, with Lemma 8.2, we have: $u - t_o h \in \mathcal{D}_\gamma$. As $\ell \in K_1(u)$, we get $\langle \ell, -t_o h\rangle \le 0$. Hence, $\langle \ell, h\rangle \ge 0$. ■

Lemma 8.4. *Let $u \in \mathcal{D}_\gamma$ and $w \in L_\gamma$ be such that $u + w \in \mathcal{D}_\gamma$, then: $u + w_+ \in \mathcal{D}_\gamma$.*

Proof. For all $\lambda > 1$, we have: $\int_\Omega \gamma(u/\lambda)\, dR < \infty$, $\int_\Omega \gamma(\frac{u+w}{\lambda})\, dR < \infty$ and $\int_{\{w\ge 0\}} \gamma(\frac{u+w_+}{\lambda})\, dR = \int_{\{w\ge 0\}} \gamma(\frac{u+w}{\lambda})\, dR \le \int_\Omega \gamma(\frac{u+w}{\lambda})\, dR$. Hence, $\int_\Omega \gamma(\frac{u+w_+}{\lambda})\, dR = \int_{\{w\ge 0\}} \gamma(\frac{u+w_+}{\lambda})\, dR + \int_{\{w<0\}} \gamma(\frac{u+w_+}{\lambda})\, dR = \int_{\{w\ge 0\}} \gamma(\frac{u+w_+}{\lambda})\, dR + \int_{\{w<0\}} \gamma(u/\lambda)\, dR \le \int_\Omega \gamma(\frac{u+w}{\lambda})\, dR + \int_\Omega \gamma(u/\lambda)\, dR < \infty$. That is $p_{\mathcal{D}_\gamma}(u + w_+) \le 1$ which is the desired result. ■

Proposition 8.5. *Let $u \in \mathcal{D}_\gamma$, $u \ge 0$. Then, $\ell \in K_1(u)$ if and only if $\ell \ge 0$ and for all $h \in L_\gamma$, $(h \ge 0$ and $\exists t > 0$, $u + th \in \mathcal{D}_\gamma) \Rightarrow \langle \ell, h\rangle = 0$.*

In some sense, it is a condition on the "support" of ℓ.

Proof.
- Necessary condition. Let $\ell \in K_1(u)$. We have: $u + th \in \mathcal{D}_\gamma \Rightarrow \langle \ell, th\rangle \le 0 \Rightarrow \langle \ell, h\rangle \le 0$. As $\ell \ge 0$ (Corollary 7.3) and $h \ge 0$, we also obtain $\langle \ell, h\rangle \ge 0$, which proves $\langle \ell, h\rangle = 0$.

- Sufficient condition. Let w be such that $u + w \in \mathcal{D}_\gamma$. Because of Lemma 8.4, we have $u + w_+ \in \mathcal{D}_\gamma$. Together with our assumption, this yields $\langle \ell, w_+\rangle = 0$. Therefore, as $\ell \ge 0$: $\langle \ell, w\rangle = \langle \ell, w_+\rangle - \langle \ell, w_-\rangle = -\langle \ell, w_-\rangle \le 0$. Hence, $\ell \in K_1(u)$. ■

A description of $K_2(\xi)$. Recall that D_γ is the $\|\cdot\|_\gamma$-closure of $E_\gamma := \{\dot u \in N_\gamma\,;\, u \in dom\ I_\gamma\}$ in N_γ. It is a convex set. Indeed, E_γ is the image of the convex set $dom\ I_\gamma$ by the canonical projection $u \mapsto \dot u$. As the $\|\cdot\|_\gamma$-interior of $dom\ I_\gamma$ is nonempty, the gauge functions of E_γ and D_γ are equal. This gauge function is $p_{D_\gamma}(v) = \inf\{\lambda > 0\,;\, v/\lambda \in D_\gamma\} = \inf_{u\in v, u\in L_\gamma} \inf\{\lambda > 0\,;\, \int_\Omega \gamma(u/\lambda)\, dR < \infty\}, v \in N_\gamma$. It is finite for all $v \in N_\gamma$.
From now on, any equivalence class $v \in N_\gamma$ is identified with one of its element $v \in L_\gamma$.

Lemma 8.6. *Let us denote $D_\gamma^+ = \{v \in D_\gamma\,;\, v \ge 0\} \subset N_\gamma$ and $\overline{D}_\gamma^+ = \{\xi \in \overline{D}_\gamma\,;\, \xi \ge 0\} \subset L_\gamma^{s\prime}$. The set D_γ^+ is $\sigma(L_\gamma^{s\prime}, L_\gamma^s)$-dense in $\overline{D}_\gamma^+$.*

Proof. (Ad absurdum). Let $\xi \in \overline{D}_\gamma^+$ be isolated from the closed convex set D_γ^+. By Hahn-Banach's theorem, there exists $\ell_o \in L_\gamma^s$ which separates ξ from $D_\gamma^+ : \forall v \in D_\gamma^+$, $\langle \ell_o, v\rangle \le \alpha < \langle \ell_o, \xi\rangle$. Since for all $v \ge 0$, $\langle \ell_o^+, v\rangle = \sup\{\langle \ell_o, w\rangle\,;\, 0 \le w \le v\}$, we have: $\sup_{v\in D_\gamma} \langle \ell_o^+, v\rangle = \sup_{v\in D_\gamma^+}\langle \ell_o, v\rangle \le \alpha < \langle \ell_o, \xi\rangle = \langle \ell_o^+, \xi\rangle - \langle \ell_o^-, \xi\rangle \le \langle \ell_o^+, \xi\rangle$. This is a contradiction, since $\xi \in \overline{D}_\gamma$ and $\ell_o^+ \in L_\gamma^s$. ■

Theorem 8.7. *Let $\xi \in \overline{D}_\gamma$, $\xi \ge 0$. Then, $\ell \in K_2(\xi)$ if and only if $\ell \ge 0$ and for all $h \in N_\gamma$, ($h \ge 0$ and $\exists t > 0$, $\xi + th \in \overline{D}_\gamma$) $\Rightarrow \langle \ell, h\rangle = 0$.*

Proof. The analogues of Lemma 8.2, Corollary 8.3, Lemma 8.4 and Proposition 8.5 hold mutatis mutandi for $(K_2(\xi), D_\gamma, N_\gamma)$ instead of $(K_1(u), \mathcal{D}_\gamma, L_\gamma)$. Let us call them Lemma 8.2',... Thanks to Lemmas 8.2' and 8.6', we obtain the following extension of Lemma 8.2': Let $\xi \in \overline{D}_\gamma$, $\xi \ge 0$, for all $h \in N_\gamma$, $h \ge 0$, there exists $t_o > 0$ such that $\xi - t_o h \in \overline{D}_\gamma$. As in Corollary 8.3', it follows that $\ell \in K_2(\xi)$, whenever $\ell \ge 0$.
As D_γ is dense in $\overline{D}_\gamma$, by Lemma 8.4', we obtain the following extension of Lemma 8.4': Let $\xi \in \overline{D}_\gamma$ and $w \in N_\gamma$ be such that $\xi + w \in \overline{D}_\gamma$, then, $\xi + w^+/2 \in \overline{D}_\gamma$. One considers $w/2$ rather than w to insure the existence of a generalized sequence (v_α) which converges towards ξ with $v_\alpha \in D_\gamma$ and $v_\alpha + w/2 \in D_\gamma$. One completes the proof of the theorem as in Proposition 8.5'. ■

For any u on the relative boundary of D_γ, by the support theorem, there exists ℓ^s in L_γ^s which is the direction of a support hyperplane of D_γ at $u : \delta^*(\ell^s \mid D_\gamma) = \langle \ell^s, u\rangle$.

Because of the discouraging appearance of the generic elements of $L_\gamma^{s\prime}$, those ℓ's which are associated with some u in L_γ as above, may be qualified as "good" forms. Part (b) of the following proposition states that these good forms are strongly dense.

Let us denote $D_{V,\gamma}$ the $\|\cdot\|_\gamma$-closure of $V \cap D_\gamma$ in N_γ and $\overline{D}_{V,\gamma}$ the $\sigma(L_\gamma^{s\prime}, L_\gamma^s)$-closure of $D_{V,\gamma}$ in $L_\gamma^{s\prime}$. Doing the identification: *dom* $\overline{\Phi} \subset L_\gamma''$, one obtains *dom* $\overline{\Phi} \cap \overline{V} = \overline{D}_{V,\gamma}$.

Proposition 8.8.
(a) For any $\ell^s \in L_\gamma^s$, there exists $\zeta \in \overline{D}_{V,\gamma}$ such that ℓ^s belongs to $K_2(\zeta)$ and $\delta^*(\ell^s \mid D_\gamma) = \langle \ell^s, \zeta\rangle$. This means that ℓ^s is the direction of a support hyperplane of $\overline{D}_\gamma$ at ζ.

(b) For any $\sigma(L_\gamma^{s\prime}, L_\gamma^s)$-neighbourhood W of ζ and any $\varepsilon > 0$, there exists u_ε in D_γ and ℓ_ε^s in L_γ^s such that u_ε stands in $W \cap D_\gamma$, $\|\ell^s - \ell_\varepsilon^s\|_\gamma^* \le \varepsilon$ and $\delta^*(\ell_\varepsilon^s \mid D_\gamma) = \langle \ell_\varepsilon^s, u_\varepsilon\rangle$. This means that ℓ_ε^s is the direction of a support hyperplane of D_γ at u_ε.

(c) For any $\sigma(L_\gamma^{s\prime}, L_\gamma^s)$-neighbourhood W of ζ and any $\varepsilon > 0$, there exists v_ε in $D_{V,\gamma}$ and k_ε in $V^* \cap L_\gamma^s$ such that v_ε stands in $W \cap D_{V,\gamma}$, $\sup_{y\in\mathcal{Y}}\{\langle \ell^s - k_\varepsilon, \langle y, \varphi\rangle\rangle\} \le \varepsilon$ and $\delta^*(k_\varepsilon \mid D_{V,\gamma}) = \langle k_\varepsilon, v_\varepsilon\rangle$. This means that k_ε is the direction at v_ε of a support hyperplane of $D_{V,\gamma}$ in the $\|\cdot\|_\gamma$-closure of V.

Comments. The important point in (b) and (c) is that u_ε and v_ε are both functions (in L_γ).
In (b), the function u_ε may not be in $D_{V,\gamma}$, so that ℓ_ε^s may not be the singular part of a minimizer. On the other hand in (c), as v_ε stands in $D_{V,\gamma}$, it is a function in L_γ which is the $\|\cdot\|_\gamma$-limit of functions in V. Furthermore, there exists some ℓ_ε^s which is the singular part of a minimizer with v_ε entering in its dual parameter, such that its restriction to the $\|\cdot\|_\gamma$-closure of V is precisely k_ε. But (c) doesn't give a full control of $\|\ell^s - \ell_\varepsilon^s\|_\gamma^*$; it only states a control in restriction to the $\|\cdot\|_\gamma$-closure of V.

Proof. Let us show (a). On the one hand, we have $\delta^*(\ell^s \mid D_\gamma) = \delta^*(\ell^s \mid \overline{D}_\gamma) = \Phi^*(\ell^s)$. On the other hand, identifying $\zeta \in \overline{D}_{V,\gamma}$ with $\hat{\zeta}(\ell) = \zeta(\ell^s)$, $\ell \in L'_\gamma$, we obtain $\ell^s \in \partial\overline{\Phi}(\hat{\zeta})$. This implies that $\Phi^*(\ell^s) = \langle \ell^s, \hat{\zeta}\rangle - \overline{\Phi}(\hat{\zeta}) = \langle \ell^s, \zeta\rangle$.

The part (b) of the proposition is a consequence of the Bishop-Phelps theorem. Without loss of generality, one can take W of the following form: $W = \{\xi \in L^{s\prime}_\gamma; |\langle \xi - \zeta, \ell_k\rangle| < \beta, \forall k \leq K\}$, $K \geq 1$, $\ell_k \in L^s_\gamma$, $k \leq K$, $\beta > 0$. Let us consider $W_\eta = \{\xi \in L^{s\prime}_\gamma; |\langle \xi - \zeta, \ell_k\rangle| < \beta/2, \forall k \leq K\} \cap \{\xi \in L^{s\prime}_\gamma; |\langle \ell^s, \xi\rangle - \delta^*(\ell^s \mid D_{V,\gamma})| < \eta\}$. As $\delta^*(\ell^s \mid D_\gamma) = \langle \ell^s, \zeta\rangle$, W_η is a $\sigma(L^{s\prime}_\gamma, L^s_\gamma)$-open neighbourhood of ζ. Since ζ is an accumulation point of D_γ, there exists $u_* \in D_\gamma \cap W_\eta$. By the Bishop-Phelps theorem, as $\langle \ell^s, u_*\rangle \geq \delta^*(\ell^s \mid D_\gamma) - \eta$, there exists $u_\eta \in D_\gamma$ and $\ell^s_\eta \in L^s_\gamma$ such that $\langle \ell^s_\eta, u_\eta\rangle = \delta^*(\ell^s_\eta \mid D_\gamma)$, $\|u_* - u_\eta\|_\gamma \leq \sqrt{\eta}$ and $\|\ell^s_\eta - \ell^s\|^*_\gamma \leq \sqrt{\eta}$. But, for any $k \leq K$, $|\langle u_\eta - \zeta, \ell_k\rangle| \leq |\langle u_\eta - u_*, \ell_k\rangle| + |\langle \zeta - u_*, \ell_k\rangle| \leq \|\ell_k\|^*_\gamma \|u_\eta - u_*\|_\gamma + \beta/2 \leq \|\ell_k\|^*_\gamma \sqrt{\eta} + \beta/2$. Thus, u_η stands in W for any small enough η. One completes the proof of (b), choosing $\eta = \varepsilon^2$.

The proof of part (c) follows the same line, replacing D_γ with $D_{V,\gamma}$, L_γ with the $\|\cdot\|_\gamma$-closure of V and $L^{s\prime}_\gamma$ with the $\sigma(L^{s\prime}_\gamma, L^s_\gamma)$-closure of V in $L^{s\prime}_\gamma$. Note also that $dom\, \overline{\Phi} \cap \overline{V} = \overline{D}_{V,\gamma}$. ■

9. Characterization of the minimizers II

In Section 7, we have obtained the characterization of minimizers under the assumption that γ is an even function. We have been able to handle the case where the constraints are subgradient. Under the additional assumption that $\gamma^*(\kappa) < \infty$, the minimizers are characterized for all admissible constraints. The constraint which are not handled by our results are nonsubgradient in the case where $\gamma^*(\kappa) = \infty$. For such an example with $\gamma^*(t) = e^{|t|} - |t| - 1$, see ([Lé2], Section 3).

In this section, γ is not supposed to be even. In order to avoid the "bad" boundary constraints, we assume that the even functions

$$\gamma_+(s) = \gamma(|s|) \text{ and } \gamma_-(s) = \gamma(-|s|)$$

which are associated with γ satisfy the following condition

$$\text{(9.1)} \qquad [\gamma_+^* \text{ is } \Delta_2 \text{ and } \gamma_+ \text{ is steep}] \text{ and } [\gamma_-^*(\kappa_-) = \gamma^*(-\kappa_-) < \infty \text{ and } \gamma_- \text{ is } \Delta_2]$$

where $\kappa_- = \lim_{s \to -\infty} \gamma(s)/|s| > 0$. As a special important example, we think of the relative entropy (see Section 10) which corresponds to $\gamma^*(t) = (t+1)\log(t+1) - t$ and $\gamma(s) = e^s - s - 1$. Note that if R is a bounded measure and $\gamma_-^*(\kappa_-) < \infty$, then γ_- is Δ_2.

Let x be an admissible constraint and ℓ a solution to the corresponding minimization problem (5.3). As γ_+^* is Δ_2, the domain of Λ_+^* has no boundary. Thus, the representation of ℓ_+ does not include an infinite force field. On the other hand, if γ_+ is not Δ_2, ℓ_+ may admit a singular part. Therefore, its representation in terms of dual parameter is of the type $(\bar{z}_1, \zeta_2)$.

As $\gamma_-^*(\kappa_-) < \infty$ and γ_- is Δ_2, ℓ_- has no singular part, but its representation may admit an infinite force field. Therefore, its representation in terms of dual parameter is of the type $(\bar{z}_1 + \infty(n), 0)$.

In order to state the characterization of the minimizers in Theorem 9.2 below, let us define the notion of admissible dual parameter under assumption (9.1).

Definition 9.1. *Let $\bar{z}_1 + \infty(n)$ be a force field. Its infinite component: $\infty(n)$, determines the measurable subsets $\mathcal{S}$, $\mathcal{T}_+$ and $\mathcal{T}_-$ (see (7.6)). Its finite component $\bar{z}_1$ is a linear form on $\mathcal{X}$.*
Under assumption (9.1), $(\bar{z}_1+\infty(n), \zeta_2)$ is said to be an admissible dual parameter, if the following properties are satisfied.

1. $\langle \bar{z}_1, \varphi\rangle$ *is a measurable function whose support is included in* $\mathcal{S}$
2. $\int_{\mathcal{S}} \gamma(\langle \bar{z}_1, \varphi\rangle)\, dR < \infty$ *and* $\int_{\mathcal{S}} \gamma^* \circ \gamma'(\langle \bar{z}_1, \varphi\rangle)\, dR < \infty$
3. *for any* $j \in \mathcal{J}$, $\int_{\cap_{i<j}\{n^i=0\}} \langle n^j, \varphi\rangle_-\, dR < \infty$
4. $R(\mathcal{T}_-) < \infty$ *and* $R(\mathcal{T}_+) = 0$
5. $\zeta_2 \in L^{s\prime}_{\gamma+}$ *and* $\zeta_2 \geq 0$
6. *for all* $\varepsilon > 0$, $K \geq 1$, $f_k \in L_{\gamma_+^*}$, $g_k \in L_{\gamma_-^*}$, $\ell_k \in L^s_{\gamma+}$, $k \leq K$, *there exists* $y \in \mathcal{Y}$ *such that* $\int_\Omega \gamma(\langle y, \varphi\rangle)\, dR < \infty$ *and for all* $k \leq K$:
$$|\int_{\mathcal{S}}(\langle \bar{z}_1, \varphi\rangle_+ - \langle y, \varphi\rangle_+) f_k\, dR| + |\int_{\mathcal{S}}(\langle \bar{z}_1, \varphi\rangle_- - \langle y, \varphi\rangle_-) g_k\, dR| \leq \varepsilon$$
$$|\langle \zeta_2 - \langle y, \varphi\rangle_+, \ell_k\rangle| \leq \varepsilon$$

As a consequence of Theorem 5.4, Propositions 6.8, 6.9, Lemma 8.1, Theorems 7.6 and 7.9, we obtain the following result.

Theorem 9.2. *Let us assume that γ satisfies (9.1).*
The set $\mathcal{S}(x)$ is nonempty if and only if x is admissible, that is $\sup_{y\in\mathcal{Y}}\{\langle x, y\rangle - \int_\Omega \gamma(\langle y, \varphi\rangle)\, dR\} < \infty$.
Let x be any admissible constraint. One can associate with it an admissible dual parameter $(\bar{z}_1 + \infty(n), \zeta_2)$ in the sense of Definition 9.1, such that

$$x = \int_{\mathcal{S}} \varphi\gamma'(\langle \bar{z}_1, \varphi\rangle)\, dR - \kappa_- \int_{\mathcal{T}_-} \varphi\, dR + \langle \varphi, \tilde{\ell}\rangle, \text{ for some } \tilde{\ell} \in K(\mathbb{1}_{\mathcal{S}}\langle \bar{z}_1, \varphi\rangle_+, \zeta_2). \tag{9.2}$$

Moreover, the set of minimizers of (5.3) is

$$\mathcal{S}(x) = \left\{[\gamma'(\langle \bar{z}_1, \varphi\rangle)\mathbb{1}_{\mathcal{S}} - \kappa_- \mathbb{1}_{\mathcal{T}_-}] \cdot R + \tilde{\ell}\,;\, \tilde{\ell} \in K(\mathbb{1}_{\mathcal{S}}\langle \bar{z}_1, \varphi\rangle_+, \zeta_2) \text{ such that (9.2) holds}\right\} \tag{9.3}$$

Furthermore, $\mathcal{S}(x)$ is a $\sigma(L'_{\gamma_o}, L_{\gamma_o})$-compact subset of L'_{γ_o}. All the elements of $\mathcal{S}(x)$ share the same absolutely continuous part: $[\gamma'(\langle \bar{z}_1, \varphi\rangle)\mathbb{1}_{\mathcal{S}} - \kappa_- \mathbb{1}_{\mathcal{T}_-}] \cdot R$ and have a nonnegative singular part: $\tilde{\ell}$, in $L^s_{\gamma+}$.
Conversely, for any admissible dual parameter $(\bar{z}_1 + \infty(n), \zeta_2)$, x being defined by (9.2), we have $\sup_{y\in\mathcal{Y}}\{\langle x, y\rangle - \int_\Omega \gamma(\langle y, \varphi\rangle)\, dR\} < \infty$ and $\mathcal{S}(x)$ is given by (9.3).

Of course, one can mix the results of Section 7 in different manners to obtain similar results under assumptions different from (9.1).

Finitely many constraints. Let us have a look to the special case of finitely many constraints. This means that $\mathcal{Y} = \mathcal{X} = \mathbb{R}^d$ with $1 \leq d < \infty$ and $\varphi = (\varphi_1, \dots, \varphi_d) : \Omega \to \mathbb{R}^d$ is measurable. It is supposed that $\varphi_i \in L_{\gamma_o}$, for all $1 \leq i \leq d$. A constraint is given by: $\langle \varphi_i, \ell\rangle = x_i \in \mathbb{R}$, $1 \leq i \leq d$. In this situation, V is the subspace of L_{γ_o} which is spanned by $(\varphi_1, \dots, \varphi_d)$ and it is clear that

$\overline{V} = V$. As a consequence, provided that γ is even (to simplify), the dual parameter $(\bar{z}_1, \zeta_2)$ of any subgradient constraint satisfies

$$\langle \bar{z}_1, \varphi \rangle \in L_{\gamma_o} \text{ and } \langle \zeta_2, \ell \rangle = \langle \langle \bar{z}_1, \varphi \rangle, \ell \rangle, \forall \ell \in L_\gamma^s \tag{9.4}$$

This is very much in contrast with (6.1). Note that if γ is Δ_2, (9.4) doesn't carry any information since L_γ^s is the null space so that $\zeta_2 \simeq 0$.

Saturated constraints. In the opposite direction, one may consider the saturated constraints $\ell = x \in \mathcal{X} = L_\gamma^*$. This corresponds to $\varphi : \omega \in \Omega \mapsto \delta_\omega \in \mathcal{X}$ and $\mathcal{Y} = \mathcal{L}_\gamma$ (the function space corresponding to L_γ without R-a.e. everywhere equality) since $\langle y, \varphi(\omega) \rangle = y(\omega)$ for any ω and $\langle \ell, \varphi \rangle = \ell$ for any $y \in L_{\gamma_o}$, $\ell \in L_{\gamma_o}^*$. To simplify, let us suppose that γ is even. The set of admissible constraints is the domain of Φ^* (see Lemma 6.1): ℓ is admissible if and only if $\ell \in L'_\gamma$, $I_{\gamma^*}(\frac{d\ell^a}{dR}) < \infty$ and $\delta^*(\ell^s \mid dom\ \Phi) < \infty$. In this situation, $(\bar{z}_1, \zeta_2)$ is an admissible dual parameter in the sense of Definition 7.5 if and only if $\bar{z}_1 \in L_\gamma$, $\int_\Omega \gamma(\bar{z}_1)\, dR < \infty$, $\int_\Omega \gamma^* \circ \gamma'(\bar{z}_1)\, dR < \infty$ and $\zeta_2 \in \overline{D}_\gamma$. As $\ell^a = \gamma'(\bar{z}_1) \cdot R$ and $\ell^s \in K(\bar{z}_1, \zeta_2)$ may be chosen indepently to make $\ell = \ell^a + \ell^s$ admissible, the corresponding dual parameters $\bar{z}_1$ and ζ_2 are not linked to each other. This is in accordance with (6.1).

10. Extended relative entropy

Let $\mathcal{P}$ be the set of all probability measures on $(\Omega, \mathcal{A})$. For any R and P in $\mathcal{P}$, the relative entropy of P with respect to R is $H(P \mid R) = \int_\Omega \log(\frac{dP}{dR})\, dP$ if $P \ll R$ and $+\infty$ otherwise. Taking $\gamma^*(t) = \begin{cases} (t+1)\log(t+1) - t & \text{if } t > -1 \\ 1 & \text{if } t = -1 \\ +\infty & \text{otherwise} \end{cases}$, one sees that $H(P \mid R) = \int_\Omega \gamma^*(\frac{d(P-R)}{dR})\, dR$. The convex conjugate of γ^* is given by

$$\gamma(s) = e^s - s - 1, s \in \mathbb{R}. \tag{10.1}$$

The corresponding function γ_o is denoted by $\tau(s) = e^{|s|} - |s| - 1$, $s \in \mathbb{R}$. Its convex conjugate is $\tau^*(t) = (|t|+1)\log(|t|+1) - |t|$, $t \in \mathbb{R}$. The effective domain of $H(\cdot \mid R)$ is included in $\mathcal{P} \cap (L_{\tau^*}(R) \cdot R)$. Considering for any $\ell \in L_{\tau^*}$, $I_a(\ell) = \int_\Omega \gamma^*(\frac{d\ell}{dR})\, dR$ with γ^* as above, we obtain that for all P in $\mathcal{P} \cap L_{\tau^*}$, $H(P \mid R) = I_a(P - R)$.

Taking our minimization problem (5.3) into account, we introduce the following function

$$I(\ell) = I_a(\ell^a) + I_s(\ell^s) \text{ where } I_s(\ell) = \sup\left\{\langle \ell, u \rangle\, ;\, u \in L_\tau, \int_\Omega (e^u - u - 1)\, dR < \infty\right\}, \ell \in L'_\tau(R).$$

In some applications, it is useful to consider I when R is nonnegative and σ-finite. Under this general assumption, we have the following

Lemma 10.1. *For any ℓ in L'_τ, we have*

$$I_s(\ell) = \begin{cases} \sup\{\langle \ell, u \rangle\, ;\, u \geq 0, \int_{\{u \leq 1\}} u^2\, dR < \infty, \int_{\{u \geq 1\}} e^u\, dR < \infty\} & \text{if } \ell \in L_\tau^s \text{ and } \ell \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

If in addition R is bounded, we have

$$I_s(\ell) = \begin{cases} \sup\{\langle \ell, u\rangle \,;\, u \geq 0, \int_\Omega e^u\, dR < \infty\} & \text{if } \ell \in L^s_\tau \text{ and } \ell \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

Proof. Let us first remark that u is in L_τ if and only if $\int_{\{|u|\leq 1\}} u^2\, dR < \infty$ and $\int_{\{|u|\geq 1\}} e^{\lambda|u|}\, dR < \infty$ for some $\lambda > 0$.

It has already been proved in Lemma 8.1 that $I_s(\ell) < \infty$ implies that ℓ is singular.

Let us show that $I_s(\ell) < \infty$ implies $\ell \geq 0$. For all $u \in L_\tau$, $u \leq 0$ implies $\int(e^u - u - 1)\, dR < \infty$. Thus, $I_s(\ell) \geq \sup\{\langle \ell, u\rangle \,;\, u \leq 0, u \in L_\tau\}$. Suppose that there exists $u_* \in L_\tau$ such that $u_* \leq 0$ and $\langle \ell, u_*\rangle > 0$. As $\beta u_* \in L_\tau$, $\beta u_* \leq 0$ for all $\beta \geq 0$, we obtain $I_s(\ell) \geq \sup_{\beta\geq 0} \beta\langle \ell, u_*\rangle = +\infty$.

The rest of the proof follows easily. ■

Let us go back to the relative entropy.

Definition 10.2. *Let R be a probability measure.*
Let us denote $\mathcal{Q}_o = \{Q \,;\, Q \in L'_\tau(R), Q \geq 0, \langle Q, \mathbb{I}\rangle = 1\}$. Clearly, $\mathcal{P} \cap L_{\tau^*} \subset \mathcal{Q}_o$. For all $Q \in \mathcal{Q}_o$,

$$\overline{H}(Q \mid R) = H(Q^a \mid R) + \sup\left\{\langle Q^s, u\rangle \,;\, u \geq 0, \int_\Omega e^u\, dR < \infty\right\}$$

is the extended relative entropy of Q with respect to R.

Comments. Because of Lemma 10.1, we have $\overline{H}(Q \mid R) = I(Q - R)$
We denote Q^s for $(Q-R)^s \in L^s_\tau$ without ambiguity, since $R^s = 0$. As Q^s is strongly R-singular and $Q \geq 0$, one can prove that Q^s is nonnegative, for any $Q \in \mathcal{Q}_o$. For all $u \in M_\tau = \{u \,;\, \forall\lambda, \int e^{\lambda|u|}\, dR < \infty\}$, we have $\langle Q^s, u\rangle = 0$. In particular, $\langle Q^s, \mathbb{I}\rangle = 0$. As $Q^s \geq 0$ on the domain of $\overline{H}(\cdot \mid R)$, this means that such a Q^s has a zero mass. Nevertheless, Q^s may not be null!
Since for any $u \in L_\tau$ and any $n \geq 1$ we have $\mathbb{I}_{\{|u|\leq n\}}u \in M_\tau$, we also obtain that $\langle Q^s, \mathbb{I}_{\{|u|\leq n\}}u\rangle = 0$. This implies that for all $n \geq 1$, $\langle Q^s, u\rangle = \langle Q^s, \mathbb{I}_{\{|u|\geq n\}}u\rangle$. Note that $\lim_{n\to\infty} R(|u| \geq n) = 0$.

A counterpart of the minimization problem (5.3) in this situation is the following one. For $x \in \mathcal{X}$,

$$\text{minimize } Q \in \mathcal{Q}_o \mapsto \overline{H}(Q \mid R) \text{ subject to } \int_\Omega \varphi\, dQ = x \tag{10.2}$$

where it is assumed that for all $y \in \mathcal{Y}$, $\langle y, \varphi\rangle$ is measurable and $\int_\Omega e^{\lambda|\langle y,\varphi\rangle|}\, dR < \infty$ for some $\lambda > 0$. Recall that the expression $\int_\Omega \varphi\, dQ = x$ is formal, it doesn't correspond to a usual integral, see (2.4).

Since $\gamma(s) = e^s - s - 1$ satisfies the assumption (9.1), the notion of admissible dual parameter for the extended entropy is in the sense of Definition 9.1. Let us write it down.

Definition 10.3. *Let $\bar{z}_1 + \infty(n)$ be a force field. Its infinite component: $\infty(n)$, determines the measurable subsets $\mathcal{S}$, $\mathcal{T}_+$ and $\mathcal{T}_-$ (see (7.6)). Its finite component $\bar{z}_1$ is a linear form on $\mathcal{X}$.*
The couple $(\bar{z}_1 + \infty(n), \zeta_2)$ is said to be an admissible dual parameter for the extended entropy $\overline{H}(\cdot \mid R)$, if the following properties are satisfied.

1. *$\langle \bar{z}_1, \varphi\rangle$ is a measurable function whose support is included in $\mathcal{S}$ with $\int_{\mathcal{S}} e^{\langle \bar{z}_1,\varphi\rangle}\, dR = 1$*
2. *$\int_{\mathcal{S}}[\langle \bar{z}_1, \varphi\rangle_+ \exp(\langle \bar{z}_1, \varphi\rangle_+) + \langle \bar{z}_1, \varphi\rangle_-]\, dR < \infty$*

3. for any $j \in \mathcal{J}$, $\int_{\cap_{i<j}\{n^i=0\}} \langle n^j, \varphi\rangle_- \, dR < \infty$
4. $R(\mathcal{T}_-) < \infty$ and $R(\mathcal{T}_+) = 0$
5. $\zeta_2 \in L_\tau^{s\prime}$ and $\zeta_2 \geq 0$
6. for all $\varepsilon > 0$, $K \geq 1$, $f_k \in L_{\tau^*}$, $g_k \in L_\infty$, $\ell_k \in L_\tau^s$, $k \leq K$, there exists $y \in \mathcal{Y}$ such that $\int_\Omega [\exp(\langle y, \varphi\rangle_+) + \langle y, \varphi\rangle_-] \, dR < \infty$ and for all $k \leq K$:
$$|\int_{\mathcal{S}} (\langle \bar{z}_1, \varphi\rangle_+ - \langle y, \varphi\rangle_+) f_k \, dR| + |\int_{\mathcal{S}} (\langle \bar{z}_1, \varphi\rangle_- - \langle y, \varphi\rangle_-) g_k \, dR| \leq \varepsilon$$
$$|\langle \zeta_2 - \langle y, \varphi\rangle_+, \ell_k\rangle| \leq \varepsilon$$

Note that in 1., it is assumed that $\mathbb{I}_{\mathcal{S}} e^{\langle \bar{z}_1, \varphi\rangle} \cdot R$ is a probability measure. This implies that $\mathbb{I}_{\mathcal{S}} e^{\langle \bar{z}_1, \varphi\rangle} \cdot R + \tilde{\ell}$ belongs to $\mathcal{Q}_o$ for any $\tilde{\ell} \in L_\tau^s$, $\tilde{\ell} \geq 0$.

Let $y \mapsto \int_\Omega e^{\langle y, \varphi\rangle} \, dR$ be the Laplace transform of the image probability measure $R \circ \varphi^{-1}$. We denote its logarithm $\Gamma(y) = \log \int_\Omega e^{\langle y, \varphi\rangle} \, dR \in]-\infty, +\infty]$, $y \in \mathcal{Y}$ and

$$\Gamma^*(x) = \sup_{y \in \mathcal{Y}} \left\{ \langle x, y\rangle - \log \int_\Omega e^{\langle y, \varphi\rangle} \, dR \right\}, \; x \in \mathcal{X}$$

the convex conjugate of Γ. The function Γ^* is also called the Cramér's transform of $R \circ \varphi^{-1}$. This transform plays a crucial role in Large Deviations (see [DZ]) and in Statistical Mechanics (see [Rue]).

With Hölder's inequality in L_p, one proves that Γ is convex. In statistical mechanics, the functions $\exp(\Gamma)$, Γ and Γ^* correspond respectively to the partition function, the pressure and the free energy of a system without interaction.

As a corollary of Theorem 9.2, we obtain the following

Theorem 10.4. *Let R be a probability measure on $(\Omega, \mathcal{A})$ and $\varphi : \Omega \to \mathcal{X}$ a function such that for all $y \in \mathcal{Y}$, $\langle y, \varphi\rangle : \Omega \to \mathbb{R}$ is measurable and there exists $\lambda > 0$ such that $\int_\Omega e^{\lambda|\langle y, \varphi\rangle|} \, dR < \infty$. Let $\mathcal{M}(x) \subset \mathcal{Q}_o$ denote the set of solutions to (10.2).*

The set $\mathcal{M}(x)$ is nonempty if and only if $\Gamma^*(x) < \infty$. It is a convex $\sigma(\mathcal{Q}_o, L_\tau)$-compact subset of $\mathcal{Q}_o$.

Let $x \in \mathcal{X}$ be such that $\Gamma^*(x) < \infty$. One can associate with it an admissible dual parameter $(\bar{z}_1 + \infty(n), \zeta_2)$ in the sense of Definition 10.3, such that

$$x = \int_{\mathcal{S}} \varphi e^{\langle \bar{z}_1, \varphi\rangle} \, dR + \langle \varphi, \tilde{\ell}\rangle, \text{ for some } \tilde{\ell} \in K(\mathbb{I}_{\mathcal{S}} \langle \bar{z}_1, \varphi\rangle_+, \zeta_2). \tag{10.3}$$

Any $\tilde{\ell}$ in the convex cone $K(\mathbb{I}_{\mathcal{S}} \langle \bar{z}_1, \varphi\rangle_+, \zeta_2)$ is nonnegative and singular. Moreover,

$$\mathcal{M}(x) = \left\{ \mathbb{I}_{\mathcal{S}} e^{\langle \bar{z}_1, \varphi\rangle} \cdot R + \tilde{\ell}; \, \tilde{\ell} \in K(\mathbb{I}_{\mathcal{S}} \langle \bar{z}_1, \varphi\rangle_+, \zeta_2) \text{ such that (10.3) holds} \right\} \tag{10.4}$$

All the elements of $\mathcal{M}(x)$ share the same absolutely continuous part: $\mathbb{I}_{\mathcal{S}} e^{\langle \bar{z}_1, \varphi\rangle} \cdot R$ (which is a probability measure) and have a nonnegative singular part: $\tilde{\ell}$, in L_τ^s.

Conversely, for any admissible dual parameter $(\bar{z}_1 + \infty(n), \zeta_2)$, x being defined by (10.3), we have $\Gamma^*(x) < \infty$ and (10.4) holds.

Proof. Let us consider the minimisation problem (5.3) associated with (10.1) and the constraint function $\psi : \omega \in \Omega \mapsto (1, \varphi(\omega)) \in \mathbb{R} \times \mathcal{X}$:

$$\text{(10.5)} \qquad \text{minimize } \Phi^*(\ell) \text{ subject to } \ell \in L'_\tau, \int_\Omega \psi \, d\ell = (0, x - \int_\Omega \varphi \, dR)$$

As for all $\ell \in L'_\tau$, $\Phi^*(\ell) = I(\ell + R) = \int_\Omega \gamma^* \left(\frac{d\ell^a}{dR} + 1\right) dR + I_s(\ell^s)$ and $dom\ \gamma^*(\cdot + 1) \in [0, \infty[$, by Lemma 10.1, we see that any solution ℓ_x of (10.5) satisfies $\ell_x + R \geq 0$. As, $\int_\Omega \psi \, d\ell = (0, x - \int_\Omega \varphi \, dR)$ $\Leftrightarrow \begin{cases} \langle \mathbb{I}, \ell + R \rangle & = 1 \\ \langle \varphi, \ell + R \rangle & = x \end{cases}$, ℓ_x satisfies $\ell_x + R \in \mathcal{Q}_o$ and $\langle \varphi, \ell_x + R \rangle = x$. On the other hand, for all $\ell \in -R + \mathcal{Q}_o$, $\Phi^*(\ell) = I(\ell + R) = \overline{H}(\ell + R \mid R)$. Therefore, putting $Q = \ell + R$, the problems (10.2) and (10.5) are equivalent.

For any $x \in \mathcal{X}, y \in dom\ \Gamma$, a simple computation yields $\inf_{\lambda \in \mathbb{R}} \{-\lambda - 1 + e^\lambda \int_\Omega e^{\langle y, \varphi \rangle} \, dR\} = \Gamma(y)$. Thus,

$$\begin{aligned}
\Lambda^*(0, x - \int_\Omega \varphi \, dR) &= \sup \left\{ \langle (\lambda, y), (0, x - \int_\Omega \varphi \, dR) \rangle - \int_\Omega \gamma(\langle (\lambda, y), \psi(\omega) \rangle \, R(d\omega) \,;\, \lambda \in \mathbb{R}, y \in \mathcal{Y} \right\} \\
&= \sup \left\{ \langle x, y \rangle - \int_\Omega \langle y, \varphi \rangle \, dR - \int_\Omega \gamma(\lambda + \langle y, \varphi(\omega) \rangle) \, R(d\omega) \,;\, \lambda \in \mathbb{R}, y \in \mathcal{Y} \right\} \\
&= \sup_{y \in dom\ \Gamma} \left\{ \langle x, y \rangle - \inf_{\lambda \in \mathbb{R}} \{-\lambda - 1 + e^\lambda \int_\Omega e^{\langle y, \varphi \rangle} \, dR\} \right\} \\
&= \sup_{y \in dom\ \Gamma} \{\langle x, y \rangle - \Gamma(y)\} = \Gamma^*(x)
\end{aligned}$$

One completes the proof of the theorem, transcripting the statement of Theorem 9.2. ■

Csiszár's example. This example has been introduced by I. Csiszár in [Cs2]. Take $\Omega = [0, \infty[$, $R(d\omega) = c \frac{e^{-y_* \omega}}{1 + \omega^3} \, d\omega$ where $y_* > 0$, c is the unspecified normalizing constant which makes R a probability measure and $d\omega$ is the Lebesgue measure on $[0, \infty[$. The constraint function is $\varphi(\omega) = \omega \in \mathcal{X} = \mathbb{R}$. Note that although φ doesn't belong to $M_\tau(R)$, it enters the assumption of Theorem 10.4 since it is in $L_\tau(R)$.
The log-Laplace transform of R is $\Gamma(y) = \log \int_{[0,\infty[} e^{(y - y_*)\omega} \frac{c \, d\omega}{1 + \omega^3} \in] - \infty, +\infty]$. Its domain is $dom\ \Gamma =] - \infty, y_*]$. We introduce the family of probability measures on $[0, \infty[$:

$$P_y(d\omega) = \exp(y\omega - \Gamma(y)) \, R(d\omega), \ \ y \in dom\ \Gamma.$$

In particular, $P_{y_*}(d\omega) = c_* \frac{d\omega}{1 + \omega^3}$ where c_* is a normalization constant. We have for any $y < y_*$, $\Gamma'(y) = \int_{[0,\infty[} \omega \, P_y(d\omega)$ and $x_* := \sup_{y < y_*} \Gamma'(y) = \int_{[0,\infty[} \omega \, P_{y_*}(d\omega) = \int_{[0,\infty[} c_* \frac{\omega}{1 + \omega^3} \, d\omega < \infty$.
Let us first consider the following minimization problem, for any $x \in \mathbb{R}$

$$\text{minimize } P \in \mathcal{P} \mapsto H(P \mid R) \text{ subject to } \int_{[0,\infty[} \omega P(d\omega) = x$$

In [Cs2], it is proved that
- for any $0 < x \leq x_*$, the infimum is attained at P_{y_x} where $y = y_x$ is the unique solution of $\Gamma'(y) = x$

- for any $x > x_*$ the infimum is not attained (in $\mathcal{P}$), but any minimizing sequence (P_n) (i.e. such that $\lim_{n\to\infty} H(P_n \mid R) = \inf\{H(P \mid R)\,;\, P \in \mathcal{P}, \int_{[0,\infty[} \omega P(d\omega) = x\}$ and $\int_{[0,\infty[} \omega P_n(d\omega) = x, \forall n \geq 1$), converges in the sense of total variation to P_{y_*}.
I. Csiszár introduced in [Cs2] the notion of generalized I-projection to take this phenomenon into account. Now, let us

$$\text{minimize } Q \in \mathcal{Q}_o \mapsto \overline{H}(Q \mid R) \text{ subject to } \int_{[0,\infty[} \varphi\, dQ = x \tag{10.6}$$

by means of Theorem 10.4. As $dom\, \Gamma^* =]0,\infty[$, x is an admissible constraint if and only if $x > 0$. For any $y \in dom\, \Gamma$, P_y corresponds to the dual parameter: $\mathcal{S} = \Omega$, $\langle \bar{z}_1, \varphi\rangle = y\varphi$. Therefore, for any $0 < x \leq x_*$, choosing $y = y_x$: the unique solution of $\Gamma'(y) = x$, one obtains that $x = \int_{[0,\infty[} \omega P_{y_x}(d\omega)$. This is (10.3), so that (10.4) leads us to $\mathcal{M}(x) = \{P_{y_x}\}$.
For any $x > x_*$, take $\mathcal{S} = \Omega$, $\langle \bar{z}_1, \varphi\rangle = \zeta_2 = u_*$ with $u_*(\omega) = y_*\omega$, $\omega \geq 0$. The equality $\langle \bar{z}_1, \varphi\rangle = \zeta_2$ is imposed by (9.4). Any $\langle \bar{z}_1, \varphi\rangle$ must be of the form $y\varphi$ to stand in $\overline{V} = V$. The choice of $y = y_*$ corresponds to the unique case where $y\varphi$ is on the relative boundary of $u \mapsto \int_{[0,\infty[} e^u\, dR$. We have no remaining choice.
Let us have a look to $K(u_*, u_*) = K_1(u_*)$. Note that $u_* \geq 0$. By Proposition 8.5, $K_1(u_*)$ is the convex cone of all $\tilde{\ell} \in L^s_\tau$, $\tilde{\ell} \geq 0$ satisfying: For any $h \in L_\tau$, $h \geq 0$, if there exists $t > 0$ such that $\int_{[0,\infty[} e^{u_*+th}\, dR < \infty$, then $\langle \tilde{\ell}, h\rangle = 0$. In particular, φ is such an h. Hence, there exists $\tilde{\ell}_o$ in the cone $K_1(u_*)$ such that $\langle \tilde{\ell}_o, \varphi\rangle = 1$ and, for all $x > x_*$, we have

$$x = \int_{[0,\infty[} \omega P_{y_*}(d\omega) + (x - x_*)\langle \tilde{\ell}_o, \varphi\rangle$$

This is (10.3) with $\tilde{\ell} = (x - x_*)\tilde{\ell}_o$. By (10.4), it comes out that $\mathcal{M}(x) = \{P_{y_*} + \tilde{\ell}\,;\, \tilde{\ell} \in K_1(u_*), \langle \tilde{\ell}, \varphi\rangle = x - x_*\}$. We have proved the following

Proposition 10.5. *For any $x \in \mathbb{R}$, (10.6) admits at least a solution if and only if $x > 0$. For any $x > 0$, $Q_x \in \mathcal{Q}_o$ is a solution to (10.6) if and only if*

$$Q_x = \begin{cases} P_{y_x} & \text{if } x \leq x_* \\ P_{y_*} + (x - x_*)\tilde{\ell}_o & \text{if } x \geq x_* \end{cases}$$

where $y = y_x$ is the unique solution of $\Gamma'(y) = x$ ($x \leq x_*$), and $\tilde{\ell}_o$ is any nonnegative element of $L^s_\tau(R)$ such that $\langle \tilde{\ell}_o, \varphi\rangle = 1$ and whose "support" is determined by $\langle \tilde{\ell}_o, h\rangle = 0$ for all nonnegative h in $L_\tau(R) \cap L_\tau(P_{y_*})$.

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