On a non linear stochastic wave equation in the plane: existence and uniqueness of the solution.

Annie MILLET [*]& Pierre-Luc MORIEN [†]

Abstract

In this paper, we investigate the existence and uniqueness of the solution for a class of stochastic wave equations in two space-dimensions containing a non-linearity of polynomial type. The method used in the proofs combines functional analysis arguments with probabilistic tools, and further estimates for the Green function associated with the classical wave equation.

Short title: Non linear stochastic wave equation

Keywords: stochastic partial differential equations, wave equation, Gaussian noise, a priori estimates.

AMS 1991 subject classification: primary 60H15, secondary 35J60

1 Introduction

Let Θ be a bounded open subset of $\mathbb{R}^n$, $T > 0$, $\rho > 0$. The following nonlinear PDE defined on $[0;T] \times \Theta$

$$\left\{ \begin{array}{lcl} \dfrac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) + |u(t,x)|^\rho \cdot u(t,x) & = & \phi(t,x), \\ u(0,x) = u_0(x), & & \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x), & & \end{array} \right. \tag{1.1}$$

which appears in relativistic quantum mechanics, has been extensively studied (see J.L. Lions [7] and the references therein for a detailed account on the subject). If $u_0 \in H_0^1(\Theta) \bigcap L^{\rho+2}(\Theta)$, $v_0 \in L^2(\Theta)$ and $\phi \in L^2(]0,T[\times\Theta)$, it is known that the Cauchy problem (1.1) admits a unique solution $u \in L^\infty\left([0,T]; H_0^1(\Theta) \bigcap L^{\rho+2}(\Theta)\right) \bigcap C([0,T]; L^2(\Theta))$.

[*]MODAL'X *and* Laboratoire de Probabilités et Modèles Aléatoires (UMR 7599).
[†]MODAL'X.

When the forcing term $\phi(t,x)$ is random and $\rho = 0$, (1.1) reduces to a linear or semilinear SPDE and has been studied by several authors. More precisely, consider the following stochastic real-valued wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) \;=\; \sigma(u(t,x))\dot{F}(t,x) + b(u(t,x)), \\ u(0,x) = u_0(x), \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \end{cases} \tag{1.2}$$

where σ, $b : \mathbb{R} \longmapsto \mathbb{R}$ are globally Lipschitz functions. When $n = 1$, R. Carmona and D. Nualart have shown in [2] that (1.2) has a unique solution when F is the space-time white noise.

For $n = 2$, the fundamental solution $S(t,x)$ to the wave equation $\dfrac{\partial^2 S}{\partial t^2}(t,x) - \Delta S(t,x) = \delta_{(0,0)}$ is still a function (while in dimension $n \geq 3$ it is only a distribution) but lacks L^2 integrability properties, which forbids to consider equation (1.2) when F is the space-time white noise. On the other hand, physical models of wave propagation in a random environment have led to Gaussian perturbations which are white in time but correlated in space (see e.g. S.K. Biswas and N.U. Ahmed [1], R.N. Miller [8]). Thus C. Mueller [11], R. Dalang and N. Frangos [4], A. Millet and M. Sanz-Solé [10] have studied existence and uniqueness of the solution of (1.2) when F is a generalized Gaussian noise $(F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2))$ with covariance

$$\mathbb{E}[F(\varphi)F\psi)] = \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \varphi(t,x) \cdot \psi(t,y) \cdot f(|x-y|)\, dxdydt, \tag{1.3}$$

where f is the Fourier transform of some positive measure μ on $\mathbb{R}^2$. In [10], it is shown that the following integrability condition

$$\int_{0^+} r f(r) \ln\left(1 + \frac{1}{r}\right) dr < \infty \tag{1.4}$$

is necessary and sufficient to obtain existence of a unique L^2- bounded solution $u(t,x)$ for

(1.2). (A similar result was proved in [11] when f is bounded and, in [4], in the linear case or for "small time" in the semilinear case.)

We remark that in dimension 1 and 2 equation (1.2) is to be considered in a weak form, with stochastic integrals with respect to the martingale measure $M_t(A) = F([0,t] \times A)$, $t \in [0,T]$, $A \in \mathcal{B}(\mathbb{R}^2)$, associated with the noise F. Equivalently, one can consider the following evolution formulation:

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^2} S(t, x-y) v_0(y) dy + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^2} S(t,x-y) u_0(y) dy\right) \\ &\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) \left[\sigma(u(s,y)) F(ds,dy) + b(u(s,y) dy ds\right] . \end{aligned} \tag{1.5}$$

S. Peszat and J. Zabczyk [13], R. Dalang [3] and S. Peszat [12] have recently studied the existence and uniqueness of the solution to (1.2) in dimension $n \geq 3$ by using Fourier transform methods and a characterization of the space covariance structure of the noise F. In [13], the authors show the existence of a unique solution u in $C([0,T]; L^2(\mu))$ where μ is a positive finite measure on $\mathbb{R}^n$. In [3], a theory of distribution-valued martingale measures is developed, which enables the author to solve the Cauchy problem (1.2) in non-Hilbert spaces.

In the present paper, we study the following nonlinear stochastic wave equation, deduced from (1.1) by replacing $\phi(t,x)$ by a random forcing term and from (1.2) by replacing $b(r)$ by the non-globally Lipschitz function $-|r|^\rho r$ for $\rho > 0$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) + |u(t,x)|^\rho \, u(t,x) = \sigma(u(t,x)) \dot{F}(t,x), \\ u(0,x) = u_0(x), \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x). \end{cases} \tag{1.6}$$

For this problem, when σ is bounded and u_0 and v_0 have compact support, we prove an existence and uniqueness result in the case of a general Gaussian noise F with covariance defined by (1.3) and satisfying certain integrability properties. We also obtain a sharper result in the particular case where the function f appearing in (1.3) is $x^{-\alpha}$ with $\alpha \in]0;2[$ (or is dominated by this function). The proofs are based on a combination of classical functional analysis and probability theory, as it can be found, for instance, in a recent paper by I. Gyöngy (see [5]) for the study of a stochastic Burgers-type equation. The solution of (1.6) is obtained by an approximation procedure *via* regularized versions of equation (1.6) and suitable a priori estimates. To this end, new regularity properties for the Green function S are proved.

The paper is organized as follows: the framework and the results are presented in the next section; in section 3, we prove the uniqueness of a solution to (1.6), while the existence is established in section 4. Finally, some technical estimates of integrals involving S are proved in the Appendix.

2 General framework and statements of the results.

Let $F(t,x)$ be a Gaussian centered noise on $\mathbb{R}_+ \times \mathbb{R}^2$ with covariance given by (1.3). We assume that the function $f :]0,+\infty[\longrightarrow \mathbb{R}_+$ is continuous and satisfies (1.4).

Let $\mathcal{E}$ denote the inner product space of measurable functions $\varphi : \mathbb{R}^2 \longmapsto \mathbb{R}$ such that

$$\int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \, |\varphi(x)| \, f(|x-y|) \, |\varphi(y)| < \infty$$

endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \, \varphi(x) | \, f(|x-y|) \, \psi(y),$$

and let $\mathcal{H}$ denote the completion of $\mathcal{E}$.

We shall say that condition $(\mathbf{H}_\beta)$ holds if there exists a constant C such that:

$$(\mathbf{H}_\beta) \qquad \int_{0^+} r^{1-\beta} f(r) dr < \infty \, ;$$

it clearly implies that (1.4) is satisfied. Consider the nonlinear stochastic wave equation defined in (1.6). Following the method of Walsh [15], a natural way to give it a rigorous meaning is in terms of the following weak formulation: given any function $\varphi \in \mathcal{D}([0,T] \times \mathbb{R}^2)$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \left(\frac{\partial \varphi}{\partial t^2} - \Delta \varphi \right)(t,x) u(t,x) dt dx + \int_0^T \int_{\mathbb{R}^2} |u(t,x)|^\rho u(t,x) \varphi(t,x) dt dx \qquad (2.1) \\
&= \int_{\mathbb{R}^2} \left(\varphi(0,x) v_0(x) - \frac{\partial \varphi}{\partial t}(0,x) u_0(x) \right) dx + \int_0^T \int_{\mathbb{R}^2} \varphi(t,x) \sigma(u(t,x)) F(dt,dx) .
\end{aligned}$$

As is classical, (2.1) can be stated equivalently in terms of the associated evolution equation:

$$\begin{aligned}
u(t,x) &= u^{(0)}(t,x) - \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) |u(s,y)|^\rho u(s,y) dy ds \\
&\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) \sigma(u(s,y)) F(ds,dy), \qquad (2.2)
\end{aligned}$$

where

$$u^{(0)}(t,x) = \int_{\mathbb{R}^2} S(t, x-y) v_0(y) dy + \frac{\partial}{\partial t} \left(\int_{\mathbb{R}^2} S(t, x-y) u_0(y) dy \right) \qquad (2.3)$$

and S is the fundamental solution of the deterministic wave equation associated to (1.6), that is:

$$S(t,x) = \frac{1}{2\pi} (t^2 - |x|^2)^{-\frac{1}{2}} \, 1_{\{|x|<t\}} . \qquad (2.4)$$

We assume the following hypotheses:

($\mathbf{C_1}$) u_0, $v_0 : \mathbb{R}^2 \longmapsto \mathbb{R}$ have compact support K.
($\mathbf{C_2}$) u_0 is of class C^1, $v_0 \in L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in]2, +\infty[$.
($\mathbf{C_3}$) $\sigma : \mathbb{R} \longmapsto \mathbb{R}$ is globally Lipschitz and bounded such that $\sigma(0) = 0$.

For any $t \in [0,T]$, set

$$D(t) = \left\{ x \in \mathbb{R}^2 : \exists y \in K, |x-y| < t \right\} .$$

Because of the definition of S, it is easy to see that if u_0 and v_0 satisfy $(\mathbf{C_1})$ and $(\mathbf{C_2})$, then

$$u^{(0)}(t,x) = 0 \text{ for } x \notin D(t). \tag{2.5}$$

Besides, consider for the time being the "Lipschitz" version of equation (1.6) (or (2.2)), that is:

$$\begin{aligned} u(t,x) &= u^{(0)}(t,x) + \int_0^t\int_{\mathbb{R}^2} S(t-s,x-y)b(u(s,y))dyds \\ &\quad + \int_0^t\int_{\mathbb{R}^2} S(t-s,x-y)\sigma(u(s,y))F(ds,dy), \end{aligned} \tag{2.6}$$

where b is globally Lipschitz and $b(0) = 0$. It is well-known that the unique solution of (2.6) can be obtained by means of the following Picard approximation procedure:

$$\left\{ \begin{array}{lcl} u^0(t,x) &=& u^{(0)}(t,x) \\ u^{k+1}(t,x) &=& u^{(0)}(t,x) + \int_0^t\int_{\mathbb{R}^2} S(t-s,x-y)b(u^k(s,y))dyds \\ && + \int_0^t\int_{\mathbb{R}^2} S(t-s,x-y)\sigma(u^k(s,y))F(ds,dy) \end{array} \right. \tag{2.7}$$

Then, by induction, one easily sees that if u_0 and v_0 satisfy $(\mathbf{C_1})$ and $(\mathbf{C_2})$, then, for all k

$$u^k(t,x) = 0 \text{ if } x \notin D(t)\,. \tag{2.8}$$

Indeed, assume (2.8) for some k and for all $t \in [0,T]$, then for a fixed time $t \in [0,T]$ and $x \notin D(t)$, one has: for every $s \in [0,t]$ and every y such that $|x-y| \le t-s$,

$$\forall z \in K,\ |z-y| \ge |z-x| - |y-x| \ge s\,.$$

The induction assumption implies that $u^k(s,y) = 0$ for all $s \in [0,t]$ and $y \notin D(s)$; since $b(0) = \sigma(0) = 0$, we deduce $u^{k+1}(t,x) = 0$ for $x \notin D(t)$, which yields (2.8) for $k+1$.

Of course, (2.8) yields the same support property for the solution u itself. This property of "propagation of the support", which will also be proved for the solution to (1.6), is very important because, by only assuming $(\mathbf{C_1})$ and $(\mathbf{C_2})$, all the integrals on $\mathbb{R}^2$ involved in (2.2) can be considered as integrals on the bounded region $\Theta := D(T)$ of $\mathbb{R}^2$, and thus one can work in spaces based on Θ. More precisely, we prove the following result:

Theorem 2.1 *Let $\rho \in]0,2]$, u_0, v_0 satisfy $(\mathbf{C_1})$ and $(\mathbf{C_2})$, and σ satisfy $(\mathbf{C_3})$. Then:*

(a) *If the function f in (1.3) satisfies $(\mathbf{H}_\beta)$ for some $\beta \in]0,2[$, then equation (1.6) has a unique solution $u \in C([0,T]; L^p(\Theta))$ for $8 < p < \frac{2(\rho+2)}{\rho}$.*

(b) *If $f(x) = x^{-\alpha}$ with $\alpha \in]0,2[$, then equation (1.6) has a unique solution $u \in C([0;T]; L^p(\Theta))$ for $2 \vee (\rho+1) \vee \left(\frac{8}{5-2\alpha}\right) < p < \frac{2(\rho+2)}{\rho}$.*

The next sections are devoted to the proof of this theorem. In the sequel, $\|\cdot\|_p$ will denote the usual norm in $L^p(\Theta)$.

3 Uniqueness and local existence of the solution.

The main result of this section is the following:

Proposition 3.1 *Suppose that the assumptions of Theorem 2.1 hold and that either condition (a) or (b) is satisfied:*

(a) *f satisfies $(\mathbf{H}_\beta)$ for some $\beta \in]0,2[$ and $p \in]8,+\infty[$.*

(b) *$f(r) = r^{-\alpha}$ for some $\alpha \in]0,2[$ and $p \in]2 \vee \left(\frac{8}{5-2\alpha}\right), +\infty[$.*

Then the Cauchy problem (1.6) has at most one solution in $C([0,T]; L^p(\Theta))$ such that for all $t \in [0,T]$ the support of $u(t,\cdot)$ is contained in $D(t)$.

Notice that the property of "propagation of support" is postulated because at this stage, we have no way to obtain it *a priori*. We will prove later on that the solution we construct possesses this property; this yields a more satisfactory uniqueness result.

Proof: The method used is adapted from that of Proposition 4.7 in [5]. Given $R > 0$, let $\chi_R : \mathbb{R} \longmapsto \mathbb{R}$ be a C^1 function such that $\chi_R(x) = 1$ for $|x| \leq R$, $\chi_R(x) = 0$ for $|x| \geq R+1$, and $\|\chi_R'\|_\infty \leq 2$. We consider the following "truncated" problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) + |u(t,x)|^\rho \, u(t,x)\, \chi_R(\|u(t,\cdot)\|_p) = \sigma(u(t,x))\dot{F}(t,x), \\ u(0,x) = u_0(x), \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x)\,. \end{cases} \tag{3.1}$$

Set $b(r) = -|r|^\rho\, r$. Let u and v be solutions to (3.1) such that, for all $t \in [0,T]$, the functions $u(t,\cdot)$ and $v(t,\cdot)$ vanish outside $D(t)$. Writing the evolution formula for (3.1) and using the support property for u and v, one obtains

$$u(t,x) - v(t,x) = A(t,x) + B(t,x),$$

where

$$\begin{aligned} A(t,x) &= \int_0^t \int_{D(s)} S(t-s, x-y)\ \big[\chi_R(\|u(s,\cdot)\|_p) b(u(s,y)) \\ &\qquad\qquad - \chi_R(\|v(s,\cdot)\|_p) b(v(s,y))\,\big]\, dyds\,, \\ B(t,x) &= \int_0^t \int_{D(s)} S(t-s,x-y)\ [\sigma(u(s,y)) - \sigma(v(s,y)] F(dy,ds). \end{aligned}$$

Burkholder's and Hölder's inequalities yield

$$\mathbb{E}\left(\|B(t,\cdot)\|^p_{L^p(D(t))}\right) \leq C_p \int_{D(t)} \mathbb{E}\left(\left| \int_0^t \|S(t-s,x-\cdot)\ [\sigma(u(s,\cdot)) - \sigma(v(s,\cdot)]\|^2_{\mathcal{H}} ds \right|^{\frac{p}{2}} dx \right).$$

Because of the hypotheses on p and the Lipschitz property of σ, Lemma A.4 implies the existence of $\gamma > -1$ such that

$$\mathbb{E}\left(\|B(t,\cdot)\|^p_{L^p(D(t))}\right) \leq C_p \int_0^t (t-s)^\gamma \|u(s,\cdot) - v(s,\cdot)\|^p_{L^p(D(s))} ds. \tag{3.2}$$

On the other hand, suppose for instance that $\|u(s,\cdot)\|_p \le \|v(s,\cdot)\|_p$. Then, setting $q = \frac{p}{\rho+1}$ and using the definition of χ_R, we have

$$\begin{aligned}
&\|\chi_R(\|u(s,\cdot)\|_p) b(u(s,\cdot) - \chi_R(\|v(s,\cdot)\|_p) b(v(s,\cdot))\|_q \\
&\le |\chi_R(\|u(s,\cdot)\|_p) - \chi_R(\|v(s,\cdot)\|_p)| \ \|b(u(s,\cdot))\|_q \\
&\quad + \chi_R(\|v(s,\cdot)\|_p) \|b(u(s,\cdot)) - b(v(s,\cdot))\|_q \\
&\le 2\|u(s,\cdot) - v(s,\cdot)\|_p \ \|u(s,\cdot)\|_p^{\rho+1} \ 1_{\{\|v(s,\cdot)\|_p \le R+1\}} \\
&\qquad + C_\rho \, \chi_R(\|v(s,\cdot)\|_p) \ \| |u(s,\cdot) - v(s,\cdot)| \sup(|u(s,\cdot)|^\rho, |v(s,\cdot)|^\rho) \|_q \\
&\le C(R) \|u(s,\cdot) - v(s,\cdot)\|_p \\
&\qquad + C_\rho \, \chi_R(\|v(s,\cdot)\|_p) \|u(s,\cdot) - v(s,\cdot)\|_p \ \left(\|u(s,\cdot)\|_p^\rho + \|v(s,\cdot)\|_p^\rho \right) \\
&\le C(R) \, \|u(s,\cdot) - v(s,\cdot)\|_p,
\end{aligned}$$

by means of Hölder's inequality used in the following way:

If $h_1 \in L^p$, $h_2 \in L^{\frac{p}{\rho}}$, then $\|h_1 \, h_2\|_{L^q} \le \|h_1\|_{L^p} \ \|h_2\|_{L^{\frac{p}{\rho}}}$.

Hence, since $p > 2 > \rho$, the inequality (A.9) in Lemma A.2 applied with $\kappa = 1 + \frac{1}{p} - \frac{1}{q} = 1 - \frac{\rho}{p} > 0$ and Hölder's inequality imply that for $t \in [0,T]$

$$\|A(t,\cdot)\|_p^p \le C(R) \int_0^t (t-s)^{2\kappa-1} \|u(s,\cdot) - v(s,\cdot)\|_p^p \, ds. \tag{3.3}$$

Thus (3.2) and (3.3) together with Gronwall's lemma yield

$$\sup_{0 \le t \le T} \mathbb{E} \left(\|u(t,\cdot) - v(t,\cdot)\|_p^p \right) = 0 \,, \tag{3.4}$$

which means that uniqueness holds for the truncated problem (3.1). Now, let $u_1, u_2 \in C([0,T]; L^p(\Theta))$ be solutions to (1.6) such that for all $t \in [0,T]$ the support of $u_1(t,\cdot)$ and $u_2(t,\cdot)$ is included in $D(t)$. For every $R > 0$ and $i = 1, 2$, define

$$\tau_R^i = \inf\{t \ge 0 : \|u_i(t,\cdot)\|_p \ge R\} \wedge T.$$

Then $\lim_{R\longrightarrow+\infty}\mathbb{P}(\tau_R^1\wedge\tau_R^2<T)=0$ while (3.4) shows that $u_1(t,x)=u_2(t,x)$ a.s. for every $t\in[0,\tau_R^1\wedge\tau_R^2]$ and almost every $x\in\Theta$; this concludes the proof. □

Using arguments similar to those of the proof of Proposition 3.1, one can also show a local existence theorem for the solution to (1.6). Indeed, let $\mathcal{R}$ denote the Banach space of $L^p(\Theta)$-valued random processes $v(t)$, $t\in[0,T]$, endowed with the norm

$$\|v\|_{\mathcal{R}}:=\sup_{t\leq T}\{\mathbb{E}(w\|v(t)\|_p^p)\}^{1/p}<\infty,$$

where $w:=\exp(-(\|u_0\|_p+\|v_0\|_p+\|\nabla v_0\|_p)$. In this argument, we may suppose that the initial conditions $u_0(.)$ and $v_0(.)$ are random processes indexed by $\mathbb{R}^2$ and independent of the noise F.

Define the operator $\mathcal{A}$ on $\mathcal{R}$ by

$$\mathcal{A}(v)(t,x):=\sum_{i=1}^4 A_i(t,x),$$

where

$$\begin{aligned}
A_1(t,x) &:= \int_{\mathbb{R}^2} S(t,x-y)v_0(y)dy,\\
A_2(t,x) &:= \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^2} S(t,x-y)u_0(y)dy\right),\\
A_3(t,x) &:= \int_0^t\int_{D(s)} S(t-s,x-y)\chi_R(\|v(s,\cdot)\|_p)b(v(s,y))\,dyds,\\
A_4(t,x) &:= \int_0^t\int_{D(s)} S(t-s,x-y)\sigma(v(s,y))F(dy,ds).
\end{aligned}$$

Clearly,

$$\mathbb{E}(w\|\mathcal{A}(v)(t,\cdot)\|_p^p)\leq 4^{p-1}\sum_{i=1}^4 T_i(t),$$

where $T_i(t)=\mathbb{E}(w\|A_i(v)(t,\cdot)\|_p^p)$. Using Young's inequality (with $q=1$), we have

$$\|A_1\|_{\mathcal{R}}^p=\sup_{t\leq T}T_1(t)\leq C_p\mathbb{E}\left(w\int_{\mathbb{R}^2}|v_0(y)|^p dy\right)=C_p<+\infty\,. \tag{3.5}$$

We have

$$\begin{aligned} A_2(t,x) &= \int_{|\xi|<1} \frac{1}{2\pi}(1-|\xi|^2)^{-\frac{1}{2}} u_0(x+t\xi)d\xi + \int_{\mathbb{R}^2} S(t,x-y)\nabla u_0(y)dy \\ &:= A_2^1 + A_2^2, \end{aligned}$$

and, using Hölder's inequality with respect to the measure $\frac{1}{2\pi}(1-|\xi|^2)^{-\frac{1}{2}}dx$ and Fubini's theorem, we obtain:

$$\begin{aligned} \|A_2^1\|_{\mathcal{R}}^p &=\leq C_p \sup_{t\leq T} \mathbb{E}\left[w \int_{\mathbb{R}^2} \int_{|\xi|<1} \frac{1}{2\pi}(1-|\xi|^2)^{-\frac{1}{2}} |u_0(x+t\xi)|^p d\xi dx \right] \\ &\leq C_p \sup_{t\leq T} \mathbb{E}\left[w \left(\int_{|\xi|<1} \frac{1}{2\pi}(1-|\xi|^2)^{-\frac{1}{2}} d\xi \right) \|u_0\|_p^p \right] \leq C_p \,. \end{aligned} \quad (3.6)$$

On the other hand, Young's inequality yields

$$\|A_2^2\|_{\mathcal{R}}^p \leq C_p w \|\nabla u_0\|_p^p \leq C_p. \quad (3.7)$$

Finally, using again (A.9), Lemma A.4 and the fact that σ is bounded, computations similar to that proving (3.2) and (3.3) show that $\|A_3\|_{\mathcal{R}}$ and $\|A_4\|_{\mathcal{R}}$ are also bounded by a constant only depending on p and R . Hence the operator $\mathcal{A}$ maps the Banach space $\mathcal{R}$ into itself.

Furthermore, let u and v belong to $\mathcal{R}$; using arguments similar to the previous ones, one proves the existence of $\beta > -1$ such that

$$\begin{aligned} &\mathbb{E}\left(w \|\mathcal{A}(u)(t,\cdot) - \mathcal{A}(v)(t,\cdot)\|_r^r \right) && (3.8) \\ &\leq C_{p,R} \sup_{t\leq T} \mathbb{E}\left(w \int_0^t (t-s)^\beta \|u(s,\cdot)-v(s,\cdot)\|_r^r ds \right) && (3.9) \\ &\leq C_{p,R,\beta} T^{\beta+1} \sup_{t\leq T} \mathbb{E}(w\|u(t,\cdot)-v(t,\cdot)\|_r^r)\,; \end{aligned}$$

hence $\mathcal{A}$ is a contraction on $\mathcal{R}$ provided $T < t_1 := C_{p,R,\beta}^{-\frac{1}{\beta+1}}$. Consequently, there exists a unique solution to (3.1) on $[0, t_1/2]$; notice that the constant $C_{p,R,\beta}$ does not depend on the initial conditions u_0 and v_0. Considering next the initial conditions $u(t_1,\cdot)$ and

$\frac{\partial u}{\partial t}(t_1, \cdot)$ at time t_1, we get a solution to (3.1) on the interval $[t_1/2, t_1]$ in the same way, with the obvious modification of the Banach space $\mathcal{R}$ and the operator $\mathcal{A}$. Iterating this procedure, we thus construct a solution to (3.1) on the whole interval $[0, T]$. Finally, if $\tau_R = \inf\{t \geq 0 : \|u(t, .)\|_p \geq R\} \wedge T$ and $\tau_\infty = \lim_{R \to +\infty} \tau_R$, we deduce the local existence (on the interval $[0, \tau_\infty[$) of a solution to equation (1.6).

The problem of global existence is addressed in the next section.

4 Global existence of a solution.

The purpose of this section is to prove the following result:

Proposition 4.1 *Under the assumptions (a) or (b) of Theorem 2.1, equation (1.6) admits a solution $u \in C([0, T]; L^p(\Theta))$ for p satisfying the requirements stated in Theorem 2.1. Moreover, for all $t \in [0, T]$, the function $u(t, \cdot)$ vanishes outside $D(t)$.*

The proof is divided into several steps.

Step 1: We first "regularize" the equation (1.6). For every $n \geq 1$, let b_n and B_n be defined as follows:

$$b_n(r) := \begin{cases} -|r|^\rho \cdot r & \text{if } |r| \leq n, \\ -n^{\rho+1} - (\rho+1)n^\rho(r - n) & \text{if } r \geq n, \\ n^{\rho+1} - (\rho+1)n^\rho(r + n) & \text{if } r \leq -n\,, \end{cases} \tag{4.1}$$

and

$$B_n(r) = \int_0^r b_n(u)du. \tag{4.2}$$

Then $-B_n$ is a non-negative even function. Let us introduce the following SPDE:

$$\begin{cases} \dfrac{\partial^2 u_n}{\partial t^2}(t, x) - \Delta u_n(t, x) - b_n(u_n(t, x)) = \sigma(u_n(t, x))\dot{F}(t, x), \\ u_n(0, x) = u_0(x), \\ \dfrac{\partial u_n}{\partial t}(0, x) = v_0(x)\,. \end{cases} \tag{4.3}$$

The properties of b_n and its antiderivative B_n are proved in Lemma A.1 of the Appendix. Since in particular b_n is globally Lipschitz on $\mathbb{R}$, Theorem 1.2. of [10] provides a unique weak solution to this equation, which is also the unique solution to the following evolution equation

$$u_n(t,x) = u^{(0)}(t,x) + \eta_n(t,x) + \int_0^t\int_{\mathbb{R}^2} S(t-s,x-y) b_n(u_n(s,y)) dyds, \tag{4.4}$$

where

$$\eta_n(t,x) = \int_0^t\int_{\mathbb{R}^2} S(t-s,x-y)\sigma(u_n(s,y))F(dy,ds). \tag{4.5}$$

We remark that, as the solution to (2.6), u_n satisfies

$$u_n(t,x) = 0 \text{ if } x \notin D(t). \tag{4.6}$$

We shall prove that $\{u_n\}_n$ admits a subsequence which converges in distribution to a solution u to (1.6) (or (2.2)). We at first study the behaviour of the stochastic integrals:

Lemma 4.1 *Let σ satisfy $(\mathbf{C_3})$, F satisfy H_β, ζ_n be a predictable random field on $[0,T]\times\Theta$ such that, for all $t \in [0,T]$, the support of $\zeta_n(t,\cdot)$ is included in $D(t)$. Then the sequence of processes*

$$I_n(t,x) := \int_0^t\int_{\mathbb{R}^2} S(t-s,x-y)\sigma(\zeta_n(s,y))F(dy,ds)$$

is uniformly tight in $C([0,T]\times\Theta)$, and hence in $C([0,T];L^p(\Theta))$ for any $p \in [1,+\infty[$. Moreover, for all $t \in [0,T]$, the support of $I_n(t,\cdot)$ is included in $D(t)$.

Proof of Lemma 4.1: The support property of I_n is clear. Given $0 \le t < t' \le T$ $x, x' \in \Theta$, the boundedness of σ, Burkholder's inequality and (A.15) imply that for $0 < \delta < \frac{1}{2}(\beta \wedge 1)$,

$$\begin{aligned} \mathbb{E}[|I_n(t,x) - I_n(t',x')|^p) &\le C \left(\int_0^T \| \sigma(u(s,.)) \left[S(t-s,x-.) - S(t'-s,x'-.) \right] \|_{\mathcal{H}}^2 \, ds \right)^{\frac{p}{2}} \\ &\le C\|\sigma\|_\infty^p \left(|t-t'| + |x-x'| \right)^{p\delta}. \end{aligned} \tag{4.7}$$

Set $D := \bigcup_{0\le t\le T}\left(\{t\}\times\overline{D(t)}\right)$; for $\gamma < \frac{1}{p}+\delta$, $E\left(\int_D\int_D\left(\frac{|I_n(\xi)-I_n(\xi')|}{|\xi-\xi'|^\gamma}\right)^p d\xi\, d\xi'\right) < +\infty$, and on $\left\{\int_D\int_D\left(\frac{|I_n(\xi)-I_n(\xi')|}{|\xi-\xi'|^\gamma}\right)^p d\xi\, d\xi' \le \lambda\right\}$, $0<\bar\delta=\gamma-\frac{4}{p}<\delta-\frac{3}{p}$, the Garsia-Rodemich-Rumsey lemma (see e.g. [14], p. 60) yields $\|I_n(\cdot,\cdot)\|_{C^{\bar\delta,\bar\delta}(D)} \le \lambda^{\frac{1}{p}}$. Hence, given $p>\frac{3}{\delta}$ and $0<\bar\delta<\delta-\frac{3}{p}$,

$$\sup_n \mathbb{P}\left(\|I_n(\cdot,\cdot)\|_{C^{\bar\delta,\bar\delta}(D)} \ge \lambda\right) \le C\lambda^{-p^2},$$

so that by Ascoli's theorem I_n is uniformly tight in $C(D)$. □

Define $\eta_n^\star := \sup_{(t,x)\in D}|\eta_n(t,x)|\vee 1$. Applying Lemma 4.1 to u_n yields in particular

$$\sup_n \mathbb{E}(\eta_n^\star) < \infty \tag{4.8}$$

and

$$\lim_{C\longrightarrow+\infty}\sup_n \mathbb{P}\left(\eta_n^\star \ge C\right) = 0. \tag{4.9}$$

Set $\xi_n(t,x)=u_n(t,x)-\eta_n(t,x)$; then ξ_n is the unique (weak) solution to the following semilinear wave equation (defined ω by ω):

$$\left\{\begin{array}{lll}\dfrac{\partial^2\xi_n}{\partial t^2}(t,x)-\Delta\xi_n(t,x)-b_n(\xi_n(t,x)+\eta_n(t,x)) & = & 0,\\ \xi_n(0,x)=u_0(x), & & \\ \dfrac{\partial u_n}{\partial t}(0,x)=v_0(x)\,. & & \end{array}\right. \tag{4.10}$$

Step 2: We now prove a suitable *a priori* estimate for the sequence $\{\xi_n\}$, and follow here the method of J.L. Lions [7]. Let $H^1(\Theta)=\{v\in L^2(\Theta):\frac{\partial v}{\partial x_i}\in L^2(\Theta), i=1,2\}$, endowed with the norm

$$\|u\|_{H^1(\Theta)} = \left(\|v\|_2^2+\sum_{i=1}^2\left\|\frac{\partial v}{\partial x_i}\right\|_2^2\right)^{\frac{1}{2}} \tag{4.11}$$

and let H_0^1 be the closure of $\mathcal{D}(\Theta)$ in $H^1(\Theta)$. Let v_i be a sequence of elements of $L^{\rho+2}(\Theta)\bigcap H_0^1(\Theta)$ which is total in this set. Given $u,v\in H_0^1(\Theta)$, set

$$a(u,v) := \sum_{j=1}^2\int_\Theta \frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_i}dx. \tag{4.12}$$

Then $\sqrt{a(u,u)}$ is a norm on $H_0^1(\Theta)$ equivalent with $\|u\|_{H^1(\Theta)}$. For each $n \geq 1$, we approximate ξ_n by the sequence $(\xi_n^k, k \geq 1)$ defined by

$$\xi_n^k = \sum_{i=1}^{k} g_{i,n}^k(t) v_i(x), \tag{4.13}$$

where the functions $(g_{i,n}^k, 1 \leq i \leq k)$ are determined by the conditions

$$\begin{cases} \left((\xi_n^k)''(t,\cdot), v_j\right) + a\left(\xi_n^k(t,\cdot), v_j\right) - \left(b_n\left(\xi_n^k(t,\cdot) + \eta_n(t,\cdot)\right), v_j\right) = 0\,,\ 1 \leq j \leq k, \\ \xi_n^k(0,x) = u_0^k(x), \\ \frac{\partial \xi_n^k}{\partial t}(0,x) = v_0^k(x)\,, \end{cases} \tag{4.14}$$

where $(\cdot,\cdot)$ denotes the usual scalar product on $L^2(\Theta)$ and

$$\begin{cases} u_0^k(x) = \sum_{i=1}^k \alpha_{i,n} v_i \quad \longrightarrow \quad u_0 \text{ in} L^{\rho+2}(\Theta) \bigcap H_0^1(\Theta) \text{ when } k \to +\infty, \\ v_0^k(x) = \sum_{i=1}^k \beta_{i,n} v_i \quad \longrightarrow \quad v_0 \text{ in } L^2(\Theta) \text{ when } k \to +\infty. \end{cases} \tag{4.15}$$

For a.e. ω, the system (4.14)-(4.15) of ordinary differential equations has a unique solution on the time interval $\left[0, t_n^k(\omega)\right]$ with $t_n^k(\omega) \leq T$. This is due to the linear independence of the functions v_i, which yields $\det((v_i, v_j), 1 \leq i,j \leq k) \neq 0$. In the sequel, we shall prove that $t_n^k = T$.

Multiplying the first line of (4.14) by $(g_{j,n}^k)'(t)$ and summing up for $1 \leq j \leq k$, we deduce

$$\frac{1}{2}\frac{d}{dt}\left[\left\|(\xi_n^k)'(t,\cdot)\right\|_2^2 + a\left(\xi_n^k(t,\cdot), \xi_n^k(t,\cdot)\right)\right] - \frac{d}{dt}\left(\int_\Theta B_n(\xi_n^k(t,x))dx\right) = D_n^k(t), \tag{4.16}$$

where

$$D_n^k(t) = \int_\Theta \left[b_n(\xi_n^k(t,x) + \eta_n(t,x)) - b_n(\xi_n^k(t,x))\right]\ (\xi_n^k)'(t,x)\, dx\,.$$

Schwarz's inequality and the Taylor formula yield

$$|D_n^k(t)| \leq \frac{1}{2}\|(\xi_n^k)'(t,\cdot)\|_2^2 + \frac{1}{2}\int_\Theta \int_0^1 \left(b_n'(\xi_n^k(t,x) + r\eta_n(t,x))\right)^2\ \eta_n^2(t,x)\, dr\, dx\,.$$

The inequality (A.4) in Lemma A.1 yields

$$|D_n^k(t)| \leq \frac{1}{2}\|(\xi_n^k)'(t,\cdot)\|_2^2 + C\int_\Theta \left[-B_n(\xi_n^k(t,x)) + |\eta_n(t,x)|^{2\rho} + 1\right]\ \eta_n^2(t,x)\, dx\,. \tag{4.17}$$

Thus, for $0 \le t \le t_n^k$, (4.16) and (4.17) imply that for any $k \ge 1$:

$$\begin{aligned}
&\frac{1}{2}\|(\xi_n^k)'(t,\cdot)\|_2^2 + C\|\xi_n^k(t,\cdot)\|_{H^1(\Theta)}^2 - \int_\Theta B_n(\xi_n^k(t,x))dx \\
&\le \ \frac{1}{2}\int_0^t \|(\xi_n^k)'(s,\cdot)\|_2^2 ds - C\eta_n^{\star 2}\int_0^t\int_\Theta B_n(\xi_n^k(s,x))\,dxds \,+ C\eta_n^{\star 2(\rho+1)} + C\eta_n^{\star 2} \,+ C(n,k),
\end{aligned}$$

where

$$\begin{aligned}
C(n,k) &= \frac{1}{2}\|(\xi_n^k)'(0,\cdot)\|^2 + C\,\|\xi_n^k(0,\cdot)\|_{H^1(\Theta)} - \int_\Theta B_n(\xi_n^k(0,x))\,dx \\
&= \frac{1}{2}\|v_0^k\|_2^2 + C\|u_0^k\|_{H_1} + \int_\Theta |u_0^k(x)|^{\rho+2}\,dx \le C
\end{aligned}$$

for some constant C which does not depend on k and n; hence Gronwall's lemma implies:

$$\sup_{0\le t\le t_n^k}\left(\|(\xi_n^k)'(t,\cdot)\|_2^2 + \|\xi_n^k(t,\cdot)\|_{H^1(\Theta)}^2 - \int_\Theta B_n(\xi_n^k(t,x))dx\right) \le C\left[1+\eta_n^{\star 2(\rho+1)}\right]\exp\left(C\eta_n^{\star 2}\right). \tag{4.18}$$

Step 3: We now extract converging subsequences. Since $-B_n$ is nonnegative, (4.18) implies that for every n

$$\sup_{k\ge 1}\ \sup_{0\le t\le t_n^k}\|\xi_n^k(t,\cdot)\|_{H^1(\Theta)}^2 < \infty\,,$$

which means that $t_n^k = T$ for all k. Recall that an Orlicz function Φ satisfies the condition $(\Delta 2)$ if for any $a>1$, $\limsup_{t\to+\infty}\frac{\Phi(at)}{\Phi(t)} < +\infty$ (see [6] for details). According to (A.3), $|B_n|$ is an Orlicz function which satisfies $(\Delta 2)$ and its conjugate function $|\tilde{B_n}|$ also satisfies $(\Delta 2)$; therefore $L^1([0,T],|\tilde{B_n}|)' \simeq L^\infty([0,T],|B_n|)$. Then (4.18) implies that there exists a subsequence $(\xi_n^{s_k})_k$ which converges to $\tilde{\xi_n}$ in $L^\infty([0,T],H_0^1(\Theta)\bigcap L_{B_n}(\Theta))$ weak-star and $(\xi_n'^{s_k})$ converges to $\tilde{\xi_n'}$ in $L^\infty([0,T],L^2(\Theta))$ weak-star (see e.g. [7]).

Since the inclusion $H^1(]0,T[\times\Theta)\hookrightarrow L^2(]0,T[\times\Theta)$ is compact, we can extract a further subsequence, still denoted by $(\xi_n^{s_k})$, such that $\xi_n^{s_k}$ converges to $\tilde{\xi_n}$ in $L^2(]0,T[\times\Theta)$ and $dt\otimes dx$ a.s. on $]0,T[\times\Theta$. Hence,

$$b_n(\xi_n^{s_k}+\eta_n) \longrightarrow b_n(\tilde{\xi_n}+\eta_n) \qquad dt\otimes dx \text{ a.s..}$$

Furthermore, (4.18) and (A.3) imply that $(b_n(\xi_n^{s_k}+\eta_n), k \geq 1)$ is uniformly integrable, since

$$\sup_k \sup_{0\leq t\leq T} \int_\Theta |b_n(\xi_n^{s_k}(t,x)+\eta_n(t,x))|^{\frac{\rho+2}{\rho+1}} dx < \infty \, .$$

Therefore, extracting a further subsequence, we obtain that $(b_n(\xi_n^{s_k}+\eta_n)\, ,\, k\geq 1)$ converges to $b_n(\tilde{\xi_n}+\eta_n)$ in $L^1(]0,T[\times\Theta)$ and to some limit l_n in $L^\infty([0,T], L^{\frac{\rho+2}{\rho+1}}(\Theta))$ weak-star. This yields that $l_n = b_n(\tilde{\xi_n}+\eta_n)$. Letting $k \longrightarrow +\infty$ in (4.14), we obtain

$$\begin{cases} \left((\tilde{\xi_n})''(t,\cdot), v_j\right) + a\left(\tilde{\xi_n}(t,\cdot), v_j\right) - \left(b_n\left(\tilde{\xi_n}(t,\cdot)+\eta_n(t,\cdot)\right), v_j\right) & = & 0, \\ \tilde{\xi_n}(0,x) & = & u_0(x), \\ \frac{\partial\tilde{\xi_n}}{\partial t}(0,x) & = & v_0(x)\, . \end{cases}$$

Since $\{v_j\}$ is total in $H_0^1(\Theta)$, we conclude that $\tilde{\xi_n}$ satisfies (4.10), which by uniqueness yields $\xi_n = \tilde{\xi_n}$.

Therefore, letting $k \longrightarrow +\infty$ in (4.18) and using Fatou's lemma, we deduce that

$$\begin{aligned} \int_0^T \int_\Theta |B_n(\xi_n(t,x))|\, dxdt &\leq T \sup_{0\leq t\leq T} \liminf_k \int_\Theta |B_n(\xi_n^{s_k}(t,x))|\, dxdt \\ &\leq C\left[1+\eta_n^{\star 2(\rho+1)}\right] \exp\left(C\eta_n^{\star 2}\right). \end{aligned}$$

Since $u_n = \xi_n + \eta_n$ and $|\eta_n|$ is bounded by $\eta_n^\star$, using (A.3) and (A.5) in Lemma A.1, we deduce that for $q = \frac{\rho+2}{\rho+1}$,

$$\int_0^T \int_\Theta |b_n(u_n(t,x))|^q dxdt \leq \left[C_1 + C_2\eta_n^{\star(\rho+2)}\right] \exp\left(C\eta_n^{\star 2}\right) \, . \tag{4.19}$$

The following result gives a tightness criterion for a sequence of convolution of random fields with the Green function.

Lemma 4.2 *Let $q \in]1,+\infty[$; for $v \in L^\infty([0,T]; L^q(\Theta))$, set*

$$J(v)(t,x) := \int_0^t \int_\Theta S(t-s, x-y) v(s,y) dyds.$$

Let $(\zeta_n(t,x), n \geq 1)$ be a sequence of random fields on $[0,T] \times \Theta$ such that for all $t \in [0,T]$, $\zeta(t,\cdot)$ vanishes outside $D(t)$ and such that there exists $\gamma \in]1,+\infty[$ and a sequence of finite random variables $(M_n; n \geq 1)$ which satisfies the following conditions:

$$\|\zeta_n\|_{L^\gamma([0,T];L^q(\Theta))} \leq M_n \,, \tag{4.20}$$

$$\lim_{C \longrightarrow +\infty} \sup_n \mathbb{P}(M_n \geq C) = 0. \tag{4.21}$$

Then, if p satisfies $0 < \frac{1}{q} - \frac{1}{p} < \frac{1}{2}$, the sequence of processes $(J(\zeta_n); n \geq 1)$ is uniformly tight in $C([0,T]; L^p(\Theta))$.

Proof of Lemma 4.2: Given $R > 0$, set

$$\Gamma_R = \left\{ J(v) : v \in L^\gamma([0,T]; L^q(\Theta)), \|\zeta_n\|_{L^\gamma([0,T];L^q(\Theta))} \leq R \right\}.$$

Lemma A.2 shows that if $0 < \frac{1}{q} - \frac{1}{p} < \frac{1}{2}$, then

$$\sup_{J(v) \in \Gamma_R} \sup_{t \in [0,T]} \|J(v(t,\cdot))\|_p = C(R) < \infty, \tag{4.22}$$

$$\limsup_{h \longrightarrow 0} \sup_{|t-s|<h, s,t \leq T} \sup_{J(v) \in \Gamma_R} \sup_{t \in [0,T]} \|J(v(t,\cdot) - J(v(s,\cdot))\|_p = 0, \tag{4.23}$$

$$\limsup_{|z| \longrightarrow 0} \sup_{J(v) \in \Gamma_R} \sup_{t \leq T} \|J(v(t,\cdot)) - J(v(t,\cdot + z))\|_p = 0. \tag{4.24}$$

Therefore Ascoli-Arzela's and Kolmogorov's theorems (see [5], Lemma 3.3) imply that the set Γ_R is relatively compact in $C([0,T], L^p(\Theta))$. Furthermore, given $\varepsilon > 0$, assumptions (4.20) and (4.21) imply the existence of some $R > 0$ such that

$$1 - \varepsilon \leq \inf_n \mathbb{P}(M_n \geq R) \leq \inf_n \mathbb{P}(J(\zeta_n) \in \Gamma_n)\,;$$

this concludes the proof. □

¿From (4.19) and Lemma 4.2 (applied with $\gamma = q = \frac{\rho+2}{\rho+1}$), we deduce that the sequence of processes

$$\int_0^t \int_\Theta S(t-s, x-y) b_n(u_n(s,y)) dy ds$$

is uniformly tight in $C([0,T];L^p(\Theta))$ for $q < p < \frac{2q}{2-q}$, that is, for $p \in \left]\frac{\rho+2}{\rho+1}, \frac{2(\rho+2)}{\rho}\right[$. On the other hand, Lemma 4.1 implies that the sequence (η_n) is uniformly tight in the same space. Hence, (4.4) implies that the sequence (u_n) itself is uniformly tight in $C([0,T];L^p(\Theta))$. Thus, by Skorohod's theorem, given subsequences (u_m) and (u_l), there exist further subsequences $(m(k), l(k))$, a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and a sequence of random elements $z_k := (\tilde{u}_k, \bar{u}_k, \hat{F}_k)$ in $C([0,T];L^p(\Theta))^2 \times C([0,T];\mathcal{D}'(\Theta))$ such that z_k converges $\hat{\mathbb{P}}$-a.s. to $z := (\tilde{u}, \bar{u}, \hat{F})$ when $k \to +\infty$, and the laws of z_k and $(u_{m(k)}, u_{l(k)}, F)$ are the same. Hence $(\hat{F}_k, \hat{\mathbb{P}})$ is a Gaussian random field such that for every $i \geq 1$:

$$\lim_k \sup_{t\in[0,T]} \left| \langle \hat{F}_k - \hat{F}, e_i \rangle (t) \right| = 0 \ \hat{\mathbb{P}} - \text{a.s.} \tag{4.25}$$

where $(e_i; i \geq 1)$ is a complete orthonormal system of $\mathcal{H}$ made of elements of $\mathcal{E}$. Using Proposition 3.1, we will prove that $\bar{u} = \tilde{u}$ by checking that both satisfy (2.1) with $\hat{F}$ instead of F. Thus, for any $\varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^2)$ with compact support included in $[0,T] \times \Theta$,

$$\begin{aligned} &\int_0^T\int_\Theta \left(\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi\right)(t,x)\tilde{u}_k(t,x)dtdx \qquad (4.26)\\ &= \int_\Theta \left(\varphi(0,x)v_0(x) - \frac{\partial\varphi}{\partial t}(0,x)u_0(x)\right)dx + \int_0^T\int_\Theta \varphi(t,x)\sigma(\tilde{u}_k(t,x))\hat{F}_k(dt,dx)\\ &\quad + \int_0^T\int_\Theta \varphi(t,x)b_{m(k)}(\tilde{u}_k(t,x))dtdx. \end{aligned}$$

Since $p > 1$ and $(\tilde{u}_k)$ is bounded in $\mathcal{C}([0,T], L^p(\Theta))$, the dominated convergence theorem implies that the left hand-side of (4.26) converges $\hat{\mathbb{P}}$-a.s. to the left hand-side of (2.1) with $\tilde{u}$ instead of u.

We now need the following technical results to study the right hand side of (4.26):

Lemma 4.3 *Let*

$$W^i(t) := \int_0^T \int_{\mathbb{R}^2} 1_{[0,t]} \otimes e_i(x) F(dx, ds), \tag{4.27}$$

$(F_n, n \geq 1)$ be Gaussian processes with the same covariance as F, W_n^i be defined like W^i (with F_n instead of F), $h_n(t,x); n \geq 1)$ (resp. $h(t,x)$) be a sequence of $(\mathcal{F}_t^n)$- adapted (resp. an $\mathcal{F}_t$-adapted) random fields. Suppose that for every $i \geq 1$,

$$\lim_n \sup_{t\in[0,T]} \left|W_n^i(t) - W^i(t)\right| = 0 \text{ in probability}, \tag{4.28}$$

$$\mathbb{E}(\|h\|^2_{L^2([0,T];\mathcal{H})}) < \infty\,, \tag{4.29}$$

$$\lim_n \mathbb{E}(\|h_n - h\|^2_{L^2([0,T];\mathcal{H})}) = 0\,. \tag{4.30}$$

Then, for any $\varepsilon > 0$,

$$\lim_n \mathbb{P}\left(\left|\int_0^T\int_{\mathbb{R}^2} h_n(s,y)F_n(dy,ds) - \int_0^T\int_{\mathbb{R}^2} h_(s,y)F(dy,ds)\right| > \varepsilon\right) = 0. \tag{4.31}$$

Lemma 4.4 *Let (v_n) and v be random fields satisfying, for some $p \in [\rho+1, +\infty[$ the following properties:*

$$\int_0^T\int_\Theta |u(t,x)|^p dxdt < \infty \quad a.s.\,, \tag{4.32}$$

$$\lim_n \int_0^T\int_\Theta |u_n(t,x) - u(t,x)|^p dxdt = 0 \quad a.s. \tag{4.33}$$

Then for any $\phi \in C^2([0,T]\times\Theta)$ with compact support

$$\lim_n \int_0^T\int_\Theta \phi(t,x)[b_n(u_n(t,x)) - b(u(t,x))]dxdt = 0 \ a.s. \tag{4.34}$$

Suppose that these two results hold. Then Lemma 4.4 implies that for $\hat{\mathbb{P}}$-almost every ω,

$$\lim_k \int_0^T\int_\Theta \phi(t,x)[b_{m(k)}(\tilde{u}_k(t,x)) - b(\tilde{u}(t,x))]dxdt = 0 \tag{4.35}$$

On the other hand, Lemma 4.3 applied with $h_k(t,x) = \varphi(t,x)\,\sigma(\tilde{u}_k(t,x))$ shows that in $\hat{\mathbb{P}}$-probability

$$\lim_k \left(\int_0^T\int_{\mathbb{R}^2} \varphi(t,x)\,\sigma(\tilde{u}_k(t,x))\hat{F}_k(dy,ds) - \int_0^T\int_{\mathbb{R}^2} \varphi(t,x)\,\sigma(\tilde{u}(t,x))\hat{F}(dy,ds)\right) = 0. \tag{4.36}$$

Therefore, letting $k \longrightarrow +\infty$ in (4.26) yields that $\tilde{u}$ solves (2.1) with $\hat{F}$ instead of F. A similar argument shows that $\bar{u}$ solves the same equation. Therefore, by Proposition 3.1, we deduce that $\tilde{u} = \bar{u}$ $\hat{\mathbb{P}}$-almost surely; hence the subsequences of $C[0,T];L^p(\Theta))$-valued random variable $(u_{m(k)})$ and $(u_{l(k)})$ converge weakly to the same limit. Using a result of Gyöngy and Krylov (see [5], Lemma 4.1), we conclude that u_n converges in $\hat{\mathbb{P}}$-probability to some random variable $u \in C[0,T];L^p(\Theta))$.

Applying again the dominated convergence theorem, Lemma 4.3 with $F_n = F$ and $h_n(t,x) = \varphi(t,x)\sigma(u_n(t,x))$, Lemma 4.4 and letting $n \longrightarrow +\infty$ in the weak formulation of (4.3), that is:

$$\begin{aligned}
&\int_0^T\int_\Theta \left(\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi\right)(t,x)u_n(t,x)dtdx \\
&= \int_\Theta \left(\varphi(0,x)v_0(x) - \frac{\partial\varphi}{\partial t}(0,x)u_0(x)\right)dx + \int_0^T\int_\Theta \varphi(t,x)\sigma(u_n(t,x))F(dt,dx) \\
&\quad + \int_0^T\int_\Theta \varphi(t,x)b_n(u_n(t,x))dtdx. \qquad (4.37)
\end{aligned}$$

we finally conclude that u solves (2.1), which concludes the proof of existence. □

It only remains to prove Lemmas 4.3 and 4.4.

Proof of Lemma 4.3: Note first that by definition $(W^i, i \geq 1)$ (resp. $(W_n^i, i \geq 1)$) are sequences of independent standard Brownian motions. Furthermore, recall that $\mathcal{H}_T := L^2([0,T];\mathcal{H})$ is isomorphic to the reproducing kernel space of F (resp. F_n) and that F can be identified with the Gaussian process $\{W(h), h \in \mathcal{H}_T\}$ defined by

$$W(h) = \sum_{j\geq 0}\int_0^T \langle h(s), e_j\rangle_{\mathcal{H}} dW^j(s).$$

Given $\varepsilon > 0$, using (4.29), we choose i_0 such that

$$\mathbb{E}\left(\sum_{i\geq i_0}\int_0^T \|h^i(s)\|_{\mathcal{H}}^2 ds\right) < \varepsilon$$

where $h^i(s) = \langle h(s,\cdot), e_i\rangle_{\mathcal{H}}$. Then using (4.30), we choose n_0 such that for $n \geq n_0$

$$\mathbb{E}\left(\sum_{i\geq i_0}\int_0^T \|h_n^i(s)\|_{\mathcal{H}}^2 ds\right) < 2\varepsilon.$$

The proof of (4.31) then reduces to checking that for any $\varepsilon > 0$,

$$\lim_n \mathbb{P}\left(\sum_{i=1}^{i_0}\left|\int_0^T h_n^i(s)dW_n^i(s) - \int_0^T h^i(s)dW^i(s)\right| > \frac{\varepsilon}{3}\right) = 0. \tag{4.38}$$

Clearly, (4.30) implies that for every $i \geq 1$

$$\lim_n \mathbb{E}\left(\int_0^T \left|h_n^i(s) - h^i(s)\right|^2 ds\right) = 0.$$

Using (4.28), a generalization of Skorohod's argument (see e.g. [5], p.282) yields that for every $i \geq 1$ and $\varepsilon > 0$

$$\lim_n \mathbb{P}\left(\left|\int_0^T h_n^i(s)dW_n^i(s) - \int_0^T h^i(s)dW^i(s)\right| > \frac{\varepsilon}{3(i_0+1)}\right) = 0.$$

This concludes the proof of (4.31). □

Proof of Lemma 4.4: To prove (4.34), it clearly suffices to check that for $\phi \in C_c^2([0,T]\times\Theta)$,

$$\lim_n \int_0^T\int_\Theta \phi(t,x)[b_n(u_n(t,x)) - b_n(u(t,x))]dxdt = 0 \text{ a.s.} \tag{4.39}$$

and

$$\lim_n \int_0^T\int_\Theta \phi(t,x)[b_n(u(t,x)) - b(u(t,x))]dxdt = 0 \text{ a.s.} \tag{4.40}$$

Using the Taylor formula, (A.1) in Lemma A.1, then Hölder's inequality with the conjugate exponents p and $p' = \frac{p}{p-1}$, (4.32) and (4.33) we obtain

$$\begin{aligned}
&\left|\int_0^T\int_\Theta \phi(t,x)[b_n(u_n(t,x)) - b_n(u(t,x))]dxdt\right| \\
&\leq\ C\|\phi\|_\infty \int_0^T\int_\Theta |u_n(t,x) - u(t,x)|\ \left(|u_n(t,x)|^\rho + |u(t,x)|^\rho\right) dxdt \\
&\leq\ C\|\phi\|_\infty \left(\int_0^T\int_\Theta |u_n(t,x) - u(t,x)|^p dxdt\right)^{\frac{1}{p}} \left(\|u_n\|^\rho_{L^{p'\rho}([0,T]\times\Theta)} + \|u\|^\rho_{L^{p'\rho}([0,T]\times\Theta)}\right) \\
&\leq\ C\|\phi\|_\infty \|u_n - u\|_{L^p([0,T]\times\Theta)}\,,
\end{aligned}$$

since $p > \rho + 1$ and $\rho \in]0, 2[$, so that $p'\rho \leq p$; this proves (4.39). Furthermore, (4.32) implies that for any $\bar{p} < p$ we have

$$|u(t,x)|^{\bar{p}} \in L^{\frac{p}{\bar{p}}}([0,T] \times \Theta)\,;$$

hence $|u(t,x)|^{\bar{p}}$ is uniformly integrable. Therefore, since $p > \rho + 1$, given $\varepsilon > 0$, we can choose $M \geq 1$ such that

$$\int\int_{|u(t,x)| \geq M} |u(t,x)|^{\rho+1} dxdt < \varepsilon.$$

Hence, using the fact $b_n(r) = b(r)$ when $|r| \leq n$ and (A.2) in Lemma A.1, we conclude that for $n \geq M$,

$$\begin{aligned} &\left| \int_0^T \int_\Theta \phi(t,x)[b_n(u(t,x)) - b(u(t,x))] dxdt \right| \\ &\leq \; C\|\phi\|_\infty \int\int_{|u(t,x)| \geq M} |u(t,x)|^{\rho+1} dxdt \leq C\, \|\phi\|_\infty \, \varepsilon. \end{aligned}$$

This concludes the proof of (4.40). □

A Appendix.

We begin this section by a technical result concerning the approximation b_n of $-|r|^\rho r$ defined in section 4.

Lemma A.1 *For each $n \geq 1$, let b_n and B_n de defined by (4.1) and (4.2) respectively. Then b_n is a C^1, globally Lipschitz function on $\mathbb{R}$, B_n is an even function and $|B_n|$ is an Orlicz function which satisfies $(\Delta 2)$. Furthermore:*

(i) There exists a constant C such that for every $r \in \mathbb{R}$,

$$\sup_n |b_n'(r)| \;\leq\; C|r|^\rho, \tag{A.1}$$

$$\sup_n |b_n(r)| \;\leq\; C|r|^{\rho+1}. \tag{A.2}$$

(ii) There exists a constant C such that for $q := \dfrac{\rho+2}{\rho+1}$ and for every $n \geq 1$ and $r \in \mathbb{R}$

$$|b_n(r)|^q \leq C\,(1 + |B_n(r)|) \leq C(1 + |r|^{\rho+2})\,. \tag{A.3}$$

(iii) There exists a constant C such that for every $n \geq 1$, r_1, $r_2 \in \mathbb{R}$,

$$|b'_n(r_1 + r_2)|^2 \leq C\left(1 + |B_n(r_1)| + |r_2|^{2\rho}\right). \tag{A.4}$$

(iv) There exists a constant C such that for every $n \geq 1$, r_1, $r_2 \in \mathbb{R}$,

$$|B_n(r_1 + r_2)| \leq C\left(|B_n(r_1)| + |r_2|^{\rho+2}\right). \tag{A.5}$$

Proof: It is clear that b_n is odd, so that b'_n and B_n are even (since $B_n(0) = 0$). On $]0, +\infty[$, the function $b(r) = -|r|^\rho r$ is negative, decreasing, so that b_n is clearly decreasing on $\mathbb{R}$, negative on $]0, +\infty[$ (resp. positive on $]-\infty, 0[$). Furthermore, $\sup_{|r|\geq n} |b'_n(r)| = |b'(n)| = (\rho+1)n^\rho$, which yields (A.1) and the fact that b_n is globally Lipschitz. As for (A.2), it is simply obtained by integration of (A.1).

Now, as B_n is even and b_n is negative on $]0, +\infty[$, $-B_n$ is non-negative on $\mathbb{R}$ and its restriction to $[0 + \infty[$ is clearly an Orlicz function which satisfies ($\Delta 2$) (see [6] for basic results on Orlicz functions). If $|r| \leq n$, inequality (A.3) reduces to

$$|r|^{q(\rho+1)} \leq C\left(1 + |r|^{\rho+2}\right),$$

which is clear given the value of q. If $|r| \geq n$, (A.3) can be deduced form

$$\left(n^{\rho+1} + (\rho+1)n^\rho(|r| - n)\right)^q \leq C\left(n^{\rho+2} + n^{\rho+1}|r| + n^\rho r^2\right),$$

which again is clear, given the value of q and the fact that $n \leq |r|$.

We now prove (A.4). We remark that the corresponding inequality for b

$$|b'(r_1+r_2)|^2 \leq C\left(1+|B(r_1)|+|r_2|^{2\rho}\right), \tag{A.6}$$

is satisfied insofar as $\rho \leq 2$ (B being the antiderivative of b which is zero at $r=0$). This fact will be used in the sequel.

- <u>If $|r_1+r_2| \leq n$</u>: Then there are 3 subcases:

(a) If $|r_1| \leq n$, then (A.6) yields

$$|b_n'(r_1+r_2)|^2 = |b'(r_1+r_2)|^2 \leq C\left(1+|B(r_1)|+|r_2|^{2\rho}\right),$$

and, as $|r_1| \leq n$, $|B(r_1)| = |B_n(r_1)|$.

(b) If $r_1 \geq n$, since $r_1+r_2 \leq n$, we have $0 \leq r_1 - n \leq -r_2$, which means in particular that $r_2 \leq 0$. Furthermore, $|B_n|$ increases on $[0,+\infty[$, so that

$$\begin{aligned}
|b_n'(r_1+r_2)|^2 &= |b'(r_1+r_2-n+n)|^2 \\
&\leq C\left(1+|B(n)|+|r_1+r_2-n|^{2\rho}\right) \\
&= C\left(1+|B_n(n)|+|r_1+r_2-n|^{2\rho}\right) \\
&\leq C\left(1+|B_n(r_1)|+2^{2\rho-1}|r_2|^{2\rho}+2^{2\rho-1}|r_1-n|^{2\rho}\right),
\end{aligned}$$

and since $|r_1-n| \leq |r_2|$, we have

$$|b_n'(r_1+r_2)|^2 \leq C\left(1+|B_n(r_1)|+|r_2|^{2\rho}\right).$$

(c) If $r_1 \leq -n$, since $-n \leq r_1+r_2$, we clearly have $r_2 \geq -n-r_1 = |r_1+n|$. This implies

$$\begin{aligned}
|b_n'(r_1+r_2)|^2 &= |b'((r_1+r_2+n)+(-n))|^2 \\
&\leq C\left(1+|B(-n)|+|r_1+r_2+n|^{2\rho}\right) \\
&\leq C\left(1+|B_n(r_1)|+2^{2\rho-1}|r_2|^{2\rho}+2^{2\rho-1}|r_1+n|^{2\rho}\right),
\end{aligned}$$

and we conclude as in case (b).

•<u>If $r_1 + r_2 \geq n$</u>: Then we have

$$|b_n'(r_1 + r_2)|^2 = |b'(n)|^2.$$

(a) If $|r_1| \leq n$, we have $0 \leq n - r_1 \leq r_2$ and (A.6) used with $r_2 = n - r_1$ yields

$$\begin{aligned} |b'(n)|^2 &\leq C\left(1 + |B(r_1)| + |n - r_1|^{2\rho}\right) = C\left(1 + |B_n(r_1)| + |n - r_1|^{2\rho}\right) \\ &\leq C\left(1 + |B_n(r_1)| + |r_2|^{2\rho}\right). \end{aligned}$$

(b) If $r_1 \geq n$, since $|B_n|$ increases on $[0, +\infty[$, using (A.6) with $r_2 = 0$ we obtain:

$$|b'(n)|^2 \leq C\left(1 + |B(n)|\right) \leq C\left(1 + |B_n(r_1)| + |r_2|^{2\rho}\right).$$

(c) Finally, if $r_1 \leq -n$, we have $r_2 \geq n - r_1 \geq 2n$; (A.6) used with $r_1 = -n$ and $r_2 = 2n$ yields

$$|b'(n)|^2 \leq C\left(1 + |B(-n)| + |2n|^{2\rho}\right) \leq C\left(1 + |B_n(-n)| + |r_2|^{2\rho}\right),$$

which the required result.

The case $r_1 + r_2 \leq -n$, which is similar, is omitted.

We finally prove (A.5). We remark that the same inequality holds trivially for B instead of B_n. As before, we divide the proof into several cases.

• <u>If $|r_1 + r_2| \leq n$</u>, then

(a) If $|r_1| \leq n$, we deduce

$$|B_n(r_1 + r_2)| = |B(r_1 + r_2)| \leq C(|B(r_1)| + |r_2|^{\rho+2}) = C(|B_n(r_1)| + |r_2|^{\rho+2}).$$

(b) If $r_1 \geq n$, we have $0 \leq r_1 - n \leq -r_2$, that is $|r_1 - n| \leq |r_2|$. Hence

$$\begin{aligned}
|B_n(r_1+r_2)| &= |B(r_1+r_2)| \leq C[|B(n)| + |r_2 + r_1 - n|^{\rho+2}] \\
&\leq C[|B_n(n)| + 2^{\rho+1}(|r_2|^{\rho+2} + |r_1 - n|^{\rho+2}] \\
&\leq C[B_n(r_1)| + 2^{\rho+2}|r_2|^{\rho+2}]
\end{aligned}$$

(since $|B_n|$ increases on $[0, +\infty[$).

(c) The case $r_1 \leq -n$ is similarly dealt with.

- If $r_1 + r_2 \geq n$, then there exists a constant C (which does not depend on n) such that

$$|B_n(r_1+r_2)| = \left| -\frac{n^{\rho+2}}{\rho+2} - n^{\rho+1}(r_1+r_2-n) - \frac{\rho+1}{2}\, n^{\rho}(r_1+r_2-n)^2 \right| \leq C\,|r_1+r_2|^{\rho+2}\,. \tag{A.7}$$

(a) If $|r_1| \leq n$, then $|B_n(r_1)| = \frac{|r_1|^{\rho+2}}{\rho+2}$ and we have

$$|B_n(r_1+r_2)| \leq C(|B_n(r_1)| + |r_2|^{\rho+2}).$$

(b) If $r_1 \geq n$, then

$$B_n(r_1+r_2) = B_n(r_1) - n^{\rho+1}\, r_2 - \frac{\rho+1}{2}\, n^{\rho}\, r_2\, [2(r_1-n) + r_2]\,.$$

Since $|B_n|$ increases on $[0, +\infty[$, if $|r_2| \leq n$ we clearly obtain

$$\begin{aligned}
|B_n(r_1+r_2)| &\leq |B_n(r_1)| + C\left(\frac{n^{\rho+2}}{\rho+2} + n^{\rho+1}\,(r_1-n)\right) \\
&\leq C|B_n(r_1)| \leq C(|B_n(r_1)| + |r_2|^{\rho+2})\,.
\end{aligned}$$

If on the contrary $|r_2| \geq n$, using Schwarz's inequality, we obtain

$$\begin{aligned}
|B_n(r_1+r_2)| &\leq |B_n(r_1)| + C|r_2|^{\rho+2} + 2n^{\rho}|r_2|\,|r_1-n| \\
&\leq |B_n(r_1)| + C|r_2|^{\rho+2} + Cn^{\rho}\left(|r_1-n|^2 + r_2^2\right) \\
&\leq C(|B_n(r_1) + |r_2|^{\rho+2})\,.
\end{aligned}$$

(c) Finally, if $r_1 \leq -n$, then $r_2 \geq 2n$ and

$$B_n(r_1) = -\frac{n^{\rho+2}}{\rho+2} + n^{\rho+1}(r_1+n) - \frac{\rho+1}{2}\, n^{\rho}(r_1+n)^2 \,.$$

Hence, we have

$$\begin{aligned} |B_n(r_1+r_2)| &\leq |B_n(r_1)| + C\, n^{\rho+1}\, |r_2 - 2n| + Cn^{\rho}\, (r_2-2n)^2 \\ &\leq C(|B_n(r_1)| + |r_2|^{\rho+2} \,. \end{aligned}$$

The last case $r_1 + r_2 \leq -n$ is similarly dealt with. This concludes the proof. □

We now prove a series of technical results on the fundamental solution S of the classical wave equation in the plane. Let $q \geq 1$, $\mathbf{V}$ be an open subset of $\mathbb{R}^2$ (not necessarily bounded), let $v \in L^{\infty}([0,T]; L^q(\mathbf{V}))$ and set

$$J(v)(t,x) := \int_0^t\!\!\int_{\mathbf{V}} S(t-s, x-y)v(s,y)dyds. \tag{A.8}$$

To lighten the notation, we shall denote by $\|\cdot\|_p$ the usual norm in $L^p(\mathbf{V})$. The following lemma provides continuity properties for the operator J.

Lemma A.2 *Let $p, q \in [1,+\infty[$ be such that $\kappa := 1 + \frac{1}{p} - \frac{1}{q} \in]\frac{1}{2}, 1[$, $T > 0$, $\gamma \in [1,+\infty[$ and $v \in L^{\gamma}([0,T]; L^q(\mathbf{V}))$. Then there exist constants C_i, $1 \leq i \leq 5$, which do not depend on $\mathbf{V}$ and such that:*

(i) For $t \in [0,T]$ and $\gamma > (2\kappa)^{-1}$,

$$\|J(v)(t,\cdot)\|_p \leq C_1 \int_0^t (t-s)^{2\kappa-1}\|v(s,\cdot)\|_q ds \leq C_2 t^{2\kappa-\frac{1}{\gamma}} \left(\int_0^t \|v(s,\cdot)\|_q^{\gamma} ds\right)^{\frac{1}{\gamma}} . \tag{A.9}$$

(ii) For $\bar{\kappa} \in]0, \kappa - \frac{1}{2}[$ and $z \in \mathbb{R}^2$,

$$\begin{aligned} \|J(v)(t,\cdot) - J(v)(t,\cdot+z)\|_p &\leq C_3|z|^{\bar{\kappa}} \int_0^t (t-s)^{\bar{\kappa}}\|v(s,\cdot\|_q ds \\ &\leq C_4|z|^{\bar{\kappa}} t^{\bar{\kappa}+1-\frac{1}{\gamma}} \left(\int_0^t \|v(s,\cdot)\|_q^{\gamma} ds\right)^{\frac{1}{\gamma}} \end{aligned} \tag{A.10}$$

(iii) For $\bar\kappa \in]0, \kappa - \frac{1}{2}[$ *and* $\gamma \in]1, \infty[$,

$$\|J(v)(t,\cdot) - J(v)(s,\cdot)\|_p \le C_5 |t-s|^{\bar\kappa \wedge \left(\frac{\gamma-1}{\gamma}\right)} \left(\int_0^{s\vee t} \|v(r,\cdot)\|_q^\gamma dr \right)^{\frac{1}{\gamma}}. \quad \text{(A.11)}$$

Remark A.2: If $\gamma = +\infty$, $p > 2\rho$ and $q = \frac{p}{\rho+1}$, we have $\kappa > \frac{1}{2}$. Thus (A.11) yields the existence of $\delta > 0$ such that J is a bounded linear operator from $L^\infty([0,T]; L^q(\mathbf{V}))$ into $C^\delta([0,T]; L^p(\mathbf{V}))$.

Proof: (i) We first remark that $\|S(t,\cdot)\|_r$ is convergent if and only if $r < 2$, and that

$$\|S(t,\cdot)\|_r^r \le C t^{2-r} \quad \text{(A.12)}$$

where the constant C does not depend on $\mathbf{V}$. Using Minkovski's inequality, then Young's inequality for $\dfrac{1}{p} = \dfrac{1}{q} + \dfrac{1}{r} - 1$, $\kappa = \frac{1}{r}$, we deduce

$$\begin{aligned} \|J(v)(t,\cdot)\|_p &\le C \int_0^t \|S(t-s,\cdot) \star v(s,\cdot)\|_p ds \le C \int_0^t \|S(t-s,\cdot)\|_r \, \|v(s,\cdot)\|_q ds \\ &\le C \int_0^t (t-s)^{2\kappa - 1} \, \|v(s,\cdot)\|_q ds. \end{aligned}$$

Then Hölder's inequality concludes the proof of (A.9).

(ii) A similar computation yields

$$\|J(v)(t,\cdot) - J(v)(t,\cdot+z)\|_p \le C \int_0^t \|S(t-s,\cdot) - S(t-s,\cdot+z)\|_r \, \|v(s,\cdot)\|_q ds.$$

Using the proof of Lemma A.4 in [10], we conclude that for $1 < r < 2$ and $0 < \bar r < 1 - \frac{r}{2}$,

$$A_1 = \int_{|y+z|<|y|<s} \left| \frac{1}{\sqrt{s^2 - |y|^2}} - \frac{1}{\sqrt{s^2 - |y+z|^2}} \right|^r dy \le C |z|^{\bar r} \, s^{\bar r}. \quad \text{(A.13)}$$

On the other hand, the triangular inequality implies that if $|y+z| > s$ and $|y| < s$, we have $(s - |z|)^+ < |y| < s$, so that

$$\begin{aligned} A_2 &= \int_{|y|<s<|y+z|} (s^2 - |y|^2)^{-\frac{r}{2}} dy \le C \int_{(s-|z|)^+}^s (s^2 - v^2)^{-\frac{r}{2}} \, v \, dv \\ &\le C s^{1-\frac{r}{2}} \, |z|^{1-\frac{r}{2}}. \end{aligned} \quad \text{(A.14)}$$

The inequalities (A.13) and (A.14) imply that for $0 < \bar{\kappa} < \frac{1}{r} - \frac{1}{2} = \kappa - \frac{1}{2}$,

$$\|S(s,.) - S(s,.+z)\|_r \le C(A_1 + A_2) \le C|z|^{\bar{\kappa}}\, s^{\bar{\kappa}}\,,$$

and hence

$$\|J(v)(t,\cdot) - J(v)(t,\cdot+z)\|_p \le C \int_0^t |z|^{\bar{\kappa}}\, (t-s)^{\bar{\kappa}} \|v(s,\cdot)\|_q\, ds\,.$$

Again, Hölder's inequality concludes the proof of (A.10).

(iii) Similar computations yield, for $0 \le s \le t \le T$

$$\begin{aligned} \|J(v)(t,\cdot) - J(v)(s,\cdot)\|_p \;\le\; & \int_0^s \|S(t-u,\cdot) - S(s-u,\cdot)\|_r\, \|v(u,\cdot)\|_q\, du \\ & + \int_s^t \|S(t-u,\cdot)\|_r\, \|v(u,\cdot)\|_q\, du. \end{aligned}$$

Fix $\lambda \in]0; \kappa - \frac{1}{2}[$; then, for $0 \le t' < t \le T$, we have

$$\begin{aligned} & \int_{|z|<t'} \left| \frac{1}{\sqrt{t'^2 - |z|^2}} - \frac{1}{\sqrt{t^2 - |z|^2}} \right|^r dz \\ \le\; & C \int_0^{t'} \left(\frac{t^2 - t'^2}{(t'^2 - v^2)^{\frac{1}{2}}\, (t^2 - v^2)^{\frac{1}{2}}\, [(t'^2 - v^2) + (t^2 - v^2)]^{\frac{1}{2}}} \right)^{\lambda r} \\ & \qquad \left(\frac{1}{(t'^2 - v^2)^{\frac{1}{2}}} + \frac{1}{(t^2 - v^2)^{\frac{1}{2}}} \right)^{(1-\lambda) r} v\, dv \\ \le\; & C\, |t - t'|^{\lambda r} \int_0^{t'} \frac{v\, dv}{(t'^2 - v^2)^{\frac{3\lambda r}{2} + \frac{(1-\lambda) r}{2}}} \\ \le\; & C\, |t - t'|^{\lambda r}\, t'^{2 - r - 2\lambda r}\,. \end{aligned}$$

Hence, using (A.12) for the second term, we deduce

$$\begin{aligned} & \|J(v)(t,\cdot) - J(v)(s,\cdot)\|_p \\ & \le\; C \left\{ \int_0^s (t-s)^{\lambda} (s-u)^{2\kappa - 1 - 2\lambda} \|v(u,\cdot)\|_q du + \int_s^t u^{2\kappa - 1} \|v(u,\cdot)\|_q\, du \right\}. \end{aligned}$$

Thus, Hölder's inequality implies that for $\gamma \in]1, +\infty[$,

$$\begin{aligned} & \|J(v)(t,\cdot) - J(v)(s,\cdot)\|_p \\ & \le\; C \left\{ (t-s)^{\lambda} \left(\int_0^s \|v(u,\cdot)\|_q^{\gamma}\, ds \right)^{\frac{1}{\gamma}} + (t-s)^{\frac{\gamma - 1}{\gamma}} \left(\int_s^t \|v(u,\cdot)\|_q^{\gamma}\, ds \right)^{\frac{1}{\gamma}} \right\}. \end{aligned}$$

This completes the proof of (A.11). $\square$

The following upper estimate for the increments of the Green function S has been proved in [9], Lemmas A.2 and A.6. Suppose that f satisfies (H_β); then for $\delta \in]0, \beta \wedge 1[$, $0 \le t \le t' \le T$, $x, x' \in \mathbb{R}^2$:

$$\int_0^T \|S(t-s, x-\cdot) - S(t'-s, x'-\cdot)\|_{\mathcal{H}}^2 \, ds \le C\left(|t-t'| + |x-x'|\right)^\delta . \tag{A.15}$$

The following lemma provides an upper estimate of an integral generalizing the function $J(s)$ introduced in [10], identity (A.1).

Lemma A.3 *For $s \in [0,T]$, $\lambda > 0$ and $p \in [1, +\infty[$, set*

$$I(s) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} S(s,y)^p f(|y-z|)^\lambda S(s,z)^p dy dz \, .$$

(a) Suppose that $f(r) = r^{-\alpha}$ for some $\alpha \in]0; 2[$. Then for $1 \le p < 2 \wedge (3 - \lambda\alpha) \wedge (4 - 2\lambda\alpha)$, one has

$$I(s) \le C\, s^{4-2p-\lambda\alpha}. \tag{A.16}$$

(b) Suppose that the function f satisfies $(\mathbf{H}_\beta)$ for $\beta \in]0, 2[$. If $\lambda \in]0, 1[$ and $1 \le p < 2 \wedge (3 - 2\lambda) \wedge [4 - 2\lambda(2-\beta)] \wedge (\frac{5}{2} - \lambda)$, then one has

$$I(s) \le C\, s^{4-2p-\lambda(2-\beta)}. \tag{A.17}$$

Proof: The change of variables $x = (u\cos(\theta_0), u\sin(\theta_0))$, $z = (v\cos(\theta+\theta_0), v\sin(\theta+\theta_0))$ and $r = \cos(\theta)$ used in the proof of Lemma A.1 in [10] and Fubini's theorem yield

$$\begin{aligned} I(s) &\le C \int_0^s \frac{udu}{(s^2-u^2)^{\frac{p}{2}}} \int_0^{2u} v f(v)^\lambda dv \int_{\frac{v}{2u}}^1 \frac{dr}{(1-r)^{\frac{1}{2}} (s^2 - u^2 - v^2 + 2uvr)^{\frac{p}{2}}} \\ &\le C\left(I_1(s) + I_2(s)\right), \end{aligned}$$

where

$$
\begin{aligned}
I_1(s) &= \int_0^{2s} v f(v)^\lambda dv \int_{\frac{v}{2}}^{s} \frac{u^{\frac{3}{2}} du}{(s^2-u^2)^{\frac{p}{2}}(u-\frac{v}{2})^{\frac{1}{2}}} \int_{\frac{v}{2u}}^{\frac{1}{2}(1+\frac{v}{2u})} \frac{dr}{(s^2-u^2-v^2+2uvr)^{\frac{p}{2}}}, \\
I_2(s) &= \int_0^{2s} v f(v)^\lambda dv \int_{\frac{v}{2}}^{s} \frac{u du}{(s^2-u^2)^{\frac{p}{2}}\left[(s^2-u^2)+v(u-\frac{v}{2})\right]^{\frac{p}{2}}} \int_{\frac{1}{2}(1+\frac{v}{2u})}^{1} \frac{dr}{(1-r)^{\frac{1}{2}}}.
\end{aligned}
$$

Since $p<2$, for $r \le \frac{1}{2}(1+\frac{v}{2u})$ one has

$$
(s^2-u^2-v^2+2uvr)^{1-\frac{p}{2}} \le \left[s^2-(u-\frac{v}{2})^2-\frac{v^2}{4}\right]^{1-\frac{p}{2}} \le s^{2-p},
$$

and hence, since $\ln(1+x)\le Cx^b$ for $x>0$ and $b\in]0,1-\frac{p}{2}[$,

$$
\begin{aligned}
I_1(s) &\le \int_0^{2s} v f(v)^\lambda\, dv \int_{\frac{v}{2}}^{s} \frac{u^{\frac{3}{2}} du}{(s^2-u^2)^{\frac{p}{2}}(u-\frac{v}{2})^{\frac{1}{2}}} \int_{\frac{v}{2u}}^{\frac{1}{2}(1+\frac{v}{2u})} \frac{s^{2-p}dr}{s^2-u^2-v^2+2uvr} \\
&\le s^{2-p}\int_0^{2s} f(v)^\lambda\, dv \int_{\frac{v}{2}}^{s} \frac{u^{\frac{1}{2}}}{(s^2-u^2)^{\frac{p}{2}}(u-\frac{v}{2})^{\frac{1}{2}}} \ln\left(1+\frac{v(u-\frac{v}{2})}{s^2-u^2}\right) du \\
&\le s^{2-p}\int_0^{2s} v^b f(v)^\lambda dv \int_{\frac{v}{2}}^{s} s^{\frac{1}{2}}\left(u-\frac{v}{2}\right)^{b-\frac{1}{2}}(s-u)^{-b-\frac{p}{2}}s^{-b-\frac{p}{2}}du \\
&\le C s^{\frac{5}{2}-\frac{3p}{2}-b}\int_0^{2s} v^b f(v)^\lambda (s-\frac{v}{2})^{\frac{1}{2}-\frac{p}{2}}\, dv\,. \qquad (A.18)
\end{aligned}
$$

In the last inequality, we have used the fact that for $x_1<x_2$,

$$
\int_{x_1}^{x_2}(x-x_1)^{r_1}\,(x_2-x)^{r_2}dx = \begin{cases} C_{r_1,r_2}\,(x_2-x_1)^{1+r_1+r_2} & \text{if } r_1>-1 \text{ and } r_2>-1\,, \\ +\infty & \text{otherwise.}\end{cases} \qquad (A.19)
$$

On the other hand, let $p-1<\gamma<\frac{3}{2}$; using again (A.19), we obtain

$$
\begin{aligned}
I_2(s) &\le C\int_0^{2s} v f(v)^\lambda\, dv \int_{\frac{v}{2}}^{s} u^{\frac{1}{2}}(u-\frac{v}{2})^{\frac{1}{2}}(s^2-u^2)^{-p+\gamma}\left[v(u-\frac{v}{2})\right]^{-\gamma} du \\
&\le C\, s^{\frac{1}{2}-p+\gamma}\int_0^{2s} v^{1-\gamma} f(v)^\lambda (s-\frac{v}{2})^{\frac{3}{2}-p} dv\,. \qquad (A.20)
\end{aligned}
$$

We then consider separately the two cases:

(a) If $f(r)=r^{-\alpha}$, from (A.19) we deduce that the right hand side of (A.18) converges if and only if $b-\lambda\alpha>-1$ and $\frac{1}{2}-\frac{p}{2}>-1$; then it is equal to $C\,s^{4-2p-\lambda\alpha}$. The constraints on b: $0 \vee (\lambda\alpha - 1) < b < 1 - \frac{p}{2}$ and $p < 3$ are compatible if and only if $p < 2 \wedge (4 - 2\lambda\alpha)$. On the other hand, the right hand side of (A.20) converges if and only if $1-\gamma-\lambda\alpha > -1$, $\frac{3}{2} - p > -1$. The constraints on γ: $p - 1 < \gamma < \frac{3}{2}$, $1 - \gamma - \lambda\alpha > -1$ and $p < \frac{5}{2}$ are compatible if and only if $p < \frac{5}{2} \wedge (3 - \lambda\alpha)$. This concludes the proof of (A.16).

(b) If $(\mathbf{H}_\beta)$ holds and $0 < \lambda < 1$, Hölder's inequality implies that

$$\int_0^{2s} v^b f(v)^\lambda (s - \frac{v}{2})^{\frac{1}{2}-\frac{p}{2}} dv \le \Big(\int_0^{2s} v^{1-\beta} f(v) dv\Big)^\lambda \times \Big(\int_0^{2s} v^{\frac{b-\lambda(1-\beta)}{1-\lambda}} (s - \frac{v}{2})^{\frac{1-p}{2(1-\lambda)}} dv\Big)^{1-\lambda}.$$

Thus (A.19) implies that the last integral converges if and only if $b - \lambda(1-\beta) > -1 + \lambda$ and $1 - p > -2 + 2\lambda$, for $0 < b < 1 - \frac{p}{2}$; then it is equal to $C s^{b-\lambda(1-\beta)+\frac{1-p}{2}+1-\lambda}$. The constraints on b, p, β are compatible if and only if $p < (3 - 2\lambda) \wedge (4 - 2\lambda(2-\beta))$, and $I_1(s)$ is dominated by $C\, s^{4-2p-\lambda(2-\beta)}$. On the other hand, using again Hölder's inequality, we obtain for $p - 1 < \gamma < \frac{3}{2}$,

$$\int_0^{2s} v^{1-\gamma} f(v)^\lambda (s - \frac{v}{2})^{\frac{3}{2}-p} dv \le \Big(\int_0^{2s} v^{1-\beta} f(v) dv\Big)^\lambda \times \Big(\int_0^{2s} v^{\frac{1-\gamma-\lambda(1-\beta)}{1-\lambda}} (s - \frac{v}{2})^{\frac{3-2p}{2(1-\lambda)}} dv\Big)^{1-\lambda}.$$

The last integral converges if and only if $1-\gamma-\lambda(1-\beta) > -1+\lambda$ and $\frac{3}{2} - p > -1 + \lambda$, and is equal to $C\, s^{1-\gamma-\lambda(1-\beta)+\frac{3}{2}-p+1-\lambda}$. The constraints on p, γ, λ are compatible for $\lambda \in]0;1[$ if $p < 2 \wedge (3 - \lambda(2-\beta)) \wedge (\frac{5}{2} - \lambda)$ and yield $I_2(s) \le C\, s^{4-2p-\lambda(2-\beta)}$. Finally, in order to obtain (A.17), we need $\lambda \in]0,1[$ and $1 \le p < 2 \wedge (3 - \lambda(2-\beta)) \wedge (\frac{5}{2} - \lambda) \wedge (4 - 2\lambda(2-\beta))$.

□

Finally, the following lemma provides a useful tool to estimate the moments of stochastic integrals with respect to F:

Lemma A.4 *Let $(\Delta(s,x); s \in [0;T], x \in \mathbb{R}^2)$ be a continuous random process such that supp$(\Delta(s,\cdot)) \subset D(s)$ for every $s \in [0,T]$. For $p \in [2, +\infty[$, set*

$$I := \int_{D(t)} dx \left| \int_0^t \|S(t-s, x-\cdot)\Delta(s,\cdot)\|_{\mathcal{H}}^2 ds \right|^{\frac{p}{2}}.$$

Then

(i) If $f(r) = r^{-\alpha}$*,* $0 < \alpha < 2$ *and* $2 \vee \left(\frac{8}{5-2\alpha}\right) < p < +\infty$*, then there exists some* $\delta > -1$ *such that*

$$I \leq C \int_0^t (t-s)^{\delta} \left(\int_{D(s)} |\Delta(s,x)|^p dx \right) ds. \tag{A.21}$$

(ii) If $(\mathbf{H}_\beta)$ *holds for some* $\beta \in]0,2[$*, then for* $p \in]8,+\infty[$*, (A.21) holds for some* $\delta > -1$*.*

Proof: Let $p_1 \in]1,+\infty[$ and $p_2 \in]1,p[$ be conjugate exponents, and let $\lambda \in]0,1[$. Hölder's inequality implies

$$I \leq \int_{D(t)} \left| \int_0^t I_1(s,x)^{\frac{1}{p_1}} \, I_2(s,x)^{\frac{1}{p_2}} ds \right|^{\frac{p}{2}} dx \,, \tag{A.22}$$

where

$$\begin{aligned}
I_1(s,x) &= \int \int S(t-s,x-y)^{p_1} f(|x-y|)^{\lambda p_1} S(t-s,x-z)^{p_1} dydz, \\
I_2(s,x) &= \int_{D(s)} \int_{D(s)} |\Delta(s,y)|^{p_2} f(|y-z|)^{(1-\lambda)p_2} |\Delta(s,z)|^{p_2} dydz.
\end{aligned}$$

Let $a := \frac{p}{p_2} \in]1,+\infty[$ and $b \in]1,+\infty[$ be such that $\frac{1}{a} + \frac{1}{b} - 1 = 1 - \frac{1}{a}$. Hölder's and Young's inequalities imply that for $s \in [0,T]$ and $x \in K$,

$$\begin{aligned}
I_2(s,x) &\leq \left(\int_{D(s)} |\Delta(s,y)|^{ap_2} dy \right)^{\frac{1}{a}} \left(\int_{D(s)} \left| \int_{D(s)} |f(|y-z|)^{(1-\lambda)p_2} |\Delta(s,z)|^{p_2} dz \right|^{\frac{a}{a-1}} dy \right)^{\frac{a-1}{a}} \\
&\leq \|\Delta(s,\cdot)\|^{2p_2}_{L^p(D(s))} \left\| f(|\cdot|)^{(1-\lambda)p_2} \right\|_{L^b(\bar{K})} ,
\end{aligned} \tag{A.23}$$

where $\bar{K} = \{x - y \,:\, x,y \in D(T)\}$ is a compact subset of $\mathbb{R}^2$ depending on T and K.

(i) If $f(r) = r^{-\alpha}$, the right hand side of (A.23) converges if and only if $\int_{0+} r\, f(r)^{(1-\lambda)p_2 b} dr < +\infty$, i.e., $\alpha(1-\lambda)bp_2 < 2$. Furthermore, if $1 \leq p_1 < 2 \wedge (3 - \lambda p_1 \alpha) \wedge (4 - 2\lambda p_1 \alpha)$, (A.16) implies that

$$I_1(s,x) \leq C(t-s)^{4-2p_1-\lambda p_1 \alpha}. \tag{A.24}$$

Therefore, using (A.22)-(A.24) we deduce that if $\delta := \frac{4}{p_1} - 2 - \lambda\alpha > -1$ and the previous constraints on λ, p_1 and p_2 are satisfied, then (A.21) holds. The requirements on p_2 and λ are gathered in the following system:

$$\begin{cases} 2 < p_2 < p < +\infty\,, \\ \lambda\alpha < 2 - \frac{3}{p_2}\,, \\ \lambda\alpha < \frac{3}{2} - \frac{2}{p_2}\,, \\ \alpha + 4\left(\frac{1}{p} - \frac{1}{p_2}\right) < \lambda\alpha < \alpha\,. \end{cases}$$

These inequalities on $\lambda\alpha \in]0, \alpha[$ are compatible if and only if

$$\begin{cases} 2 < p_2 < p < +\infty\,, \\ \frac{4}{p} < 2 - \alpha + \frac{1}{p_2} \\ \frac{4}{p} < \frac{3}{2} - \alpha + \frac{2}{p_2}\,, \end{cases}$$

which in turn are compatible if and only if $2 \vee \left(\frac{8}{5-2\alpha}\right) < p < +\infty$.

(ii) Suppose that $(\mathbf{H}_\beta)$ holds for some $\beta \in]0, 2[$. Let $q = (1-\lambda)bp_2$; if $q = 1$, $\int_{0+} rf(r)^q dr < +\infty$. Furthermore, if $0 < q < 1$, Hölder's inequality applied with respect to the measure $r^{1-\beta}dr$ implies that for every $R > 0$,

$$\int_0^R r\, f(r)^q dr \le \left(\int_0^R r^{1-\beta}\, f(r)dr\right)^q \left(\int_0^R r^{1-\beta+\frac{\beta}{1-q}}\, dr\right)^{1-q} < +\infty\,.$$

On the other hand, if $\lambda p_1 \le 1$, $p_1 < 2 \wedge (3 - 2\lambda p_1) \wedge [4 - 2\lambda p_1(2-\beta)] \wedge (\frac{5}{2} - \lambda p_1)$, then (A.17) implies that

$$I_1(s,x) \le C(t-s)^{4-2p_1-\lambda p_1(2-\beta)}. \tag{A.25}$$

Therefore, using (A.22), (A.23) and (A.25), we see that if the previous requirements on λ, p_1, p_2 and β are satisfied, then (A.21) holds if $\delta := \frac{4}{p_1} - 2 - \lambda(2-\beta) > -1$. The constraints on λ and p_1 are summarized in the following system:

$$\begin{cases} 2 < p_2 < p < +\infty\,, \\ 0 < \lambda < 1 - \frac{3}{2\,p_2}\,, \\ \lambda > 1 + 2\left(\frac{1}{p} - \frac{1}{p_2}\right)\,, \\ \lambda < \frac{3}{2(2-\beta)} - \frac{2}{2-\beta} \cdot \frac{1}{p_2}\,, \\ \lambda < \frac{3}{2} - \frac{5}{2\,p_2}\,. \end{cases}$$

Since for $p_2 > 2$ one has $1 - \frac{3}{2\,p_2} < \frac{3}{2} - \frac{5}{2\,p_2}$, these inequalities are compatible if and only if

$$\begin{cases} 2 < p_2 < p < +\infty\,, \\ \frac{2}{p} < \frac{1}{2p_2}\,, \\ \frac{2}{p} < \frac{2\beta-1}{2(2-\beta)} + \frac{2(1-\beta)}{(2-\beta)\,p_2}\,. \end{cases}$$

This system is equivalent to $2 < p_2$, $4p_2 < p < +\infty$ and $p > \frac{4(2-\beta)p_2}{p_2(2\beta-1)+4(1-\beta)} > 0$ If $\frac{1}{2} \le \beta < 2$, $p_2(2\beta-1)+4(1-\beta) > 0$ always holds, while if $0 < \beta < \frac{1}{2}$, this inequality is equivalent with $p_2 < \frac{4(1-\beta)}{1-2\beta}$ (note that in this case $2 < \frac{4(1-\beta)}{1-2\beta}$).

- If $0 < \beta \le 1$, the map $p_2 \longmapsto \frac{4(2-\beta)p_2}{p_2(2\beta-1)+4(1-\beta)}$ is increasing and the system is compatible (for $p_2 \sim 2$) if $p > 8$.
- If $1 \le \beta < 2$, the same map is decreasing and (for $p_2 \sim \frac{p}{4}$ and $p > 8$) the system is compatible if $p > 8$ and $p > \frac{4(2-\beta)p}{p(2\beta-1)+16(1-\beta)}$, that is $p > 8 \vee \left(\frac{4(3\beta-2)}{2\beta-1}\right) = 8$.

This concludes the proof of the lemma. □

Remark A.4: If $f(r) = r^{-\alpha}$ with $0 < \alpha \le \frac{1}{2}$, (A.21) holds for $p > 2$, and if $\frac{1}{2} < \alpha < 2$, (A.21) holds for $p > \frac{8}{5-2\alpha}$. Finally, $\sup\limits_{0\le\alpha<2} \dfrac{8}{5-2\alpha} = 8$ gives the lower limit of p in case (ii).

References

[1] S.K. Biswas, N.U. Ahmed *Stabilization of systems governed by the wave equation in the presence of distributed white noise*, IEEE Transactions on Automatic Control AC-30 (1985), pp. 1043-1045.

[2] R. Carmona, D. Nualart *Random nonlinear wave equations: smoothness of the solution*, Prob. Th. and Rel. Fields 79 (1988), pp. 469-508.

[3] R. Dalang *Extending the martingale measure stochastic integrals to spatially homogeneous spde's*, Electronic Journal of Probability, 4 (1999), http://www.math.washington.edu/∼ejpecp/EjpVol4/paper6.abs.html

[4] R. Dalang, N. E. Frangos *The stochastic wave equation in two spatial dimensions*, Annals of Probab. 26 (1998), 187-212.

[5] I. Gyöngy *Existence and uniqueness results for semilinear stochastic partial differential equations*, Stochastic Processes and their Applications 73 (1998), pp. 271-299.

[6] M.A. Krasnosel'skii, Ya. B. Rustickii *Convex functions and Orlicz spaces*, Groningen, Noordorff, 1961.

[7] J.L. Lions *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod - Gauthier Villars, 1969.

[8] R.N. Miller *Tropical data assimilation with simulated data: the impact of the tropical ocean and global atmosphere thermal array for the ocean*, Journal of Geophysical Research 96 (1990), 11, 461-482.

[9] A. Millet, P.L. Morien *A stochastic wave equation in two space dimensions: regularity of the solution and its density*, Stochastic Processes and their Applications 86 (2000), 141-162.

[10] A. Millet, M. Sanz-Solé *A stochastic wave equation in two space dimensions: smoothness of the law*, Annals of Probab. 27 (1999), 803-844.

[11] C. Mueller *Long-time existence for the wave equation with a noise term*, Annals of Probability 25 (1997), pp. 133- 151.

[12] S. Peszat *The Cauchy problem for a nonlinear stochastic wave equation in any dimension*, Preprint (1999).

[13] S. Peszat, J. Zabczyk *Nonlinear stochastic wave and heat equations*, Preprint (1998).

[14] D.W. Stroock, S.R.S. Varadhan, *Multidimensional Diffusion Processes*, Springer Verlag, 1979.

[15] J.B. Walsh *An introduction to stochastic partial differential equations*, Ecole d'été de Probabilités de Saint- Flour, Lecture notes in Math. 1180, pp. 266-437, Springer Verlag, 1986.

Annie MILLET
amil@ccr.jussieu.fr
MODAL'X and Laboratoire de
Probabilités et Modèles Aléatoires
Université Paris 6
4, Place Jussieu
75252 PARIS Cedex 05
FRANCE

Pierre-Luc MORIEN
morien@modalx.u-paris10.fr
MODAL'X
Université Paris 10
200, Avenue de la République
92001 NANTERRE Cedex
FRANCE