A random space-time birth particle method for 2d vortex equations with L^1-external field

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Abstract

We consider a class of 2d Navier-Stokes equations with external non-conservative forces. We develop a probabilistic interpretation based on a vortex equation with external integrable field. We relate the latter to a nonlinear process with random space-time birth, providing a probabilistic description of the creation of vorticity. The initial data and external field are only assumed to satisfy integrability properties. Initially, a regularized version of the process is obtained, replacing the singular Biot and Savart kernel by some Lipschitz continuous regularization. Then, we remove the regularization parameter and deduce the existence, uniqueness and regularity of a mild solution of the vortex equation with external field, and thus the existence of the nonlinear process. We define interacting particle systems with space-time random births and propose a stochastic numerical particle method for the vorticity and also for the velocity field. We obtain either pathwise or weak convergence results, depending on the integrability of the initial data and of the external field. We finally illustrate our results with simulations.

1 Introduction

The Navier-Stokes equation for an homogeneous and incompressible fluid in the whole plane subject to an external force field $\mathbf{f}$, is given by

$$\begin{aligned} &\tfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla\mathbf{p} + \mathbf{f}; \\ &div\ \mathbf{u}(t,x) = 0; \quad \mathbf{u}(t,x)\to 0 \text{ as } |x|\to\infty. \end{aligned} \tag{1}$$

Here, $\mathbf{u}$ denotes the velocity field, $\mathbf{p}$ is the pressure function and $\nu > 0$ is the viscosity (constant) coefficient.
In absence of the external force field, (or more generally, when $\mathbf{f} = \nabla\Psi$ is a conservative field), a probabilistic interpretation of (1) is known since the work of Marchioro and Pulvirenti [15]. The probabilistic approach to (1) is based on the associated vortex equation, i.e. the equation satisfied by the (scalar) vorticity field $w := curl\mathbf{u}$, which is interpreted as a generalized McKean-Vlasov equation associated with a nonlinear diffusion process. This

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process can also be obtained as the limit of interacting particle systems in mean field interaction, and this fact provides stochastic approximations of the vortex equation associated with (1). Convergence on the path space of these particles (or equivalently, propagation of chaos for the system) has been proved in more recent works of Méléard [16] and [17].

In this work, we will extend such approach to the Navier-Stokes equation with external force field (1). We will permanently combine analytical and probabilistic arguments.
The non-conservative external force gives an additional field $g = curl f$ in the vortex equation. More precisely, the vorticity field $w = curl\ \mathbf{u}$ satisfies the scalar equation

$$\frac{\partial w}{\partial t} + (K * w \cdot \nabla)w = \nu \Delta w + g;$$

where $K(x) = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|x|^2}$ is the so-called Biot-Savart kernel in $\mathbb{R}^2$.

The external field g is physically interpreted as creation of vorticity. In order to provide a probabilistic description of this phenomenon, we relate this equation to a nonlinear process with random birth in space and time, according to a law related to the initial vorticity w_0 and the external field g. A similar idea is developed in the work of Jourdain and Méléard [13] in the context of a vortex equation on a bounded domain with Neumann's condition on the boundary.

A minimal assumption for the probabilistic study of the vortex equation, is that w_0 and $g(t, \cdot)$ are integrable functions for each t. Our first goal is to prove existence and uniqueness both for the vortex equation and for the nonlinear process under this assumption. The main difficulties in this study are the singularity of the kernel K and the lack of continuity of the convolution operator $K * w$ for $w \in L^1$.
Therefore, we will first consider a mollified setting, working with regularized versions of the Biot-Savart kernel. We will adapt the classic McKean-Vlasov techniques to prove the pathwise existence and uniqueness of a mollified nonlinear process. The family of its time-marginal laws weighted by some function of the space-time initial data gives a solution of the mollified vortex equation. By construction these solutions are in L^1, uniformly on the regularization parameter.
We construct a stochastic interacting particle system with space-time births and prove propagation of chaos and its convergence to the mollified nonlinear process.
In order to remove the regularization parameter, we will assume in a first step that the initial vorticity and external field belong to $L^1 \cap L^p$, for $p > \frac{4}{3}$. This choice is suggested by the continuity properties of the Biot and Savart operator. To obtain uniform L^p estimates, we introduce original techniques that take advantage of the volume preserving property of the stochastic flow associated with the mollified nonlinear process. We deduce the existence of a global mild solution of the vortex equation. By analytic techniques we prove uniqueness and regularity of this solution, and then the pathwise existence and uniqueness of the nonlinear process. Moreover, we obtain pathwise convergence for the particle system in a strong norm, and deduce an approximation result for the velocity field $\mathbf{u}$ at an explicit rate.
In a second step, we extend our results to L^1 initial condition and external field. The analytical part of our study generalizes to the case $g \neq 0$ some compacity arguments of Ben-Artzi [2] and Brezis [5] when $g = 0$. Hence, we obtain existence, uniqueness and regularity of the mild solution of the vortex equation. We deduce existence and uniqueness in law for the nonlinear process, by proving the convergence of some martingale problems, and convergence in law of the particle approximations.

The explosion of the solution at time 0 prevents us in this case from obtaining pathwise results and a stronger convergence.
Our results improve the functional $(L^1 \cap L^\infty)$ assumption required in [15] and [16], and also the smallness condition on the norm $\|w_0\|_1$ partially needed in [17].
We finally illustrate our results with numerical simulations.

1.1 Notation

- $C^{1,2}$ is the set of real valued functions on $[0,T] \times \mathbb{R}^2$ with continuous derivatives up to the first order in $t \in [0,T]$ and up to the second order in $x \in \mathbb{R}$. $C_b^{1,2}$ is the subspace of bounded functions in $C^{1,2}$ with bounded derivatives.
- $\mathcal{D}$ is the space of of infinitely differentiable functions on $\mathbb{R}^2$ having compact support.
- For all $1 \le p \le \infty$ we denote by L^p the space $L^p(\mathbb{R}^2)$ of real valued functions on $\mathbb{R}^2$. By $\|\cdot\|_p$ we denote the corresponding norm and p^* stands for the Hölder conjugate of p. We write $W^{i,p} = W^{i,p}(\mathbb{R}^2)$ for the Sobolev space of functions in L^p with partial derivatives up to the i-th order in L^p.
- C and $C(T)$ are finite positive constants that may change from line to line.

The next two elementary results will be used throughout.

Lemma 1.1 *Let ε, θ be strictly positive constants and $\beta(\theta,\varepsilon) = \int_0^1 (1-s)^{\theta-1} s^{\varepsilon-1} ds$ be the Beta function of parameters θ and ε. Then, for all $t > 0$,*

$$\int_0^t (t-s)^{\theta-1} s^{\varepsilon-1} ds = t^{\theta+\varepsilon-1} \beta(\theta,\varepsilon).$$

The following is a version of Gronwall's lemma proved for instance in [9].

Lemma 1.2 *Let $k : [0,T] \to \mathbb{R}_+$ be a bounded nonnegative measurable function and suppose that there are constants $C, A \ge 0$ and $\theta > 0$ such that, for all $t \le T$,*

$$k(t) \le A + C \int_0^t (t-s)^{\theta-1} k(s)\, ds.$$

Then,

$$\sup_{t \le T} k(t) \le C_T A,$$

where the constant C_T does not depend on A.

2 The vortex equation with external force and its probabilistic interpretation

The vortex equation associated with the Navier-Stokes equation with external force (1) is the equation satisfied by $w = curl\ \mathbf{u}$, that is

$$\begin{aligned} &\tfrac{\partial w}{\partial t} + (\mathbf{u} \cdot \nabla) w = \nu \Delta w + g; \\ &w_0(x) = curl\ \mathbf{u}(0,x) \end{aligned} \tag{2}$$

where $\mathbf{u}$ is the velocity field solution of (1) and

$$g = curl\ \mathbf{f}. \tag{3}$$

Thanks to divergence free property of $\mathbf{u}$ and the Biot-Savart law, we can write

$$\mathbf{u} = K * w \tag{4}$$

where

$$K(x) = \frac{1}{2\pi}\frac{(-x_2, x_1)}{|x|^2}, \qquad x \in \mathbb{R}^2 \backslash \{0\}$$

is the so-called Biot-Savart kernel in $\mathbb{R}^2$.

We will fix for all the sequel an arbitrary finite time interval $[0, T]$.

In view of our probabilistic interpretation of Equation (2), it is natural and necessary to assume that the functions $w_0 : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the minimal integrability hypothesis:

- $w_0 \in L^1(\mathbb{R}^2)$.
- $g \in L^1([0, T] \times \mathbb{R}^2)$.

We denote by $\|g\|_{1,T}$ the L^1-norm of g on $[0, T] \times \mathbb{R}^2$:

$$\|g\|_{1,T} = \int_0^T \int_{\mathbb{R}^2} |g(s, x)| dx\ ds.$$

We are interested in weak solutions of (2) defined as follows.

Definition 2.1 *A measurable function $w : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a weak solution of the vortex equation (2) with initial condition w_0 and external field g on the interval $[0, T]$, if*

$$\int_{[0,T]\times\mathbb{R}^2} |K * w_t(x)||w_t(x)| dx dt < \infty, \tag{5}$$

and for every function $\phi \in C_b^{1,2}([0, T] \times \mathbb{R}^2)$ and $t \le T$,

$$\begin{aligned}\int_{\mathbb{R}^2} \phi(t, x) w_t(x) dx &= \int_{\mathbb{R}^2} \phi(0, x) w_0(x) dx + \int_0^t \int_{\mathbb{R}^2} \phi(s, x) g_s(x) dx\ ds \\ &+ \int_0^t \int_{\mathbb{R}^2} \left[\frac{\partial \phi(s, x)}{\partial s} + \nu \triangle \phi(s, x) + (K * w_s)(x) \nabla \phi(s, x)\right] w_s(x) dx.\end{aligned} \tag{6}$$

Even if our probabilistic approach naturally leads to this type of solution, for analytic purpose we need to deal with mild solution of Equation (2). We denote by

$$G_t^\nu(x) := (4\pi\nu t)^{-1} e^{-|x|^2/4\nu t}$$

the heat kernel in $\mathbb{R}^2$. The following are well known estimates.

Lemma 2.2 *Let $m \in [1,\infty]$ and $l \geq m$. There exist constants $c(m,l), c'(m,l) > 0$ such that for all $f \in L^m$*

$$\|G_t^\nu * f\|_l \leq c(m,l) t^{\frac{1}{l}-\frac{1}{m}} \|f\|_m \qquad \text{and} \qquad \|\nabla G_t^\nu * f\|_l \leq c'(m,l) t^{-\frac{1}{2}+\frac{1}{l}-\frac{1}{m}} \|f\|_m.$$

Definition 2.3 *A measurable function $w : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is called a mild solution of the vortex equation with external field g if condition (5) holds and*

$$w_t(x) = G_t^\nu * w_0(x) + \int_0^t G_{t-s}^\nu * g_s(x)\ ds + \int_0^t \nabla G_{t-s}^\nu * [(K * w_s) w_s](x)\ ds \tag{7}$$

for all $t \in [0,T]$.

Remark 2.4 *A weak solution is always a mild solution. This is easily seen by taking for each fixed $t > 0$ and $\psi \in \mathcal{D}$ in Equation (6) the function $\phi_t(s,x) := G_{t-s}^\nu * \psi(x)$ (which solves on $[0,t[\times\mathbb{R}^2$ the heat equation with final condition ψ). Using Fubini's theorem (thanks to (5)) yields (7). The converse is immediate.*

2.1 The nonlinear process with random space-time birth

In the case $g = 0$, Equation (2) can be seen as a generalized McKean-Vlasov equation, associated with a nonlinear stochastic process. When $g \neq 0$, an additional "free" term appears in the weak formulation of the equation (Definition 2.1). We interpret this term as creation of vorticity, associating with Equation (2) a nonlinear process with random space-time birth. An analogous approach has been developed in Jourdain and Méléard [13]. In that work, a vortex equation on a bounded domain with Neumann's condition on the boundary is associated with a nonlinear process with space-time random birth located at the boundary.

Let us define the probability measure $P_0(dt,dx)$ on $[0,T] \times \mathbb{R}^2$ by

$$P_0(dt,dx) = \delta_0(dt) \frac{|w_0(x)|}{\|w_0\|_1 + \|g\|_{1,T}} dx + \frac{|g_t(x)|}{\|w_0\|_1 + \|g\|_{1,T}} dx\ dt, \tag{8}$$

together with the scalar weight function

$$h(t,x) = \mathbf{1}_{\{t=0\}} \frac{w_0(x)}{|w_0(x)|} (\|w_0\|_1 + \|g\|_{1,T}) + \frac{g_t(x)}{|g_t(x)|} (\|w_0\|_1 + \|g\|_{1,T}) \mathbf{1}_{\{t>0\}} \tag{9}$$

with the convention "$\frac{0}{0} = 0$" and $\mathbf{1}$ denoting the indicator function. Hence, h takes values in $\{-(\|w_0\|_1 + \|g\|_{1,T}), 0, \|w_0\|_1 + \|g\|_{1,T}\}$.

Remark 2.5 *For any measurable bounded function ϕ on $[0,T] \times \mathbb{R}^2$,*

$$\int_{[0,T]\times\mathbb{R}^2} \phi(t,x) h(t,x) P_0(dx,dt) = \int_{\mathbb{R}^2} \phi(0,x) w_0(x) dx + \int_{[0,T]\times\mathbb{R}^2} \phi(t,x) g_t(x) dx\ dt$$

Let now $(\tau, (X_t)_{t\in[0,T]})$ denote the canonical process on the space $\mathcal{C}_T := [0,T] \times C([0,T],\mathbb{R}^2)$. With each probability measure Q on $\mathcal{C}_T$ we associate the flow of signed measures $(\tilde{Q}_t)_{t\in[0,T]}$ on $\mathbb{R}^2$, defined for all bounded measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{Q}_t(f) = E^Q\big(f(X_t) h(\tau, X_0) \mathbf{1}_{t\geq\tau}\big). \tag{10}$$

Clearly, for each $t \in [0,T]$ the total mass of $\tilde{Q}_t$ is bounded by $\|w_0\|_1 + \|g\|_{1,T}$.
Notice furthermore that if $Q \circ (X_t)^{-1}$ has a density, say ρ_t, then so does $\tilde{Q}_t$. We then denote the density of $\tilde{Q}_t$ by

$$\tilde{\rho}_t(x)$$

We always take versions of $(t,x) \mapsto \rho_t(x)$ and $(t,x) \mapsto \tilde{\rho}_t(x)$ that are measurable in the pair of variables (t,x), if such versions exist.

Definition 2.6 *A probability measure P on $\mathcal{C}_T$ is a solution to the nonlinear martingale problem* **(MP)** *if*

- *$P \circ (\tau, X_0)^{-1} = P_0$ and $\tilde{P}_t$ has bi-measurable densities $\tilde{\rho}_t(x)$ for $(t,x) \in [0,T] \times \mathbb{R}^2$*
- $f(t,X_t) - f(\tau,X_0) - \int_0^t \left[\frac{\partial f}{\partial s}(s,X_s) + \nu \triangle f(s,X_s) + K * \tilde{\rho}_s(X_s)\nabla f(s,X_s)\right] \mathbf{1}_{s \geq \tau} ds,$

  $0 \leq t \leq T$, is a continuous P-martingale for all $f \in \mathcal{C}_b^{1,2}$ w.r.t. the filtration $\mathcal{F}_t = \sigma(\tau, (X_s), s \leq t)$.

The link between this problem and Equation (2) is the following.

Lemma 2.7 *Assume that the problem* **(MP)** *has a solution P which satisfies*

$$\int_{[0,T]\times\mathbb{R}^2} |K * \tilde{\rho}_t(x)||\tilde{\rho}_t(x)| dx dt < \infty.$$

Then $w := \tilde{\rho}$ is a weak solution of the vortex equation with external force field (2).

Proof: Since the variable $h(\tau, X_0)\mathbf{1}_{\{\tau \leq t\}}$ is measurable with respect to $\mathcal{F}_0$, the process

$$f(t,X_t)h(\tau,X_0)\mathbf{1}_{\{\tau\leq t\}} - f(\tau,X_0)h(\tau,X_0)\mathbf{1}_{\{\tau\leq t\}} \\ - \int_0^t \left[\frac{\partial f}{\partial s}(s,X_s) + \nu\triangle f(s,X_s)ds + K * \tilde{\rho}_s(X_s)\nabla f(s,X_s)\right] h(\tau,X_0)\mathbf{1}_{\{s\geq\tau\}} ds$$

is a P-martingale for all $f \in \mathcal{C}_b^{1,2}$ w.r.t. $(\mathcal{F}_t)$. We take expectation and use Fubini's theorem, and we conclude by Remark 2.5 and the definition of $\tilde{\rho}$.

□

By a standard argument using the semi-martingale decomposition of the coordinate processes X^i and their products $X^i X^j$, we obtain that for $f \in C_b^{1,2}$ the martingale part of $f(t,X_t)$ in **(MP)** is given by the stochastic integral

$$\sqrt{2\nu} \int_0^t \nabla f(s,X_s)\mathbf{1}_{\{s\geq\tau\}} dB_s,$$

with respect to a Brownian motion B defined on some extension of the canonical space. Consequently, on the random interval $[0,\tau]$, the martingales in **(MP)** are null and $X_t = X_0$.

Remark 2.8 *It follows that the second condition in* **(MP)** *is equivalent to the fact that*

$$f(t,X_t) - f(0,X_0) - \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds - \int_0^t [\nu\triangle f(s,X_s) + K * \tilde{\rho}_s(X_s)\nabla f(s,X_s)] \mathbf{1}_{\{s\geq\tau\}} ds \tag{11}$$

is a continuous P-martingale with respect to $(\mathcal{F}_t)$ for all $f \in C_b^{1,2}$.

3 The mollified problem

In a first stage we deal with a regularized version of the kernel K. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded and smooth function with bounded derivatives, satisfying $\|\varphi\|_1 = 1$. For $\varepsilon > 0$ we define $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\varphi(\frac{x}{\varepsilon})$, and

$$K_\varepsilon = K * \varphi_\varepsilon.$$

The function K_ε is bounded and smooth, and has bounded derivatives. We denote by M_ε its sup-norm on $\mathbb{R}^2$ and by L_ε a Lipschitz constant, that respectively behave like $\frac{1}{\varepsilon^2}$ and $\frac{1}{\varepsilon^3}$ when $\varepsilon << 1$. Notice that $div\ K_\varepsilon = (div\ K) * \varphi_\varepsilon = 0$.

In this section, we fix the parameter ε, and we consider the mollified equation obtained from Equation (2) by replacing K by K_ε:

$$\frac{\partial v}{\partial t} + (K_\varepsilon * v \cdot \nabla)v = \nu\Delta v + g. \tag{12}$$

We will adapt the usual McKean-Vlasov approach to give a probabilistic interpretation to (12) and construct some approximating stochastic particle system.

3.1 The nonlinear process.

Consider on some given probability space a 2-dimensional Brownian motion B and a $\mathbb{R}_+ \times \mathbb{R}^2$ valued random variable (τ, X_0) independent of B with law P_0.

Theorem 3.1 *There is existence and uniqueness, trajectorial and in law, for the following nonlinear stochastic differential equation in the sense of McKean*

$$X_t^\varepsilon = X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{s\geq\tau} dB_s + \int_0^t K_\varepsilon * \tilde{P}_s^\varepsilon(X_s^\varepsilon)\mathbf{1}_{s\geq\tau} ds, \quad t > 0, \tag{13}$$

under the conditions: $law(\tau, X_0) = P_0$ *and* $law(\tau, X^\varepsilon) = P^\varepsilon$.

Proof : The proof is easily adapted from Theorem 1.1 in [20]. Denote by d_T the Kantorovich-Wasserstein distance on $\mathcal{C}_T$

$$d_T(Q^1, Q^2) := \inf\left\{ \int_{(\mathcal{C}_T)^2} \left[\sup_{t\in[0,T]} (|x(t) - y(t)| \wedge 1) + |\alpha - \beta|\right]\Pi(d\alpha, dx, d\beta, dy) : \right.$$
$$\left. \Pi \text{ has marginal laws } Q^1 \text{ and } Q^2 \right\},$$

and by $\mathcal{C}_T^0$ the closed subspace $\mathcal{C}_T^0 = \{Q \in \mathcal{C}_T : Q\circ(\tau, X_0)^{-1} = P_0\}$. Define a mapping $\Theta : \mathcal{C}_T^0 \to \mathcal{C}_T^0$ associating with Q the law $\Theta(Q)$ of the unique solution of

$$X_t^Q = X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{s\geq\tau} dB_s + \int_0^t K_\varepsilon * \tilde{Q}_s(X_s^Q)\mathbf{1}_{s\geq\tau} ds.$$

By trajectorial considerations, one can show that for each $t \leq T$,

$$d_t(\Theta(Q^1), \Theta(Q^2)) \leq C(T)\int_0^t d_s(Q^1, Q^2) ds$$

(with $d_t(Q^1, Q^2)$ the distance between the projections of Q^1 and Q^2 to $\mathcal{C}_t$). We deduce the existence of a unique fixed point for Θ and hence a unique solution in law. The trajectorial statement then follows from the Lipschitz property of K_ε (see [20] for details).

□

It will be convenient to introduce the stochastic flow associated with the nonlinear process (13), that is, the three parameter process

$$\xi^\varepsilon_{s,t}(x) = x + \sqrt{2\nu}(B_t - B_s) + \int_s^t K_\varepsilon * \tilde{P}^\varepsilon_r(\xi^\varepsilon_{s,r}(x))dr. \tag{14}$$

The function $(s,x) \mapsto K_\varepsilon * \tilde{P}^\varepsilon_s(x)$ is continuous, and Lipschitz continuous in x uniformly in time, as well as all its spatial derivatives. This implies that there is a continuous version $(s,t,x) \mapsto \xi^\varepsilon_{s,t}(x)$ such that $x \mapsto \xi^\varepsilon_{s,t}(x)$ is continuously differentiable for all (s,t) (cf. [14]). We denote by

$$G^\varepsilon(s,x;t,y), \quad (s,x,t,y) \in (\mathbb{R}_+ \times \mathbb{R}^2)^2, s < t$$

the density of $\xi^\varepsilon_{s,t}(x)$, which is a continuous function of (s,x,t,y) (see [10]).
Since $X^\varepsilon_t = X_0$ for all $t \le \tau$, we have that

$$X^\varepsilon_t = \xi^\varepsilon_{\tau,t}(X_0)\mathbf{1}_{\{t\ge\tau\}} + X_0\mathbf{1}_{\{t<\tau\}}.$$

Hence, conditioning with respect to (τ, X_0), we obtain for bounded functions f that

$$\begin{aligned} E(f(X^\varepsilon_t)) =& E(f(X^\varepsilon_t)\mathbf{1}_{\{t\ge\tau\}}) + E(f(X^\varepsilon_t)\mathbf{1}_{\{t<\tau\}}) \\ =& E\left[f(\xi^\varepsilon_{\tau,t}(X_\tau))\mathbf{1}_{\{t\ge\tau\}}\right] + E\left[f(X_0)\mathbf{1}_{\{t<\tau\}}\right] \\ =& \int_0^t \int_{(\mathbb{R}^2)^2} f(y)G^\varepsilon(s,x;y,t)dyP_0(ds,dx) + \int_t^T \int_{\mathbb{R}^2} f(x)P_0(ds,dx), \\ =& \int_{\mathbb{R}^2} f(x)\bar{w}_0(x)dx + \int_0^t \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} f(y)G^\varepsilon(s,x;t,y)dy\right] \bar{g}_s(x)dx\ ds \\ &+ \int_t^T \int_{\mathbb{R}^2} f(x)\bar{g}_s(x)dxds, \end{aligned}$$

where we have introduced the notation

$$\bar{w}_0(x) = \frac{|w_0(x)|}{\|w_0\|_1 + \|g\|_{1,T}} \quad \text{and} \quad \bar{g}_s(x) = \frac{|g_s(x)|}{\|w_0\|_1 + \|g\|_{1,T}}.$$

By Fubini's theorem, we deduce that for each $t \in]0,T]$, X^ε_t has a bi-measurable density

$$(t,y) \mapsto \rho^\varepsilon_t(y).$$

Similarly, we have

$$\begin{aligned} \int_{\mathbb{R}^2} f(x)\tilde{P}^\varepsilon_t(dx) =& E[f(\xi^\varepsilon_{\tau,t}(X_0))h(\tau,X_0)\mathbf{1}_{\{\tau\le t\}}] + E[f(X_0)h(\tau,X_0)\mathbf{1}_{\{\tau>t\}}] \\ =& \int_{\mathbb{R}^2} f(x)w_0(x)dx + \int_0^t \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} f(y)G^\varepsilon(s,x;t,y)dy\right] g_s(x)dx\ ds \\ &+ \int_t^T \int_{\mathbb{R}^2} f(x)g_s(x)dxds \end{aligned}$$

and then, $\tilde{P}^\varepsilon_t(dy)$ has a bi-measurable density family, that we denote by

$$(t,y) \mapsto \tilde{\rho}^\varepsilon_t(y).$$

Remark 3.2 *By construction, we obtain*

$$\sup_{\varepsilon>0}\sup_{t\in[0,T]}\|\tilde{\rho}_t^\varepsilon\|_1 \le \|w_0\|_1 + \|g\|_{1,T} \tag{15}$$

We deduce the following result about Equation (12).

Corollary 3.3 *The function $\tilde{\rho}^\varepsilon$ is the unique weak solution of Equation (12) in the space $L^\infty([0,T],L^1)$.*

Proof: We write Itô's formula for $\phi(t,X_t^\varepsilon)$ and proceed as in Lemma 2.7 (boundedness of K_ε provides us now an analogous integrability condition as required therein.) We obtain that $\tilde{\rho}_t^\varepsilon \in L^\infty([0,T],L^1)$ is a solution of the weak McKean-Vlasov type equation

$$\begin{aligned}\int_{\mathbb{R}^2}\phi(t,x)\tilde{\rho}_t^\varepsilon(x)dx &= \int_{\mathbb{R}^2}\phi(0,x)w_0(x)dx + \int_0^t\int_{\mathbb{R}^2}\phi(s,x)g_s(x)dx\ ds\\ &+ \int_0^t\int_{\mathbb{R}^2}\left[\frac{\partial\phi(s,x)}{\partial s} + \nu\triangle\phi(s,x) + K_\varepsilon * \tilde{\rho}_s^\varepsilon(x)\nabla\phi(s,x)\right]\tilde{\rho}_s^\varepsilon(x)dxds\end{aligned} \tag{16}$$

for all $\phi \in C_b^{1,2}([0,T]\times\mathbb{R}^2)$.

Let us now prove uniqueness. Using boundedness of K_ε and proceeding as in Remark 2.4, we check that $\tilde{\rho}^\varepsilon$ is a solution in $L^\infty([0,T],L^1)$ of the *mollified mild equation*

$$\tilde{\rho}_t^\varepsilon(x) = G_t^\nu * w_0(x) + \int_0^t G_{t-s}^\nu * g_s(x)\ ds + \int_0^t \nabla G_{t-s}^\nu * \left[(K_\varepsilon * \tilde{\rho}_s^\varepsilon)\tilde{\rho}_s^\varepsilon\right](x)\ ds. \tag{17}$$

If v is another solution of (17), we obtain, thanks to Lemma 2.2 with $l=\infty$ and $m=1$ that

$$\|\tilde{\rho}_t^\varepsilon - v_t\|_1 \le C(\varepsilon)\int_0^t (t-s)^{-\frac{1}{2}}\|\tilde{\rho}_s^\varepsilon - v_s\|_1 ds$$

and conclude with Lemma 1.2.

□

3.2 Stochastic particle approximations

We now define an interacting particle system which is naturally associated with the non-linear process studied above. The system takes into account the random space-time births. Its pathwise existence and uniqueness can be proved by adapting standard arguments.

Definition 3.4 *Consider a sequence $(B^i)_{i\in\mathbb{N}}$ of independent Brownian motions on $\mathbb{R}^2$ and a sequence of independent variables $(\tau^i,X_0^i)_{i\in\mathbb{N}}$ with values in $[0,T]\times\mathbb{R}^2$ distributed according to P_0, and independent of the Brownian motions. For a fixed $\varepsilon>0$, for each $n\in\mathbb{N}^*$, let us consider the interacting processes defined for $1\le i\le n$ by*

$$X_t^{in,\varepsilon} = X_0^i + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\ge\tau^i\}}dB_s^i + \int_0^t \mathbf{1}_{\{s\ge\tau^i\}}K_\varepsilon * \tilde{\mu}_s^{n,\varepsilon}(X_s^{in,\varepsilon})ds \tag{18}$$

where

$$\tilde{\mu}_s^{n,\varepsilon} = \frac{1}{n}\sum_{j=1}^n h(\tau^j,X_0^j)\mathbf{1}_{\{s\ge\tau^j\}}\delta_{X_s^{jn,\varepsilon}}$$

is the weighted empirical measure of the system at time s and

$$K_\varepsilon * \tilde{\mu}_s^{n,\varepsilon}(z) = \frac{1}{n}\sum_{j=1}^n h(\tau^j, X_0^j)\mathbf{1}_{\{s\geq\tau^j\}}K_\varepsilon(z - X_s^{jn,\varepsilon}).$$

Observe that particles either have birth at time 0 or at a random time, and evolve as soon as they are born as diffusive particles that interact following a mean field depending on the parameter ε. We introduce a coupling between these interacting processes and some independent copies of the limiting process defined in (13), as follows:

Definition 3.5 *For $i \geq 1$, we define $\bar{X}^{i,\varepsilon}$ by*

$$\bar{X}_t^{i,\varepsilon} = X_0^i + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\geq\tau^i\}}dB_s^i + \int_0^t \mathbf{1}_{\{s\geq\tau^i\}}K_\varepsilon * \tilde{\rho}_s^\varepsilon(\bar{X}_s^{i,\varepsilon})ds. \tag{19}$$

We have the following estimates for each $\varepsilon > 0$.

Proposition 3.6 *There exist positive constants C_1, C_2 such that for all $n \in \mathbb{N}$, $1 \leq i \leq n$ and $\varepsilon \in]0,1[$,*

$$E(\sup_{t\leq T}|X_t^{in,\varepsilon} - \bar{X}_t^{i,\varepsilon}|) \leq \frac{C_1\varepsilon}{\sqrt{n}}\exp(C_2(\|w_0\|_1 + \|g\|_{1,T})(\varepsilon^{-2})T). \tag{20}$$

Proof. The proof is an adaptation of the proof of Proposition 2.2 in [11]. We have

$$\begin{aligned}\sup_{s\leq t}|X_s^{in,\varepsilon} - \bar{X}_s^{i,\varepsilon}| \leq \quad & \int_0^t \frac{1}{n}\sum_{j=1}^n \|h\|_\infty\, L_\varepsilon\left(|X_s^{in,\varepsilon} - \bar{X}_s^{i,\varepsilon}| + |X_s^{jn,\varepsilon} - \bar{X}_s^{j,\varepsilon}|\right)\, ds \\ & + \int_0^t \left|\frac{1}{n}\sum_{j=1}^n h(\tau^j, X_0^j)K_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - K_\varepsilon * \tilde{\rho}_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\right|\, ds.\end{aligned}$$

Since the sequence $(\tau^i, X^{in,\varepsilon}, \bar{X}^{i,\varepsilon})_{1\leq i\leq n}$ is exchangeable,

$$E\Big[\sup_{s\leq t}|X_s^{in,\varepsilon} - \bar{X}_s^{i,\varepsilon}|\Big] \leq 2\|h\|_\infty\, L_\varepsilon\int_0^t E\Big[\sup_{u\leq s}|X_u^{in,\varepsilon} - \bar{X}_u^{i,\varepsilon}|\Big]\, ds$$

$$+\int_0^t\left[E\left(\frac{1}{n}\sum_{j=1}^n h(\tau^j, X_0^j)K_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - K_\varepsilon * \tilde{\rho}_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\right)^2\right]^{1/2} ds$$

The expectation in the last term above is equal to $\frac{1}{n^2}$ times a double sum of terms

$$E\Big[\Big(h(\tau^j, X_0^j)K_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - K_\varepsilon * \tilde{\rho}_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\Big)\Big(h(\tau^k, X_0^k)K_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{k,\varepsilon}) - K_\varepsilon * \tilde{\rho}_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\Big)\Big], \tag{21}$$

$k, j = 1\ldots n$. Observe that for each $x \in \mathbb{R}^2$ the random variable $h(\tau^j, X_0^j)K_\varepsilon(x - \bar{X}_s^{j,\varepsilon}) - K_\varepsilon * \tilde{\rho}_s^\varepsilon(x)$ is centered, from definition of $\tilde{\rho}_s^\varepsilon$. By independence of $(\tau^m, \bar{X}_s^{m,\varepsilon})_{1\leq m\leq n}$, we deduce that if $j \neq k$,

$$E\left[h(\tau^j, X_0^j)K_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - K_\varepsilon * \tilde{\rho}_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\middle|(\tau^i, \bar{X}_s^{i,\varepsilon}), (\tau^k, \bar{X}_s^{k,\varepsilon})\right] = 0,$$

and consequently, the expression (21) vanishes. Otherwise, it is bounded by $\frac{4}{n}M_\varepsilon^2\|h\|_\infty^2$. Thus, we obtain that

$$E\left[\sup_{s\le t}|X_s^{in,\varepsilon}-\bar{X}_s^{i,\varepsilon}|\right]\le 2\|h\|_\infty L_\varepsilon\int_0^t E\left[\sup_{s\le t}|X_s^{in,\varepsilon}-\bar{X}_s^{i,\varepsilon}|\right]ds+\frac{2tM_\varepsilon\|h\|_\infty}{\sqrt{n}}.$$

Remembering that $\|h\|_\infty=\|w_0\|_1+\|g\|_{1,T}$, we conclude with Gronwall's lemma that

$$E(\sup_{t\le T}|X_t^{in,\varepsilon}-\bar{X}_t^{i,\varepsilon}|)\le\frac{CM_\varepsilon}{L_\varepsilon\sqrt{n}}\exp(2(\|w_0\|_1+\|g\|_{1,T})L_\varepsilon T).$$

□

Remark 3.7 *Since the function h is bounded, we can easily deduce from the previous theorem that for all continuous bounded $f:\mathbb{R}^2\to\mathbb{R}$ and $\varepsilon>0$,*

$$E\left|\langle\tilde{\mu}_t^{n,\varepsilon},f\rangle-\int_{\mathbb{R}^2}f(x)\tilde{\rho}_t^\varepsilon(x)dx\right|\to 0$$

when $n\to\infty$.

3.3 Density estimates

Constructing the nonlinear process gave us existence for each $\varepsilon>0$ of a weak solution of (12). This probabilistic approach has naturally provided uniform (in ε) L^1 estimates for the solution. In order to make $\varepsilon\to 0$, and because of the bad behavior of K in the space L^1, it will be necessary to additionally obtain uniform L^p estimates for some p strictly greater than 1. The stochastic flow will be the fundamental tool for this purpose.
The diffusion coefficient in (14) being constant, the following "stochastic version" of Liouville's theorem can be proven in a similar way as the standard one (e.g. [8] Ch. 1).

Lemma 3.8 *Let $J\xi_{s,t}^\varepsilon=|det(\nabla_x\xi_{s,t}^\varepsilon)|$ be the Jacobian of the function $\xi_{s,t}^\varepsilon:\mathbb{R}^2\to\mathbb{R}^2$. Then*

$$J\xi_{s,t}^\varepsilon(x)=1+\int_s^t div\left[K_\varepsilon*\tilde{P}_r^\varepsilon(\xi_{s,r}^\varepsilon(x))\right]J\xi_{s,r}^\varepsilon dr.$$

Since $div\ K_\varepsilon*\tilde{P}_r^\varepsilon(\xi_{s,r}^\varepsilon(x))=0$ we conclude that for all (s,t,x),

$$J\xi_{s,t}^\varepsilon(x)=1.$$

Lemma 3.9 *Let ρ^ε and $\tilde{\rho}^\varepsilon$ be respectively the family of densities of X under P^ε and the family of weighted densities associated with P^ε through (10). Let $p\in[1,\infty]$ and assume that $w_0\in L^p$ and $g\in L^1([0,T],L^p)$. Then, we have*

$$i)\quad\|\tilde{\rho}_t^\varepsilon\|_p\le\|w_0\|_p+\int_0^t\|g_s\|_p\,ds$$

$$ii)\quad\|\rho_t^\varepsilon\|_p\le\frac{\|w_0\|_p+\int_0^T\|g_s\|_p\,ds}{\|w_0\|_1+\int_0^T\|g_s\|_1\,ds}$$

for all $\varepsilon>0$ and $t\in[0,T]$.

Proof: Consider a fixed function $\psi \in \mathcal{D}$ and $t > 0$. By the Feynman-Kac formula, the function $\phi^t(s,x) := E(\psi(\xi^\varepsilon_{s,t}(x)))$ is the unique solution of the Cauchy problem

$$\begin{aligned} \tfrac{\partial \phi(s,x)}{\partial s} + \nu \triangle \phi(s,x) + K_\varepsilon * \tilde{\rho}^\varepsilon_s(x) \nabla \phi(s,x) &= 0 \quad \text{for all } (s,x) \in [0,t[\times \mathbb{R}^2 \\ \phi(t,x) &= \psi. \end{aligned}$$

Replacing ϕ^t in the weak equation (16) and using Fubini's theorem, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} \psi(x) \tilde{\rho}^\varepsilon_t(x) dx = & \int_{\mathbb{R}^2} \phi^t(0,x) w_0(x) dx + \int_0^t \int_{\mathbb{R}^2} \phi^t(s,x) g_s(x) dx\ ds \\ = & E\left(\int_{\mathbb{R}^2} \left[\psi(\xi^\varepsilon_{0,t}(x)) w_0(x) \right] dx \right) + \int_0^t E\left(\int_{\mathbb{R}^2} \left[\psi(\xi^\varepsilon_{s,t}(x)) g_s(x) \right] dx \right)\ ds \end{aligned}$$

and so

$$\left| \int_{\mathbb{R}^2} \psi(x) \tilde{\rho}^\varepsilon_t(x) dx \right| \leq E\left[\|\psi(\xi^\varepsilon_{0,t}(\cdot\ ))\|_{p^*} \right] \|w_0\|_p + \int_0^t E\left[\|\psi(\xi^\varepsilon_{s,t}(\cdot\ ))\|_{p^*} \right] \|g_s\|_p ds.$$

Thanks to Lemma 3.8, we conclude that

$$\left| \int_{\mathbb{R}^2} \psi(x) \tilde{\rho}^\varepsilon_t(x) dx \right| \leq \|\psi\|_{p^*} \left(\|w_0\|_p + \int_0^t \|g_s\|_p ds \right).$$

which proves *i)*. To prove *ii)*, define a sub-probability density $\hat{\rho}^\varepsilon_t$ by $\int_{\mathbb{R}^2} \psi(x) \hat{\rho}^\varepsilon_t(x) dx = E(\psi(X^\varepsilon_t) \mathbf{1}_{\{t \geq \tau\}})$. Writing Itô's formula for $f(t, X^\varepsilon_t)$, multiplying by $\mathbf{1}_{\{t \geq \tau\}}$ and taking expectations, we check that

$$\begin{aligned} \int_{\mathbb{R}^2} \phi_t(x) \hat{\rho}^\varepsilon_t(x) dx = & \int_{\mathbb{R}^2} \phi(0,x) \bar{w}_0(x) dx + \int_0^t \int_{\mathbb{R}^2} \phi(s,x) \bar{g}_s(x) dx\ ds \\ & + \int_0^t \int_{\mathbb{R}^2} \left[\frac{\partial \phi(s,x)}{\partial s} + \nu \triangle \phi(s,x) + K_\varepsilon * \tilde{\rho}^\varepsilon_s(x) \nabla \phi(s,x) \right] \hat{\rho}^\varepsilon_s(x) dx ds. \end{aligned}$$

We deduce as previously that

$$\left| \int_{\mathbb{R}^2} \psi(x) \hat{\rho}^\varepsilon_t(x) dx \right| \leq \frac{\|\psi\|_{p^*}}{\|w_0\|_1 + \|g\|_{1,T}} \left(\|w_0\|_p + \int_0^t \|g_s\|_p ds \right).$$

The desired estimate for ρ^ε_t follows from here, since $E(\psi(X^\varepsilon_t) \mathbf{1}_{\{t < \tau\}}) = E(\psi(X_0) \mathbf{1}_{\{t < \tau\}}) = \int_t^T \int_{\mathbb{R}^2} \psi(x) \frac{|g_s(x)|}{\|w_0\|_1 + \|g\|_{1,T}} dx\ ds$.

□

In [6], Busnello also relied on the stochastic flow to obtain uniform in time estimates for some solutions to the vortex equation. However, her argument needs regularity of the initial condition and does not consider external force fields.

4 $L^1 \cap L^p$ data: existence, uniqueness and pathwise approximation

Our goal now is to make ε go to 0. The singular kernel K has a bad behavior in the space L^1. However, it satisfies the following fundamental continuity properties.

Lemma 4.1 *Let $p \in (1,2)$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. For each $f \in L^p$ and $x \in \mathbb{R}^2$ the integral $K * f(x)$ is absolutely convergent. Furthermore, there is a constant $C_{p,q} > 0$ such that*

i)

$$\|K * f\|_q \leq C_{p,q} \|f\|_p \qquad \textit{for all } \in f \in L^p. \tag{22}$$

ii)

$$\|K * f\|_{W^{1,q}} \leq C_{p,q} \|f\|_{W^{1,p}} \qquad \textit{for all } \in f \in W^{1,p} \tag{23}$$

Proof: The absolute convergence of $K * f(x)$, and statement *i)* follow from the analogous results for the Riesz transform

$$f \in L^p \mapsto \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|} dy \in L^q(dx), \tag{24}$$

(cf. Theorem 1, Ch. 5 in Stein [19]). To prove *ii)*, by using the latter and a density argument, it is enough to check that the operator $K*$ commutes with derivatives when acting on $\mathcal{D}$. But this follows by taking derivatives under the integral sign by dominated convergence.

□

Remark 4.2 *Lemma 4.1 with the same constants applies to each mollified kernel K_ε.*

We now introduce the adequate spaces to work in. For measurable $w : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ and real numbers $p \in [1, \infty]$ and $r \geq p$ we introduce the norms

- $\|w\|_{0,p,T} = \sup_{0 \leq t \leq T} \|w_t\|_p$
- $\|w\|_{0,r,(T;p)} = \sup_{0 \leq t \leq T} \left\{ t^{\frac{1}{p} - \frac{1}{r}} \|w_t\|_r \right\}$.

and we denote the associated Banach spaces respectively by

$$F_{0,p,T} \quad \text{and} \quad F_{0,r,(T;p)}.$$

For analytical purposes, we will treat in a unified way the mollified and non-mollified equations. We write $K_0 = K$, and for each $\varepsilon \geq 0$, we define the bilinear operator $\mathcal{B}^\varepsilon$ on measurable functions $v, w : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, by

$$\mathcal{B}^\varepsilon(v,w)(t,x) = \int_0^t \int_{\mathbb{R}^2} \nabla G^\nu_{t-s}(x-y) \cdot K_\varepsilon * v_s(y) w_s(y) dy\, ds \tag{25}$$

Accordingly, we also write $\mathcal{B} = \mathcal{B}^0$.
Finally, we denote by W_0 the function

$$W_0(t,x) = G^\nu_t * w_0(x) + \int_0^t G^\nu_{t-s} * g_s(x) ds.$$

Lemma 4.3 *i) Let $p \in [1, \infty]$ and assume $w_0 \in L^p$ and $g \in F_{0,p,T}$. Then, we have*

$$W_0 \in F_{0,r,(T;p)} \quad \textit{for all } r \geq p.$$

ii) *For each* $r \geq \frac{4}{3}$, $v, w \in F_{0,r,T}$, *and each* $t \leq T$, *we have*

$$\sup_{\varepsilon \geq 0} \|K_\varepsilon * v(t)w(t)\|_{\frac{2r}{4-r}} \leq C\|v(t)\|_r \|w(t)\|_r. \tag{26}$$

iii) *If* $\frac{4}{3} \leq p < 2$, $p \leq r < 2$ *and* $\frac{2r}{4-r} \leq r' < \frac{r}{2-r}$, *then* $\mathcal{B}^\varepsilon : (F_{0,r,(T;p)})^2 \to F_{0,r',(T;p)}$ *is well defined for each* $\varepsilon \geq 0$, *and*

$$\sup_{\varepsilon \geq 0} \|\!|\mathcal{B}^\varepsilon(v, w)\|\!|_{0,r',(T;p)} \leq C(T) \|\!|v\|\!|_{0,r,(T;p)} \|\!|w\|\!|_{0,r,(T;p)}$$

for all $v, w \in F_{0,r,(T;p)}$.

iv) *If* $\frac{4}{3} \leq p < 2$, $w_0 \in L^p$ *and* $g \in F_{0,p,T}$, *we have for all* $r \in [p, \frac{2p}{2-p})$

$$\sup_{\varepsilon \geq 0} \|\!|\tilde{\rho}^\varepsilon\|\!|_{0,r,(T;p)} < \infty.$$

Proof : Part *i)* follows from Lemma 2.2, and the estimate

$$\left\| \int_0^t G^\nu_{t-s} * g_s \, ds \right\|_r \leq C(p,r) t^{1 + \frac{1}{r} - \frac{1}{p}} \left(\sup_{t \in [0,T]} \|g_t\|_p \right) \tag{27}$$

for some constant $C(p,r) > 0$ since $1/p < 1/r + 1$.

ii) Notice that $1 \leq \frac{2r}{4-r}$. Equation (26) is immediately obtained from Lemma 4.1, Remark 4.2 and Hölder's inequality.

iii) By (26), noticing that $1 \leq \frac{2r}{4-r} \leq r'$ and by Lemma 2.2 and Lemma 4.1, we have

$$\begin{aligned} \|\mathcal{B}^\varepsilon(v,w)_t\|_{r'} \leq & C \int_0^t (t-s)^{\frac{1}{r'} - \frac{2}{r}} \|v_s\|_r \|w_s\|_r \, ds \\ \leq & C \|\!|v\|\!|_{0,r,(T;p)} \|\!|w\|\!|_{0,r,(T;p)} \int_0^t (t-s)^{\frac{1}{r'} - \frac{2}{r}} s^{\frac{2}{r} - \frac{2}{p}} ds \\ = & C t^{1 + \frac{1}{r'} - \frac{2}{p}} \|\!|v\|\!|_{0,r,(T;p)} \|\!|w\|\!|_{0,r,(T;p)} \end{aligned} \tag{28}$$

with constants that do not depend on $\varepsilon \geq 0$. In the last step we have used the fact that $\frac{1}{r} - \frac{1}{p} > -\frac{1}{2}$ because $r < 2 < \frac{2p}{2-p}$. The statement follows.

iv) By Lemma 3.9, we have

$$\sup_{\varepsilon \geq 0} \|\!|\tilde{\rho}^\varepsilon\|\!|_{0,p,T} < \infty$$

Observe that $\frac{p}{2-p} \geq 2$. We define $p_1 := \frac{4p}{2+p} \in (p, 2)$ and apply *iii)* to $r = p$ and $r' = p_1$ which yields $\sup_{\varepsilon \geq 0} \|\!|\tilde{\rho}^\varepsilon\|\!|_{0,p_1,(T;p)} < \infty$, considering Equation (17) and *i)*. We now apply *iii)* to $r = p_1$ and some $r' \in [\frac{2p_1}{4-p_1}, \frac{p_1}{2-p_1}) = [p, \frac{2p}{2-p})$ and conclude in a similar way. □

Throughout the sequel, we make following type of assumption on the initial condition and the external field:

$(\mathbf{H_p})$:

- $w_0 \in L^p(\mathbb{R}^2)$ and

- $g \in F_{0,p,T}$.

Remark 4.4 *In view of the continuity property of the Biot-Savart operator, and of part iii) of the previous lemma, we will always consider*

$$p = 1 \text{ or } p \in [\frac{4}{3}, 2).$$

4.1 Convergence of the mollified solutions for $L^1 \cap L^p$ data

For technical reasons, we will make a particular choice of approximating kernels $K_\varepsilon(x) = K * \varphi_\varepsilon(x)$. We will assume that $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\varphi(\frac{\cdot}{\varepsilon})$, with φ a cutoff function with radial symmetry. The following function has been given by Raviart [18] in a general context of approximations, and proposed by Bossy [1] for a numerical study of the vortex algorithm:

$$\varphi(x) = \frac{2(2-r^2)}{\pi(1+r^2)^4}, \quad r = |x|.$$

It is proven in [17] that in this case, we have

$$K_\varepsilon(x) = \frac{4\varepsilon^4 + (r^2 + 3\varepsilon^2)r^2}{2\pi(r^2+\varepsilon^2)^3}(-x_2, x_1). \tag{29}$$

Lemma 4.5 *For each $l \in [1, 2)$ we have*

$$\|K_\varepsilon - K\|_l \leq C\varepsilon^{\frac{2-l}{l}}$$

where the constant C depends only on l.

Proof. We have

$$K_\varepsilon(x) - K(x) = \frac{\varepsilon^4}{2\pi}\frac{r^2-\varepsilon^2}{(r^2+\varepsilon^2)^3}\frac{1}{r^2}(-x_2, x_1).$$

Then, for $l \geq 1$,

$$\begin{aligned}
\|K_\varepsilon - K\|_l^l &\leq \frac{\varepsilon^{4l}}{(2\pi)^{l-1}}\int_0^{+\infty}\frac{(r^2-\varepsilon^2)^l}{(r^2+\varepsilon^2)^{3l}r^{l-1}}dr \\
&\leq \varepsilon^{2-l}\frac{1}{(2\pi)^{l-1}}\int_0^{+\infty}\frac{(\alpha^2-1)^l}{(\alpha^2+1)^{3l}\alpha^{l-1}}d\alpha \\
&\leq C\varepsilon^{2-l}, \text{ for } l < 2.
\end{aligned}$$

□

Proposition 4.6 *Assume that $(\mathbf{H_p})$ holds, with $p \in [\frac{4}{3}, 2)$. Then, for any $l \in (1, 2)$ and $\varepsilon > \varepsilon' > 0$ we have*

$$\sup_{t\leq T}\|\tilde{\rho}_t^\varepsilon - \tilde{\rho}_t^{\varepsilon'}\|_p \leq C_l\varepsilon^{\frac{2-l}{l}}$$

for some positive constant C_l. We deduce that the sequence $\tilde{\rho}^\varepsilon$ converges, when ε tends to 0, to some function $w \in F_{0,p,T}$ solving the mild vortex equation (7), and

$$\sup_{t\leq T}\|\tilde{\rho}_t^\varepsilon - w_t\|_p \leq C_l\varepsilon^{\frac{2-l}{l}}. \tag{30}$$

Proof: By Lemma 4.5, we get for $1 < l < 2$ that

$$\|K_\varepsilon - K_{\varepsilon'}\|_l \leq C\varepsilon^{\frac{2-l}{l}}.$$

In view of (17), Lemma 2.2 and Lemma 4.3, we have, for $t \leq T$,

$$\begin{aligned}
\|\tilde{\rho}_t^\varepsilon - \tilde{\rho}_t^{\varepsilon'}\|_p &\leq \int_0^t \|\nabla G_{t-s}^\nu * [(K_\varepsilon * \tilde{\rho}_s^\varepsilon)\tilde{\rho}_s^\varepsilon - (K_{\varepsilon'} * \tilde{\rho}_s^{\varepsilon'})\tilde{\rho}_s^{\varepsilon'}\|_p \, ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{p}} \|(K_\varepsilon * \tilde{\rho}_s^\varepsilon)\tilde{\rho}_s^\varepsilon - (K_{\varepsilon'} * \tilde{\rho}_s^{\varepsilon'})\tilde{\rho}_s^{\varepsilon'}\|_{\frac{2p}{4-p}} \, ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{p}} (\|(K_\varepsilon * \tilde{\rho}_s^\varepsilon)\tilde{\rho}_s^\varepsilon - (K_{\varepsilon'} * \tilde{\rho}_s^\varepsilon)\tilde{\rho}_s^\varepsilon\|_{\frac{2p}{4-p}} \\
&\qquad + \|(K_{\varepsilon'} * \tilde{\rho}_s^\varepsilon)\tilde{\rho}_s^\varepsilon - (K_{\varepsilon'} * \tilde{\rho}_s^{\varepsilon'})\tilde{\rho}_s^\varepsilon\|_{\frac{2p}{4-p}} + \|(K_{\varepsilon'} * \tilde{\rho}_s^{\varepsilon'})(\tilde{\rho}_s^\varepsilon - \tilde{\rho}_s^{\varepsilon'})\|_{\frac{2p}{4-p}}) \, ds \\
&\leq C \|\tilde{\rho}^\varepsilon\|_{0,p,T} \|K_{\varepsilon'} - K_\varepsilon\|_l \|\tilde{\rho}^\varepsilon\|_{0,r,(T;p)} \int_0^t (t-s)^{-\frac{1}{p}} s^{\frac{1}{r} - \frac{1}{p}} ds \\
&\qquad + C\Big(\|\tilde{\rho}^\varepsilon\|_{0,p,T} + \|\tilde{\rho}^{\varepsilon'}\|_{0,p,T}\Big) \int_0^t (t-s)^{-\frac{1}{p}} \|\tilde{\rho}_s^\varepsilon - \tilde{\rho}_s^{\varepsilon'}\|_p \, ds, \qquad (31)
\end{aligned}$$

with $r \in (p, \frac{2p}{2-p})$ given by the relation $\frac{1}{p} - \frac{1}{2} = \frac{1}{r} + \frac{1}{l} - 1$. The last inequality follows from Young's inequality for the first term and from (26) for the last two terms. Notice that $\frac{1}{r} - \frac{1}{p} + 1 > 0$, and so Lemmas 4.3 *iv)* and Lemma 4.5 finally imply that

$$\|\tilde{\rho}_t^\varepsilon - \tilde{\rho}_t^{\varepsilon'}\|_p \leq C \sup_\varepsilon \|\tilde{\rho}^\varepsilon\|_{0,p,T} \left(\varepsilon^{\frac{2-l}{l}} \sup_\varepsilon \|\tilde{\rho}^\varepsilon\|_{0,r,(T;p)} + C \int_0^t (t-s)^{-\frac{1}{p}} \|\tilde{\rho}_s^\varepsilon - \tilde{\rho}_s^{\varepsilon'}\|_p \, ds \right) \quad (32)$$

which together with Lemma 1.2 implies that

$$\sup_{t \leq T} \|\tilde{\rho}_t^\varepsilon - \tilde{\rho}_t^{\varepsilon'}\|_p \leq C\varepsilon^{\frac{2-l}{l}} .$$

The sequence is hence Cauchy in $F_{0,p,T}$. By similar arguments it is immediate to see that its limit $w \in F_{0,p,T}$ is a solution of (7) .

□

We finally obtain the following existence and uniqueness result:

Theorem 4.7 *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_p})$ *with* $p \in [\frac{4}{3}, 2)$*. Then, the mild vortex equation with external force field (7) has a unique solution* w *in the space* $F_{0,p,T} \cap F_{0,1,T}$*. The solution satisfies*

$$\|w(t)\|_p \leq \|w_0\|_p + \int_0^t \|g_s\|_p ds. \qquad (33)$$

Proof: Existence of a solution $w \in F_{0,p,T}$ has been proved in Proposition 4.6. The upper-bound (33) follows from the convergence statement therein and from Lemma 3.9.
Let us check that $w \in F_{0,1,T}$. Consider the sequence defined by $\gamma_0 = 1$, $\gamma_{n+1} = \frac{4\gamma_n}{2+\gamma_n}$, which is strictly increasing, converges to 2 and satisfies $\gamma_n = \frac{2\gamma_{n+1}}{4-\gamma_{n+1}}$. Take $N \in \mathbb{N}$ such that $\gamma_N \leq p < \gamma_{N+1}$. We have $\gamma_N \in [\frac{2p}{4-p}, p]$, and so from Lemma 4.3 *iii)* and the fact that $W_0 \in F_{0,\gamma_N,T}$ we deduce that $w \in F_{0,\gamma_N,T}$. Since furthermore $W_0 \in F_{0,\gamma_{N-1},T}$, we obtain that $w \in F_{0,\gamma_{N-1},T}$. We iterate this argument $N-1$ times and conclude that $w \in F_{0,1,T}$.

Finally, if $w, w' \in F_{0,p,T}$ are two solutions of (17), as in the proof of Lemma 4.3, *ii)* (with $p = r = r'$) we get

$$\|w_t - w'_t\|_p \le C\left(\|\|w\|\|_{0,p,T} + \|\|w'\|\|_{0,p,T}\right)\int_0^t (t-s)^{-\frac{1}{p}}\|w_s - w'_s\|_p ds$$

We conclude uniqueness by Lemma 1.2.

∎

In the sequel, the family of equations

$$w_t^\varepsilon(x) = G_t^\nu * w_0(x) + \int_0^t G_{t-s}^\nu * g_s(x)\, ds + \int_0^t \nabla G_{t-s}^\nu * \left[(K_\varepsilon * w_s^\varepsilon)w_s^\varepsilon\right](x)\, ds, \qquad (34)$$

with $\varepsilon \ge 0$, will be refereed as the *mollified mild equations*. Notice that with this notation, Equation (34) is Equation (7) if $\varepsilon = 0$ and Equation (17) if $\varepsilon > 0$.

Remark 4.8 *With similar arguments as in Theorem 4.7, we can also prove uniqueness for each of the mollified mild equations (34) in the space $F_{0,p,T}$, $p \in [\frac{4}{3}, 2)$. We deduce that under the assumptions of Theorem 4.7, for each $\varepsilon \ge 0$ Equation (34) has a unique solution $w^\varepsilon \in F_{0,p,T} \cap F_{0,1,T}$, given by $w^\varepsilon = \tilde{\rho}^\varepsilon$ if $\varepsilon > 0$, or by $w^0 = w$.*

4.2 Regularity estimates

We have so far proved existence and uniqueness of a solution w of the vortex equation in $F_{0,p,T} \cap F_{0,1,T}$. But we still need minimal regularity properties of w and $K * w$ (such as boundedness of the latter) in order to construct a solution of the nonlinear martingale problem.

Therefore, we now prove some uniform (in ε) regularity properties for functions $\tilde{\rho}^\varepsilon$. These results will imply strong existence and uniqueness for the limiting process and moreover, under assumptions $(\mathbf{H_1})$ and $(\mathbf{H_p})$, pathwise convergence of the mollified processes when ε tends to 0.

For $T > 0$ and $r \ge p$ we introduce some additional norms

- $\|\|v\|\|_{1,r,(T;p)} = \sup_{0\le t\le T}\left\{t^{\frac{1}{p}-\frac{1}{r}}\|v_t\|_r + t^{\frac{1}{2}+(\frac{1}{p}-\frac{1}{r})}\|\nabla v_t\|_r\right\}$.
- $\|\|v\|\|_{1,p,T} = \|\|v\|\|_{1,p,(T;p)} = \sup_{0\le t\le T}\left\{\|v_t\|_p + t^{\frac{1}{2}}\|\nabla v_t\|_p\right\}$

and the associated Banach spaces

$$F_{1,r,(T;p)} \quad \text{and} \quad F_{1,p,T}.$$

Lemma 4.9 *i) Assume $(\mathbf{H_p})$ with $\frac{4}{3} \le p < 2$. Then we have*

$$W_0 \in F_{1,r,(T;p)} \quad \textit{for all } r \in [p, \frac{2p}{2-p}).$$

ii) If $\frac{4}{3} \le p < 2$, $p \le r < 2$ and $\frac{2r}{4-r} \le r' < \frac{r}{2-r}$, then $\mathcal{B}^\varepsilon : (F_{1,r,(T;p)})^2 \to F_{1,r',(T;p)}$ is well defined for each $\varepsilon \ge 0$ and

$$\sup_{\varepsilon\ge 0}\|\|\mathcal{B}^\varepsilon(v,u)\|\|_{1,r',(T;p)} \le C\|\|v\|\|_{1,r,(T;p)}\|\|u\|\|_{1,r,(T;p)}.$$

Proof: *i)* For $p \in [1,2)$ the function $t \mapsto t^{-\frac{1}{2}+\frac{1}{r}-\frac{1}{p}}$ is integrable in 0 if and only if $r < \frac{2p}{2-p}$. Taking the gradient of $G^{\nu}_{t-s} * g_s$ under the time integral and using Lemma 2.2 we obtain

$$\left\| \nabla \left(\int_0^t G^{\nu}_{t-s} * g_s \, ds \right) \right\|_r \leq C'(p,r) t^{\frac{1}{2}+\frac{1}{r}-\frac{1}{p}} \left(\sup_{t \in [0,T]} \|g_t\|_p \right) \tag{35}$$

for some constant $C'(p,r) > 0$, which implies that $W_0 \in F_{1,r,(T;p)}$.
ii) In view of Remark 4.2, it is enough to check the continuity properties for $\mathcal{B}$. If $v, u \in F_{1,r,(T;p)}$ the function $(K * v_t)u_t$ belongs to $W^{1,\frac{2r}{4-r}}$, and so by integration by parts,

$$\mathcal{B}(v,u)(t,x) = \int_0^t \int_{\mathbb{R}^2} G^{\nu}_{t-s}(x-y)(K * v_s)(y) \cdot \nabla u_s(y) dy \, ds$$

(recall that $div\ K * u = 0$). Next, for any $\psi \in \mathcal{D}$ it holds that

$$\begin{aligned} &\int_0^t \int_{(\mathbb{R}^2)^2} G^{\nu}_{t-s}(x-y)|\psi(x)||K * v_s(y)||\nabla u_s(y)| dx \, dy \, ds \\ &\qquad \leq C \|\psi\|_{(r')^*} \|\!|v|\!\|_{0,r,(T;p)} \|\!|u|\!\|_{1,r,(T;p)} \int_0^t (t-s)^{\frac{1}{r'}-\frac{2}{r}+\frac{1}{2}} s^{\frac{2}{r}-\frac{2}{p}-\frac{1}{2}} ds < \infty, \end{aligned}$$

where $(r')^*$ is the conjugate of r'. Thus, by using Fubini's Theorem and integration by parts,

$$\int_{\mathbb{R}^2} \mathcal{B}(v,u)_t(x) \frac{\partial \psi}{\partial x_i}(x) dx = -\int_0^t \int_{(\mathbb{R}^2)^2} \frac{\partial G^{\nu}_{t-s}}{\partial x_i}(x-y)\psi(x)(K * v_s)(y) \nabla u_s(y) dx \, dy \, ds,$$

$(i = 1,2)$ from where we deduce that

$$\begin{aligned} \|\nabla \mathcal{B}(v,u)_t\|_{r'} &\leq C \|\!|v|\!\|_{0,r,(T;p)} \|\!|u|\!\|_{1,r,(T;p)} \int_0^t (t-s)^{\frac{1}{r'}-\frac{2}{r}} s^{\frac{2}{r}-\frac{2}{p}-\frac{1}{2}} ds \\ &= C \|\!|v|\!\|_{0,r,(T;p)} \|\!|u|\!\|_{1,r,(T;p)} t^{\frac{1}{2}+\frac{1}{r'}-\frac{2}{p}} \end{aligned}$$

(we use here the fact that $\frac{2}{r} - \frac{2}{p} - \frac{1}{2} > -1$). From this estimate and (28) we conclude *ii)*.
∎

Remark 4.10 *By the previous lemma, and since*

$$\frac{2p}{4-p} \leq p < \frac{p}{2-p},$$

Equation (34) in the space $F_{1,p,T}$, with $\frac{4}{3} \leq p < 2$, is equivalent to the abstract equation

$$w^{\varepsilon} = W_0 + \mathcal{B}^{\varepsilon}(w^{\varepsilon}, w^{\varepsilon}). \tag{36}$$

To obtain the required additional regularity, we will prove a local existence result of regular mild solutions. We need the following lemma (see e.g. Cannone [7], Ch.1).

Lemma 4.11 *Let* $(F, \|\!| \cdot \|\!|)$ *be a Banach space,* $\Lambda \in \mathbb{R}$ *a positive constant and* $\mathbf{B} : F^2 \to F$ *a continuous bilinear operator such that*

$$\|\!|\mathbf{B}(\mathbf{x}_1, \mathbf{x}_2)\|\!| \le \Lambda \|\!|\mathbf{x}_1\|\!| \ \|\!|\mathbf{x}_2\|\!|$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in F$*. If* $\mathbf{y} \in F$ *is such that* $4\Lambda\|\!|\mathbf{y}\|\!| < 1$*, there exists a unique solution* $x \in F$ *of*

$$\mathbf{x} = \mathbf{y} + \mathbf{B}(\mathbf{x}, \mathbf{x})$$

in the centered ball of radius $R_\gamma := \frac{1-\sqrt{1-4\Lambda\|\!|\mathbf{y}\|\!|}}{2\Lambda} \le 2\|\!|\mathbf{y}\|\!|$.

Proposition 4.12 *Let* $p \in [\frac{4}{3}, 2)$*. There is a constant* $\lambda_p > 0$ *independent of* $\varepsilon \ge 0$ *such that for all* $\theta > 0$ *and* $w_0 \in L^p$*,* $g \in F_{0,p,\theta}$ *satisfying*

$$\theta^{1-\frac{1}{p}} \left(\|w_0\|_p + \theta \|\!|g\|\!|_{0,p,\theta} \right) < \lambda_p,$$

Equations (34) with $\varepsilon \ge 0$*, have a unique solution in* $F_{1,p,\theta}$ *such that* $\|\!|w^\varepsilon\|\!|_{1,p,\theta} \le 2\|\!|W_0\|\!|_{1,p,\theta}$.

Proof: From Lemma 4.9 *ii)* (with $r = r' = p' = p$), the operators $\mathcal{B}^\varepsilon : (F_{1,p,\theta})^2 \to F_{1,p,\theta}$ are continuous with norm bounded by $\theta^{1-\frac{1}{p}}$ times a constant $C(p) > 0$ not depending on θ or $\varepsilon \ge 0$. Furthermore, from the proofs of Lemma 2.2 and Lemmas 4.3 *i)* and 4.9 *i)* there is $\tilde{C}(p) > 0$ such that

$$\|\!|W_0\|\!|_{1,p,\theta} \le \tilde{C}(p) \left(\|w_0\|_p + \theta \|\!|g\|\!|_{0,p,\theta} \right).$$

Hence, by the previous lemma, a solution $w \in F_{1,p,\theta}$ to the abstract equation (36) exists as soon as

$$4C(p)\theta^{1-\frac{1}{p}} \left(\|w_0\|_p + \theta \|\!|g\|\!|_{0,p,\theta} \right) \tilde{C}(p) < 1.$$

□

Theorem 4.13 *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_p})$ *with* $p \in [\frac{4}{3}, 2)$ *and let* $w^\varepsilon \in F_{0,p,T} \cap F_{0,1,T}$*,* $\varepsilon \ge 0$*, be the solution of (34).*

i) We have

$$\sup_{\varepsilon \ge 0} \|\!|w^\varepsilon\|\!|_{1,p,T} < \infty.$$

ii) For each $r \in [p, \frac{2p}{2-p})$

$$\sup_{\varepsilon \ge 0} \|\!|w^\varepsilon\|\!|_{1,r,(T;p)} < \infty.$$

Proof: *i)* We follow a similar argument as in Lemma **4.4** in [9] to deal with the bad behavior of w^ε at time 0. Firstly, we prove that for each $r \in [0, T]$, the shifted function $(s, x) \to w^\varepsilon_{r+s}(x)$ coincides on some small time interval $[0, \theta_0]$ with a function in F_{1,p,θ_0}. By the semigroup property of G^ν_t and the estimates of Lemma 2.2, it is checked that

$$w^\varepsilon_{r+t}(x) = G^\nu_t * w^\varepsilon_r(x) + \int_0^t G^\nu_{t-s} * g_s + r(x)\, ds + \int_0^t \nabla G^\nu_{t-s} * \left[(K_\varepsilon * w^\varepsilon_{r+s}) w^\varepsilon_{r+s} \right](x)\, ds, \quad (37)$$

for all $t \in (0, T - r]$. We write $W^\varepsilon_{(r)}(t, x) := G^\nu_t * w^\varepsilon_r(x) + \int_0^t G^\nu_{t-s} * g_{s+r}(x)\, ds$. Then, as in the proof of Proposition 4.12 and with the same constant $\tilde{C}(p)$, we have

$$\begin{aligned} \|\!|W^\varepsilon_{(r)}\|\!|_{1,p,\theta} &\le \tilde{C}(p) \left(\|w^\varepsilon_r\|_p + \theta \|\!|g_{\cdot+r}\|\!|_{0,p,\theta} \right) \\ &\le \tilde{C}(p) \left(\|w_0\|_p + 2T \|\!|g\|\!|_{0,p,T} \right) \end{aligned} \quad (38)$$

for all $\theta \in [0, T-r]$, the last inequality due to Lemma 3.9.
Let $\lambda_p > 0$ be the constant given in Proposition 4.12, and take $\theta_0 \in [0, T]$ such that

$$\theta_0^{1-\frac{1}{p}} (\|w_0\|_p + 2T \|\!| g \|\!|_{0,p,T}) < \lambda_p.$$

Then, for each $r \in [0, T]$ such that $\theta_0 \le T - r$, we have

$$\theta_0^{1-\frac{1}{p}} (\|w_r^\varepsilon\|_p + \theta_0 \|\!| g_{\cdot+r} \|\!|_{0,p,\theta_0}) \le \theta_0^{1-\frac{1}{p}} (\|w_0\|_p + 2T \|\!| g \|\!|_{0,p,T}) < \lambda_p,$$

and consequently, by Proposition 4.12, each equation (37) has a solution, say $v_{(r)}^\varepsilon$, in the space F_{1,p,θ_0}. Since uniqueness holds for (37) in the space $F_{0,p,\theta_0 \wedge (T-r)}$ for each r, we have $w_{r+\cdot}^\varepsilon(\cdot) = v_{(r)}^\varepsilon \in F_{1,p,\theta_0\wedge(T-r)}$ and

$$\|\!| w_{r+\cdot}^\varepsilon(\cdot) \|\!|_{1,p,\theta_0} \le 2 \|\!| W_{(r)}^\varepsilon \|\!|_{1,p,\theta_0}, \tag{39}$$

also by Proposition 4.12. This clearly implies that $w_t^\varepsilon \in W^{1,p}$ for strictly positive t.
Notice now that, by choosing $r_k := k\frac{\theta_0}{2}$, $k \in \{0 \dots, [\frac{2T}{\theta_0}]\}$, we get $w_{r_k+t}^\varepsilon = w_{r_{k-1}+\frac{\theta_0}{2}+t}^\varepsilon$ for $t \in [0, \frac{\theta_0}{2}]$ and $k \in \{1 \dots, [\frac{2T}{\theta_0}]\}$. Consequently, for such t and k we have

$$\begin{aligned}
(r_k + t)^{\frac{1}{2}} \|\nabla w_{r_k+t}^\varepsilon\|_p &\le \theta_0^{-\frac{1}{2}} (r_k+t)^{\frac{1}{2}} (t + \frac{\theta_0}{2})^{\frac{1}{2}} \|\nabla w^\varepsilon_{(r_{k-1})+t+\frac{\theta_0}{2}}\|_p \\
&\le C \left(\frac{T}{\theta_0}\right)^{\frac{1}{2}} \|\!| W_{(r_{k-1})} \|\!|_{1,p,\theta_0} \quad \text{by (39)}, \\
&\le C \left(\frac{T}{\theta_0}\right)^{\frac{1}{2}} (\|w_0\|_p + 2T \|\!| g \|\!|_{0,p,T}),
\end{aligned}$$

the last inequality by (38). This and (39) with $r = 0$ yield the desired upper bound for w^ε. Finally, using Lemma 4.9, the proof of *ii)* is done in a similar way as in Lemma 4.3 *iv)*.

□

Corollary 4.14 *Denote by $\mathcal{C}^\alpha$ the space of Hölder continuous functions $\mathbb{R}^2 \to \mathbb{R}^2$ of index $\alpha \in (0,1)$. Under the assumptions of Theorem 4.13, we have*

i)

$$\sup_{\varepsilon \ge 0} \sup_{t \in [0,T]} \left\{ t^{\frac{1}{2}} \left(\|K_\varepsilon * w_t^\varepsilon\|_\infty + \|K_\varepsilon * w_t^\varepsilon\|_{\mathcal{C}^{2-\frac{2}{p}}} \right) \right\} < \infty \quad ;$$

ii) for all $r \in (2, \frac{2p}{2-p})$,

$$\sup_{\varepsilon \ge 0} \sup_{t \in [0,T]} \left\{ t^{\frac{1}{2} - \frac{1}{r} + \frac{1}{p}} \left(\|\nabla K_\varepsilon * w_t^\varepsilon\|_\infty + \|\nabla K_\varepsilon * w_t^\varepsilon\|_{\mathcal{C}^{1-\frac{2}{r}}} \right) \right\} < \infty.$$

Proof: Recall that for each $m \in (2, \infty]$, the Sobolev space $W^{1,m}(\mathbb{R}^2)$ is continuously embedded into $L^\infty(\mathbb{R}^2) \cap \mathcal{C}^{1-\frac{2}{m}}$ (see e.g. [4]). We obtain part *i)* using the equi-continuity of the family of operators $\{K_\varepsilon : W^{1,p} \to W_2^{1,\frac{2p}{2-p}}\}_{\varepsilon \ge 0}$ for $p \in (1,2)$, the fact that the w^ε's are uniformly bounded in $F_{1,p,T}$, and the refereed embedding result for $m = q$.

To prove *ii)* we use the following fact (see Bertozzi and Majda [3], p.76 for a proof): each distributional derivative of the velocity field $K * w$ is obtained by applying some singular integral operator on w .
As a consequence, for each $i, j \in \{1, 2\}$, the mapping $f \mapsto \frac{\partial}{\partial x_i} K_j * f$ defines a continuous operator $L^r \to L^r$ for all $r \in (1, \infty)$. There exists moreover a homogeneous function $m_{i,j} : \mathbb{R}^2 \to \mathbb{R}$ of degree 0 such that for $f \in L^2$, the following relation on Fourier transforms holds

$$\mathcal{F}\left(\frac{\partial}{\partial x_i} K_j * f\right)(\xi) = \mathcal{F}(m_{i,j})\mathcal{F}(f)(\xi),$$

with $\mathcal{F}(m_{i,j})$ a bounded function (see [19] Ch. 1 for all these facts). This implies that

$$\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_i} K_j * f\right) = \frac{\partial}{\partial x_i}\left(K_j * \frac{\partial f}{\partial x_k}\right), \quad k = 1, 2$$

for all $f \in W^{1,2}$, and then for all $f \in W^{1,r}$ and all $r \in (1, \infty)$ by density and continuity.
By the previous commutation relation and continuity of $\frac{\partial}{\partial x_i} K_j *$ in L^r, the operators $\frac{\partial}{\partial x_i} K_j * : W^{1,r} \to W^{1,r}$ are continuous. Moreover, it is not hard to check that the Fourier transform of $\frac{\partial}{\partial x_i}(K_\varepsilon)_j$ equals that of $\frac{\partial}{\partial x_i} K_j$ times some function in L^∞ with norm smaller than 1. It follows that the operators $\frac{\partial}{\partial x_i}(K_\varepsilon)_j * : W^{1,r} \to W^{1,r}$ are equi-continuous in $\varepsilon \geq 0$. We conclude *ii)* using the latter, the uniform estimate for w^ε in $F_{1,r,(T;p)}$ when $r \in (2, \frac{2p}{2-p})$ and the above mentioned embedding with $m = r$.

□

4.3 Pathwise convergence of the mollified processes

Definition 4.15 *We denote by $\mathcal{P}_{p,T}$ the space of probability measures on $C_T = [0, T] \times C([0, T], \mathbb{R}^2)$ such that for each $t \in [0, T]$, the signed measure $\tilde{P}_t$ has a density $\tilde{\rho}_t$ with respect to the Lebesgue measure and $\tilde{\rho} \in F_{0,p,T} \cap F_{0,1,T}$.*

Theorem 4.16 *Assume that $(\mathbf{H_1})$ and $(\mathbf{H_p})$ hold with $p \in [\frac{4}{3}, 2)$. Consider a $\mathbb{R}^2$-Brownian motion B and a random variable (τ, X_0) with values in $[0, T] \times \mathbb{R}^2$ and distributed according to P_0, independent of the Brownian motion.*

a) There exists in the class $\mathcal{P}_{p,T}$ a unique solution P to the nonlinear martingale problem $(\mathbf{MP})$. The corresponding function $\tilde{\rho}$ is equal to the unique solution w of the mild equation (7) in the space $F_{0,p,T} \cap F_{0,1,T}$.

b) There is a unique pathwise solution $((\tau, X), P)$ of the nonlinear stochastic differential equation $(\mathbf{E})$:

i) The law P of (τ, X) belongs to $\mathcal{P}_{p,T}$ and $\tilde{P}_t(dx) = \tilde{\rho}_t(x)dx$

$$ii)\ X_t = X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\geq\tau\}} dB_s + \int_0^t \mathbf{1}_{\{s\geq\tau\}} K * \tilde{\rho}_s(X_s) ds\ . \tag{40}$$

c) For each $\varepsilon > 0$, let X^ε be the mollified nonlinear processes constructed in the same probability space as B and (τ, X_0). Then, X^ε converges in $L^1_T := \{Y, E(\sup_{t\leq T} |Y_t|) < +\infty\}$ to X, with moreover

$$E\left(\sup_{t\leq T} |X^\varepsilon_t - X_t|\right) \leq C(p, r)\varepsilon^{2(\frac{1}{p}-\frac{1}{r})}$$

for each $r \in (p, \frac{2p}{2-p})$.

Proof: Part *a)* easily follows from *b)*. Indeed, the fact that a solution of (**E**) is a solution of the nonlinear martingale problem (**MP**) is easily verified by writing Itô's formula for $f(t, X_t)$ and using Remark 2.8. With Lemma 2.7 we deduce that $\tilde{\rho}$ is a solution of the weak equation. To check that it is also a mild solution, observe that $\tilde{\rho}$ belongs to $F_{0,\frac{4}{3},T}$ by interpolation. Therefore, $K * \tilde{\rho} \in F_{0,4,T}$ and

$$\int_0^T \int_{\mathbb{R}^2} |K * \tilde{\rho}_s(x)||\tilde{\rho}_s(x)| dx\ ds < \infty \tag{41}$$

by Hölder's inequality and Lemma 4.1. We conclude by Remark 2.4 that $\tilde{\rho}$ is a mild solution in $F_{0,p,T}$.

For the rest of the proof, we proceed in several steps.
Pathwise uniqueness for (E).
Consider, given B and (τ, X_0), two pathwise solutions (τ, Z^1) and (τ, Z^2) of (**E**). We respectively denote by P^1 and P^2 the laws of (τ, Z^1) and (τ, Z^2) which belong to $\mathcal{P}_{p,T}$. We deduce as previously that $\tilde{\rho}^1 = \tilde{\rho}^2 = w$. Hence (τ, Z^1) and (τ, Z^2) are both solutions of a stochastic differential equation ($\mathbf{E}^w$) defined like (**E**), but with the known drift coefficient $K * w_s$ instead of $K * \tilde{\rho}_s^1$ or $K * \tilde{\rho}_s^2$ in (40).
Then, using the Lipschitz property of $K * w$ obtained in Corollary 4.14, we get for all $r \in (2, \frac{2p}{2-p})$ and $t \leq T$ that

$$\begin{aligned} E(\sup_{u \leq t} |Z_u^1 - Z_u^2|) &= E\left(\sup_{u \leq t} \left| \int_0^u (K * w_s(Z_s^1) - K * w_s(Z_s^2)) ds \right| \right) \\ &\leq \int_0^t s^{\frac{1}{r} - \frac{1}{p} - \frac{1}{2}} E(\sup_{u \leq s} |Z_u^1 - Z_u^2|) ds. \end{aligned} \tag{42}$$

By Gronwall's lemma, we deduce that $E(\sup_{t \leq T} |Z_t^1 - Z_t^2|) = 0$ i.e. the processes Z^1 and Z^2 are indistinguishable.

Pathwise convergence
We will now prove that the sequence (X^ε) is Cauchy in L_T^1, and that it converges as ε tends to 0 to a process X, such that (τ, X) is solution of the nonlinear stochastic differential equation (**E**).
We denote as usual by q the number defined by $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. We choose $r \in (p, \frac{2p}{2-p})$ and $l \in (1, 2)$ such that $\frac{1}{p} - \frac{1}{2} = \frac{1}{q} = \frac{1}{r} + \frac{1}{l} - 1$. We firstly prove the following estimate: for each $\varepsilon > \varepsilon' \geq 0$,

$$E\left(\int_0^T \left| K_\varepsilon * \tilde{\rho}_s^\varepsilon(X_s^\varepsilon) - K_{\varepsilon'} * \tilde{\rho}_s^{\varepsilon'}(X_s^\varepsilon) \right| ds \right) \leq C\varepsilon^{\frac{2-l}{l}}. \tag{43}$$

Write $\frac{1}{q^*} = 1 - \frac{1}{q}$ and observe that $q^* \in [1, p]$ so that by Lemma 3.9,

$$\sup_{\varepsilon > 0} \|\rho^\varepsilon\|_{0,q^*,T} < \infty.$$

Therefore, the left hand side of (43) writes

$$\begin{aligned}\int_0^T E|K_\varepsilon * \tilde\rho^\varepsilon_s(X^\varepsilon_s) - K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_s(X^\varepsilon_s)|ds &= \int_0^T \int_{\mathbb{R}^2} |K_\varepsilon * \tilde\rho^\varepsilon_s(x) - K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_s(x)|\rho^\varepsilon_s(x)dxds \\ &\le \int_0^T \|K_\varepsilon * \tilde\rho^\varepsilon_s - K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_s\|_q \|\rho^\varepsilon_s\|_{q^*} ds \\ &\le C\int_0^T \|K_\varepsilon * \tilde\rho^\varepsilon_s - K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_s\|_q \; ds \\ &\le C\int_0^T \|K_\varepsilon * \tilde\rho^\varepsilon_t - K_{\varepsilon'} * \tilde\rho^{\varepsilon}_t\|_q + \|K_{\varepsilon'} * \tilde\rho^\varepsilon_t - K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_t\|_q ds\end{aligned}$$

By Young's inequality, Lemma 4.5, and Lemma 4.3 *iv)* the first term in the last integral is bounded by

$$\|K_\varepsilon - K_{\varepsilon'}\|_l \|\tilde\rho^\varepsilon_t\|_r \le C\varepsilon^{\frac{2-l}{l}} \|\tilde\rho^\varepsilon_t\|_r \le C\varepsilon^{\frac{2-l}{l}} t^{\frac{1}{r}-\frac{1}{p}}.$$

On the other hand, by Lemma 4.1 and Proposition 4.6 we get that the second term is bounded by $C\varepsilon^{\frac{2-l}{l}}$. From these estimates, (43) follows since $\frac{1}{r} - \frac{1}{p} + 1 > 0$.
Now, for $u \le T$, we have

$$\begin{aligned}&E\left(\sup_{v\le u}|X^\varepsilon_v - X^{\varepsilon'}_v|\right) \le E\left(\int_0^u |K_\varepsilon * \tilde\rho^\varepsilon_s(X^\varepsilon_s) - K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_s(X^{\varepsilon'}_s)|ds\right) \\ &\le \int_0^u \Big(E|K_\varepsilon * \tilde\rho^\varepsilon_s(X^\varepsilon_s) - K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_s(X^\varepsilon_s)| + E|K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_s(X^\varepsilon_s) - K_{\varepsilon'} * \tilde\rho^{\varepsilon'}_s(X^{\varepsilon'}_s)|\Big)ds \\ &\le C\varepsilon^{\frac{2-l}{l}} + \int_0^u s^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}} E(\sup_{v\le s}|X^\varepsilon_v - X^{\varepsilon'}_v|)ds \qquad (44)\end{aligned}$$

by (43) and Corollary 4.14. We conclude by Gronwall's Lemma (since $\frac{1}{r} - \frac{1}{p} - \frac{1}{2} > -1$) that

$$E\left(\sup_{t\le T}|X^\varepsilon_t - X^{\varepsilon'}_t|\right) \le C\varepsilon^{\frac{2-l}{l}}. \qquad (45)$$

The sequence (X^ε) is hence Cauchy in the space $L^1_T := \{Y : E(\sup_{t\in[0,T]}|Y_t|) < +\infty\}$. Thus, it converges in L^1_T at speed $C(p,r)\varepsilon^{2(\frac{1}{p}-\frac{1}{r})}$, to some process X^w. The final step is

Identification of the limit as a solution of (E)

Taking $\varepsilon' = 0$ in the previous estimates easily leads to the fact that (τ, X^w) is solution of the stochastic differential equation

$$X^w_t = X_0 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\ge\tau\}}dB_s + \int_0^t \mathbf{1}_{\{s\ge\tau\}} K * w_s(X^w_s)ds \qquad (46)$$

Denote by P^w the law of X^w. To finish the proof we just need to verify that X^w is a solution of the nonlinear stochastic differential equation (**E**). This amounts to check that each of the signed measures $\tilde P^w_t$ has a density which is equal to w. We have, for each $f \in \mathcal{D}$ and $t \in [0,T]$ that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} f(x)\tilde\rho^\varepsilon_t(x)dx - \tilde P^w_t(f)\right| &= \left|E(f(X^\varepsilon_t)h(\tau,X_0)\mathbf{1}_{\{\tau\le t\}}) - E(f(X^w_t)h(\tau,X_0)\mathbf{1}_{\{\tau\le t\}})\right| \\ &\le C\|\nabla f\|_\infty E|X^\varepsilon_t - X^w_t| \to 0 \text{ when } \varepsilon\to 0\end{aligned}$$

and since $\int_{\mathbb{R}^2} f(x)\tilde\rho^\varepsilon_t(x)dx \to \int_{\mathbb{R}^2} f(x)w_t(x)dx$ from Proposition 4.6, this concludes the proof.

□

4.4 The stochastic vortex method

From the results in the previous sections we readily deduce

Corollary 4.17 *Let $T > 0$ and assume that $(\mathbf{H_1})$ and $(\mathbf{H_p})$ hold with $p \in [\frac{4}{3}, 2)$. Consider a sequence $\varepsilon_n \to 0$ in such way that*

$$\frac{C_1\varepsilon_n}{\sqrt{n}} \exp(C_2(\|w_0\|_1 + \|g\|_{1,T})(\varepsilon_n^{-2})T) \to 0$$

when $n \to \infty$, where C_1 and C_2 are the positive constants given in Proposition 3.6. With the notation (3.4), we define for all $n \in \mathbb{N}$ and $i = 1 \dots n$ the system of particles

$$Z^{in} := X^{in,\varepsilon_n}$$

and consider on the same probability space the sequence of i.i.d processes $(\bar{X}^i)_{i\in\mathbb{N}}$, with $\bar{X}^i$ the unique strong solution of

$$\begin{aligned} &i) \text{ the law } P \text{ of } (\tau^i, \bar{X}^i) \text{ belongs to } \mathcal{P}_{p,T} \text{ and } \tilde{P}_t(dx) = \tilde{\rho}_t(x)dx \\ &ii) \ \bar{X}_t^i = X_0^i + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{s\geq\tau^i\}}dB_s^i + \int_0^t \mathbf{1}_{\{s\geq\tau^i\}}K * \tilde{\rho}_s(\bar{X}_s^i)ds\ . \end{aligned} \tag{47}$$

Then, for all $k \in \mathbb{N}$ and any $r \in (p, \frac{2p}{2-p})$, we have

$$E\left(\sup_{t\in[0,T]}\sum_{i=1}^k |Z_t^{in} - \bar{X}_t^i|\right) \leq kC\varepsilon_n^{2(\frac{1}{p}-\frac{1}{r})} + k\frac{C_1\varepsilon_n}{\sqrt{n}}\exp(C_2(\|w_0\|_1 + \|g\|_{1,T})(\varepsilon_n^{-2})T) \to 0$$

(the constant C depending on p, r and T).

Corollary 4.18 *Consider $\alpha \in]0, \frac{1}{2}[$ and the sequence (ε_n) given by*

$$\varepsilon_n := \left(\frac{C_2\|h\|_\infty T}{\alpha \ln n}\right)^{\frac{1}{2}},$$

with a constant $C_2 > 0$ as in Corollary 4.17. Consider moreover the weighted empirical process

$$\tilde{\mu}_s^{n,\varepsilon_n} = \frac{1}{n}\sum_{j=1}^n h(\tau^j, X_0^j)\mathbf{1}_{\{s\geq\tau^j\}}\delta_{Z_s^{jn}}$$

and the approximate velocity field

$$K_{\varepsilon_n} * \tilde{\mu}_s^{n,\varepsilon_n}(z) = \frac{1}{n}\sum_{j=1}^n h(\tau^j, X_0^j)\mathbf{1}_{\{s\geq\tau^j\}}K_{\varepsilon_n}(z - Z_s^{jn}).$$

Then, under the assumptions of Theorem 4.16, for all $l \in (1, 2)$ we have

$$\sup_{x\in\mathbb{R}^2} E\left(\sup_{t\in[0,T]} t^{\frac{1}{l}}|K_{\varepsilon_n} * \tilde{\mu}_t^{n,\varepsilon_n}(x) - u(t,x)|\right) \leq C(l,\alpha,T)\left(\frac{\ln n}{n^{\frac{1}{2}-\alpha}} + \frac{1}{(\alpha\ln n)^{\frac{2-l}{2l}}}\right)$$

Proof: Firstly we prove that for all $l \in (1,2)$, for some constant $C(T)$ depending on l and T it holds that for $p \in [4/3, 2)$,

$$\sup_{t \leq T} t^{\frac{1}{2}} \|w_t^\varepsilon - w_t\|_{W^{1,p}} \leq C(T) \varepsilon^{\frac{2-l}{l}}. \tag{48}$$

From Lemma 4.9 *ii)*, for $\varepsilon \geq 0$ the function ∇w^ε satisfies

$$\begin{aligned} \frac{\partial w_t^\varepsilon}{\partial x_i}(x) =& \frac{\partial}{\partial x_i} G_t^\nu * w_0(x) + \frac{\partial}{\partial x_i} \int_0^t G_{t-s}^\nu * g_s(x) ds \\ &+ \int_0^t \int_{\mathbb{R}^2} \frac{\partial G_{t-s}^\nu}{\partial x_i}(x-y)(K_\varepsilon * w_s^\varepsilon)(y) \cdot \nabla w_s^\varepsilon(y) dy\, ds, \end{aligned} \tag{49}$$

since $div K_\varepsilon = 0$. Proceeding as in Proposition 4.6, we deduce for $r \in (p, \frac{2p}{2-p})$ given by $\frac{1}{p} - \frac{1}{2} = \frac{1}{r} + \frac{1}{l} - 1$ that

$$\begin{aligned} \|\nabla w_t^\varepsilon - \nabla w_t\|_p \quad &\leq \quad C \int_0^t (t-s)^{-\frac{1}{p}} (\|(K_\varepsilon * w_s^\varepsilon)\nabla w_s^\varepsilon - (K * w_s^\varepsilon)\nabla w_s^\varepsilon\|_{\frac{2p}{4-p}} \\ &\quad + \|(K * w_s^\varepsilon)\nabla w_s^\varepsilon - (K * w_s)\nabla w_s^\varepsilon\|_{\frac{2p}{4-p}} + \|(K * w_s)(\nabla w_s^\varepsilon - \nabla w_s\|_{\frac{2p}{4-p}}) \, ds \\ &\leq \quad C \|w^\varepsilon\|_{0,p,T} \|K - K_\varepsilon\|_l \|w^\varepsilon\|_{1,r,(T;p)} \int_0^t (t-s)^{-\frac{1}{p}} s^{\frac{1}{r} - \frac{1}{p} - \frac{1}{2}} ds \\ &\quad + C \|w^\varepsilon\|_{1,p,T} \int_0^t (t-s)^{-\frac{1}{p}} s^{-\frac{1}{2}} \|w_s^\varepsilon - w_s\|_p \, ds \\ &\quad + C \|w\|_{0,p,T} \int_0^t (t-s)^{-\frac{1}{p}} \|\nabla w_s^\varepsilon - \nabla w_s\|_p \, ds. \end{aligned} \tag{50}$$

Notice that $\frac{1}{r} - \frac{1}{p} + \frac{1}{2} > 0$. With help of Lemma 1.1, Lemma 4.5 and Proposition 4.6 (since $\tilde{\rho}^\varepsilon = w^\varepsilon$), we deduce for all $\theta \leq T$ and $t \leq \theta$ that

$$t^{\frac{1}{2}} \|\nabla w_t^\varepsilon - \nabla w_t\|_p \leq C \varepsilon^{\frac{2-l}{l}} t^{-\frac{2}{p} + \frac{1}{r} + 1} + C \varepsilon^{\frac{2-l}{l}} t^{1 - \frac{1}{p}} + t^{1 - \frac{1}{p}} C \left\{ \sup_{s \leq \theta} s^{\frac{1}{2}} \|\nabla w_s^\varepsilon - \nabla w_s\|_p \right\},$$

where all powers of t are non negative. It follows that

$$\left\{ \sup_{s \leq \theta} s^{\frac{1}{2}} \|\nabla w_s^\varepsilon - \nabla w_s\|_p \right\} \leq C(T) \varepsilon^{\frac{2-l}{l}} + C \theta^{1 - \frac{1}{p}} \left\{ \sup_{s \leq \theta} s^{\frac{1}{2}} \|\nabla w_s^\varepsilon - \nabla w_s\|_p \right\},$$

from where $\sup_{s \leq \theta_0} s^{\frac{1}{2}} \|\nabla w_s^\varepsilon - \nabla w_s\|_p \leq C(T) \varepsilon^{\frac{2-l}{l}}$ for $\theta_0 > 0$ small enough.
By similar steps as before, starting from the equations satisfied by $w^\varepsilon_{\cdot+\theta_0}$ and $w_{\cdot+\theta_0}$, and noting that these functions and their gradients are bounded in $F_{0,p,T-\theta_0}$ and $F_{0,r,(T;p)}$ uniformly in $\varepsilon \geq 0$, we obtain now

$$\begin{aligned} \|\nabla w_{\theta_0+t}^\varepsilon - \nabla w_{\theta_0+t}\|_p \quad &\leq \quad C \varepsilon^{\frac{2-l}{l}} + C \varepsilon^{\frac{2-l}{l}} \int_0^t (t-s)^{-\frac{1}{p}} s^{\frac{1}{r} - \frac{1}{p}} ds \\ &\quad + C \int_0^t (t-s)^{-\frac{1}{p}} \|w_{\theta_0+s}^\varepsilon - w_{\theta_0+s}\|_p \, ds \\ &\quad + C \int_0^t (t-s)^{-\frac{1}{p}} \|\nabla w_{\theta_0+s}^\varepsilon - \nabla w_{\theta_0+s}\|_p \, ds. \end{aligned} \tag{51}$$

Using Proposition 4.6 and Lemma 1.2 we deduce that

$$\sup_{t\leq T-\theta_0}\|\nabla w^{\varepsilon}_{\theta_0+t}-\nabla w_{\theta_0+t}\|_p\leq C(T)\varepsilon^{\frac{2-l}{l}}$$

From the previous estimates and Proposition 4.6 we deduce that

$$|||w^{\varepsilon}-w|||_{1,p,T}\leq C\varepsilon^{\frac{2-l}{l}},$$

from where (48) follows. Next, we have

$$\begin{aligned}|K_{\varepsilon_n}*\tilde{\mu}^{n,\varepsilon_n}_t(x)-u(t,x)|\leq&\left|K_{\varepsilon_n}*\tilde{\mu}^{n,\varepsilon_n}_t(x)-\frac{1}{n}\sum_{i=1}^{n}K_{\varepsilon_n}(\bar{X}^{i,\varepsilon_n}_t-x)h(\tau^i,X^i_0)\mathbf{1}_{\{s\geq\tau^i\}}\right|\\&+\left|\frac{1}{n}\sum_{i=1}^{n}K_{\varepsilon_n}(\bar{X}^{i,\varepsilon_n}_t-x)h(\tau^i,X^i_0)\mathbf{1}_{\{s\geq\tau^j\}}-K_{\varepsilon_n}*w^{\varepsilon_n}_t(x)\right|\\&+|K_{\varepsilon_n}*w^{\varepsilon_n}_t(x)-K*w_t(x)|\end{aligned}\tag{52}$$

Let L_{ε_n} and M_{ε_n} respectively be a Lipschitz constant for K_{ε_n} and an upper bound for its sup-norm. Recall that there exists some constant C such that for n large enough, $M_{\varepsilon_n}\leq\frac{C}{\varepsilon_n^2}$ and $L_{\varepsilon_n}\leq\frac{C}{\varepsilon_n^3}$. By our choice of (ε_n), and thanks to Proposition 3.6, the expectation of the first term can be bounded by

$$L_{\varepsilon_n}C_1\frac{\varepsilon_n}{\sqrt{n}}\exp(C_2\|h\|_\infty(\varepsilon_n^{-2})T)\leq C\frac{n^\alpha}{\sqrt{n}\varepsilon_n^2}\leq C\frac{\alpha\ln n}{\|h\|_\infty Tn^{\frac{1}{2}-\alpha}},$$

where $\|h\|_\infty=\|w_0\|_1+\|g\|_{1,T}$. On the other hand, independence of the processes $(\tau^i,X^{i\varepsilon_n})$ implies that the expectation of the second term on the r.h.s is bounded above by

$$\frac{1}{\sqrt{n}}2M_{\varepsilon_n}\|h\|_\infty\leq C\frac{\alpha\ln n}{Tn^{\frac{1}{2}}}.$$

For the last term, notice that by similar arguments as in the proof Corollary 4.14,

$$\begin{aligned}|K_{\varepsilon_n}*w^{\varepsilon_n}_t(x)-K*w_t(x)|\leq&C\|K_{\varepsilon_n}*w^{\varepsilon_n}_t-K_{\varepsilon_n}*w_t\|_{W^{1,\frac{2p}{2-p}}}+C\|K_{\varepsilon_n}*w_t-K*w_t\|_{W^{1,\frac{2p}{2-p}}}\\\leq&C\|w^{\varepsilon_n}_t-w_t\|_{W^{1,p}}+\|K_{\varepsilon_n}-K\|_l\|w_t\|_{W^{1,r}}\\\leq&C\varepsilon_n^{\frac{2-l}{l}}t^{-\frac{1}{2}}+C\varepsilon_n^{\frac{2-l}{l}}t^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}}|||w|||_{1,r,(T;p)}.\end{aligned}$$

where l and r are chosen as before. Since $\frac{1}{r}-\frac{1}{p}-\frac{1}{2}=\frac{1}{l}$, we conclude that

$$\begin{aligned}\sup_{x\in\mathbb{R}^2}E\left(\sup_{t\in[0,T]}t^{\frac{1}{l}}|K_{\varepsilon_n}*\tilde{\mu}^{n,\varepsilon_n}_t(x)-u(t,x)|\right)\leq&C\left(\frac{\ln n}{n^{\frac{1}{2}-\alpha}}+\varepsilon_n^{\frac{2-l}{l}}\right)\\\leq&C(l,\alpha,T)\left(\frac{\ln n}{n^{\frac{1}{2}-\alpha}}+\frac{1}{(\alpha\ln n)^{\frac{2-l}{2l}}}\right)\end{aligned}$$

□

5 Extension to L^1 initial condition and force field

In this section, we shall extend the previous results to the case when w_0 and $g(t)$ belong only to L^1. The bad behavior in L^1-norm of the Biot-Savart operator and its derivative prevents us from working with solutions that are only L^1 functions. We introduce the adequate spaces for embedding the mild vortex equation with L^1 data.

5.1 Analytical framework and results

For $p \in [1, \infty]$ and a measurable function $v : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$, we define the norms

- $\|v\|^{\sharp}_{0,p,T} = \sup_{0 \le t \le T} \left\{ t^{1-\frac{1}{p}} \|v_t\|_p \right\}$
- $\|v\|^{\sharp}_{1,p,T} = \sup_{0 \le t \le T} \left\{ t^{1-\frac{1}{p}} \|v_t\|_p + t^{\frac{3}{2}-\frac{1}{p}} \|\nabla v_t\|_p \right\}$

and write respectively $F^{\sharp}_{0,p,T}$ and $F^{\sharp}_{1,p,T}$ for the associated Banach spaces.

Remark 5.1 *i) If* $(\mathbf{H_1})$ *holds, then by Lemma 2.2 we have* $G^{\nu}_t * w_0 \in F_{0,1,T} \cap F^{\sharp}_{1,p,T}$ *for all* p, *and* $(t, x) \mapsto \int_0^t G^{\nu}_{t-s} * g(s, x) ds$ *belongs to* $F_{0,1,T} \cap F^{\sharp}_{1,p,T}$ *for* $p < 2$.

ii) Proceeding as in Lemma 4.3, one can check for $\frac{4}{3} \le p < 2$ *and* $\frac{2p}{4-p} \le p' < \frac{p}{2-p}$ *that*

$$\sup_{\varepsilon \ge 0} \|\mathcal{B}^{\varepsilon}(v, u)\|^{\sharp}_{0,p',T} \le C \|v\|^{\sharp}_{0,p,T} \|u\|^{\sharp}_{0,p,T}$$

for all $v, u \in F^{\sharp}_{0,p,T}$ *and for some constant* $C > 0$. *Moreover, the norm of* $\mathcal{B}^{\varepsilon}$: $(F^{\sharp}_{0,p,T})^2 \to F^{\sharp}_{0,p',T}$ *does not depend on* T.

We shall prove below existence of a solution in $F_{0,1,T} \cap F^{\sharp}_{0,p,T}$ for $p \in [\frac{4}{3}, 2)$ and arbitrary $T > 0$, by an approximation argument by mean of $F_{0,1,T} \cap F_{0,p,T}$ solutions. We follow ideas of Ben-Artzi [2] who has studied the vortex equation with L^1 initial condition but without external field. The following two lemmas will be crucial.

Lemma 5.2 *Let* $\Gamma \subseteq L^1(\mathbb{R}^2)$ *be a pre-compact set and* $\Gamma_T \subseteq F_{0,1,T}$ *a bounded set. Then, for each* $p \in (1, 2)$ *there is an increasing function* $\theta \mapsto \delta(\theta, p, \Gamma, \Gamma_T)$, *going to* 0 *with* θ *such that*

$$\sup_{\psi \in \Gamma, \phi \in \Gamma_T} \|G^{\nu}_{\cdot} * \psi + \int_0^{\cdot} G^{\nu}_{\cdot - s} * \phi(s) ds\|^{\sharp}_{0,p,\theta} \le \delta(\theta, p, \Gamma, \Gamma_T).$$

Proof: Since Γ is pre-compact and $L^p \cap L^1$ is dense in L^1, for each $\epsilon > 0$ there is a finite set $\Gamma^{\epsilon} \subseteq L^p \cap L^1$ such that the L^1-balls of radius ϵ and centered in Γ^{ϵ} cover Γ. Hence, for each $\psi \in \Gamma$ there exists some $\psi^{\epsilon} \in \Gamma^{\epsilon}$ such that

$$\begin{aligned} \|G^{\nu}_t * \psi\|_p \le& \|G^{\nu}_t * (\psi - \psi^{\epsilon})\|_p + \|G^{\nu}_t * \psi^{\epsilon}\|_p \\ \le& C(p) t^{\frac{1}{p}-1} \|\psi - \psi^{\epsilon}\|_1 + \|\psi^{\epsilon}\|_p \end{aligned}$$

by Young's inequality and the L^p estimate for G^ν_t. Writing $M(\epsilon) = \sup_{\bar\psi \in \Gamma^\epsilon} \|\bar\psi\|_p$, we have

$$\sup_{\psi\in\Gamma} \|G^\nu_\cdot * \psi\|^\sharp_{0,p,\theta} \leq C\epsilon + \theta^{1-\frac{1}{p}} M(\epsilon),$$

and so the l.h.s term goes to 0 with θ. Also by Young's inequality we have $\|G^\nu_{t-s} * \phi(s)\|_p \leq \sup_{\phi\in\Gamma_T} \|\phi\|_{0,1,T}(t-s)^{\frac{1}{p}-1}$, from which we obtain the following estimate:

$$\sup_{\phi\in\Gamma_T} \| \int_0^\cdot G^\nu_{\cdot-s} * \phi(s) ds\|^\sharp_{0,p,\theta} \leq C(\Gamma_T)\theta.$$

The function of θ defined by

$$\delta(\theta, p, \Gamma, \Gamma_T) = \sup_{\theta'\leq\theta} \left\{ \sup_{\psi\in\Gamma} \|G^\nu_\cdot * \psi\|^\sharp_{0,p,\theta'} + \sup_{\phi\in\Gamma_T} \| \int_0^\cdot G^\nu_{\cdot-s} * \phi(s) ds\|^\sharp_{0,p,\theta'} \right\}$$

has the required properties. □

Lemma 5.3 *Let $\Gamma \subseteq L^1$ and $\Gamma_T \subseteq F_{0,1,T}$ respectively be a pre-compact set and a bounded set, and assume that moreover there is $p \in [\frac{4}{3}, 2)$ such that $\Gamma \subseteq L^p(\mathbb{R}^2)$ and $\Gamma_T \subseteq F_{0,p,T}$. For each $\varepsilon \geq 0$, each initial condition $\psi \in \Gamma$ and each external field $\phi \in \Gamma_T$, let $w^{\varepsilon,\psi,\phi}$ be the unique solution in $F_{0,1,T} \cap F_{0,p,T}$ of the associated mollified mild equation (34).*
Then, there exists $T_0 > 0$ such that for all $\theta \leq T_0$,

$$\sup_{\varepsilon\geq 0} \sup_{\psi\in\Gamma,\phi\in\Gamma_T} \|w^{\varepsilon,\psi,\phi}\|^\sharp_{0,p,\theta} \leq 2\delta(\theta, p, \Gamma, \Gamma_T).$$

Proof: Consider $t \leq \theta \leq T$ and proceed as in Lemma 4.3 *iii)*, with $r = r' = p$ to get

$$\begin{aligned} \|\mathcal{B}^\varepsilon(w^{\varepsilon,\psi,\phi}, w^{\varepsilon,\psi,\phi})_t\|_p \leq & C \int_0^t (t-s)^{-\frac{1}{p}} \|w^{\varepsilon,\psi,\phi}_s\|_p^2 \, ds \\ \leq & C(\|w^{\varepsilon,\psi,\phi}\|^\sharp_{0,p,\theta})^2 \int_0^t (t-s)^{-\frac{1}{p}} s^{\frac{2}{p}-2} ds \\ = & C_p t^{\frac{1}{p}-1} (\|w^{\varepsilon,\psi,\phi}\|^\sharp_{0,p,\theta})^2, \text{by Lemma 1.1.} \end{aligned} \tag{53}$$

Consider the real function $f_\theta(s) := C_p s^2 - s + \delta(\theta, p, \Gamma, \Gamma_T)$. By inequalities (53), Lemma 5.2 and the definition of $w^{\varepsilon,\psi,\phi}$, we have for any $\theta \leq T$ that $f_\theta(\|w^{\varepsilon,\psi,\phi}\|_{0,p,\theta}) \geq 0$. Moreover

$$0 \leq f_{\theta'}(\|w^{\varepsilon,\psi,\phi}\|_{0,p,\theta'}) \leq f_\theta(\|w^{\varepsilon,\psi,\phi}\|_{0,p,\theta'}) \text{ for all } \theta' \leq \theta \leq T \tag{54}$$

since $\delta(\cdot, p, \Gamma, \Gamma_T)$ is increasing.
Let T_0 be such that $\delta(T_0, p, \Gamma, \Gamma_T) < \frac{1}{4C_p}$ and $\theta \in (0, T_0]$. Then, $\delta(\theta, p, \Gamma, \Gamma_T) < \frac{1}{4C_p}$ and f_θ has two positive real roots, say $0 < s_1(\theta) < s_2(\theta)$, and it is strictly negative in between. Notice that from the fact that $w^{\varepsilon,\psi,\phi} \in F_{0,p,T}$, the function $\theta' \mapsto \|w^{\varepsilon,\psi,\phi}\|^\sharp_{0,p,\theta'}$ is continuous and goes to 0 when $\theta' \to 0$. Hence, from (54) we must have $\|w^{\varepsilon,\psi,\phi}\|^\sharp_{0,p,\theta'} \leq s_1(\theta)$ for all $\theta' \leq \theta$. In particular,

$$\|w^{\varepsilon,\psi,\phi}\|^\sharp_{0,p,\theta} \leq s_1(\theta) = \frac{1 - \sqrt{1 - 4C_p\delta(\theta, p, \Gamma, \Gamma_T)}}{2C_p} \leq 2\delta(\theta, \Gamma, \Gamma_T).$$

□

Following arguments of Ben-Artzi [2] and Brezis [5], we deduce uniqueness for L^1 data w_0 and g, with help of the previous lemmas, and under an additional continuity assumption.

Proposition 5.4 *Let w_0 and g satisfy assumption $(\mathbf{H_1})$. Then, for each $p \in [\frac{4}{3}, 2)$ and $\varepsilon \geq 0$, equation (34) has at most one solution in the space $C([0,T], L^1) \cap C(]0,T], L^p)$.*

Proof: We prove it for $\varepsilon = 0$, the case $\varepsilon > 0$ being done identically. Let $w \in C([0,T], L^1) \cap C(]0,T], L^p)$ be a mild solution. Recall that, by definition,

$$\int_{[0,T]\times\mathbb{R}^2} |(K * w_s)(x)||w_s(x)| dx\ ds < \infty.$$

Then, the shifted function $w_{r+\cdot}(\cdot) \in C([0,T-r], L^1) \cap C(]0,T-r], L^p)$ solves the mild vortex equation with initial condition w_r and external field $g_{r+\cdot}(\cdot)$. Since $w \in C([0,T], L^1)$, the set

$$\Gamma := \{w_r\}_{r\in[0,\frac{T}{2}]} \subseteq L^1$$

is pre-compact. On the other hand,

$$\Gamma_{T/2} := \{g_{r+\cdot}(\cdot) : [0,T/2] \times \mathbb{R}^2 \to \mathbb{R}\}_{r\in[0,T/2]}$$

is bounded in $F_{0,1,\frac{T}{2}}$. By Lemma 5.3, there is an increasing function that we denote by $\delta(\theta)$ for short, which does not depend on $r \in [0,T/2]$, goes to 0 with θ, and satisfies

$$t^{1-\frac{1}{p}} \|w_{r+t}\|_p \leq \delta(\theta)$$

for small enough θ and all $t \in (0,\theta]$. Letting $r \to 0$, we deduce that

$$\sup_{t\in[0,\theta]} t^{1-\frac{1}{p}} \|w_t\|_p \leq \delta(\theta).$$

Let w' be a second solution and define $\delta'(\theta)$ analogously. Proceeding as in Lemma 5.3, we deduce that

$$|\!|\!|w - w'|\!|\!|^{\sharp}_{0,p,\theta} \leq C(\delta(\theta) + \delta'(\theta))|\!|\!|w - w'|\!|\!|^{\sharp}_{0,p,\theta}$$

and so $|\!|\!|w - w'|\!|\!|^{\sharp}_{0,p,\theta} = 0$ for small enough $\theta > 0$. Hence, $w_{(\theta)}$ and $w'_{(\theta)}$ solve the mild equation in $F_{0,p,T-\theta}$ with same data $w(\theta)$ and $g(\theta + \cdot)$. We conclude from Theorem 4.7 that $w = w'$ in $[0,T]$.

□

In view of the previous result, and in order to have a complete (existence and uniqueness) statement for the mild equation with L^1 data, we will slightly strengthen hypothesis $(\mathbf{H_1})$, assuming

$(\mathbf{H_1'})$:

- $w_0 \in L^1(\mathbb{R}^2)$ and
- $g \in C([0,T], L^1(\mathbb{R}^2))$.

Next lemma will allow us to construct mild solutions with the required continuity property.

Lemma 5.5 *Assume $(\mathbf{H_1'})$ and $(\mathbf{H_p})$ with $p \in [\frac{4}{3}, 2)$. Then, for each $\varepsilon \geq 0$ the unique solution $w^\varepsilon \in F_{0,1,T} \cap F_{0,p,T}$ of the mild equation (34) belongs to $C([0,T], L^p \cap L^1)$*

Proof: First notice that if $\frac{4}{3} \leq p < 2$ and $\frac{2p}{4-p} \leq p' < \frac{p}{2-p}$, we have for any $\varepsilon \geq 0$ that

$$\begin{aligned}\|\mathcal{B}^\varepsilon(v,u)_t - \mathcal{B}^\varepsilon(v,u)_{t'}\|_{p'} \leq & C\left(\|\!|v|\!\|_{0,p,T} + \|\!|u|\!\|_{0,p,T}\right) \int_0^{t\wedge t'} s^{\frac{1}{p'}-\frac{2}{p}} \Big[\|u_{t-s} - u_{t'-s}\|_p \\ & + \|v_{t-s} - v_{t'-s}\|_p \Big] ds \\ & + C\left(\|\!|v|\!\|_{0,p,T} + \|\!|u|\!\|_{0,p,T}\right)^2 [(t\vee t')^{1+\frac{1}{p'}-\frac{2}{p}} - (t\wedge t')^{1+\frac{1}{p'}-\frac{2}{p}}],\end{aligned}$$

and the right hand side goes to 0 when $t \to t'$ by dominated convergence for $v, w \in C([0,T], L^p)$. This and Lemma 4.3 imply the continuity of the operators

$$\mathcal{B}^\varepsilon : (C([0,T], L^p))^2 \to (C([0,T], L^{p'})).$$

On the other hand, by Young's inequality we have for $r \in [1,p]$

$$\begin{aligned}\left\| \int_0^t G^\nu_{t-s} * g_s ds - \int_0^{t'} G^\nu_{t'-s} * g_s \, ds \right\|_r \leq & \int_0^{t\wedge t'} s^{\frac{1}{r}-1} \|g_{t-s} - g_{t'-s}\|_1 \, ds \\ & + \int_{t\wedge t'}^{t\vee t'} s^{\frac{1}{r}-1} \|g_{(t\vee t')-s}\|_1 \, ds,\end{aligned}$$

and so from continuity of $s \mapsto g_s \in L^1$ we deduce that $W_0 \in C([0,T], L^r)$.
Proceeding as in Section **3.4** (using Lemma 4.11 and Proposition 4.12) we deduce a local existence statement for (34) in the space $C([0,T], L^p)$. From uniqueness in $F_{0,p,T}$ for the equation satisfied by $w^\varepsilon(\theta + \cdot)$ we conclude that $w^\varepsilon \in C([0,T], L^p)$.
Finally, repeating the arguments of Theorem 4.7 in the spaces $C([0,T], L^r)$ yields that $w^\varepsilon \in C([0,T], L^1)$.

$\square$

Now we can prove

Theorem 5.6 *Let $p \in [\frac{4}{3}, 2)$ be fixed and w_0 and g be functions satisfying* $(\mathbf{H_1'})$.

a) For each $\varepsilon \geq 0$, there exists a unique solution w^ε to the mild vortex equations (34) in the space

$$C([0,T], L^1) \cap C(]0,T], L^p).$$

This solution also belongs to $C([0,T], L^1) \cap C(]0,T], L^{\frac{4}{3}})$.

In particular, for $\varepsilon = 0$ there exists under $(\mathbf{H_1'})$ a global solution $w = w^0$ to the mild equation (7).

Moreover, we have

$$\sup_{\varepsilon \geq 0} \|\!|w^\varepsilon|\!\|^\sharp_{0,p,\theta} \to 0 \textit{ when } \theta \to 0.$$

b) For all $t \in]0,T]$, we have $w^\varepsilon_t \to w_t$ in L^p, and the following estimate holds:

$$\sup_{t\in[0,T]} \left(t^{\frac{1}{2}} \|w^\varepsilon_t - w_t\|_p \right) \leq C(p,T)\varepsilon^{\frac{2-p}{p}}.$$

Proof: Let w_0^n and g^n be sequences respectively in $L^1 \cap L^p$ and in $C([0,T], L^1 \cap L^p)$, and such that $w_0^n \to w_0$ in L^1 and $g^n \to g$ in $C([0,T], L^1)$ when $n \to \infty$. Observe that

$$\Gamma = \{w_0^n\} \text{ and } \Gamma_T = \{g^n\}$$

satisfy the hypothesis of Lemma 5.3. For each $\varepsilon \geq 0$ and $n \in \mathbb{N}$ we denote by $w^{\varepsilon,n} \in F_{0,p,T}$ the solution of the mild equation with data w_0^n and g^n.

We split the proof of *a)* in several parts. Uniqueness has already been proved in Proposition 5.4.

Convergence of $w^{\varepsilon,n}$ to a mild solution $w^\varepsilon \in F^\sharp_{0,p,T}$
Using standard L^p estimates for $G_t^\nu * w_0^n$ and $G_{t-s}^\nu * g_s^n$, and similar arguments as those in (53), we check that

$$\begin{aligned}\|w_t^{\varepsilon,n} - w_t^{\varepsilon,m}\|_p \leq Ct^{\frac{1}{p}-1}\|w_0^n - w_0^m\|_1 + Ct^{\frac{1}{p}}\|\!|g^m - g^n|\!\|_{0,1,T} \\ + Ct^{\frac{1}{p}-1}(\|\!|w^{\varepsilon,n} - w^{\varepsilon,m}|\!\|^\sharp_{0,p,\theta})^2\end{aligned}$$

for all $t \in [0,\theta]$. Thanks to Lemma 5.3, for θ small enough we have for all $n, m \in \mathbb{N}$

$$\begin{aligned}\|\!|w^{\varepsilon,n} - w^{\varepsilon,m}|\!\|^\sharp_{0,p,\theta} \leq & C\|w_0^n - w_0^m\|_1 + CT\|\!|g^m - g^n|\!\|_{0,1,T} \\ & + C\delta(\theta, p, \Gamma, \Gamma_T)\|\!|w^{\varepsilon,n} - w^{\varepsilon,m}|\!\|^\sharp_{0,p,\theta}\end{aligned}$$

(the constants are independent of ε, n and m). Therefore, for each $\varepsilon \geq 0$, the sequence $w^{\varepsilon,n}$ is Cauchy in the space $F^\sharp_{0,p,T}$ if θ is small enough.
Next, from the mild equation satisfied by the function $w^{\varepsilon,n}_{\theta+\cdot}(\cdot)$ we deduce that

$$\begin{aligned}\|w^{\varepsilon,n}_{\theta+t} - w^{\varepsilon,m}_{\theta+t}\|_p \leq & C\|w^n(\theta) - w^m(\theta)\|_p + C(T)\|\!|g^n - g^m|\!\|_{0,1,T} \\ & + C(T)\int_0^t \|w^{\varepsilon,n}_{\theta+s} - w^{\varepsilon,m}_{\theta+s}\|_p ds\end{aligned}$$

for all $t \in]\theta, T]$. It follows that $\{w^{\varepsilon,n}_{\theta+\cdot}\}_{n\in\mathbb{N}}$ is Cauchy in the space $F_{0,p,T-\theta}$, and consequently $\{w^{\varepsilon,n}\}_{n\in\mathbb{N}}$ converges in $F^\sharp_{0,p,T}$ for each $\varepsilon \geq 0$. We denote by w^ε the limit in $F^\sharp_{0,p,T}$, and set $w = w^0$.
Using continuity of $\mathcal{B}^\varepsilon$ in the space $F^\sharp_{0,p,T}$ (cf. Remark 5.1 *ii)* with $p = p'$) we easily check for each $\varepsilon \geq 0$ that w^ε is a solution of the mild vortex equation (34).

Continuity of $t \mapsto w_t^\varepsilon \in L^1 \cap L^p$ on $]0,T]$
By Lemma 5.5, $t \to w^{\varepsilon,n}_{\theta+t}$ is a continuous L^p-valued function on $[0, T-\theta]$ for each n and $\theta \in (0,T]$. This clearly implies that $w^\varepsilon \in C(]0,T], L^p)$.
To prove that $w^\varepsilon \in C(]0,T], L^1)$, we notice that by similar arguments as in Lemma 5.5, we can establish that $\mathcal{B}^\varepsilon : (F^\sharp_{0,r,T} \cap C(]0,T], L^r))^2 \to F^\sharp_{0,r',T} \cap C(]0,T], L^{r'})$ is continuous when $\frac{4}{3} \leq r \leq p$, $\frac{2r}{4-r} \leq r' < \frac{r}{2-r}$. Indeed, we have for all $t, t' \in]0,T]$ that

$$\begin{aligned}\|\mathcal{B}^\varepsilon(v,u)_t - \mathcal{B}^\varepsilon(v,u)_{t'}\|_{r'} \leq & C\left(\|\!|v|\!\|^\sharp_{0,r,T} + \|\!|u|\!\|^\sharp_{0,r,T}\right)\int_0^{t\wedge t'} s^{\frac{1}{r'}-\frac{2}{r}}(t-s)^{\frac{1}{r}-1} \\ & \left[\|u_{t-s} - u_{t'-s}\|_r + \|v_{t-s} - v_{t'-s}\|_r\right]ds \\ & + C\left(\|\!|v|\!\|^\sharp_{0,r,T} + \|\!|u|\!\|^\sharp_{0,r,T}\right)^2 [(t\vee t')^{\frac{1}{r'}-1} - (t\wedge t')^{\frac{1}{r'}-1}],\end{aligned}$$

which together with Lebesgue's theorem yields the asserted continuity property. On the other hand, it is not hard to check that $W_0 \in C(]0,T],L^r)$ for all $r \in [1,p]$, from where by a standard argument (see the proof of Theorem 4.7) it follows that $w^\varepsilon \in C(]0,T],L^1)$.

Behavior at $\theta = 0$ of $\|\!|w^\varepsilon|\!\|^\sharp_{0,p,\theta}$
¿From Remark 5.1, *ii)* we deduce for each $\frac{4}{3} \leq r \leq p$, $\frac{2r}{4-r} \leq r' < \frac{r}{2-r}$, and $\theta \leq T$ that

$$\|\!|w^\varepsilon|\!\|^\sharp_{0,r',\theta} \leq \|\!|W_0|\!\|^\sharp_{0,r',\theta} + C_{r,r'}(\|\!|w^\varepsilon|\!\|^\sharp_{0,r,\theta})^2,$$

for a constant $C_{r,r'}$ not depending on $\varepsilon \geq 0$. By Lemma 5.2, if furthermore $r' \neq 1$, we obtain for small enough $\theta > 0$ that

$$\|\!|w^\varepsilon|\!\|^\sharp_{0,r',\theta} \leq \delta(\theta, r', \Gamma, \Gamma_T) + C_{r,r'}(\|\!|w^\varepsilon|\!\|^\sharp_{0,r,\theta})^2. \tag{55}$$

Taking $r = r' = p$ and proceeding as in the proof of Lemma 5.3, we conclude that

$$\sup_{\varepsilon \geq 0} \|\!|w^\varepsilon|\!\|^\sharp_{0,p,\theta} \leq 2\delta(\theta, p, \Gamma, \Gamma_T) \tag{56}$$

for small enough θ.
Continuity of $t \mapsto w^\varepsilon(t) \in L^1$ in $t = 0$
We now prove that $w^\varepsilon_t \to w_0$ in L^1 when $t \to 0$. Notice that by (55), if $\|\!|w^\varepsilon|\!\|^\sharp_{0,r,\theta} \to 0$ when $\theta \to 0$, then also $\|\!|w^\varepsilon|\!\|^\sharp_{0,r',\theta} \to 0$. Thus, by an iterative argument using (55), starting from (56) and suitably choosing consequent values of r and r', we deduce that $\|\!|w^\varepsilon|\!\|^\sharp_{0,\frac{4}{3},\theta} \to 0$. Taking in Remark 5.1 *ii)* $p' = 1$ and the value $\frac{4}{3}$ in place of p yields

$$\|w^\varepsilon(t) - w_0\|_1 \leq \|G^\nu_t * w_0 - w_0\|_1 + t\|\!|g|\!\|_{0,1,T} + C(\|\!|w^\varepsilon|\!\|^\sharp_{0,\frac{4}{3},t})^2.$$

Making $t \to 0$ we conclude the asserted convergence.

Finally, it is clear by interpolation that a solution in $C([0,T],L^1) \cap C(]0,T],L^p)$ also belongs to $C([0,T],L^1) \cap C(]0,T],L^{\frac{4}{3}})$

b) Notice that

$$\sup_{\varepsilon \geq 0} \|\!|w^\varepsilon|\!\|^\sharp_{0,2,T} < \infty. \tag{57}$$

This follows by using once Remark 5.1 *ii)* (with $p' = 2$) if $p \in (\frac{4}{3}, 2)$, and using it twice (with some $p' \in (\frac{4}{3}, 2)$ and then with $p'' = 2$) if $p = \frac{4}{3}$. Consequently, taking in Lemma 4.5 $l = p$ and using Young's inequality we obtain, for $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, that

$$\|K_\varepsilon * w^\varepsilon_s - K * w^\varepsilon_s\|_q \leq \|K_\varepsilon - K\|_p \|w^\varepsilon_s\|_2 \leq C s^{-\frac{1}{2}} \varepsilon^{\frac{2-p}{p}} \tag{58}$$

for a constant not depending on $\varepsilon \geq 0$.

By standard estimates and the previous considerations together with (56), we deduce that for $t \leq \theta$

$$
\begin{aligned}
\|w_t^\varepsilon - w_t\|_p \leq & C \int_0^t (t-s)^{-\frac{1}{p}} \|w_s^\varepsilon - w_s\|_p \left(\|w_s^\varepsilon\|_p + \|w_s\|_p\right) ds \\
& + C \int_0^t (t-s)^{-\frac{1}{p}} \|K_\varepsilon * w_s - K * w_s\|_q \left(\|w_s^\varepsilon\|_p + \|w_s\|_p\right) ds \\
\leq & C\delta(\theta, p, \Gamma, \Gamma_T) \int_0^t (t-s)^{-\frac{1}{p}} s^{\frac{1}{p}-1} \|w_s^\varepsilon - w_s\|_p ds \\
& + C\delta(\theta, p, \Gamma, \Gamma_T) \varepsilon^{\frac{2-p}{p}} \int_0^t (t-s)^{-\frac{1}{p}} s^{(-\frac{1}{2})+(\frac{1}{p}-1)} ds \\
\leq & C\delta(\theta, p, \Gamma, \Gamma_T) t^{-\frac{1}{2}} \sup_{s \in [0,\theta]} \left(s^{\frac{1}{2}} \|w_s^\varepsilon - w_s\|_p \right) + C\delta(\theta, p, \Gamma, \Gamma_T) \varepsilon^{\frac{2-p}{p}} t^{-\frac{1}{2}}.
\end{aligned}
$$

In the last step we used that $\int_0^t (t-s)^{-\frac{1}{p}} s^{\frac{1}{p}-\frac{3}{2}} ds = \beta(\delta, \theta) t^{-\frac{1}{2}}$. Therefore,

$$
\sup_{t \in [0,\theta]} \left(t^{\frac{1}{2}} \|w_t^\varepsilon - w_t\|_p \right) \leq C\delta(\theta, p, \Gamma, \Gamma_T) \left[\sup_{t \in [0,\theta]} \left(t^{\frac{1}{2}} \|w_t^\varepsilon - w_t\|_p \right) + \varepsilon^{\frac{2-p}{p}} \right],
$$

and so for some small enough $\theta > 0$ we have

$$
\sup_{t \in [0,\theta]} \left(t^{\frac{1}{2}} \|w_t^\varepsilon - w_t\|_p \right) \leq C\delta(\theta) \varepsilon^{\frac{2-p}{p}}.
$$

Using this L^p estimate for $w_\theta^\varepsilon - w_\theta$, the mild equations satisfied by $w_{\theta+\cdot}^\varepsilon$ and $w_{\theta+\cdot}$, and similar arguments as in Proposition 4.6, we deduce that

$$
\|\!|w_{\theta+\cdot}^\varepsilon - w_{\theta+\cdot}\|\!|_{0,p,T-\theta} \leq C\varepsilon^{\frac{2-p}{p}}.
$$

The two previous estimates prove *b)*.

□

We provide now additional regularity properties.

Theorem 5.7 *Under assumption* $(\mathbf{H_1'})$*, for each* $p \in [\frac{4}{3}, 2)$ *we have*

$$
\sup_{\varepsilon \geq 0} \|\!|w^\varepsilon\|\!|_{1,p,T}^\sharp < \infty
$$

Proof: The proof is similar as in the case of $L^1 \cap L^p$ data, with help of a local existence result. There are however important differences.
First, we need to slightly modify some lines of Proposition 4.12, since the estimates estimates valid under $(\mathbf{H_p})$ do no longer hold. For data $\hat{w}_0 \in L^p$ and $\hat{g} \in F_{0,1,\hat{T}}$, $\hat{T} > 0$, we have by Hölder's inequality the estimate

$$
\|\!|\hat{W}_0\|\!|_{1,p,\hat{T}} \leq C'(p) \left(\|\hat{w}_0\|_p + \hat{T}^{\frac{1}{p}} \|\!|\hat{g}\|\!|_{0,1,\hat{T}} \right)
$$

(with the obvious meaning of $\hat{W}_0$) for some constant $C'(p) > 0$. From this and from Lemma 4.11, we deduce that a solution $\hat{w}^\varepsilon \in F_{1,p,\theta}$ to the mild vortex equation exists, as soon as $\theta \in (0, \hat{T}]$ satisfies

$$
\theta^{1-\frac{1}{p}} \left(\|\hat{w}_0\|_p + \hat{T}^{\frac{1}{p}} \|\!|\hat{g}\|\!|_{0,1,\hat{T}} \right) < \lambda_p', \tag{59}
$$

for certain constant $\lambda_p' > 0$ independent of $\varepsilon \geq 0$. This local solution $\hat{w}^\varepsilon$ satisfies

$$\|\hat{w}^\varepsilon\|_{1,p,\theta} \leq 2C'(p)\left(\|\hat{w}_0\|_p + \hat{T}^{\frac{1}{p}}\|\hat{g}\|_{0,1,\hat{T}}\right). \tag{60}$$

Recall now that for each $r > 0$ the function $w^\varepsilon(r + \cdot)$ solves on $[0, T-r]$ the mild equation with data $\hat{w}_0 = w_r^\varepsilon$ and $\hat{g} = g_{r+\cdot}$. Since now $w^\varepsilon \notin F_{0,p,T}$, we cannot expect (59) to hold for some $\theta > 0$ uniformly in the initial conditions $\hat{w}_0 = w_r^\varepsilon$, $r \in [0,T]$. Nevertheless, Theorem 5.6 *a)* implies that $\sup_{\varepsilon\geq 0}\theta^{1-\frac{1}{p}}\|w_\theta^\varepsilon\|_p \to 0$ when $\to 0$, so that there is $\theta_0 > 0$ small enough such that for all $\theta \in]0,\theta_0]$,

$$\sup_{\varepsilon\geq 0}\theta^{1-\frac{1}{p}}\left(\|w_\theta^\varepsilon\|_p + T^{\frac{1}{p}}\|g\|_{0,1,T}\right) < \lambda_p'.$$

Consequently, by the previous existence argument there is a solution $\hat{w}^\varepsilon \in F_{1,p,\theta}$ for the data $\hat{w}_0 = w_\theta^\varepsilon$ and $\hat{g} = g_{\theta+\cdot}$ to the same equation satisfied by $w_{\theta+\cdot}^\varepsilon$ in $F_{0,p,T-\theta}$. Using uniqueness in $F_{0,p,\theta\wedge(T-\theta)}$ and estimate (60), we deduce that

$$\|w_{\theta+\cdot}^\varepsilon(\cdot)\|_{1,p,\theta} \leq 2C'(p)\left(\|w_\theta^\varepsilon\|_p + T^{\frac{1}{p}}\|g\|_{0,1,T}\right).$$

It follows that for each $s \in [0,\theta]$

$$s^{\frac{1}{2}}\|\nabla w_{\theta+s}^\varepsilon\|_p \leq C(\|w_\theta^\varepsilon\|_p + T^{\frac{1}{p}}\|g\|_{0,1,T})$$

with a constant C not depending on $\varepsilon \geq 0$. This yields for any $\theta \in (0,\theta_0]$

$$\begin{aligned}\theta^{\frac{3}{2}-\frac{1}{p}}\|\nabla w_\theta^\varepsilon\|_p \leq& C(\theta/2)^{1-\frac{1}{p}}(\theta/2)^{\frac{1}{2}}\|\nabla w_{\theta/2+\theta/2}^\varepsilon\|_p\\ \leq& C(\theta/2)^{1-\frac{1}{p}}(\|w_{\theta/2}^\varepsilon\|_p + T^{\frac{1}{p}}\|g\|_{0,1,T})\\ \leq& C(\|w^\varepsilon\|_{0,p,T}^\sharp + T\|g\|_{0,1,T}).\end{aligned}$$

We deduce that $\sup_{\varepsilon\geq 0}\|w^\varepsilon\|_{1,p,\theta_0}^\sharp < \infty$.
To obtain an upper bound in the whole interval $[0,T]$, notice that for any $r \in [\theta_0/2, T]$ and $\theta > 0$, we have

$$\theta^{1-\frac{1}{p}}\|w_r^\varepsilon\|_p \leq \left(\frac{2\theta}{\theta_0}\right)^{1-\frac{1}{p}}\left(\sup_{\varepsilon\geq 0}\|w^\varepsilon\|_{0,p,T}^\sharp\right) \leq C\theta^{1-\frac{1}{p}}.$$

Therefore, there exists $\theta_1 > 0$ such that for all $r \in [\theta_0/2, T]$

$$\sup_{\varepsilon\geq 0}\theta_1^{1-\frac{1}{p}}\left(\|w_r^\varepsilon\|_p + T^{\frac{1}{p}}\|g\|_{0,1,T}\right) < \lambda_p'.$$

We deduce as before, that for all such r it holds

$$\|w_{r+\cdot}^\varepsilon(\cdot)\|_{1,p,\theta_1} \leq 2C'(p)\left(\|w_r^\varepsilon\|_p + T^{\frac{1}{p}}\|g\|_{0,1,T}\right).$$

We can now proceed as in the last part of the proof of Theorem 4.13 *i)* (i.e. by suitably splitting the interval $[\theta/2, T]$ and applying the previous local estimate), and conclude that

$$\sup_{\varepsilon\geq 0}\|w_{\theta/2+\cdot}^\varepsilon(\cdot)\|_{1,p,T-\theta/2} < \infty.$$

This fact achieves the proof.

□

By similar arguments as in Corollary 4.14 *i)*, we deduce the proof of

Corollary 5.8 *Assume that* $(\mathbf{H_1'})$ *holds and let* w^ε *with* $\varepsilon \geq 0$ *be the unique solution to the mild equation (34) in* $C([0,T],L^1) \cap C(]0,T],L^p)$, $p \in [\frac{4}{3},2)$ *given by Theorem 5.6. We have*

$$\sup_{\varepsilon\geq 0}\sup_{t\in[0,T]}\left\{t^{\frac{3}{2}-\frac{1}{p}}\left(\|K_\varepsilon * w_t^\varepsilon\|_\infty + \|K_\varepsilon * w_t^\varepsilon\|_{\mathcal{C}^{2-\frac{2}{p}}}\right)\right\} < \infty.$$

5.2 The nonlinear process and particle approximations

We now proceed to prove, under assumption $(\mathbf{H_1'})$, convergence of the mollified nonlinear processes, and existence and uniqueness for the nonlinear martingale problem (**MP**).

Definition 5.9 *For* $p \in [\frac{4}{3},2)$*, we denote by* $\mathcal{P}'_{p,T}$ *the space of probability measures on* $C_T = [0,T]\times C([0,T],\mathbb{R}^2)$ *such that for each* $t\in[0,T]$*, the signed measure* $\tilde{P}_t$ *has a density* $\tilde{\rho}_t$ *with respect to the Lebesgue measure and* $\tilde{\rho} \in C([0,T],L^1)\cap C(]0,T],L^p)\cap F^\sharp_{0,p,T}$.

Theorem 5.10 *Assume* $(\mathbf{H_1'})$.

a) *For each* $p \in [\frac{4}{3},2)$*, there exists in the class* $\mathcal{P}'_{p,T}$ *a unique solution* P *to the nonlinear martingale problem* (**MP**)*. The corresponding function* $\tilde{\rho}$ *is equal to the unique solution* $w \in C([0,T],L^1)\cap C(]0,T],L^p)\cap F^\sharp_{0,p,T}$ *of the mild equation (7).*

b) *The solution* $P \in \mathcal{P}'_{p,T}$ *is the limit in law when* $\varepsilon\to 0$ *of the laws* P_ε *of the mollified processes* (X^ε).

Proof: We proceed in several steps.
Uniqueness. Let $P\in\mathcal{P}'_{p,T}$ be a solution of (**MP**). Since $\frac{4}{3}\leq p<2$, for $f:\mathbb{R}^2\to\mathbb{R}$ the interpolation inequality $\|f\|_{\frac{4}{3}}^{\frac{4}{3}} \leq \|f\|_1 + \|f\|_p^p$ holds (cf. $|f|^{\frac{4}{3}} \leq |f|\mathbf{1}_{|f|\leq 1} + |f|^p\mathbf{1}_{|f|>1}$). Taking $f = t\tilde{\rho}_t$ and multiplying by t^{-1}, we deduce that $\tilde{\rho}\in F^\sharp_{0,\frac{4}{3},T}$. Therefore, as for (41), we obtain

$$\int_{[0,T]\times\mathbb{R}^2} |K*\tilde{\rho}_t(x)||\tilde{\rho}_t(x)|dxdt < \infty.$$

Also by standard arguments we deduce that $\tilde{\rho}$ is a mild solution of (7) in the space $C([0,T],L^1)\cap C(]0,T],L^{\frac{4}{3}})\cap F^\sharp_{0,\frac{4}{3},T}$. Consequently, if P^1 and P^2 are two solutions, the associated functions $\tilde{\rho}^1$ and $\tilde{\rho}^2$ are equal by Theorem 5.6. We set $w = \tilde{\rho}^2 = \tilde{\rho}^1$.
Let us now define a family $(\hat{P}^i_t)_{t\in[0,T]}$ of sub-probability measures $\hat{P}^i_t$ on $\mathbb{R}^2$ by

$$\int_{\mathbb{R}^2} f(x)\hat{P}^i_t(dx) = E^{P^i}\left(f(X_t)\mathbf{1}_{\{\tau\leq t\}}\right) \tag{61}$$

with (τ,X) the canonical process. Notice that the drift coefficient is not bounded, so it is not immediate whether each $\hat{P}^i_t$ has a density. Denote by D_n, $n\in\mathbb{N}\backslash\{0\}$ the shift operator defined in the canonical space $[0,T]\times C([0,T],\mathbb{R}^2)$ by

$$D_n((\tau,X)) = \left((\tau-\frac{1}{n})_+, X_{\cdot+\frac{1}{n}}\right).$$

Under the laws $Q_n^i := P^i \circ D_n^{-1}$, $i = 1, 2$, the canonical variable (τ, X_0) has law $P^i_{\frac{1}{n}}$ given by

$$\int_{\mathbb{R}^2\times[0,T-\frac{1}{n}]} f(t,x)\ P^i_{\frac{1}{n}}(dt,dx) = E^{P^i}\left(f((\tau-\frac{1}{n})_+, X_{\frac{1}{n}})\right)$$
$$= \int_{\frac{1}{n}}^T \int_{\mathbb{R}^2} f(t-\frac{1}{n}, x)P_0(dt,dx) + \int_{\mathbb{R}^2} f(0,x)\hat{P}^i_{\frac{1}{n}}(dx).$$

Then, under Q_n^i, the canonical process (τ, X) solves the martingale problem

- $Q \circ (\tau, X_0)^{-1} = P^i_{\frac{1}{n}}$
- $f(t,X_t) - f(\tau, X_0) - \int_0^t \left[\frac{\partial f}{\partial s}(s,X_s) + \nu\triangle f(s,X_s) + K * w_{s+\frac{1}{n}}(X_s)\nabla f(s,X_s)\right]\mathbf{1}_{s\geq\tau}ds,$ (62)

$0 \leq t \leq T - \frac{1}{n}$, is a continuous Q-martingale for all $f \in \mathcal{C}_b^{1,2}$ w.r.t. the filtration $\mathcal{G}_t = \mathcal{F}_{t+\frac{1}{n}}$.

Notice that we cannot ensure by the moment that the "initial condition " $Q_n^i \circ (\tau, X_0)^{-1}$ is uniquely determined. On the other hand, if that fact is established, we deduce that (**MP**) has a unique solution as follows. First, we remark that the drift coefficient $K * w_{s+\frac{1}{n}}$ is bounded by Corollary 5.8. Then, we can adapt standard results on martingale problems to deduce that $P^1 \circ D_n^{-1} = P^2 \circ D_n^{-1}$ for all $n \in \mathbb{N}$. Since both probability measures converge as $n \to \infty$, respectively to P^1 and P^2, this is enough to conclude that $P^1 = P^2$.
So we proceed to check that $P^1_{\frac{1}{n}} = P^2_{\frac{1}{n}}$, which by (61) is equivalent to $\hat{P}^1_{\frac{1}{n}} = \hat{P}^2_{\frac{1}{n}}$. First we prove that these two probability measures on $\mathbb{R}^2$ have densities, or more generally, that $\hat{P}^i_t, i = 1, 2$ have densities for each $t > 0$.
Observe that for $t > 0$ the indicator function in the definition (61) can be replaced by that of the event $\{\tau < t\}$. Thus, it is not hard to check that for $t > \frac{1}{n}$ it holds that

$$\int f(x)\hat{P}^i_t(dx) = E^{Q_n^i}(f(X_{t-\frac{1}{n}})\mathbf{1}_{\{\tau<t-\frac{1}{n}\}}). \tag{63}$$

On the other hand, since $K * w_{s+\frac{1}{n}}$ is bounded, by a standard argument based on Girsanov's theorem we can check that Q_n^i is absolutely continuous (on $[0, T - \frac{1}{n}] \times C([0, T - \frac{1}{n}], \mathbb{R}^2)$) w.r.t. the law of the process $(\tau, X_0 + \int_0^t \mathbf{1}_{s\geq\tau}dB_s)$, where (τ, X_0) has distribution $P^i_{\frac{1}{n}}$ and B is an independent Brownian motion. From this and (63) it follows that $\hat{P}^i_t(dx)$ has a density (independently of whether $P^i_{\frac{1}{n}}$ does or not). Hence, $\hat{P}^i_t$ has a density for all t.
We denote the density of $\hat{P}^i_t$ by $\hat{\rho}^i_t$. We just have to prove that $\hat{\rho}^1_t = \hat{\rho}^2_t$.
Following similar arguments as in the proof of Lemma 3.9 *ii)*, and using the fact that

$$\int_{[0,T]\times\mathbb{R}^2} |K * w_t(x)|\hat{\rho}^i_t(x)dx\ dt < \int_0^T \|K * w_t\|_\infty dt < \infty$$

by Corollary 5.8, we deduce that

$$\hat{\rho}^i_t(x) = G^\nu_t * \bar{w}_0(x) + \int_0^t G^\nu_{t-s} * \bar{g}_s(x)\ ds + \int_0^t \nabla G^\nu_{t-s} * \left[(K * w_s)\hat{\rho}^i_s\right](x)\ ds$$

for all $t \in [0,T]$, where $\bar{w}_0(x) = \frac{|w_0(x)|}{\|w_0\|_1 + \|g\|_{1,T}}$ and $\bar{g}_s(x) = \frac{|g_s(x)|}{\|w_0\|_1 + \|g\|_{1,T}}$. We take $L^{\frac{4}{3}}$ norm and use the estimate $\|K_\varepsilon * w(s)\hat{\rho}^i_s\|_1 \le C s^{-\frac{3}{4}}$ (following from Corollary 5.8) to get that

$$\|\hat{\rho}^i_t\|_{\frac{4}{3}} \le C t^{-\frac{1}{4}} + C + C \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{3}{4}} ds = C t^{-\frac{1}{4}} + C + C t^{-\frac{1}{2}}.$$

Consequently, we have $\sup_{t\in[0,T]} \left(t^{\frac{1}{2}} \|\hat{\rho}^i_t\|_{\frac{4}{3}} \right) < \infty$, and then, by similar steps as in the proof of Theorem 5.6 *b)*, we obtain

$$\sup_{t\in[0,\theta]} \left(t^{\frac{1}{2}} \|\hat{\rho}^1_t - \hat{\rho}^2_t\|_{\frac{4}{3}} \right) \le C\delta(\theta) \sup_{t\in[0,\theta]} \left(t^{\frac{1}{2}} \|\hat{\rho}^1_t - \hat{\rho}^2_t\|_{\frac{4}{3}} \right)$$

for small enough θ, and $\delta(\theta)$ a function associated to w as in Theorem 5.6 *a)*, satisfying thus $\delta(\theta) \to 0$ when $\theta \to 0$. We conclude that $\hat{\rho}^1_t = \hat{\rho}^2_t$ for small enough t, and then for all t by looking at the equations satisfied by $\hat{\rho}^i_{\frac{1}{n}+t}(x)$ in $F_{0,\frac{4}{3},T-\frac{1}{n}}$. Uniqueness is proved.

Estimates for time-marginal laws of P^ε

Consider $\varepsilon > 0$ and let $\tilde{\rho}^\varepsilon$ be the weighted density associated with the law P^ε of the mollified process X^ε, and $\hat{\rho}^\varepsilon$ be the density of $f \mapsto E(f(X^\varepsilon_t)\mathbf{1}_{\{\tau \le t\}})$. For an arbitrary $p \in [\frac{4}{3}, 2)$, we take the L^p norm in the mild equations satisfied by $\tilde{\rho}^\varepsilon$. From the fact that $\sup_{t\in[0,T]} \|(K_\varepsilon * \tilde{\rho}^\varepsilon_t)\tilde{\rho}^\varepsilon_t\|_1 < C(\varepsilon) < \infty$, and using Lemma 2.2 together with Young's inequality, we deduce that

$$\sup_{t\in[0,T]} t^{1-\frac{1}{p}} \|\tilde{\rho}^\varepsilon_t\|_p < \infty.$$

Similarly, starting from the mild equation satisfied by $\hat{\rho}^\varepsilon$,

$$\hat{\rho}^\varepsilon_t(x) = G^\nu_t * \bar{w}_0(x) + \int_0^t G^\nu_{t-s} * \bar{g}_s(x)\, ds + \int_0^t \nabla G^\nu_{t-s} * \left[(K_\varepsilon * \tilde{\rho}^\varepsilon_s)\hat{\rho}^\varepsilon_s\right](x)\, ds, \tag{64}$$

and since $\sup_{t\in[0,T]} \|(K_\varepsilon * \tilde{\rho}^\varepsilon_t)\hat{\rho}^\varepsilon_t\|_1 < C'(\varepsilon) < \infty$, we deduce that

$$\sup_{t\in[0,T]} t^{1-\frac{1}{p}} \|\hat{\rho}^\varepsilon_t\|_p < \infty.$$

By standard arguments, the function $\tilde{\rho}^\varepsilon(t + \cdot) \in F_{0,p,T-t}$ solves the mollified mild equation with data satisfying $(\mathbf{H'_1})$ and $(\mathbf{H_p})$. From Lemma 5.5 we deduce that $\tilde{\rho}^\varepsilon \in C([0,T], L^1) \cap C(]0,T], L^p)$ and therefore, by Theorem 5.6, $\tilde{\rho}^\varepsilon$ equals the unique solution w^ε given therein. In particular, if we define

$$\tilde{\delta}(\theta) := \sup_{\varepsilon>0} \sup_{t\in[0,\theta]} t^{1-\frac{1}{p}} \|\tilde{\rho}^\varepsilon_t\|_p,$$

then $\tilde{\delta}(\theta)$ converges to 0 when θ tends to 0. Moreover, taking L^p norm in (64) and using Remark 5.1 we get

$$|\!|\!|\hat{\rho}^\varepsilon|\!|\!|^\sharp_{0,p,\theta} \le |\!|\!|\bar{W}_0|\!|\!|^\sharp_{0,p,\theta} + C\tilde{\delta}(\theta)|\!|\!|\hat{\rho}^\varepsilon|\!|\!|^\sharp_{0,p,\theta}$$

with $\bar{W}_0$ defined in the natural way in terms of $\bar{w}_0$ anf $\bar{g}$. It follows that

$$|\!|\!|\hat{\rho}^\varepsilon|\!|\!|^\sharp_{0,p,\theta_0} \le 2|\!|\!|\bar{W}_0|\!|\!|^\sharp_{0,p,\theta_0} \tag{65}$$

for $\theta_0 > 0$ small enough. Since $\sup_{\varepsilon\geq 0} \|\tilde{\rho}^{\varepsilon}_{\theta_0+\cdot}(\cdot)\|_{0,p,T-\theta_0} < \infty$ by Theorem 5.6, by looking at the mild equations satisfied by the functions $\hat{\rho}^{\varepsilon}_{\theta_0+\cdot}$, $\varepsilon \geq 0$, which have initial conditions $\hat{\rho}^{\varepsilon}_{\theta_0}(x)$ that are bounded in L^p uniformly in ε by (65), we conclude that

$$\|\hat{\rho}^{\varepsilon}\|^{\sharp}_{0,p,T} < \infty.$$

(This estimate will be used below in the particular case $p = \frac{4}{3}$).

Tightness of the family (P_ε)**:** From Corollary 5.8, if $0 < \eta < 1$ and S, R are stopping times in the filtration of (τ, X^ε) such that $S \leq R \leq T$ and $R - S \leq \eta$, we have

$$\int_S^R |K_\varepsilon * \tilde{\rho}^{\varepsilon}_t(X^{\varepsilon}_t)| dt < C\eta^{\frac{1}{p}-\frac{1}{2}}$$

for a constant $C > 0$ independent of $\varepsilon > 0$. Tightness follows from Aldous criterion ($p < 2$).

Identification of accumulation points as solutions of (MP)

Let P be an accumulation point. By suitably approximating the function h by continuous functions (cf. [12]), one can check that $\int_{\mathbb{R}^2} \psi(x)\tilde{\rho}^{\varepsilon}_t(x)dx = E(\psi(X^{\varepsilon}_t)h(\tau, X_0)\mathbf{1}_{\{t\geq\tau\}})$ converges to $E^P(\psi(X_t)h(\tau, X_0)\mathbf{1}_{\{t\geq\tau\}})$ when ε tends to 0 for every $\psi \in \mathcal{D}$. Consequently, since $\tilde{\rho}^\varepsilon = w^\varepsilon$, we have by Theorem 5.6 *b)* that

$$\tilde{P}_t(dx) = w_t(x)dx,$$

with w the unique solution of the mild vortex equation in $C([0,T], L^1) \cap C(]0,T], L^p)$.
Let us take $f \in C^{1,2}_b$, $0 \leq s_1 \leq \cdots \leq s_m \leq s < t \leq T$ and $\lambda : [0,T] \times \mathbb{R}^{2m} \to \mathbb{R}$ a continuous bounded function. To show that P is a solution of (**MP**), it is enough to prove that

$$E^P\Bigg[\Bigg(\int_s^t \left\{\frac{\partial f}{\partial r}(r, X_r) + \nu\Delta f(r, X_r) + K * w_r(X_r)\nabla f(r, X_r)\right\} \mathbf{1}_{\{r\geq\tau\}} dr \\ + f(t, X_t) - f(s, X_s)\Bigg) \times \lambda(\tau, X_{s_1}, \ldots, X_{s_m})\Bigg] = 0, \quad (66)$$

with (τ, X) being the canonical process. Define a function $\kappa : [0,T] \times C([0,T], \mathbb{R}^2) \to \mathbb{R}$ by

$$\kappa(\theta, \xi) = \Bigg(\int_s^t \left\{\frac{\partial f}{\partial r}(r, \xi(r)) + \nu\Delta f(r, \xi(r)) + K * w_r(\xi(r))\nabla f(r, \xi(r))\right\} \mathbf{1}_{\{r\geq\theta\}}\, dr \\ + f(t, \xi(t)) - f(s, \xi(s))\Bigg) \times \lambda(\theta, \xi(s_1), \ldots, \xi(s_m)). \quad (67)$$

Thanks to Corollary 5.8, κ is continuous and bounded, and consequently,

$$E^P(\kappa(\tau, X)) = \lim_{\varepsilon'\to 0} E(\kappa(\tau, X^{\varepsilon'}))$$

for $P^{\varepsilon'}$ the subsequence converging to P. We conclude by showing that this limit is 0.
From the martingale problem satisfied by $P^{\varepsilon'}$ we deduce that

$$\begin{aligned}\left|E(\kappa(\tau, X^{\varepsilon'}))\right| &\leq CE\left[\int_0^T \left|K_{\varepsilon'} * \tilde{\rho}^{\varepsilon'}_s(X^{\varepsilon'}_s) - K * w_s(X^{\varepsilon'}_s)\right| \mathbf{1}_{\{s\geq\tau\}} ds\right] \\ &\leq C\int_0^T \int_{\mathbb{R}^2} \left|K_{\varepsilon'} * \tilde{\rho}^{\varepsilon'}_s(x) - K * w_s(x)\right| \hat{\rho}^{\varepsilon'}_s(x) dx\, ds \\ &\leq C\int_0^T \left(\|K_{\varepsilon'} * \tilde{\rho}^{\varepsilon'}_s - K_{\varepsilon'} * w_s\|_4 + \|K_{\varepsilon'} * w_s - K * w_s\|_4\right) \|\hat{\rho}^{\varepsilon'}(s)\|_{\frac{4}{3}}\, ds\end{aligned}$$

by Hölder's inequality. By Lemma 4.1 with $p = \frac{4}{3}$ and $q = 4$, Young's inequality and Lemma 4.5 with $l = \frac{4}{3}$, the latter is bounded above by

$$C \sup_{\varepsilon>0} \|\hat{\rho}^{\varepsilon'}\|_{0,\frac{4}{3},T} \int_0^T s^{-\frac{1}{4}} \left[\|\tilde{\rho}_s^{\varepsilon'} - w_s\|_{\frac{4}{3}} + \|K - K_{\varepsilon'}\|_{\frac{4}{3}} \|w_s\|_2\right] ds$$

$$\leq C \int_0^T s^{-\frac{1}{4}} \left[\|\tilde{\rho}_s^{\varepsilon'} - w_s\|_{\frac{4}{3}} + \|K - K_{\varepsilon'}\|_{\frac{4}{3}} \|w_s\|_2\right] ds$$

the last inequality owing to (58) and to Theorem 5.6 *b)*. Thus, we have

$$\left|E(\kappa(\tau, X^{\varepsilon'}))\right| \leq C(\varepsilon')^{\frac{1}{2}} \int_0^T s^{-\frac{1}{4}} s^{-\frac{1}{2}} ds = CT^{\frac{1}{4}} (\varepsilon')^{\frac{1}{2}}$$

which tends to 0. This concludes the proof.

□

We finally deduce the following approximation results in the L^1 setting.

Corollary 5.11 *Let $T > 0$ and assume that $(\mathbf{H_1'})$ holds. Consider a sequence $\varepsilon_n \to 0$ and the system of particles $(Z^{in})_{n\in\mathbb{N}, i=1\ldots n}$ defined as in Corollary 4.17.*
Let $p \in [\frac{4}{3}, 2)$ and $P \in \mathcal{P'}_{p,T}$ be the law of the nonlinear processes associated with the unique solution $w \in C([0,T], L^1) \cap C(]0,T], L^p)$ of the vortex equation. Then, for each $k \in \mathbb{N}$,

$$law(Z^{1n}, \ldots, Z^{kn}) \Longrightarrow P^{\otimes k} \quad \text{when } n \to \infty.$$

Proposition 5.12 *Assume that $(\mathbf{H_1'})$ holds, and recall that $u(t,x) = K * w_t(x)$. Let the weighted empirical process $\tilde{\mu}_s^{n,\varepsilon_n}$ and the sequence ε_n be defined as in Corollary 4.18. Then, the sequence*

$$\sup_{x\in\mathbb{R}^2} E\big(|K_{\varepsilon_n} * \tilde{\mu}_t^{n,\varepsilon_n}(x) - u(t,x)|\big) \to 0$$

for all fixed $t \in]0,T]$, as n tends to infinity, from which we deduce that

$$\int_0^T \sup_{x\in\mathbb{R}^2} E\big(|K_{\varepsilon_n} * \tilde{\mu}_t^{n,\varepsilon_n}(x) - u(t,x)|\big) dt \to 0.$$

Proof: We essentially follow the proof of Corollary 4.18, in which the continuity of the bilinear operator $\mathcal{B}$ in the spaces $F_{1,r,(T;p)}$ is needed. Under the weaker assumption $(\mathbf{H_1'})$, our solution only belongs to $F^{\sharp}_{1,p,T}$ (see Theorem 5.7), and in this setting we are not able to extend the continuity properties of $\mathcal{B}$ to a space analogous to $F_{1,r,(T;p)}$.
We thus consider the family of shifted solutions $\{w^{\varepsilon}_{r+\cdot}\}_{\varepsilon\geq 0}$ for $r \in]0,T]$. By similar computations as in (50) we obtain that

$$\|w^{\varepsilon}_{r+\cdot} - w_{r+\cdot}\|_{1,p,T} \leq C(r,T)\varepsilon^{\frac{2-l}{l}} + C'(r,T) r^{-\frac{1}{2}} \varepsilon^{\frac{2-p}{p}},$$

where the second term in the r.h.s. is due to the difference of the initial conditions in the shifted versions of 49, controlled by Theorem 5.6 *b)*.
From this we deduce that the third term on the r.h.s. of (52) goes to 0 for each fixed $t \in]0,T]$. As in the proof of Corollary 4.18, the expectations of the two other terms are bounded and go to 0 uniformly in $x \in \mathbb{R}^2$ and $t \in [0,T]$. This yields the first convergence result. To apply the dominated convergence theorem we use moreover the fact that $\|K_\varepsilon * w^{\varepsilon}_t\|_\infty \leq C\|w^{\varepsilon}_t\|_{W^{1,p}}$ and Theorem 5.7.

□

6 Numerical results

We consider data w_0 and g satisfying the second moment conditions $\int_{\mathbb{R}^2} |x|^2 |w_0(x)| dx < +\infty$ and $\int_0^T \int_{\mathbb{R}^2} |x|^2 |g(s,x)| dx\ ds < +\infty$. Taking $\phi(s,x) = 1$ and $\phi(s,x) = |x|^2$ in Equation (6), and using the fact that $div\ K * w(s,x) = 0$ for the associated solution w, we can check that for all $t \in [0,T]$,

$$\int_{\mathbb{R}^2} w_t(x)dx = \int_{\mathbb{R}^2} w_0(x)dx + \int_0^t \int_{\mathbb{R}^2} g_s(x)dx\ ds,$$

and

$$\int_{\mathbb{R}^2} |x|^2 w_t(x)dx = \int_{\mathbb{R}^2} |x|^2 w_0(x)dx + \int_0^t \int_{\mathbb{R}^2} |x|^2 g_s(x)dx\ ds + 4\nu \int_0^t \int_{\mathbb{R}^2} w_s(x)dxds.$$

The two previous quantities are respectively called "total flux of vorticity" (TFV) and "moment of fluid impulse" (MFI).
For the simulations, we take an initial condition w_0 equal to the density of the centered normal distribution on $\mathbb{R}^2$, with given variance $m_0 = 2$, and $g(s,x) := \gamma w_0(x)$ with $\gamma \neq 0$ a constant to be fixed. Thus, we have $\|w_0\|_1 + \|g\|_{1,T} = 1 + |\gamma|T$. Notice also that Equation (6) implies that the "barycenter" is null: $\int_{\mathbb{R}^2} x_i w_t(x)dx = \int_{\mathbb{R}^2} x_i w_0(x)dx = 0$ for $i = 1,2$.

Let us now consider the equi-spaced partition $\{t_k\}_{k=0}^N$ of $[0,T]$ in N subintervals. Through simple computations using (8) we obtain for $k = 1,\ldots,N$ that

$$P(\tau = 0) = \frac{1}{1+|\gamma|T} \quad ; \quad P(\tau \in]t_{k-1}, t_k]) = \frac{|\gamma|T}{N(1+|\gamma|T)}$$

and

$$P(X_0 \in dx \big| \tau = 0) = P(X_0 \in dx \big| \tau \in]t_{k-1}, t_k]) = w_0(x)dx.$$

Moreover, we have

$$h(0,x) = 1 + |\gamma|T \text{ and } h(t,x) = sign(\gamma)(1+|\gamma|T) \text{ for } t \in]0,T].$$

Taking as parameter $p = P(\tau = 0)$, we have: $|\gamma| = \frac{1-p}{pT}$, $P(\tau \in]t_{k-1}, t_k]) = \frac{(1-p)}{N}$, $h(0,x) = \frac{1}{p}$ and $h(t,x) = sign(\gamma)\frac{1}{p}$ for $t \in]0,T]$. Hence we obtain

$$\int_{\mathbb{R}^2} w_t(x)dx = 1 + sign(\gamma)\left(\frac{1-p}{pT}\right)t,$$

and

$$m(t) := \int_{\mathbb{R}^2} |x|^2 w_t(x)dx = 2 + \left(2sign(\gamma)\left(\frac{1-p}{pT}\right) + 4\nu\right)t + 2\nu sign(\gamma)\left(\frac{1-p}{pT}\right)t^2.$$

We compute these two quantities at each time t_k using the particle vortex method. Choosing $sign(\gamma) = -1$, we simulate the Euler scheme of the trajectory of each particle $X_t^{i,n} = (X_{t,1}^{i,n}, X_{t,2}^{i,n})$ defined in (18). We thus obtain the data $(X_{t_k,1}^{i,n}, X_{t_k,2}^{i,n})_{\{1\leq i\leq n\,;\,1\leq k\leq N\}}$.

The empirical values of TFV and MFI will be given by

$$\frac{1}{pn}\sum_{i=1}^n \mathbf{1}_{\{t_k \geq \tau^i\}}(\mathbf{1}_{\{\tau^i=0\}} - \mathbf{1}_{\{\tau^i\neq 0\}})$$

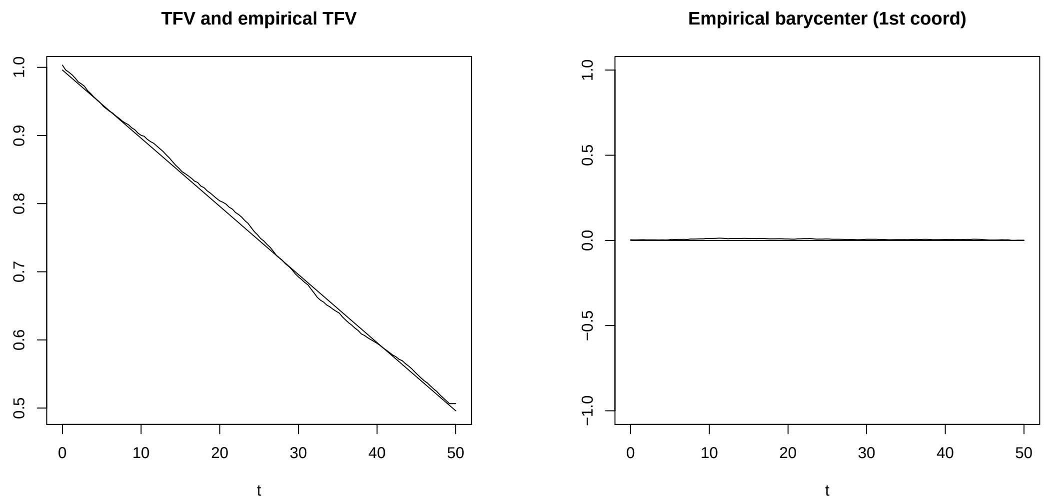

Figure 1: $\varepsilon = 10^{-4}, \nu = 5 \times 10^{-7}, n = 6000, T = 50, \triangle t = 0.8, p = \frac{2}{3}, sign(\gamma) = -1$

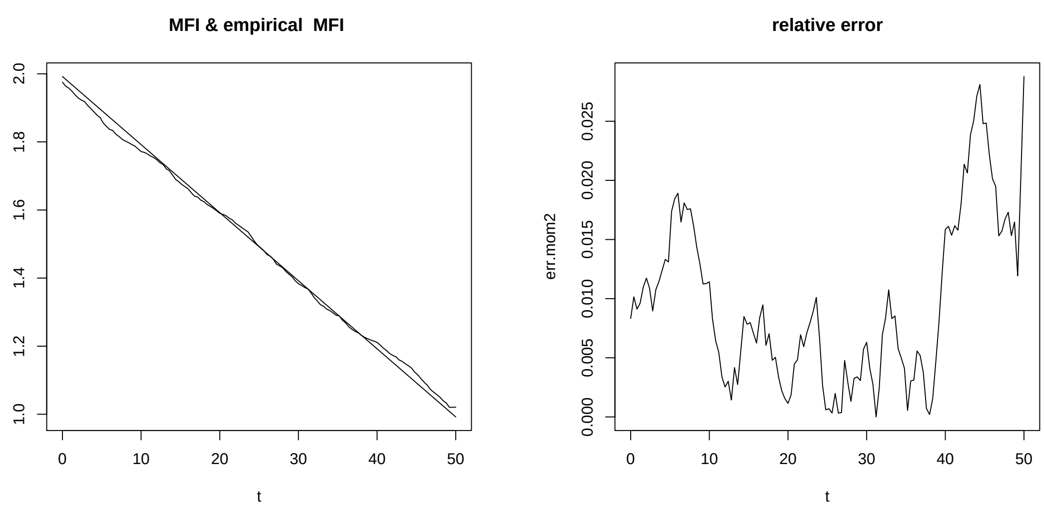

Figure 2: $\varepsilon = 10^{-4}, \nu = 5 \times 10^{-7}, n = 6000, T = 50, \triangle t = 0.8, p = \frac{2}{3}, sign(\gamma) = -1$

and

$$M(t_k) := \frac{1}{pn} \sum_{i=1}^{n} (|X_{t_k,1}^{i,n}|^2 + |X_{t_k,2}^{i,n}|^2) \mathbf{1}_{\{t_k \geq \tau^i\}} (\mathbf{1}_{\{\tau^i = 0\}} - \mathbf{1}_{\{\tau^i \neq 0\}}).$$

Notice that the Total Flux of Vorticity does not depend on particles positions, but only on the number of vortices "alive" at each time and on their "sign" . Therefore, the first graphic in Figure 1. illustrates the law of large numbers for the random time birth τ. The second graphic in Figure 1. illustrates the fact that the barycenter is null.

Figure 2. shows the theoretical and empirical Moment of Fluid Impulse and the relative error, computed as $\frac{|m(t_k) - M(t_k)|}{M(t_k)}$. Let us remark that the probabilistic vortex approach is robust for very small viscosities.

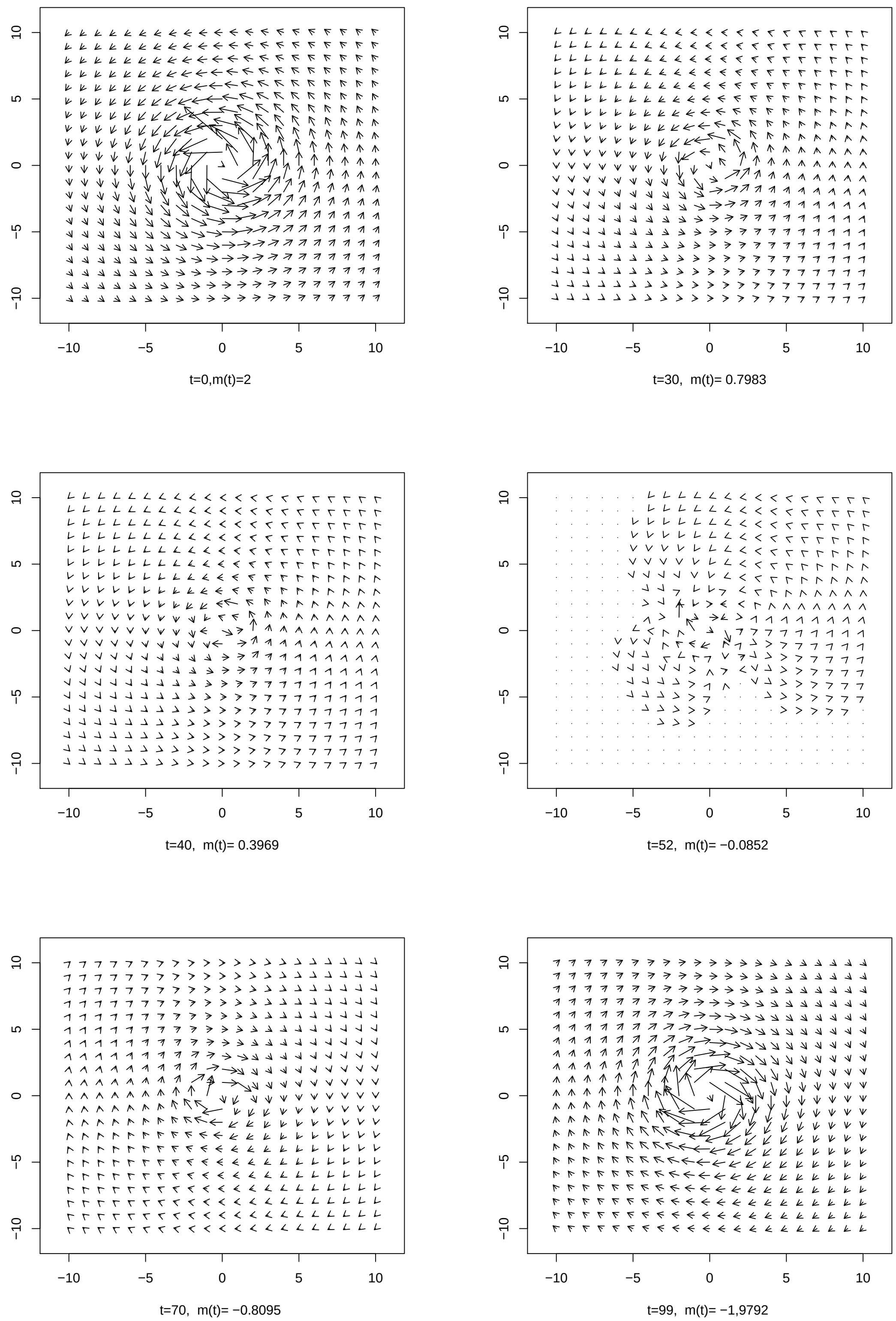

Figure 3: $\varepsilon = 10^{-4},\ \nu = 5 \times 10^{-7},\ n = 1000,\ T = 100,\ \triangle t = 1,\ p = \frac{1}{3},\ sign(\gamma) = -1$

In Figure 3. we show the time evolution of the velocity field in a regular grid. At time $t = 0$ all vortices have positive sign, and then new vortices with negative signs randomly appear. Observe that at each point the norm of the velocity field progressively decreases, attains 0 and then increases, while its direction is progressively reversed.

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References

[1] Bossy, M.: *Vitesse de convergence d'algorithmes particulaires stochastiques et application à l'équation de Burgers*, Thèse Université de Provence (1995).

[2] Ben-Artzi M.: *Global solutions of two-dimensional Navier-Stokes and Euler equations*, Arch. Anal. Rational Mech. **128**, 329-358, (1994).

[3] Bertozzi A., Majda A.: *Vorticity and incompressible flow*, Cambridge University Press, (2002).

[4] Brezis H.: *Analyse fonctionelle*, Masson, Paris, (1983).

[5] Brezis H. *Remarks on the preceding paper by M. Ben-Artzi: "Global solutions of two-dimensional Navier-Stokes and Euler equations" [Arch. Rational Mech. Anal. 128 (1994), no. 4, 329–358].* Arch. Rational Mech. Anal. 128 (1994), no. 4, 359–360.

[6] Busnello B.: *A probabilistic approach to the two dimensional Navier-Stokes equation*, Ann. Probab. **27**, 1750-1780, (1999).

[7] Cannone M.: *Ondelettes, paraproduits et Navier-Stokes*, Diderot Editeur, Paris (1995).

[8] Chorin A.J., Marsden J.E.: *A mathematical introduction to fluid mechanics*, Texts in Applied Mathematics 4, Springer Verlag (1993).

[9] Fontbona J.: *Nonlinear martingale problems involving singular integrals*, J. Funct. Anal. **200**, 198-236 (2003).

[10] Friedman A.: *Partial differential equations of parabolic type*, Prentice-Hall, Inc. (1964).

[11] Jourdain, B.; Méléard, S.: *Propagation of chaos and fluctuations for a moderate model with smooth initial data*, Ann. Inst. Henri Poincaré Probab. 34 no 6, 727-766 (1998).

[12] Jourdain B.: *Diffusion processes associated with nonlinear evolution equations for signed measures*, Methodol. Comput. Appl. Probab. **2**, 69-91, (2000).

[13] Jourdain B., Méléard S.: *Probabilistic interpretation and particle method for vortex equations with Neumann's boundary condition*, Proc. Edinburgh Math. Soc. **47**, 597-624, (2004).

[14] Kunita H.: *Stochastic differential equations and stochastic flows of diffeomorphisms*, Ecole d'été de probabilités de Saint-Flour XII-1982, L.N. in Math. 1097, Springer (1984).

[15] Marchioro C., Pulvirenti, M: *Hydrodynamics in two dimensions and vortex theory*, Commun. Maths. Phys **84**, 483-503 (1982).

[16] Méléard S.: *A trajectorial proof of the vortex method for the 2d Navier-Stokes equation*, Ann. of Appl. Prob. **10**, 1197-1211, (2000).

[17] Méléard S.: *Monte-Carlo approximations of the solution of the 2d Navier-Stokes equation with finite measure initial data*, Prob. Th. Rel. Fields. **121**, 367-388, (2001).

[18] Raviart, P.A.: *An analysis of particle methods*, L.N. in Math. 1127, 243-324, Springer (1985).

[19] Stein E.: *Singular integrals and differentiability properties of functions*, Princeton University Press (1970).

[20] Sznitman A.S.: *Topics in propagation of chaos*, Ecole d'été de probabilités de Saint-Flour XIX-1989, L.N. in Math. 1464, Springer (1991).